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Harmonic generation from laser-driven vacuum

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We investigate the feasibility that, in the field of a superstrong standing laser wave, high-order harmonics of the pumping laser are generated from vacuum. Analytical calculations employing adiabatic perturbation theory show that, for laser electric fields larger than $E_{\mathrm{cr}} = m^2c^3/\hbar e = 1.3 \times 10^{16}$ V/cm, high-order harmonics are generated. The harmonic spectrum shows a wide plateau followed by a cutoff. The cutoff starts approximately at photon energies $\hbar\omega_M \sim \sqrt{\hbar ecE_L}$, with E_L being the amplitude of the laser field. In the opposite limit $E_L \ll E_{\mathrm{cr}}$, the emission of high harmonics is very unlikely. In this case, a feasibility analysis for the experimental observation of the photon-photon scattering process using x-ray free electron lasers shows that the requirements are much less restrictive than those required to observe electron-positron pair creation.

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I. INTRODUCTION

Strong laser radiation is a powerful tool opening exciting perspectives. Available tabletop terawatt and petawatt lasers are employed for laser acceleration of electrons and ions [1], for the creation of new radiation sources in the x-ray and γ-ray domains [2], for the initiation of nuclear reactions [3], as well as for advanced fusion concepts [4]. Moreover, proposals are in a stage to advance laser intensities further [5] even to achieve the fantastic Schwinger field limit of $E_{\mathrm{cr}} = m^2/e = 1.3 \times 10^{16}$ V/cm, with $-e < 0$ and m being the electron charge and mass, respectively (throughout this paper, conventional units with $\hbar = c = 1$ are used if it is not stated otherwise). The "critical" field E_{cr} provides an electron with an energy m in the Compton wavelength $\lambda_c = 1/m$, enabling spontaneous electron-positron pair creation from vacuum and making vacuum unstable [6]. Below the critical field vacuum is stable but still can exhibit nonlinear properties due to the virtual electron-positron pair creation. In the electromagnetic field $(\mathbf{E}_L, \mathbf{B}_L)$ of a plane wave, vacuum is linear because both of the electromagnetic invariants $E_L^2 - B_L^2$ and $\mathbf{E}_L \cdot \mathbf{B}_L$ are zero in this field [6]. The vacuum nonlinearity in the field of two photons, that in the lowest order is the photon-photon scattering, is known since Euler's seminal work [7–9]. Another effect of vacuum polarization is the scattering of a photon in a Coulomb field, and it is known as Delbrück scattering [10]. Since then, a large number of light-by-light scattering processes has been considered. A variety of this process is the photon splitting in a strong external field. The photon splitting process in a strong constant magnetic field was investigated in Ref. [11]. For low photon energies $\omega \ll m$ and arbitrary field strengths, the Euler-Heisenberg (E-H) effective Lagrangian [12] has been used [13,14], while for higher photon energies the exact Green function of the electron in the external magnetic field must be considered [15]. The same process is treated in Ref. [16] by means of the operator diagram technique. Also, the photon splitting process in crossed constant electric and magnetic fields was considered in Ref. [17]. Finally, the same process in a laser field with strength E_L was studied in Ref. [18], employing expansions in the parameters E_L/E_{cr} and ω/m, with ω the initial photon energy. When the strong magnetic field is time-dependent, it can create photons directly from vacuum. Thus, photon production in a rotating magnetic field was considered in Ref. [19], employing adiabatic perturbation theory and the E-H effective Lagrangian approach, which is reliable for created photon energies $\omega \ll m$. The photon creation process in a strong electric field is distinguished by the fact of the vacuum instability. Particularly, the production of a photon becomes possible in a constant and uniform electric field when it is accompanied by the electron-positron pair production [15]. Instead, this process is forbidden in the presence only of a constant and uniform magnetic field.

Of course, the cross section of the light-by-light scattering process is much smaller than the photon scattering by atoms. In fact, if E and ω are the typical electric field strength and photon energy in the process, the light-by-light scattering cross section is proportional to some powers (depending on the particular process) of the parameters E/E_{cr} and ω/m that are usually very small in experimental situations. The experimental observation of vacuum polarization effects has been succeeded so far only for the Delbrück scattering of γ rays on a heavy nucleus [20] and for the connected process of the γ-photon splitting in the field of a heavy nucleus [21]. An experiment on the photon-photon scattering at optical energies [22] was able to determine only an upper limit of the cross section with 95% of confidence level. We should also mention the PVLAS project being under way for the experimental investigation of vacuum nonlinearities in a constant magnetic field [23]. Experimental results from the PVLAS

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project about the rotation of light polarization in vacuum have been already reported in Ref. [24].

The present paper considers harmonic generation in vacuum in the strong field of two counterpropagating laser waves: The photon-photon scattering would be the lowest order process in which two laser photons simply interact, giving rise to two other photons with the same frequency but propagating in different directions. Our formalism can be applied for any generated frequencies and also for laser electric fields $E_L \gtrsim E_{\mathrm{cr}}$. The main approximation used is the adiabatic perturbation theory, which is valid when the laser frequency ω_L is much smaller than the characteristic energy m and the characteristic frequency ω of the emitted photons. In the ultrastrong field regime, n photons from each laser wave can be absorbed (merged) to generate two counterpropagating high-energy photons with frequency $\omega_n = n\omega_L$ [25]. In this view, harmonic generation is the inverse process of photon splitting in vacuum. We investigate the discrete spectrum of the vacuum harmonic generation. As we will see, the typical electric field strengths needed to observe the vacuum high-order harmonic generation (VHHG) are larger than the critical field E_{cr}. Though VHHG due to light-light nonlinear interaction and atomic high-order harmonic generation (AHHG) due to atom-light nonlinear interaction are very different in the scale of nonlinearity, nevertheless, as we will see, similar features of AHHG spectra can be traced in VHHG spectra. Namely, VHHG spectra also show a plateaulike structure followed by a cutoff, as in AHHG [27]. The physical interpretation of the cutoff position can be considered analogously to AHHG. In fact, as we will see, VHHG is due to the absorption of a large number of laser photons by an electron-positron virtual pair and the successive emission of two high-energy photons through the pair annihilation, and, analogously to AHHG, the cutoff position corresponds to the maximal energy that the virtual pair can acquire in the laser field before the "rescattering" annihilation.

In the presence of such strong electric fields $E_L \gtrsim E_{\mathrm{cr}}$ required for VHHG, spontaneous electron-positron pair creation from vacuum takes place, and, from this point of view, we always assume that somehow the intensity of the laser field is kept constant. As we have mentioned, in the presence of an electric field, there is an additional channel for the photon production different from VHHG: the creation of a photon simultaneously with an electron-positron pair [15]. The photon spectrum via this channel is continuous in contrast to the discrete VHHG spectrum. Moreover, in this process one photon is generated, while in VHHG a pair of correlated photons is created. Therefore, the VHHG spectrum, in which our concerns lie, is distinguishable from the photon spectrum generated simultaneously with the electron-positron pair production.

In the weak laser field regime $E_L \ll E_{\mathrm{cr}}$, which still corresponds to the electric fields of the strongest available lasers, the VHHG spectra have perturbative nature. We analyze in this regime the feasibility of the experimental observation of the scattering of two counterpropagating laser beams using x-ray-free-electron lasers and optical lasers.

The paper is organized as follows. In the next two sections, we will describe the theoretical approach we have followed to calculate the VHHG spectra. In Sec. IV, we will use the E-H effective Lagrangian approach to deal with the low-photon energy region ($\omega \ll m$) of the spectrum: Both the cases of weak ($E_L \ll E_{\mathrm{cr}}$) and strong ($E_L \gg E_{\mathrm{cr}}$) laser fields are considered. In Sec. V, the general approach is followed to deal with larger harmonic orders in the strong field regime. In Sec. VI, we study qualitatively the other process leading to one photon plus one electron-positron pair production in the presence of a strong constant and uniform electric field. Finally, in Sec. VII, we summarize our conclusions, and in the two appendixes, we give the details of some calculations that would have made the main text heavier.

II. THEORETICAL MODEL

We consider the interaction of two strong counterpropagating laser beams in vacuum. The two monochromatic waves with the same amplitude $E_L/2$ and the same frequency ω_L propagate along the x axis. By indicating the resulting electromagnetic field as $(\mathbf{E}(x,t), \mathbf{B}(x,t))$, then

$$\mathbf{E}(x,t) = \left[\frac{E_L}{2}\cos(k_L x - \omega_L t) + \frac{E_L}{2}\cos(k_L x + \omega_L t)\right]\hat{\mathbf{z}} = E_L \cos k_L x \cos\omega_L t\hat{\mathbf{z}}, \tag{2.1}$$

$$\mathbf{B}(x,t) = \left[-\frac{E_L}{2}\cos(k_L x - \omega_L t) + \frac{E_L}{2}\cos(k_L x + \omega_L t)\right]\hat{\mathbf{y}} = -E_L \sin k_L x \sin\omega_L t\hat{\mathbf{y}}, \tag{2.2}$$

with $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ the units vectors in the y and z directions, respectively. In general, we are interested in the yield of photons with frequency ω much larger than ω_L. These photons are created in a volume with a typical length $\lambda = 2\pi/\omega$ much smaller than $\lambda_L = 2\pi/\omega_L$, and, if we imagine that the photon production process takes place around the x-axis origin, then $|k_L x| \ll 1$ in Eqs. (2.1) and (2.2). At the zero order we can write these equations as

$$\mathbf{E}\,(x,t) \simeq E_L \cos\omega_L t\hat{\mathbf{z}} \equiv E_L(t)\hat{\mathbf{z}}, \tag{2.3}$$

$$\mathbf{B}\,(x,t) \simeq \mathbf{0}. \tag{2.4}$$

Another consequence of the previous approximation is that the electric field $E_L(t)$ changes slowly with time with respect to the field describing the photons with which we want to deal. Now we have to write a suitable Lagrangian density to describe the process in which a certain number of laser photons merge to give at least two higher energy photons. In general, this Lagrangian density has to account for the interaction between a strong classical electromagnetic field of the laser waves and a quantized radiation field of the emerging photons. We indicate with $A^\mu(x)$ and $\mathcal{A}^\mu(x)$ the four-potentials describing the strong classical field and the radiation field, respectively [28]. In Appendix A, we show that, by starting from the one-loop effective action of the total electromagnetic field $A^\mu(x) + \mathcal{A}^\mu(x)$ and by neglecting the self-interactions of the radiation field, it is possible to obtain a quadratic effective action of the radiation field itself that takes into account exactly its interaction with the classical field. We are interested in frequencies of the radiation field much larger than the frequency ω_L of the electric field $E_L(t)$, and we will take into account the time dependence of $E_L(t)$ via adiabatic perturbation theory. Consequently, the zero-order effective Lagrangian density of the radiation field is assumed to contain the strong classical field as a constant and uniform electric field E_L. Then the corresponding effective Lagrangian density of the radiation electromagnetic field $\mathcal{F}^{\mu\nu}(x) = \partial^\mu \mathcal{A}^\nu(x) - \partial^\nu \mathcal{A}^\mu(x)$ is given by:

$$\mathcal{L}(x;\rho_L) = i\,\mathrm{Im}(\mathcal{L}^{(\mathrm{E-H})}(\rho_L)) + \mathcal{L}_M(x) + \delta\mathcal{L}(x;\rho_L), \tag{2.5}$$

where $\rho_L = E_L/E_{\mathrm{cr}}$ and where

$$\mathrm{Im}\,(\mathcal{L}^{(\mathrm{E-H})}(\rho_L)) = \frac{m^4}{(2\pi)^3}\rho_L^2 \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n\pi/\rho_L}, \tag{2.6}$$

$$\mathcal{L}_M(x) = -\frac{1}{4}\mathcal{F}^{\mu\nu}(x)\mathcal{F}_{\mu\nu}(x), \tag{2.7}$$

$$\delta\mathcal{L}(x;\rho_L) = -\frac{1}{2}\int dx' \mathcal{A}_\mu(x)\Pi^{\mu\nu}(x-x';\rho_L)\mathcal{A}_\nu(x'), \tag{2.8}$$

with $\Pi^{\mu\nu}(x-x';\rho_L)$ the photon polarization tensor in the presence of the external electric field $E_L = \rho_L E_{\mathrm{cr}}$. Before giving the explicit expression of the photon polarization tensor, we make some observations about Eqs. (2.5), (2.6), (2.7), and (2.8). First, it is worth stressing that we have not made any assumption concerning the frequencies of the photon field with which we can deal. When in treating nonlinear QED effects one starts from the E-H Lagrangian density [12], which is valid for constant and uniform fields, then one is forced to consider frequencies much less than the electron mass. As a consequence, our interaction Lagrangian density (2.8) is, unlike the E-H Lagrangian density, nonlocal in space-time. In Sec. IV, we will show that for photon energies much less than m our results reduce to those obtained by starting from the E-H Lagrangian density. Second, the term $i\,\mathrm{Im}(\mathcal{L}^{(\mathrm{E-H})}(\rho_L))$ in Eq. (2.5) results from the one-loop E-H Lagrangian of the constant field E_L [13]. It is a constant term but, since it is imaginary, it cannot be dropped because it has a physical meaning: It takes into account the fact that, in the presence of the electric field E_L, spontaneous pair production from vacuum occurs.

We can now give the expression of the photon polarization tensor $\Pi^{\mu\nu}(x-x';\rho_L)$ in Eq. (2.8), and we refer to the notation used in Ref. [14] [see also the original papers [29,30]]. As we have pointed out, since the external field is constant and uniform, the polarization tensor depends only on the 4-coordinates difference $x - x'$; then by putting

$$\Pi^{\mu\nu}(x-x';\rho_L) = \int \frac{dk}{(2\pi)^4} e^{ik(x-x')}\Pi^{\mu\nu}(k;\rho_L), \tag{2.9}$$

it can be shown that $\Pi^{\mu\nu}(k;\rho_L)$ can be written in the form [13]

$$\Pi^{\mu\nu}(k;\rho_L) = c(k;\rho_L)P^{\mu\nu}(k) + c_\perp(k;\rho_L)P_\perp^{\mu\nu}(k) + c_\parallel(k;\rho_L)P_\parallel^{\mu\nu}(k). \tag{2.10}$$

In this expression, we have defined the four-dimensional matrices

$$P^{\mu\nu}(k) = g^{\mu\nu}k^2 - k^\mu k^\nu, \tag{2.11}$$

$$P_\perp^{\mu\nu}(k) = g_\perp^{\mu\nu}k_\perp^2 - k_\perp^\mu k_\perp^\nu, \tag{2.12}$$

$$P_\parallel^{\mu\nu}(k) = g_\parallel^{\mu\nu}k_\parallel^2 - k_\parallel^\mu k_\parallel^\nu, \tag{2.13}$$

with

$$g^{\mu\nu} = \mathrm{diag}(-1,1,1,1), \qquad k^\mu = (k^0,k^1,k^2,k^3), \tag{2.14}$$

$$g_\parallel^{\mu\nu} = \mathrm{diag}(-1,0,0,1), \qquad k_\parallel^\mu = (k^0,0,0,k^3), \tag{2.15}$$

$$g_\perp^{\mu\nu} = \mathrm{diag}(0,1,1,0), \qquad k_\perp^\mu = (0,k^1,k^2,0), \tag{2.16}$$

and the coefficients

$$c(k;\rho_L) = \frac{\alpha}{2\pi}\int_0^\infty \frac{ds}{s}\int_{-1}^1 \frac{d\nu}{2}[e^{-i(s/\rho_L)\phi(s,\nu,k)}F(s,\nu) - e^{-is/\rho_L}(1-\nu^2)], \tag{2.17}$$

$$c_\perp(k;\rho_L) = \frac{\alpha}{2\pi}\int_0^\infty \frac{ds}{s}\int_{-1}^1 \frac{d\nu}{2} e^{-i(s/\rho_L)\phi(s,\nu,k)}F_\perp(s,\nu), \tag{2.18}$$

$$c_\parallel(k;\rho_L) = \frac{\alpha}{2\pi}\int_0^\infty \frac{ds}{s}\int_{-1}^1 \frac{d\nu}{2} e^{-i(s/\rho_L)\phi(s,\nu,k)}F_\parallel(s,\nu), \tag{2.19}$$

with $\alpha = e^2/4\pi$ the fine-structure constant, and

$$\phi(s,\nu,k) = 1 + \frac{k_\perp^2}{2m^2}\frac{1-\nu^2}{2} + \frac{k_\parallel^2}{2m^2}\frac{\cosh s - \cosh \nu s}{s\sinh s}, \tag{2.20}$$

$$F(s,\nu) = \frac{s}{\sinh s}(\cosh\nu s - \nu\sinh\nu s\coth s), \tag{2.21}$$

$$F_{\perp}(s,\nu)=\frac{s}{\sinh s}[(1-\nu^2)\cosh s-\cosh\nu s+\nu\sinh\nu s\coth s], \tag{2.22}$$

$$F_{\parallel}(s,\nu)=\frac{s}{\sinh s}\left(2\frac{\cosh s-\cosh\nu s}{\sinh^2 s}-\cosh\nu s+\nu\sinh\nu s\coth s\right). \tag{2.23}$$

Before going on and introducing the time dependence in the electric field, we also express the interaction Lagrangian density (2.8) like the Maxwell Lagrangian density in terms of the electromagnetic field $\mathcal{F}^{\mu\nu}(x)$ instead of the four-potential $\mathcal{A}^{\mu}(x)$. Also, this will render it easier to recover the results in the low-energy limit, where the E-H effective Lagrangian holds, because it is also expressed in terms of the electromagnetic field. Now we observe that the action corresponding to the interaction Lagrangian density (2.8) is given by

$$\delta\Gamma[A;\rho_L]\equiv\int dx\delta\mathcal{L}(x;\rho_L)=-\frac{1}{2}\int dxdx'\mathcal{A}_{\mu}(x)\Pi^{\mu\nu}(x-x';\rho_L)\mathcal{A}_{\nu}(x'). \tag{2.24}$$

By performing two integrations by parts, for example, in the term containing $P^{\mu\nu}(k)$ [see Eq. (2.10)], we obtain

$$\int dxdx'e^{ik(x-x')}\mathcal{A}_{\mu}(x)P^{\mu\nu}(k)\mathcal{A}_{\nu}(x')=\frac{1}{2}\int dxdx'e^{ik(x-x')}\mathcal{F}_{\mu\nu}(x)\mathcal{F}^{\mu\nu}(x'). \tag{2.25}$$

By using the same argument for the terms containing $P^{\mu\nu}_{\perp}(k)$ and $P^{\mu\nu}_{\parallel}(k)$, it is easy to show that the action in Eq. (2.24) is completely equivalent to the following one (we use the same symbol to denote it):

$$\delta\Gamma[\mathcal{F};\rho_L]=-\frac{1}{4}\int dxdx'\mathcal{F}_{\mu\nu}(x)\Pi^{\mu\nu\alpha\beta}(x-x';\rho_L)\mathcal{F}_{\alpha\beta}(x'), \tag{2.26}$$

where

$$\Pi^{\mu\nu\alpha\beta}(x-x';\rho_L)=\int\frac{dk}{(2\pi)^4}e^{ik(x-x')}\Pi^{\mu\nu\alpha\beta}(k;\rho_L) \tag{2.27}$$

and

$$\Pi^{\mu\nu\alpha\beta}(k;\rho_L)=c(k;\rho_L)P^{\mu\nu\alpha\beta}+c_{\perp}(k;\rho_L)P^{\mu\nu\alpha\beta}_{\perp}+c_{\parallel}(k;\rho_L)P^{\mu\nu\alpha\beta}_{\parallel}, \tag{2.28}$$

with

$$P^{\mu\nu\alpha\beta}=g^{\mu\alpha}g^{\nu\beta}, \tag{2.29}$$

$$P^{\mu\nu\alpha\beta}_{\perp}=g^{\mu\alpha}_{\perp}g^{\nu\beta}_{\perp}, \tag{2.30}$$

$$P^{\mu\nu\alpha\beta}_{\parallel}=g^{\mu\alpha}_{\parallel}g^{\nu\beta}_{\parallel}. \tag{2.31}$$

Now we can introduce the time dependence of the external field and we will do that in the following section.

III. CALCULATION OF THE SPECTRUM

As we have said, to take into account the time dependence of the external electric field, we apply adiabatic perturbation theory. We have to be careful as our Lagrangian density corresponding to the action (2.26) is nonlocal in space-time. The Lagrangian density has two characteristic time scales. One corresponds to the virtual pair creation; it is of order $1/m$ and it is expressed via the quantity $t-t'$ in the argument of the polarization operator. Another time scale is the period of the slowly varying external electric field expressed via the time dependence of the field $\rho_L(t)=E_L(t)/E_{\mathrm{cr}}$. If we want to consider created photons with arbitrarily high frequencies, then we have to take into account exactly the dependence of the Lagrangian on the first time scale. Meanwhile, the slow variation of the external field, i.e. $\omega_L\ll\omega$, m, allows the application of the adiabatic approximation on the second time scale [31,32]. Thus, we proceed by introducing the Fourier transform of the radiation field

$$\mathcal{F}_{\mu\nu}(x)=\int\frac{dk}{(2\pi)^4}e^{ikx}\mathcal{F}_{\mu\nu}(k) \tag{3.1}$$

and by making the change of variable $x_{\pm}=(x\pm x')/2$. Then Eq. (2.26) reads

$$\begin{aligned}\delta\Gamma[\mathcal{F};\rho_L]&=-4\int dx_-dx_+\frac{dk_1}{(2\pi)^4}\frac{dk_2}{(2\pi)^4}e^{ik_1(x_++x_-)}\mathcal{F}_{\mu\nu}(k_1)\Pi^{\mu\nu\alpha\beta}(2x_-;\rho_L)\mathcal{F}_{\alpha\beta}(k_2)e^{ik_2(x_+-x_-)}\\&=-\frac{1}{4}\int dx\frac{dk_1}{(2\pi)^4}\frac{dk_2}{(2\pi)^4}e^{i(k_1+k_2)x}\mathcal{F}_{\mu\nu}(k_1)\Pi^{\mu\nu\alpha\beta}((k_2-k_1)/2;\rho_L)\mathcal{F}_{\alpha\beta}(k_2).\end{aligned} \tag{3.2}$$

In this way, from the relation $\delta\Gamma[\mathcal{F};\rho_L]\equiv\int dx\delta\mathcal{L}(x;\rho_L)$, we obtain the following expression of the interaction Lagrangian density which is local in space-time but not in momentum space:

$$\delta\mathcal{L}(x;\rho_L) = -\frac{1}{4}\int \frac{dk_1}{(2\pi)^4}\frac{dk_2}{(2\pi)^4} e^{i(k_1+k_2)x} \times \mathcal{F}_{\mu\nu}(k_1)\Pi^{\mu\nu\alpha\beta}((k_2-k_1)/2;\rho_L)\mathcal{F}_{\alpha\beta}(k_2). \tag{3.3}$$

The Hamiltonian density arising from the total Lagrangian density (2.5) with $\delta\mathcal{L}(x;\rho_L)$ given by the previous equation has to be calculated by means of the usual Legendre transformation. If we restrict to laser amplitudes such that $E_L \ll (\pi/\alpha)E_{\mathrm{cr}} \simeq 5.6\times 10^{18}$ V/cm, it can be shown that the interaction Lagrangian density $\delta\mathcal{L}(x;\rho_L)$ can be treated as a small perturbation of the Maxwell Lagrangian density $\mathcal{L}_M(x)$ [14]. In this not very restrictive assumption, the corresponding Hamiltonian density is given by

$$\mathcal{H}(x;\rho_L) = -i\,\mathrm{Im}(L^{(\mathrm{E-H})}(\rho_L)) + \frac{1}{2}(\mathcal{E}^2+\mathcal{B}^2) - \delta\mathcal{L}(x;\rho_L). \tag{3.4}$$

Also, the amplitude of the production from vacuum of two photons in the states $\gamma \equiv (\mathbf{k},\lambda)$ and $\gamma' \equiv (\mathbf{k}',\lambda')$ in the presence of the oscillating field $E_L(t) = E_L\cos\omega_L t$ is given up to first order both in the laser frequency ω_L and in $\delta\mathcal{L}(x;|\rho_L(t)|)$ by

$$A_P(\gamma,\gamma';\rho_L) = -e^{-N(\rho_L)/2}\int dx \frac{1}{\omega+\omega'} \times \langle\gamma\gamma'|[\partial_{\rho_L}\delta\mathcal{L}(x;|\rho_L(t)|)]|0\rangle\dot{\rho}_L(t)e^{i(\omega+\omega')t}, \tag{3.5}$$

where $\omega = |\mathbf{k}|$ and $\omega' = |\mathbf{k}'|$ and where the absolute value of $\rho_L(t)$ reminds us that the polarization tensor depends on the *amplitude* of the external electric field [14]. In the previous equation, we have used the fact that, when the external electric field E_L in Eq. (3.4) is assumed to be time-depending, then $\delta\mathcal{L}(x;|E_L(t)|)$ is the only operator that depends explicitly on time and that can account for the transition from the vacuum to the two photons state. Therefore, in the present approximation, up to first order in $\delta\mathcal{L}(x;|\rho_L(t)|)$, all the other quantities in the amplitude (3.5) can be calculated by quantizing the photon field starting from the zero-order Lagrangian density $i\,\mathrm{Im}(\mathcal{L}^{(\mathrm{E-H})}(|\rho_L(t)|)) + \mathcal{L}_M(x)$ [see Eqs. (2.6), (2.7), and (2.8)]. In this respect, the presence of the term $i\,\mathrm{Im}(\mathcal{L}^{(\mathrm{E-H})}(|\rho_L(t)|))$ in the Lagrangian density results in the exponential $\exp(-N(\rho_L)/2)$, where $N(\rho_L)$ is given by [see Eq. (2.6)]

$$\begin{aligned} N(\rho_L) &= 2\int dx\,\mathrm{Im}(\mathcal{L}^{(\mathrm{E-H})}(|\rho_L(t)|)) \\ &= V\frac{m^4}{4\pi^3}\rho_L^2\sum_{n=1}^{\infty}\frac{1}{n^2}\int_{-T/2}^{T/2}dt\cos^2\omega_L t e^{-n\pi/\rho_L\cos\omega_L t} \\ &= 4VT\left(\frac{m}{2\pi}\right)^4\rho_L^2 \\ &\quad\times\sum_{n=1}^{\infty}\frac{1}{n^2}\int_{-\pi/2}^{\pi/2}d\eta\cos^2\eta e^{-n\pi/\rho_L\cos\eta}, \end{aligned} \tag{3.6}$$

with VT the space-time volume occupied by the laser electric field [see discussion at the end of Sec. IV A]. In fact, in the presence of an external electric field, the vacuum state becomes unstable and $\exp(-N(\rho_L))$ corresponds exactly to the vacuum-vacuum persistence probability as given in Ref. [15] for constant fields. For what it concerns the process under consideration, this implies that many new transition channels open like the creation of the two photons and one electron-positron pair, the creation of the two photons and two electron-positron pairs, and so on. In this respect, Eq. (3.5) has to be interpreted as the amplitude that two photons with quantum numbers γ and γ' are created *without* any electron-positron pair. By indicating the corresponding probability as $W(\gamma,\gamma',0)$, it is expressed as the product of the vacuum persistence probability $W(0) = \exp(-N(\rho_L))$ and the probability of the photon production $W(\gamma,\gamma') \equiv W(\gamma,\gamma',0)/W(0)$. Likewise, one can deduce that the probability $W(\gamma,\gamma',1)$ of the second process mentioned, i.e. the creation of the two photons and one electron-positron pair, will be expressed as a product of the probability $W(1)$ of one pair creation and the probability of the two photon production: $W(\gamma,\gamma',1) \simeq W(\gamma,\gamma')\times W(1)$. Since the total probability of the pair production must be one, that is, $\sum_n W(n) = 1$, then $W(\gamma,\gamma')$ can be interpreted as the *total* probability of two photon production irrespective of the number of created electron-positron pairs. This probability is given by the square modulus of the amplitude (3.5) without the exponential factor, that is,

$$A(\gamma,\gamma';\rho_L) = -\int dx\frac{1}{\omega+\omega'} \times\langle\gamma\gamma'|[\partial_{\rho_L}\delta\mathcal{L}(x;\rho_L(t))]|0\rangle\dot{\rho}_L(t)e^{i(\omega+\omega')t}. \tag{3.7}$$

Now if the external field vanishes both in the remote past and in the remote future, then

$$A(\gamma,\gamma';\rho_L) = i\int dx\langle\gamma\gamma'|\delta\mathcal{L}(x;|\rho_L(t)|)|0\rangle e^{i(\omega+\omega')t}. \tag{3.8}$$

By using the inverse Fourier transform of the radiation field [see Eq. (3.1)], it is easy to write the amplitude (3.8) as

$$\begin{aligned} A(\gamma,\gamma';\rho_L) = -\frac{i}{4}\int dx dx' e^{i(\omega+\omega')\bar{t}} &\times\langle\gamma\gamma'|\mathcal{F}_{\mu\nu}(x)\Pi^{\mu\nu\alpha\beta}(x-x';|\rho_L(\bar{t})|) \\ &\times\mathcal{F}_{\alpha\beta}(x')|0\rangle, \end{aligned} \tag{3.9}$$

with $\bar{t} = (t+t')/2$ the mean instant between t and t'. By quantizing the electromagnetic field in a finite volume V, it can be shown that

$$\begin{aligned}A(\gamma,\gamma';\rho_L) &= \frac{i}{2}\int dxdx'\frac{(e_\gamma)_\mu}{\sqrt{2V\omega}}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega\bar{t})}\int\frac{dp}{(2\pi)^4}e^{ip(x-x')}[c(p,|\rho_L(\bar{t})|)(g^{\mu\nu}k_\beta k'^\beta - k^\nu k'^\mu)\\ &\quad + c_\perp(p,|\rho_L(\bar{t})|)(g_\perp^{\mu\nu}k_{\perp\beta}k_\perp'^\beta - k_\perp^\nu k_\perp'^\mu) + c_\parallel(p,|\rho_L(\bar{t})|)(g_\parallel^{\mu\nu}k_{\parallel\beta}k_\parallel'^\beta - k_\parallel^\nu k_\parallel'^\mu)]\frac{(e_{\gamma'})_\nu}{\sqrt{2V\omega}}e^{-i(\mathbf{k}'\cdot\mathbf{r}'-\omega'\bar{t})} + \gamma\leftrightarrow\gamma'\\ &= \frac{i}{2}\frac{\delta_{\mathbf{k},-\mathbf{k}'}}{2\omega}\int_{-\infty}^{\infty}dte^{2i\omega t}(e_{\mathbf{k},\lambda})_\mu[c(\tilde{k},|\rho_L(t)|)(g^{\mu\nu}k_\beta k'^\beta - k^\nu k'^\mu) + c_\perp(\tilde{k},|\rho_L(t)|)(g_\perp^{\mu\nu}k_{\perp\beta}k_\perp'^\beta - k_\perp^\nu k_\perp'^\mu)\\ &\quad + c_\parallel(\tilde{k},|\rho_L(t)|)(g_\parallel^{\mu\nu}k_{\parallel\beta}k_\parallel'^\beta - k_\parallel^\nu k_\parallel'^\mu)](e_{-\mathbf{k},\lambda'})_\nu + \gamma\leftrightarrow\gamma', \end{aligned} \tag{3.10}$$

where $(e_\gamma)^\mu$ and $(e_{\gamma'})^\mu$ are the polarizations of the two produced photons, and $\tilde{k}^\mu = (k^\mu - k'^\mu)/2 = (0,\mathbf{k})$ is the difference of their four-momenta $k^\mu = (\omega,\mathbf{k})$ and $k'^\mu = (\omega,-\mathbf{k})$ divided by two. In fact, since the external electric field is uniform, the two emitted photons have equal and opposite momenta $\mathbf{k}$ and $\mathbf{k}' = -\mathbf{k}$ and then the same energies $\omega = \omega'$. Finally, the notation $+\gamma\leftrightarrow\gamma'$ indicates that the same expression on its left has to be summed but with γ and γ' exchanged.

By choosing the vector $\mathbf{k}$ as $\mathbf{k} = \omega(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ with ϑ and φ the polar angles and the polarizations of the photons as $(e_{\mathbf{k},\lambda})^\mu = (0,\hat{\mathbf{e}}_{\mathbf{k},\lambda})$ with

$$\hat{\mathbf{e}}_{\mathbf{k},\perp} = \frac{\hat{\mathbf{z}}\times\mathbf{k}}{|\hat{\mathbf{z}}\times\mathbf{k}|} = (-\sin\varphi, \cos\varphi, 0), \tag{3.11}$$

$$\hat{\mathbf{e}}_{\mathbf{k},\parallel} = \frac{\mathbf{k}\times\hat{\mathbf{e}}_{\mathbf{k},\perp}}{|\mathbf{k}\times\hat{\mathbf{e}}_{\mathbf{k},\perp}|} = (-\cos\vartheta\cos\varphi, -\cos\vartheta\sin\varphi, \sin\vartheta), \tag{3.12}$$

one sees that the two photons have to be created with the same polarization. In particular, they can be produced both with the electric field perpendicular to the plane determined by the two vectors $\hat{\mathbf{z}}$ (where the laser electric field lies) and $\mathbf{k}$ or with the electric field in this plane. These two modes are called "perpendicular" and "parallel," respectively, and the two relative production amplitudes are given by

$$\begin{aligned}A_\perp(\mathbf{k},\mathbf{k}';\beta_L,\rho_L) &= \frac{i}{2}\omega\delta_{\mathbf{k},-\mathbf{k}'}\int_{-\infty}^{\infty}dte^{2i\omega t}[2c(\tilde{k},|\rho_L(t)|)\\ &\quad + \sin^2\vartheta c_\perp(\tilde{k},|\rho_L(t)|)],\end{aligned} \tag{3.13}$$

$$\begin{aligned}A_\parallel(\mathbf{k},\mathbf{k}';\beta_L,\rho_L) &= -\frac{i}{2}\omega\delta_{\mathbf{k},-\mathbf{k}'}\int_{-\infty}^{\infty}dte^{2i\omega t}[2c(\tilde{k},|\rho_L(t)|)\\ &\quad + \sin^2\vartheta c_\parallel(\tilde{k},|\rho_L(t)|)],\end{aligned} \tag{3.14}$$

where we have pointed out also the dependence on the laser frequency ω_L [which is actually hidden in $\rho_L(t) = \rho_L\cos\omega_L t$] through the adimensional variable $\beta_L = \omega_L/m$. Also, the total production probability is obtained by integrating on the photon momenta, summing on the polarizations, and dividing by two to take into account that the created photons are identical particles:

$$\begin{aligned}P(\beta_L,\rho_L) &= \frac{1}{2}\left[\frac{V}{(2\pi)^3}\right]^2\int d\mathbf{k}d\mathbf{k}'[|A_\parallel(\mathbf{k},\mathbf{k}';\beta_L,\rho_L)|^2\\ &\quad + |A_\perp(\mathbf{k},\mathbf{k}';\beta_L,\rho_L)|^2].\end{aligned} \tag{3.15}$$

By recalling that the external electric field is periodic in time, we can expand the coefficients $c(\tilde{k},|\rho_L(t)|)$, $c_\perp(\tilde{k},|\rho_L(t)|)$, and $c_\parallel(\tilde{k},|\rho_L(t)|)$ in Fourier series. By doing that, it is easy to show that the total *number* of photons produced per unit volume and unit time can be written as

$$\frac{dN(\beta_L,\rho_L)}{dVdt} = \sum_{q=1}^{\infty}\frac{dN_q(\beta_L,\rho_L)}{dVdt}, \tag{3.16}$$

where

$$\begin{aligned}\frac{dN_q(\beta_L,\rho_L)}{dVdt} &= \frac{(qm\beta_L)^4}{8\pi}\int_0^{\pi/2}d\vartheta\sin\vartheta[|2c_{2q}(\vartheta;\beta_L,\rho_L)\\ &\quad + \sin^2\vartheta c_{\perp,2q}(\vartheta;\beta_L,\rho_L)|^2 + |2c_{2q}(\vartheta;\beta_L,\rho_L)\\ &\quad + \sin^2\vartheta c_{\parallel,2q}(\vartheta;\beta_L,\rho_L)|^2]\end{aligned} \tag{3.17}$$

is the number of photons produced per unit volume and unit time with frequency $\omega_q = q\omega_L$. In this expression, we have introduced the Fourier coefficients

$$c_q(\vartheta;\beta_L,\rho_L) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}d\eta e^{iq\eta}c(\vartheta,q\beta_L;\rho_L\cos\eta), \tag{3.18}$$

$$c_{\perp,q}(\vartheta;\beta_L,\rho_L) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}d\eta e^{iq\eta}c_\perp(\vartheta,q\beta_L;\rho_L\cos\eta), \tag{3.19}$$

$$c_{\parallel,q}(\vartheta;\beta_L,\rho_L) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}d\eta e^{iq\eta}c_\parallel(\vartheta,q\beta_L;\rho_L\cos\eta), \tag{3.20}$$

where $c(\vartheta,q\beta_L;\rho_L)$, $c_\perp(\vartheta,q\beta_L;\rho_L)$, and $c_\parallel(\vartheta,q\beta_L;\rho_L)$ are determined by Eqs. (2.17), (2.18), and (2.19). For later convenience, we introduce the function $h(s,\nu,\vartheta)$ defined as

$$\begin{aligned}h(s,\nu,\vartheta) &= -is\bigg(\sin^2\vartheta\frac{1-\nu^2}{4} + \frac{\cos^2\vartheta}{2}\\ &\quad \times\frac{\cosh s - \cosh\nu s}{s\sinh s}\bigg),\end{aligned} \tag{3.21}$$

and we observe that [see Eqs. (2.17), (2.18), (2.19), and (2.20)]

$$-i\frac{s}{\rho_L}\phi(s,\nu,\tilde{k})\to -i\frac{s}{\rho_L}+\frac{\beta^2}{\rho_L}h(s,\nu,\vartheta), \tag{3.22}$$

with $\beta=\omega/m$. It is worth writing an equivalent expression of the coefficient (2.17) because, even using the prescription $m^2\to m^2-i\varepsilon$, the integral $\int_0^\infty \frac{ds}{s}e^{-is/\rho_L}$ diverges logarithmically in the limit $s\to 0$. Now, if $m^2\to m^2-i\varepsilon$, we can introduce the exponential integral function [33]

$$\mathrm{Ei}\left(-\frac{is}{\rho_L}\right)\equiv -\int_s^\infty \frac{ds'}{s'}e^{-is'/\rho_L}. \tag{3.23}$$

By performing an integration by parts in the variable s in Eq. (2.17), we obtain

$$\begin{aligned} c(\vartheta,q\beta_L;\rho_L) &= \frac{\alpha}{2\pi}\lim_{\delta\to 0}\int_{-1}^1\frac{d\nu}{2}\left[\int_\delta^\infty\frac{ds}{s}e^{-is/\rho_L}e^{[(q\beta_L)^2/\rho_L]h(s,\nu,\vartheta)}F(s,\nu)-(1-\nu^2)\int_\delta^\infty\frac{ds}{s}e^{-is/\rho_L}\right]\\ &= -\frac{\alpha}{2\pi}\int_0^\infty ds\mathrm{Ei}\left(-\frac{is}{\rho_L}\right)\int_{-1}^1\frac{d\nu}{2}e^{[(q\beta_L)^2/\rho_L]h(s,\nu,\vartheta)}\left[\frac{(q\beta_L)^2}{\rho_L}F(s,\nu)\frac{\partial h(s,\nu,\vartheta)}{\partial s}+\frac{\partial F(s,\nu)}{\partial s}\right]. \end{aligned} \tag{3.24}$$

Now, as is evident from Eq. (3.16), the produced photons are emitted with a frequency multiple of the laser frequency ω_L. This is due to the fact that in the process only two photons are created and necessarily with the same energy. As a consequence, each final photon has exactly one-half of the energy of all the initial photons absorbed. In this respect, it is worth recalling that because of Furry theorem the number of initial photons absorbed is even [26]. Also, we find that both even and odd harmonics of the laser field are present in VHHG spectra. This is a difference with respect to the typical AHHG spectra that show only odd harmonics. This feature of AHHG spectra essentially originates from the fact that atomic levels have definite parity. Here, from the expression of the interaction Lagrangian density (2.8) and with the help of the last Feynman diagram in Fig. 6, we can conclude that the mechanism leading to the generation of the qth harmonic is the creation of a virtual electron-positron pair, which absorbs $2q$ photons and then annihilates emitting two photons with energy $q\omega_L$. In this picture, there are no selection rules that force the virtual pair to emit only odd harmonics of the pumping laser.

Finally, the previous expression of the total number of photons created is quite general but very unwieldy. For this reason, in the following we will consider different physical situations where the expressions (2.18), (2.19), and (3.24) can be simplified.

IV. LOW FREQUENCIES: EFFECTIVE LAGRANGIAN APPROACH

Let us consider the limit of low-order harmonic generation $q\omega_L\ll m$, i.e. $q\beta_L\ll 1$. Then we can make the approximation

$$e^{-is/\rho_L+[(q\beta_L)^2/\rho_L]h(s,\nu,\vartheta)}\simeq e^{-is/\rho_L} \tag{4.1}$$

in Eqs. (2.17), (2.18), and (2.19). In this approximation, the integral in the variable ν in the coefficients $c_q(\vartheta;\beta_L,\rho_L)$, $c_{\perp,q}(\vartheta;\beta_L,\rho_L)$, and $c_{\parallel,q}(\vartheta;\beta_L,\rho_L)$ can be performed exactly, and Eqs. (3.18), (3.19), and (3.20) can be written as

$$\begin{aligned} c_q(\vartheta;\beta_L,\rho_L) &\simeq \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}d\eta e^{iq\eta}c^{(\mathrm{E-H})}(\rho_L\cos\eta)\\ &\equiv c_q^{(\mathrm{E-H})}(\rho_L), \end{aligned} \tag{4.2}$$

$$\begin{aligned} c_{\perp,q}(\vartheta;\beta_L,\rho_L) &\simeq \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}d\eta e^{iq\eta}c_\perp^{(\mathrm{E-H})}(\rho_L\cos\eta)\\ &\equiv c_{\perp,q}^{(\mathrm{E-H})}(\rho_L), \end{aligned} \tag{4.3}$$

$$\begin{aligned} c_{\parallel,q}(\vartheta;\beta_L,\rho_L) &\simeq \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}d\eta e^{iq\eta}c_\parallel^{(\mathrm{E-H})}(\rho_L\cos\eta)\\ &\equiv c_{\parallel,q}^{(\mathrm{E-H})}(\rho_L), \end{aligned} \tag{4.4}$$

with [see also [34]]

$$\begin{aligned} c^{(\mathrm{E-H})}(\rho_L\cos\eta) &= \frac{\alpha}{2\pi}\int_0^\infty\frac{ds}{s}e^{-is/\rho_L\cos\eta}\\ &\quad\times\left(\frac{\coth s}{s}-\frac{1}{\sinh^2 s}-\frac{2}{3}\right), \end{aligned} \tag{4.5}$$

$$\begin{aligned} c_\perp^{(\mathrm{E-H})}(\rho_L\cos\eta) &= \frac{\alpha}{2\pi}\int_0^\infty\frac{ds}{s}e^{-is/\rho_L\cos\eta}\\ &\quad\times\left[\frac{1}{\sinh^2 s}-\left(\frac{1}{s}-\frac{2}{3}s\right)\coth s\right], \end{aligned} \tag{4.6}$$

$$\begin{aligned} c_\parallel^{(\mathrm{E-H})}(\rho_L\cos\eta) &= \frac{\alpha}{2\pi}\int_0^\infty\frac{ds}{s}e^{-is/\rho_L\cos\eta}\\ &\quad\times\left[\frac{2s\coth s-1}{\sinh^2 s}-\frac{\coth s}{s}\right], \end{aligned} \tag{4.7}$$

independent of both ϑ and β_L. It is not surprising that the same result would have been obtained by starting from the E-H effective Lagrangian density [13]. In fact, the E-H Lagrangian density is rigorously applicable only for constant and uniform electromagnetic fields. Nevertheless, the same Lagrangian density can be used for the total electromagnetic field

$$\mathbf{E}_T(\mathbf{r},t)=\boldsymbol{\mathcal{E}}(\mathbf{r},t)+E_L\hat{\mathbf{z}}, \tag{4.8}$$

$$\mathbf{B}_T(\mathbf{r}, t) = \boldsymbol{\mathcal{B}}(\mathbf{r}, t), \tag{4.9}$$

by assuming that it varies not much in a Compton length; i.e. its typical frequencies are much less than the electron mass. In this way, one starts by writing the E-H Lagrangian density $\mathcal{L}^{(\mathrm{E-H})}(F_T(\mathbf{r}, t), G_T^2(\mathbf{r}, t))$. Being the E-H Lagrangian density a true Lorentz scalar, it can depend only on the two electromagnetic invariants of the total field

$$F_T(\mathbf{r}, t) = \frac{1}{2}[E_T^2(\mathbf{r}, t) - B_T^2(\mathbf{r}, t)], \tag{4.10}$$

$$G_T^2(\mathbf{r}, t) = (\mathbf{E}_T(\mathbf{r}, t) \cdot \mathbf{B}_T(\mathbf{r}, t))^2. \tag{4.11}$$

Finally, by expanding $\mathcal{L}^{(\mathrm{E-H})}(F_T(\mathbf{r}, t), G_T^2(\mathbf{r}, t))$ up to second order in the radiation field $(\boldsymbol{\mathcal{E}}(\mathbf{r}, t), \boldsymbol{\mathcal{B}}(\mathbf{r}, t))$, one finds that the coefficients $c^{(\mathrm{E-H})}(\rho_L)$, $c_\parallel^{(\mathrm{E-H})}(\rho_L)$, and $c_\perp^{(\mathrm{E-H})}(\rho_L)$ can be expressed in terms of the derivatives

$$c^{(\mathrm{E-H})}(\rho_L) = -\frac{\partial\mathcal{L}^{(\mathrm{E-H})}(F_T, G_T^2)}{\partial F_T}\bigg|_{F_T(\mathbf{r},t)=-E_L^2/2, G_T^2(\mathbf{r},t)=0}, \tag{4.12}$$

$$c_\perp^{(\mathrm{E-H})}(\rho_L) = -2E_L^2 \times \frac{\partial\mathcal{L}^{(\mathrm{E-H})}(F_T, G_T^2)}{\partial G_T^2}\bigg|_{F_T(\mathbf{r},t)=-E_L^2/2, G_T^2(\mathbf{r},t)=0}, \tag{4.13}$$

$$c_\parallel^{(\mathrm{E-H})}(\rho_L) = E_L^2\frac{\partial^2\mathcal{L}^{(\mathrm{E-H})}(F_T, G_T^2)}{\partial F_T^2}\bigg|_{F_T(\mathbf{r},t)=-E_L^2/2, G_T^2(\mathbf{r},t)=0}. \tag{4.14}$$

The complete quadratic Lagrangian density, corresponding to the quadratic Lagrangian density (2.5), is given in this case by

$$\begin{aligned}\mathcal{L}^{(\mathrm{E-H})}(x; \rho_L) = {}& i\,\mathrm{Im}(\mathcal{L}^{(\mathrm{E-H})}(\rho_L)) + \mathcal{L}_M(x) \\ &+ \frac{1}{2}[c^{(\mathrm{E-H})}(\rho_L)(\mathcal{E}^2 - \mathcal{B}^2) \\ &- c_\perp^{(\mathrm{E-H})}(\rho_L)(\hat{\mathbf{z}}\cdot\boldsymbol{\mathcal{B}})^2 \\ &+ c_\parallel^{(\mathrm{E-H})}(\rho_L)(\hat{\mathbf{z}}\cdot\boldsymbol{\mathcal{E}})^2],\end{aligned} \tag{4.15}$$

with $\mathrm{Im}(\mathcal{L}^{(\mathrm{E-H})}(\rho_L))$ and $\mathcal{L}_M(x)$ given in Eqs. (2.6) and (2.7), respectively. To obtain the other terms from the general interaction Lagrangian density (2.8), it is easier, actually, to start from the expression (2.26) of the interaction action expressed in terms of the electromagnetic field instead of the four-potential $\mathcal{A}^\mu(x)$.

So far the procedure is similar to that used in Ref. [19] in dealing with an external magnetic field. But in the present case, because of the possible spontaneous vacuum decay, the coefficients (4.5), (4.6), and (4.7) are, in general, complex numbers. For this reason, it would be impossible to proceed as in Ref. [19] by quantizing the photon field in the presence of the external static field E_L and then by applying adiabatic perturbation theory with $E_L \to E_L \cos\omega_L t$. In that case, there would not be any limitation on the strength of the external field. Instead, here, as in the general case treated before, we are forced to consider field strengths less than $(\pi/\alpha)E_{\mathrm{cr}}$. This condition guarantees here that

$$|c^{(\mathrm{E-H})}(\rho_L)|, |c_\perp^{(\mathrm{E-H})}(\rho_L)|, |c_\parallel^{(\mathrm{E-H})}(\rho_L)| \ll 1 \tag{4.16}$$

in such a way that the corresponding terms in Eq. (4.15) can be treated perturbatively and the fact that they are complex can be dealt with [35,36]. Not surprisingly, the final expression of the amplitude of the qth harmonic is [see Eq. (3.17)]

$$\begin{aligned}\frac{dN_q^{(\mathrm{E-H})}(\beta_L, \rho_L)}{dVdt} = {}& \frac{(qm\beta_L)^4}{8\pi}\int_0^{\pi/2} d\vartheta \sin\vartheta[|2c_{2q}^{(\mathrm{E-H})}(\rho_L) \\ &+ \sin^2\vartheta c_{\perp,2q}^{(\mathrm{E-H})}(\rho_L)|^2 \\ &+ |2c_{2q}^{(\mathrm{E-H})}(\rho_L) \\ &+ \sin^2\vartheta c_{\parallel,2q}^{(\mathrm{E-H})}(\rho_L)|^2],\end{aligned} \tag{4.17}$$

with $c_q^{(\mathrm{E-H})}(\rho_L)$, $c_{\perp,q}^{(\mathrm{E-H})}(\rho_L)$, and $c_{\parallel,q}^{(\mathrm{E-H})}(\rho_L)$ given in Eqs. (4.2), (4.3), and (4.4). The physical meaning of the terms contributing to the amplitude (4.17) of the qth harmonic can be understood by looking at the expression of the quadratic effective Lagrangian density (4.15). In fact, the contribution of the coefficient $c_{2q}^{(\mathrm{E-H})}(\rho_L)$ arises because the energies of the photons undergo a correction due to the presence of the laser field, and, in general, this correction is complex due to the fact that the electric field can prime spontaneous pair creation from vacuum. Meanwhile, the other two coefficients $c_{\perp,2q}^{(\mathrm{E-H})}(\rho_L)$ and $c_{\parallel,2q}^{(\mathrm{E-H})}(\rho_L)$ are more directly connected with the production of the photons in the two different perpendicular and parallel polarizations [see Eqs. (3.11) and (3.12)].

We want to recall here that the general expression of the harmonic amplitude (3.17) is calculated by taking into account the nonlocality of the interaction between the external laser field and the radiation field [see Eqs. (2.24) and (3.2)]. Including nonlocality means taking into account also memory effects that introduce into the expression of the harmonic amplitude (3.17) the dependence on the frequency of the emitted harmonic [see the papers [37] for a discussion about memory effects in electron-positron pair production]. In fact, on the one hand, Eq. (3.17) is proportional to ω_L^4, but this dependence originates from the phase space dependence and from the use of the adiabatic perturbation theory. On the other hand, it also depends through the coefficients $c_{2q}(\vartheta; \beta_L, \rho_L)$, $c_{\perp,2q}(\vartheta; \beta_L, \rho_L)$, and $c_{\parallel,2q}(\vartheta; \beta_L, \rho_L)$ on the harmonic frequency $q\omega_L$, and this dependence originates from taking into account memory effects. Instead, since the number of photons expressed by Eq. (4.17) is calculated employing the E-H effective Lagrangian, then the interaction is assumed to have local character, and, consequently, no memory effects are retained in Eq. (4.17). That is expressed by the fact that the coefficients $c_{2q}^{(\mathrm{E-H})}(\rho_L)$, $c_{\perp,2q}^{(\mathrm{E-H})}(\rho_L)$, $c_{\parallel,2q}^{(\mathrm{E-H})}(\rho_L)$ depend only on the harmonic number q and the laser field strength ρ_L but neither on the laser frequency ω_L nor on the harmonic frequency $q\omega_L$. This is analogue to the case of absence of dispersion for a wave propagation in a medium with very fast relaxation time. Concerning these memory effects, it must be also pointed out that both Eq. (4.17) and the general expression (3.17) do not depend on the number of photons already created in the past. In fact, this comes from the use of the *first order* adiabatic perturbation theory in which the transition amplitude at a given time depends on the other amplitudes but calculated at the initial time [32]. Up to second order adiabatic perturbation theory, we would have taken into account the "depletion" of the vacuum due to the photon production itself. In this case, the harmonic amplitude would be less than Eq. (3.17), the correction being $\omega_L/m \ll 1$ smaller than Eq. (3.17).

In the low-frequency limit, we want to study here various representations can be given of the coefficients $c_q^{(\mathrm{E-H})}(\rho_L)$, $c_{\parallel,q}^{(\mathrm{E-H})}(\rho_L)$, and $c_{\perp,q}^{(\mathrm{E-H})}(\rho_L)$ that are more useful depending on the strength of the laser electric field that is on $\rho_L = E_L/E_{\mathrm{cr}}$. Finally, it is worth pointing out that the time-independent counterparts of the coefficients $c^{(\mathrm{E-H})}(\rho_L\cos\eta)$, $c_{\parallel}^{(\mathrm{E-H})}(\rho_L\cos\eta)$, and $c_{\perp}^{(\mathrm{E-H})}(\rho_L\cos\eta)$ [see Eqs. (4.5), (4.6), and (4.7)] have been already calculated [34,38]. The novelty of our approach is to use their analogous time-depending expressions to evaluate via the Fourier transforms $c_q^{(\mathrm{E-H})}(\rho_L)$, $c_{\parallel,q}^{(\mathrm{E-H})}(\rho_L)$, and $c_{\perp,q}^{(\mathrm{E-H})}(\rho_L)$ the photon yield from vacuum.

A. Weak electric fields: $\rho_L \ll 1$

If $\rho_L \ll 1$, the integrals in Eqs. (4.5), (4.6), and (4.7) can be evaluated as asymptotic series in powers of ρ_L. To do this, one can start from the well known expansion [33]

$$\coth s = \frac{1}{s} + \sum_{k=1}^{\infty} \frac{2B_{2k}}{(2k)!}(2s)^{2k-1}, \qquad |s| < \pi \tag{4.18}$$

with the Bernoulli numbers B_{2k}, along with the analogous expansion for $1/\sinh^2 s$. Although in Eqs. (4.5), (4.6), and (4.7) the variable s runs from zero to infinity, one can apply the above mentioned expansions, and, by exploiting the $m^2 - i\varepsilon$ prescription, one obtains the following asymptotic expansions [see also [38]]:

$$c^{(\mathrm{E-H})}(\rho_L) \sim \frac{\alpha}{2\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{B_{2(k+1)}}{2k+1} (2\rho_L)^{2k}, \tag{4.19}$$

$$c_{\perp}^{(\mathrm{E-H})}(\rho_L) \sim \frac{\alpha}{2\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\frac{B_{2k}}{3} - \frac{2B_{2(k+1)}}{2k+1}\right)(2\rho_L)^{2k}, \tag{4.20}$$

$$c_{\parallel}^{(\mathrm{E-H})}(\rho_L) \sim \frac{2\alpha}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} B_{2(k+1)}(2\rho_L)^{2k}. \tag{4.21}$$

For notational simplicity, we have written the time-independent counterpart of the coefficients because the time-dependent ones are simply obtained by substituting $\rho_L \to \rho_L \cos\eta$ with $\eta = \omega_L t$. Furthermore, by using the integral

$$\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos^{2k}\eta\, e^{2iq\eta} = \begin{cases} 0 & \text{if } k < q \\ \frac{1}{4^k}\binom{2k}{k-q} & \text{if } k \geq q \end{cases} \tag{4.22}$$

one can also obtain the corresponding asymptotic expansions of the Fourier amplitudes (4.2), (4.3), and (4.4) in the form

$$c_{2q}^{(\mathrm{E-H})}(\rho_L) \sim \frac{\alpha}{\pi} \sum_{k=q}^{\infty} \frac{(-1)^k}{k} \frac{B_{2(k+1)}}{2k+1} \binom{2k}{k-q} \rho_L^{2k}, \tag{4.23}$$

$$c_{\perp,2q}^{(\mathrm{E-H})}(\rho_L) \sim \frac{\alpha}{2\pi} \sum_{k=q}^{\infty} \frac{(-1)^k}{k} \left(\frac{B_{2k}}{3} - \frac{2B_{2(k+1)}}{2k+1}\right) \binom{2k}{k-q} \rho_L^{2k}, \tag{4.24}$$

$$c_{\parallel,2q}^{(\mathrm{E-H})}(\rho_L) \sim \frac{2\alpha}{\pi} \sum_{k=q}^{\infty} \frac{(-1)^k}{2k+1} B_{2(k+1)} \binom{2k}{k-q} \rho_L^{2k}. \tag{4.25}$$

It can easily be shown that the previous series are non-alternating and diverging, and this is an indication of the fact that the original integrals in Eqs. (4.5), (4.6), and (4.7) contain an exponentially small and imaginary contribution nonperturbative in ρ_L [39]. In fact, by performing a clockwise Wick rotation of an angle $\pi/2$ in Eqs. (4.5), (4.6), and (4.7), we have to take into account the presence of the poles of the resulting integrands at $s_n = -n\pi$, with n a positive integer. By using the residue method, one obtains the following expressions for the imaginary parts of the coefficients $c^{(\mathrm{E-H})}(\rho_L)$, $c_{\perp}^{(\mathrm{E-H})}(\rho_L)$, and $c_{\parallel}^{(\mathrm{E-H})}(\rho_L)$:

$$\mathrm{Im}\,(c^{(\mathrm{E-H})}(\rho_L)) = \frac{\alpha}{2} \sum_{n=1}^{\infty} \left[\frac{1}{n\pi\rho_L} + \frac{2}{(n\pi)^2}\right] e^{-n\pi/\rho_L}, \tag{4.26}$$

$$\mathrm{Im}\,(c_{\perp}^{(\mathrm{E-H})}(\rho_L)) = -\frac{\alpha}{6} \sum_{n=1}^{\infty} \left[2 + \frac{3}{n\pi\rho_L} + \frac{6}{(n\pi)^2}\right] e^{-n\pi/\rho_L}, \tag{4.27}$$

$$\mathrm{Im}\,(c_{\parallel}^{(\mathrm{E-H})}(\rho_L)) = \frac{\alpha}{2} \sum_{n=1}^{\infty} \left(\frac{1}{\rho_L^2} + \frac{1}{n\pi\rho_L}\right) e^{-n\pi/\rho_L}, \tag{4.28}$$

that contain, in fact, nonperturbative exponentials in ρ_L. Despite the previous expansions in Eqs. (4.19), (4.20), and (4.21), these series are clearly converging for any finite value of ρ_L, but they are very useful in the weak field limit $\rho_L \ll 1$ because the first term is exponentially larger than the others then

$$\mathrm{Im}\,(c^{(\mathrm{E-H})}(\rho_L)) \simeq \frac{\alpha}{2\pi}\frac{1}{\rho_L}e^{-\pi/\rho_L}, \tag{4.29}$$

$$\mathrm{Im}\,(c_\perp^{(\mathrm{E-H})}(\rho_L)) \simeq -\frac{\alpha}{2\pi}\frac{1}{\rho_L}e^{-\pi/\rho_L}, \tag{4.30}$$

$$\mathrm{Im}\,(c_\parallel^{(\mathrm{E-H})}(\rho_L)) \simeq \frac{\alpha}{2}\frac{1}{\rho_L^2}e^{-\pi/\rho_L} \tag{4.31}$$

and, correspondingly,

$$\begin{aligned}\mathrm{Im}(c_{2q}^{(\mathrm{E-H})}(\rho_L)) &\simeq \frac{\alpha}{2\pi^2}\int_{-\pi/2}^{\pi/2}d\eta\frac{1}{\rho_L\cos\eta}e^{-\pi/\rho_L\cos\eta+i2q\eta}\\ &\sim \frac{\alpha}{2\pi^2}\sqrt{\frac{2}{\rho_L}}e^{-\pi/\rho_L+2\rho_Lq^2/\pi},\end{aligned} \tag{4.32}$$

$$\begin{aligned}\mathrm{Im}(c_{\perp,2q}^{(\mathrm{E-H})}(\rho_L)) &\simeq -\frac{\alpha}{2\pi}\int_{-\pi/2}^{\pi/2}d\eta\frac{1}{\rho_L\cos\eta}\\ &\quad\times e^{-\pi/\rho_L\cos\eta+i2q\eta}\\ &\sim -\frac{\alpha}{2\pi^2}\sqrt{\frac{2}{\rho_L}}e^{-\pi/\rho_L+2\rho_Lq^2/\pi},\end{aligned} \tag{4.33}$$

$$\begin{aligned}\mathrm{Im}(c_{\parallel,2q}^{(\mathrm{E-H})}(\rho_L)) &\simeq \frac{\alpha}{2\pi}\int_{-\pi/2}^{\pi/2}d\eta\frac{1}{(\rho_L\cos\eta)^2}\\ &\quad\times e^{-\pi/\rho_L\cos\eta+i2q\eta}\\ &\sim \frac{\alpha}{2\pi}\sqrt{\frac{2}{\rho_L^3}}e^{-\pi/\rho_L+2\rho_Lq^2/\pi},\end{aligned} \tag{4.34}$$

where the stationary phase method has been used to give the asymptotic estimate of the integrals. By summarizing, in the weak field case we obtain

$$\begin{aligned}c_{2q}^{(\mathrm{E-H})}(\rho_L) &\sim \frac{\alpha}{\pi}\frac{(-1)^q}{q}\frac{B_{2(q+1)}}{2q+1}\rho_L^{2q}\\ &\quad+i\frac{\alpha}{2\pi^2}\sqrt{\frac{2}{\rho_L}}e^{-\pi/\rho_L+2\rho_Lq^2/\pi},\end{aligned} \tag{4.35}$$

$$\begin{aligned}c_{\perp,2q}^{(\mathrm{E-H})}(\rho_L) &\sim \frac{\alpha}{2\pi}\frac{(-1)^q}{q}\left(\frac{B_{2q}}{3}-\frac{2B_{2(q+1)}}{2q+1}\right)\rho_L^{2q}\\ &\quad-i\frac{\alpha}{2\pi^2}\sqrt{\frac{2}{\rho_L}}e^{-\pi/\rho_L+2\rho_Lq^2/\pi},\end{aligned} \tag{4.36}$$

$$\begin{aligned}c_{\parallel,2q}^{(\mathrm{E-H})}(\rho_L) &\sim \frac{2\alpha}{\pi}\frac{(-1)^q}{2q+1}B_{2(q+1)}\rho_L^{2q}\\ &\quad+i\frac{\alpha}{2\pi}\sqrt{\frac{2}{\rho_L^3}}e^{-\pi/\rho_L+2\rho_Lq^2/\pi},\end{aligned} \tag{4.37}$$

where we have kept only the first term in the asymptotic expansions (4.23), (4.24), and (4.25). In this case, the form of the spectrum is determined mainly by the real part of the coefficients because the vacuum is stable with a very good approximation and the imaginary parts of the coefficients $c_{2q}^{(\mathrm{E-H})}(\rho_L)$, $c_{\perp,2q}^{(\mathrm{E-H})}(\rho_L)$ and $c_{\parallel,2q}^{(\mathrm{E-H})}(\rho_L)$ are tiny. By performing the remaining elementary integral on the angle ϑ in Eq. (4.17), we obtain the following expression of the amplitude of the qth harmonics:

$$\begin{aligned}\frac{dN_q^{(\mathrm{E-H})}(\beta_L,\rho_L)}{dVdt} \sim \frac{(qm\beta_L)^4}{16\pi}\left(\frac{\alpha}{\pi}\right)^2\rho_L^{4q}\Bigg\{&\left(\frac{4B_{2(q+1)}}{q(2q+1)}\right)^2+\frac{16}{15}\left[\frac{1}{q^2}\left(\frac{B_{2q}}{6}-\frac{B_{2(q+1)}}{2q+1}\right)^2+\left(\frac{2B_{2(q+1)}}{2q+1}\right)^2\right]\\ &+\frac{8}{3}\left[\left(\frac{2B_{2(q+1)}}{q(2q+1)}\right)^2+\frac{2B_{2(q+1)}}{q^2(2q+1)}\left(\frac{B_{2q}}{6}-\frac{B_{2(q+1)}}{2q+1}\right)\right]\Bigg\}.\end{aligned} \tag{4.38}$$

The spectrum corresponding to $\omega_L = 2.4\times10^{-6}m = 1.2$ eV and $E_L = 10^{-4}E_{\mathrm{cr}} = 1.3\times10^{12}$ V/cm is shown in Fig. 1 and, in fact, it has the typical shape of a perturbative spectrum. The amplitude of the harmonics decreases monotonically as the harmonic order increases. Before going on, we only quote that we have also studied the angular distribution of the harmonic emission with respect to the polar angle ϑ. Actually, the results are not so interesting: All the harmonics show a symmetric angular distribution around $\vartheta = \pi/2$ with a maximum at this point and two minima at $\vartheta = 0$ and $\vartheta = \pi$ in which there is no emission.

It is clear that the production of high harmonics in this regime is very unlikely. For this reason, we concentrate on the $q = 1$ term. We have to remind that, as we have pointed out below Eqs. (2.1) and (2.2), our theory is valid for $q \gg 1$. Nevertheless, we have shown explicitly that for the $q = 1$ term, by using the exact field (2.1) and (2.2), we obtain for the probability per unit volume and unit time the expression

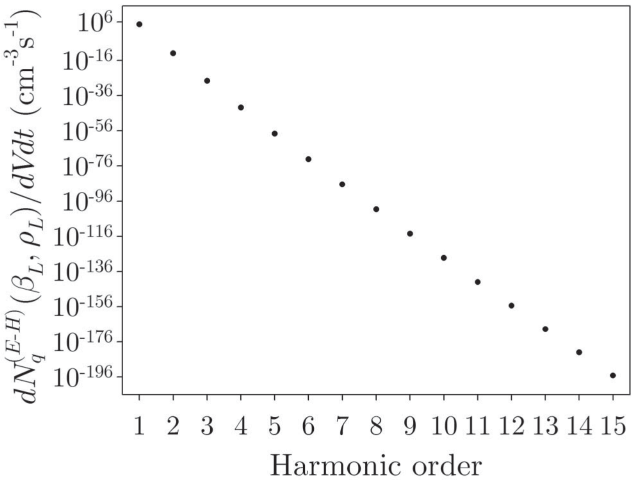

FIG. 1. Harmonic spectrum as given by Eq. (4.38) in cm^{-3} s^{-1} for $\omega_L = 2.4 \times 10^{-6} m = 1.2$ eV and $E_L = 10^{-4} E_{\rm cr} = 1.3 \times 10^{12}$ V/cm.

$$\frac{dP^{\rm (E-H)}_{\rm sw,1}(\beta_L, \rho_L)}{dVdt} \sim \frac{47}{288000\pi}(m\beta_L)^4\left(\frac{\alpha}{\pi}\right)^2 \rho_L^4, \quad (4.39)$$

where the index "sw" reminds us that we have used the exact "standing wave" configuration as external field. It is worth stressing that Eq. (4.39) coincides with the probability of the photon-photon scattering in the low-frequency limit and for initial photons with the same polarizations as calculated in Refs. [7–9,14] by means of the usual perturbation theory.

1. Experimental observation of photon-photon scattering

We want to analyze if the photon-photon scattering is experimentally observable by using the presently available or near future laser technology. Because of the dependence of the probability (4.39) on β_L^4, one realizes that x-ray lasers are more suitable than optical lasers. Also, as has been observed in Ref. [40], the main problem with x-ray free electron lasers is the focusing. For this reason, as an indicator, we estimate the minimum laser beam size $\sigma_{L,\rm min}$, at which at least one scattering event is possible during the interaction, the other laser parameters being fixed. To do this, we have to give an estimate of the space-time VT where the laser electric field amplitude is not substantially reduced from the maximal focused value E_L. We assume $T = \tau$, with τ the laser pulse duration, and $V = \pi\sigma_L^2 \times c\tau_c$, with τ_c the laser coherence time and σ_L the laser beam size (for the sake of clarity, we will use here below cgs units). In this way, the minimum laser beam size needed to obtain one photon-photon scattering event is obtained by multiplying Eq. (4.39) times VT and putting the result equal to one, then

$$\sigma_{L,\rm min} \simeq 10^{-7}\left(\frac{P_L}{1\,\rm GW}\right)\left(\frac{1\,\rm nm}{\lambda_L}\right)^2 \sqrt{\left(\frac{\tau}{1\,\rm fs}\right)\left(\frac{\tau_c}{1\,\rm fs}\right)}\,\rm nm, \quad (4.40)$$

where P_L is the laser peak power expressed in gigawatts. By referring, for example, to the laser parameters indicated as "goal" in Table 1 in Ref. [40]: $\lambda_L = 0.15$ nm, $\tau_c = 80$ fs, and $P_L = 5$ TW, we obtain for a pulse duration of $\tau = 100$ fs that the photon-photon scattering can be obtained already with a beam size $\sigma_{L,\rm min} = 2$ nm $\simeq 13.3\lambda_L$. This value is more than one order of magnitude larger than that required for electron-positron pair creation ($\sigma_L \simeq \lambda_L$), and this makes our result very interesting because, as is pointed out in Ref. [40], the main problem in achieving the goal parameters is just the focusing of the x-ray laser beam up to λ_L.

We can obtain an even better result by making three lasers collide. The resulting process is a sort of "laser-assisted" or "laser-stimulated" photon-photon scattering [22]. If we consider, for simplicity, three equal lasers, the expression of the number of photons scattered per unit volume and unit time is obtained by Eq. (4.39) times the number of photons of the third "assisting" laser. In this way, we can obtain the minimum peak power $P_{L,\rm min}$ that each laser must have to observe at least one photon scattered as

$$P_{L,\rm min} \simeq 33.5\left(\frac{\lambda_L}{1\,\rm nm}\right)\left(\frac{\sigma_L}{1\,\rm nm}\right)^{2/3}\left(\frac{1\,\rm fs}{\tau}\right)^{1/3}\left(\frac{1\,\rm fs}{\tau_c}\right)^{2/3}\,\rm GW. \quad (4.41)$$

By substituting the laser parameters of a typical focused optical laser: $\sigma_L = \lambda_L = 1\ \mu$m and $\tau = \tau_c = 100$ fs, we obtain $P_{L,\rm min} \simeq 30$ TW, which is a value today available for optical lasers. From this point of view, assisted photon-photon scattering could be observed today but a *caveat* is in order: By using three identical lasers, the scattered photon would have the same frequency of the laser photons, making its detection impossible. Nevertheless, this problem can be overcome by changing the frequency of the assisting laser and the angle between the other two lasers [see also [22]]. Finally, by substituting in Eq. (4.41) the laser parameters of a focused x-ray-free-electron laser [40]: $\sigma_L = \lambda_L = 0.4$ nm, $\tau = 100$ fs, with a coherence time of $\tau_c = 80$ fs [as given in the goal column in Table 1 in Ref. [40]], we obtain in this case $P_{L,\rm min} \simeq 1.2$ GW, which is slightly larger than the power of the planned x-ray-free-electron laser self-amplified spontaneous emission (SASE-5) at Deutsches Elektronen-Synchrotron equal to 1.1 GW [40].

B. Strong electric fields: $\rho_L \gg 1$

The previous representations (4.19), (4.20), (4.21), (4.26), (4.27), and (4.28) of the real and imaginary parts of the coefficients $c^{\rm (E-H)}(\rho_L)$, $c_\perp^{\rm (E-H)}(\rho_L)$, and $c_\parallel^{\rm (E-H)}(\rho_L)$ are not suitable to deal with the strong field case. Instead, by using the integrals given in Appendix D in Ref. [14], one obtains the following exact representations of the coefficients:

$$\mathrm{Re}(c^{(\mathrm{E-H})}(\rho_L)) = \frac{\alpha}{2\pi}\left\{-\frac{2}{3}\log\rho_L - \frac{1}{3} + 8L_1 - \frac{2}{3}\log 2 - \frac{\pi}{2\rho_L} + \frac{1}{2\rho_L^2} + 4\sum_{n=1}^{\infty}\left[n\log\left(1+\frac{1}{4n^2\rho_L^2}\right) - \frac{1}{2\rho_L}\arctan\frac{1}{2n\rho_L}\right]\right\}, \quad (4.42)$$

$$\mathrm{Re}(c_{\perp}^{(\mathrm{E-H})}(\rho_L)) = \frac{\alpha}{2\pi}\left\{\frac{1}{3} + \frac{2\gamma}{3} - 8L_1 + \frac{\pi}{2\rho_L} - \frac{1}{2\rho_L^2} - \sum_{n=1}^{\infty}\left[\frac{2}{3n}\frac{1}{1+4n^2\rho_L^2} + 4n\log\left(1+\frac{1}{4n^2\rho_L^2}\right) - \frac{2}{\rho_L}\arctan\frac{1}{2n\rho_L}\right]\right\}, \quad (4.43)$$

$$\mathrm{Re}(c_{\parallel}^{(\mathrm{E-H})}(\rho_L)) = \frac{\alpha}{2\pi}\left\{-\frac{2}{3} + \frac{\pi}{2\rho_L} - \frac{1}{\rho_L^2} - \sum_{n=1}^{\infty}\left[\frac{4n}{1+4n^2\rho_L^2} - \frac{2}{\rho_L}\arctan\frac{1}{2n\rho_L}\right]\right\}, \quad (4.44)$$

with $\gamma = 0.57721\ldots$ the Euler constant and $L_1 = 0.24875\ldots$ and

$$\mathrm{Im}\,(c^{(\mathrm{E-H})}(\rho_L)) = \frac{\alpha}{2\pi}\left[-\frac{1}{\rho_L}\log(1-e^{-\pi/\rho_L}) - 2\int_{1/\rho_L}^{\infty} dt\log(1-e^{-\pi t})\right], \quad (4.45)$$

$$\mathrm{Im}(c_{\perp}^{(\mathrm{E-H})}(\rho_L)) = \frac{\alpha}{2\pi}\left[\frac{1}{\rho_L}\log(1-e^{-\pi/\rho_L}) + 2\int_{1/\rho_L}^{\infty} dt\log(1-e^{-\pi t}) - \frac{\pi}{3}\left(\coth\frac{\pi}{2\rho_L} - 1\right)\right], \quad (4.46)$$

$$\mathrm{Im}(c_{\parallel}^{(\mathrm{E-H})}(\rho_L)) = \frac{\alpha}{2\pi}\left[-\frac{1}{\rho_L}\log(1-e^{-\pi/\rho_L}) + \frac{\pi}{2\rho_L^2}\left(\coth\frac{\pi}{2\rho_L} - 1\right)\right]. \quad (4.47)$$

Actually, these imaginary parts can also be obtained by summing the series (4.26), (4.27), and (4.28). In this respect, a derivation of the coefficients $c_{\perp}^{(\mathrm{E-H})}(\rho_L)$ and $c_{\parallel}^{(\mathrm{E-H})}(\rho_L)$ can be found in Refs. [34,38], and, since $c^{(\mathrm{E-H})}(\rho_L)$ can be obtained analogously, we have given only the final results. In particular, the expansions of the real parts come from the identity concerning the phase of the gamma function [41]

$$\arg(\Gamma(a+ib)) = b\psi(a) + \sum_{n=0}^{\infty}\left(\frac{b}{a+n} - \arctan\frac{b}{a+n}\right), \quad (4.48)$$

with a and b real numbers and $\psi(a) = \Gamma'(a)/\Gamma(a)$.

From the previous expressions, it is easy to obtain the asymptotic expressions for the coefficients in the strong field regime:

$$c^{(\mathrm{E-H})}(\rho_L) \sim \frac{\alpha}{2\pi}\left(-\frac{2}{3}\log\rho_L + 8L_1 - \frac{1}{3} - \frac{2}{3}\log 2\right) + i\frac{\alpha}{6}, \quad (4.49)$$

$$c_{\perp}^{(\mathrm{E-H})}(\rho_L) \sim \frac{\alpha}{2\pi}\left(\frac{1}{3} + \frac{2}{3}\gamma - 8L_1\right) - i\frac{\alpha}{3\pi}\rho_L, \quad (4.50)$$

$$c_{\parallel}^{(\mathrm{E-H})}(\rho_L) \sim -\frac{\alpha}{3\pi} + i\frac{\alpha}{2\pi}\frac{1}{\rho_L}\left(1 - \log\pi + \log\rho_L\right). \quad (4.51)$$

The spectrum in this regime is determined mostly by the imaginary part of $c_{\perp}^{(\mathrm{E-H})}(\rho_L\cos\eta)$, which is linear in the laser field amplitude ρ_L. By performing its Fourier transform, we obtain [see Eqs. (4.3) and (4.17)]

$$\frac{dN_q^{(\mathrm{E-H})}(\beta_L,\rho_L)}{dVdt} \sim \frac{(qm\beta_L)^4}{15\pi}\left(\frac{\alpha\rho_L}{3\pi}\right)^2\left|\int_{-\pi/2}^{\pi/2}\cos\eta e^{2iq\eta}\right|^2 = \frac{4\alpha^2}{135\pi^5}(m\beta_L)^4\rho_L^2\frac{q^4}{(4q^2-1)^2}. \quad (4.52)$$

In this case, the spectrum becomes flat for higher frequencies. If this formula held for any q, the total emitted energy would be infinite. But we have to recall that, actually, it is valid only for harmonic orders such that $q\omega_L \ll m$. The previous result indicates that, in the strong field regime, the VHHG spectra show the presence of a plateau, and they share this feature with AHHG. In the next section, we will also see that, for very high harmonics, also the VHHG spectra show a rapid cutoff of the emission like AHHG spectra.

Before concluding this low-energy part, we want to make two final remarks. First, we quote that the production of harmonics in this particular low-energy regime has also been studied in Refs. [42,43]. The authors consider the exact initial field configuration (2.1) and (2.2) with different mutual polarizations of the two plane waves, but they evaluate the harmonic amplitudes only for the first harmonics. The second remark is more technical. In fact, the imaginary parts of the coefficients $c^{(\mathrm{E-H})}(\rho_L)$, $c_\perp^{(\mathrm{E-H})}(\rho_L)$, and $c_\parallel^{(\mathrm{E-H})}(\rho_L)$ in the series form (4.26), (4.27), and (4.28) or in the integral form are very suitable for their numerical evaluation. Instead, apart from in the small field regime, the asymptotic series (4.19), (4.20), and (4.21) of the real parts of the coefficients themselves cannot be used for their numerical evaluation. Also the series (4.42), (4.43), and (4.44) are quite problematic to be dealt with numerically. For this reason, we want to conclude by deriving another converging series of the real parts of $c^{(\mathrm{E-H})}(\rho_L)$, $c_\perp^{(\mathrm{E-H})}(\rho_L)$, and $c_\parallel^{(\mathrm{E-H})}(\rho_L)$. To do this, we consider a generic constant and uniform electromagnetic field $(\mathbf{E}, \mathbf{B})$ described by the two invariants F and G^2 defined analogously to Eqs. (4.10) and (4.11). Then we consider the following *converging* series of the real part of the E-H Lagrangian density [44–46]:

$$\begin{aligned}\mathrm{Re}(\mathcal{L}^{(\mathrm{E-H})}(a,b)) = \frac{1}{2}(a^2-b^2) - \frac{\alpha}{\pi^2}ab\sum_{n=1}^{\infty}\frac{\coth n\pi b/a}{n}\left[\mathrm{Ci}\left(\frac{n\pi E_{\mathrm{cr}}}{a}\right)\cos\left(\frac{n\pi E_{\mathrm{cr}}}{a}\right) + \mathrm{Si}\left(\frac{n\pi E_{\mathrm{cr}}}{a}\right)\sin\left(\frac{n\pi E_{\mathrm{cr}}}{a}\right)\right]\\ - \frac{\coth n\pi a/b}{2n}\left[\mathrm{Ei}\left(-\frac{n\pi E_{\mathrm{cr}}}{b}\right)e^{n\pi E_{\mathrm{cr}}/b} + \mathrm{Ei}\left(\frac{n\pi E_{\mathrm{cr}}}{b}\right)e^{-n\pi E_{\mathrm{cr}}/b}\right],\end{aligned} \tag{4.53}$$

which is expressed in terms of the secular invariants a and b defined as $a + ib = \sqrt{F + iG}$ and of the sine, cosine, and exponential integrals [41]. At this point, by using the general definitions (4.12), (4.13), and (4.14) of the coefficients $c^{(\mathrm{E-H})}(\rho_L)$, $c_\perp^{(\mathrm{E-H})}(\rho_L)$, and $c_\parallel^{(\mathrm{E-H})}(\rho_L)$ in terms of the derivatives of the E-H Lagrangian density, we obtain

$$\begin{aligned}\mathrm{Re}(c^{(\mathrm{E-H})}(\rho_L)) = -\frac{\alpha}{6\pi} + \frac{\alpha}{\pi}\sum_{n=1}^{\infty}\frac{1}{(n\pi)^2}\left\{\mathrm{Ei}\left(-\frac{n\pi}{\rho_L}\right)e^{n\pi/\rho_L} + \mathrm{Ei}\left(\frac{n\pi}{\rho_L}\right)e^{-n\pi/\rho_L}\right.\\ \left. - \frac{n\pi}{2\rho_L}\left[\mathrm{Ei}\left(-\frac{n\pi}{\rho_L}\right)e^{n\pi/\rho_L} - \mathrm{Ei}\left(\frac{n\pi}{\rho_L}\right)e^{-n\pi/\rho_L}\right]\right\},\end{aligned} \tag{4.54}$$

$$\begin{aligned}\mathrm{Re}(c_\perp^{(\mathrm{E-H})}(\rho_L)) = \frac{\alpha}{6\pi} - \frac{\alpha}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{3} + \frac{1}{(n\pi)^2}\right]\left[\mathrm{Ei}\left(-\frac{n\pi}{\rho_L}\right)e^{n\pi/\rho_L} + \mathrm{Ei}\left(\frac{n\pi}{\rho_L}\right)e^{-n\pi/\rho_L}\right]\\ - \frac{n\pi}{2\rho_L}\left[\mathrm{Ei}\left(-\frac{n\pi}{\rho_L}\right)e^{n\pi/\rho_L} - \mathrm{Ei}\left(\frac{n\pi}{\rho_L}\right)e^{-n\pi/\rho_L}\right],\end{aligned} \tag{4.55}$$

$$\begin{aligned}\mathrm{Re}(c_\parallel^{(\mathrm{E-H})}(\rho_L)) = -\frac{\alpha}{3\pi} + \frac{\alpha}{2\pi\rho_L}\sum_{n=1}^{\infty}\frac{1}{n\pi}\left\{\frac{n\pi}{\rho_L}\left[\mathrm{Ei}\left(-\frac{n\pi}{\rho_L}\right)e^{n\pi/\rho_L} + \mathrm{Ei}\left(\frac{n\pi}{\rho_L}\right)e^{-n\pi/\rho_L}\right]\right.\\ \left. - \left[\mathrm{Ei}\left(-\frac{n\pi}{\rho_L}\right)e^{n\pi/\rho_L} - \mathrm{Ei}\left(\frac{n\pi}{\rho_L}\right)e^{-n\pi/\rho_L}\right]\right\},\end{aligned} \tag{4.56}$$

that are valid for any value of the external electric field. In particular, in Ref. [45] it is possible to find a deep numerical analysis on how to deal with the series in Eq. (4.53) and on how to accelerate its convergence. Analogous techniques can be used to evaluate numerically the series (4.54), (4.55), and (4.56).

V. INTERMEDIATE AND HIGH FREQUENCIES

We have seen in the previous section that, for fields much smaller than E_{cr}, the harmonic spectrum is monotonically decreasing, as perturbation theory predicts. Instead, for fields larger than E_{cr}, the nonperturbative effects dominate and the spectrum is flat. Obviously, this cannot hold for larger and larger frequencies because, otherwise, an infinite amount of energy would be produced. In fact, the results of the previous section are valid for harmonics with energies less the electron rest mass. We want to consider now the general case in which no assumptions are made on the harmonic frequencies. Taking into account that for weak laser fields the generation of higher harmonics with energy larger than m is completely negligible, we will consider only the strong field case.

In the following we will distinguish the two cases of intermediate frequencies $(q\beta_L)^2 \ll \rho_L$ and of high frequencies $(q\beta_L)^2 \gg \rho_L$. For the sake of clarity, we recall that $\beta_L = \omega_L/m$ and $\rho_L = E_L/E_{\mathrm{cr}}$.

A. Intermediate frequencies: Strong fields

In this subsection, we limit ourselves to the generation of photons whose frequencies are not necessarily much less than the electron mass but nevertheless are restricted by the following condition: $(q\beta_L)^2 \ll \rho_L$. Taking into account that the low harmonics $q\beta_L \ll 1$ have been treated in the previous section and recalling the general assumption

$\rho_L \ll \pi/\alpha \simeq 430$ [see discussion below Eq. (3.3)], we realize that the number of harmonics belonging to the present frequency "window" is not very large. In any case, it is important to consider this case because one sees how the harmonic spectrum changes with respect to the low-frequency region. Also, the relevant frequency region of gamma rays $q\omega_L \sim 1$ MeV can be dealt with in the present formalism.

We start from Eqs. (2.18), (2.19), and (3.24) and perform the change of variable $s \to s\rho_L \cos\eta$. Then one sees from the resulting expressions (2.21), (2.22), and (2.23) of the functions $F(s\rho_L\cos\eta, \nu)$, $F_\perp(s\rho_L\cos\eta, \nu)$, and $F_\parallel(s\rho_L\cos\eta, \nu)$ that the largest contribution in the limit $\rho_L \gg 1$ comes from $F_\perp(s\rho_L\cos\eta, \nu)$, which goes linearly with the laser electric field. In this approximated picture, we also neglect the terms proportional to $\exp(-(1-|\nu|)s\rho_L\cos\eta)$ because the absolute value of the variable ν is always less than one except that at the integration limits. As a result, we take into account only the contribution of the coefficient $c_{\perp,2q}(\vartheta;\beta_L,\rho_L)$ that in this limit is given by

$$c_{\perp,2q}(\vartheta;\beta_L,\rho_L) = \frac{1}{\pi}\frac{\alpha}{2\pi}\rho_L\int_{-\pi/2}^{\pi/2} d\eta e^{2iq\eta}\cos\eta \times \int_0^1 d\nu(1-\nu^2)\int_0^\infty ds \times\exp\left\{-i\left[1+(q\beta_L\sin\vartheta)^2\frac{1-\nu^2}{4}\right]s\right\}. \tag{5.1}$$

It is worth pointing out that the condition $(q\beta_L)^2 \ll \rho_L$ has been used to make the approximation [see Eqs. (2.18), (3.21), and (3.22)]

$$e^{[(q\beta_L)^2/\rho_L\cos\eta]h(s\rho_L\cos\eta,\nu,\vartheta)} \simeq e^{-is(q\beta_L\sin\vartheta)^2(1-\nu^2)/4}. \tag{5.2}$$

By performing the integrals in Eq. (5.1) on s by means of the usual Wick rotation and on η, we obtain

$$c_{\perp,2q}(\vartheta;\beta_L,\rho_L) = \frac{i}{\pi}\frac{\alpha}{\pi}\rho_L\frac{(-1)^q}{4q^2-1} \times \int_0^1 d\nu\frac{1-\nu^2}{1+(q\beta_L\sin\vartheta)^2\frac{1-\nu^2}{4}}. \tag{5.3}$$

This remaining integral can also be done exactly:

$$I(y) \equiv \int_0^1 d\nu\frac{1-\nu^2}{1+y^2(1-\nu^2)} = \frac{1}{y^3}\left(y-\frac{1}{\sqrt{1+y^2}}\operatorname{arcsinh}y\right) \tag{5.4}$$

and then, in the present approximations, the amplitude of the qth harmonic is given by [see also Eq. (3.17)]

$$\frac{dN_q(\beta_L,\rho_L)}{dVdt} \simeq \frac{1}{8\pi}(qm\beta_L)^4 \times\int_0^{\pi/2} d\vartheta\sin^5\vartheta|c_{\perp,2q}(\vartheta;\beta_L,\rho_L)|^2 \simeq \frac{\alpha^2}{8\pi^5}(m\beta_L)^4\rho_L^2\frac{q^4}{(4q^2-1)^2} \times\int_0^{\pi/2} d\vartheta\sin^5\vartheta I^2(q\beta_L\sin\vartheta/2). \tag{5.5}$$

In this regime, the form of the spectrum does not depend on the amplitude of the laser field. Also, for small harmonic orders such that $q\beta_L \ll 1$, we recover the effective Lagrangian result Eq. (4.52). We underline that the region of applicability of Eq. (5.5) is quite restricted because in our approximations, as we have said, $\rho_L \ll \pi/\alpha \simeq 430$. Further, we derive useful information on the general behavior of the spectrum in this regime by calculating the first two corrections to the effective E-H Lagrangian result. First, we introduce the function $f_q(\beta_L)$ defined as

$$f_q(\beta_L) \equiv \frac{135}{32}\int_0^{\pi/2} d\vartheta\sin^5\vartheta I^2(q\beta_L\sin\vartheta/2), \tag{5.6}$$

then [see Eq. (4.52)]

$$\frac{dN_q(\beta_L,\rho_L)}{dVdt} = f_q(\beta_L)\frac{dN_q^{(\text{E-H})}(\beta_L,\rho_L)}{dVdt}. \tag{5.7}$$

The first corrections to the E-H result for harmonic orders such that $q\beta_L \ll 1$ are obtained through the expansion

$$f_q(\beta_L) \simeq 1-\frac{12}{35}(q\beta_L)^2+\frac{1408}{3675}(q\beta_L)^4. \tag{5.8}$$

As expected from Eq. (5.3), the first correction is negative, then the spectrum decreases. Nevertheless, the second

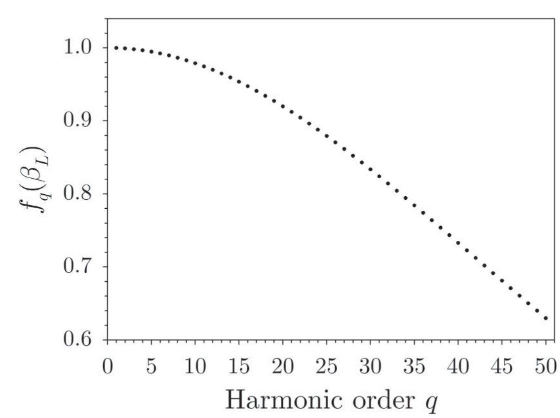

FIG. 2. Function $f_q(\beta_L)$ [see Eq. (5.6)] with $\beta_L = 0.025$ corresponding to $\omega_L = 12.5$ KeV.

correction is positive so that the decreasing of the harmonic yield is slow. This is confirmed by Fig. 2, where we show the function $f_q(\beta_L)$ with $\beta_L = 0.025$ corresponding to $\omega_L = 12.5$ KeV. Finally, as in the previous section, also in this regime the harmonics show a symmetric angular distribution around $\vartheta = \pi/2$. Also, here the photons are emitted mostly in the x-y plane $\vartheta = \pi/2$, while there is no photon emission along the laser electric field $\vartheta = 0$ and $\vartheta = \pi$.

B. High frequencies: Strong fields

In this subsection, we study the VHHG spectra in the high frequency region $(q\beta_L)^2 \gg \rho_L$ in the strong field limit $\rho_L \gg 1$. We find the asymptotic behavior of the harmonic yield and show that it is exponentially decaying. We have to calculate the relevant amplitudes [see Eq. (3.17)]

$$A_{\perp,2q}(\vartheta;\beta_L,\rho_L) \equiv 2c_{2q}(\vartheta;\beta_L,\rho_L) + \sin^2\vartheta c_{\perp,2q}(\vartheta;\beta_L,\rho_L), \quad (5.9)$$

$$A_{\parallel,2q}(\vartheta;\beta_L,\rho_L) \equiv 2c_{2q}(\vartheta;\beta_L,\rho_L) + \sin^2\vartheta c_{\parallel,2q}(\vartheta;\beta_L,\rho_L). \quad (5.10)$$

We derive explicitly only the expression of the coefficient $c_{2q}(\vartheta;\beta_L,\rho_L)$, because the expressions of the other coefficients can be derived analogously. We start from the expression (3.24) of the coefficient $c(\vartheta, q\beta_L;\rho_L)$ which is valid for a constant field ρ_L. Since $(q\beta_L)^2 \gg \rho_L$, we can evaluate the double integral in Eq. (3.24) by using the saddle point method. The calculation is lengthy, and it is presented in Appendix B. We obtain the following asymptotic estimate of the coefficient $c(\vartheta;\beta_L,\rho_L)$ [see Eq. (B14)]:

$$\begin{aligned} c(\vartheta, q\beta_L;\rho_L) \sim & -\alpha\frac{\rho_L}{(q\beta_L)^2} \\ & \times \frac{e^{[(q\beta_L)^2/\rho_L]h(s^*,\nu^*,\vartheta)}e^{-i\pi/4}e^{-i[\pi+\chi(\vartheta)]/2}}{\sqrt{\sin\vartheta\tan\vartheta K(\vartheta)}} \\ & \times \bigg\{\frac{\partial F(s^*,\nu^*)}{\partial s} - \frac{i}{2}\frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial^2 G(s^*,\nu^*,\vartheta;\rho_L)}{\partial s^2} \\ & - \frac{e^{-i\chi(\vartheta)}}{K(\vartheta)}\frac{\partial^2 G(s^*,\nu^*,\vartheta;\rho_L)}{\partial \nu^2}\bigg\}, \end{aligned} \quad (5.11)$$

with [see Eqs. (B13), (B15), and (B16)]

$$\chi(\vartheta) = \arctan\left[\frac{2x^* + (\pi^2 - x^{*2}\cos\vartheta)}{2\pi(1 - x^*\cos\vartheta)}\right], \quad (5.12)$$

$$G(s,\nu,\vartheta;\rho_L) = \mathrm{Ei}\left(-\frac{is}{\rho_L}\right)F(s,\nu)\frac{\partial h(s,\nu,\vartheta)}{\partial s}, \quad (5.13)$$

$$K(\vartheta) = \sqrt{\frac{x^{*2}+\pi^2}{16}}\left|\sin^2\vartheta + \cos^2\vartheta\frac{s^*}{\sinh s^*}\right|, \quad (5.14)$$

and $s^* = x^* - i\pi$ and $\nu^* = 0$ the saddle points of the function $h(s,\nu,\vartheta)$ [see Eqs. (B4), (B5), and (B8)]. Finally, by performing the Fourier integral (3.18), we obtain that the asymptotic behavior of $c_{2q}(\vartheta;\beta_L,\rho_L)$ is given by

$$\begin{aligned} c_{2q}(\vartheta;\beta_L,\rho_L) &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} d\eta e^{2iq\eta}c(\vartheta, q\beta_L;\rho_L\cos\eta) \\ &\sim \frac{1}{\pi}c(\vartheta, q\beta_L;\rho_L) \\ &\quad \times \int_{-\infty}^{\infty} d\eta e^{[(q\beta_L)^2/\rho_L]h(s^*,\nu^*,\vartheta)\eta^2/2} \\ &= \frac{1}{\pi}c(\vartheta, q\beta_L;\rho_L)\sqrt{-\frac{2\pi\rho_L}{q^2\beta_L^2 h(s^*,\nu^*,\vartheta)}}. \end{aligned} \quad (5.15)$$

Also, the asymptotic estimates of the other two coefficients $c_{\perp,2q}(\vartheta;\beta_L,\rho_L)$ and $c_{\parallel,2q}(\vartheta;\beta_L,\rho_L)$ can be obtained in the same way:

$$c_{\perp,2q}(\vartheta;\beta_L,\rho_L) \sim \frac{1}{\pi}c_\perp(\vartheta, q\beta_L;\rho_L) \times\sqrt{-\frac{2\pi\rho_L}{(q\beta_L)^2 h(s^*,\nu^*,\vartheta)}}, \quad (5.16)$$

$$c_{\parallel,2q}(\vartheta;\beta_L,\rho_L) \sim \frac{1}{\pi}c_\parallel(\vartheta, q\beta_L;\rho_L) \times\sqrt{-\frac{2\pi\rho_L}{(q\beta_L)^2 h(s^*,\nu^*,\vartheta)}}, \quad (5.17)$$

with

$$\begin{aligned} c_\perp(\vartheta, q\beta_L;\rho_L) \sim & \frac{\alpha}{2}\frac{\rho_L}{(q\beta_L)^2} \\ & \times \frac{e^{[(q\beta_L)^2/\rho_L]h(s^*,\nu^*,\vartheta)}e^{-i\pi/4}e^{-i[\pi+\chi(\vartheta)]/2}}{\sqrt{\sin\vartheta\tan\vartheta K(\vartheta)}} \\ & \times \frac{\cosh s^* - 1}{\sinh s^*}e^{-is^*/\rho_L}, \end{aligned} \quad (5.18)$$

$$\begin{aligned} c_\parallel(\vartheta, q\beta_L;\rho_L) \sim & \frac{\alpha}{2}\frac{\rho_L}{(q\beta_L)^2} \\ & \times \frac{e^{[(q\beta_L)^2/\rho_L]h(s^*,\nu^*,\vartheta)}e^{-i\pi/4}e^{-i[\pi+\chi(\vartheta)]/2}}{\sqrt{\sin\vartheta\tan\vartheta K(\vartheta)}} \\ & \times\left(2\frac{\cosh s^* - 1}{\sinh^2 s^*} - 1\right)\frac{e^{-is^*/\rho_L}}{\sinh s^*}. \end{aligned} \quad (5.19)$$

In this way, we observe that in the strong field limit $\rho_L \gg 1$ we can write the two amplitudes (5.9) and (5.10) in the form

$$A_{\perp,2q}(\vartheta;\beta_L,\rho_L)\sim\alpha\left(\frac{\rho_L}{q^2\beta_L^2}\right)^{3/2}[B(\vartheta)\log\rho_L + B_\perp(\vartheta)]e^{[(q\beta_L)^2/\rho_L]h(s^*,\nu^*,\vartheta)}, \tag{5.20}$$

$$A_{\parallel,2q}(\vartheta;\beta_L,\rho_L)\sim\alpha\left(\frac{\rho_L}{q^2\beta_L^2}\right)^{3/2}[B(\vartheta)\log\rho_L + B_\parallel(\vartheta)]e^{[(q\beta_L)^2/\rho_L]h(s^*,\nu^*,\vartheta)}. \tag{5.21}$$

First, we observe that the term proportional to $\log\rho_L$ arises from the expansion of the function $\mathrm{Ei}(-is/\rho_L)$ contained in $G(s,\nu,\vartheta;\rho_L)$ in the limit $\rho_L\gg 1$ [41]. Second, we have not quoted the exact expressions of the very involved complex functions $B(\vartheta)$, $B_\perp(\vartheta)$, and $B_\parallel(\vartheta)$. In fact, while for $\vartheta\sim\pi/4$ it results $|B(\vartheta)|\sim|B_\perp(\vartheta)|\sim|B_\parallel(\vartheta)|\sim 1$, instead the final integral on the polar angle ϑ in Eq. (3.17) results to be diverging near $\vartheta=0$ and $\vartheta=\pi/2$, and it cannot be performed neither numerically. Actually, this is not surprising. In fact, in general, in applying the saddle point method to evaluate an integral one tacitly assumes that all the other parameters entering the integral are fixed or, at least, that they run in intervals such that the method is always applicable. In our case, in particular, one sees from Eq. (3.21) that the method cannot be applied in the limit cases with $\vartheta=0$ and $\vartheta=\pi/2$. These two cases have to be treated separately. Actually, the case $\vartheta=0$ is not very interesting: It covers the production of photons in the direction of the laser electric field. Also, an exact evaluation of the coefficients $c_{2q}(\vartheta=0;\beta_L,\rho_L)$, $c_{\perp,2q}(\vartheta=0;\beta_L,\rho_L)$, and $c_{\parallel,2q}(\vartheta=0;\beta_L,\rho_L)$ would give obviously a finite result. Then, since in Eq. (3.17), they are multiplied by $\sin^n\vartheta$ with, at least, $n=1$, we can conclude that there is no photon emission in that direction. Finally, the case $\vartheta=\pi/2$ covers the production of photons in the x-y plane. Starting from Eqs. (3.21) and (3.24), we can say qualitatively that, if $\vartheta=\pi/2$ and if $(q\beta_L)^2/\rho_L\gg 1$, the situation is formally analogous to that encountered in Sec. IVA in dealing with small fields and small frequencies with the substitution $\rho_L\to\rho_L/(q\beta_L)^2$. From this point of view, we can conclude that the number of photons emitted in the x-y plane with energy $q\omega_L$ scales qualitatively as $(\rho_L/q^2\beta_L^2)^{4q}$.

Finally, by using the two expressions (5.20) and (5.21), it is easy to show that the final number of photons produced with frequency $q\omega_L$ per unit volume and unit time in the direction ϑ can be written as

$$\frac{dN_q(\vartheta;\beta_L,\rho_L)}{dVdtd\vartheta}\sim\frac{\alpha^2}{16\pi}(qm\beta_L)^4\sin\vartheta\left(\frac{\rho_L}{q^2\beta_L^2}\right)^3\times e^{-[\pi(q\beta_L)^2/\rho_L]\sin^2\vartheta}[|B(\vartheta)\log\rho_L + B_\perp(\vartheta)|^2+|B(\vartheta)\log\rho_L+B_\parallel(\vartheta)|^2]. \tag{5.22}$$

In contrast to the E-H effective Lagrangian case, the number of photons with high frequency $q\omega_L\gtrsim m$ in the previous equation and in Eq. (5.5) is calculated by taking into account the nonlocality of the interaction between the external laser field and the radiation field. Accordingly, the amplitude of the qth harmonic in Eqs. (5.5) and (5.22) depends not only on the harmonic number q, as in Eq. (4.17), but explicitly on the emitted frequency $q\omega_L$. Instead, since the laser field again is assumed to be slowly varying ($\omega_L\ll m, q\omega_L$), the memory effects connected with the number of photons already created are not included.

From Eq. (5.22) it is clear that in the present situation the yield of the harmonics is exponentially decreasing. From this point of view, we can conclude that the cutoff of the emission is roughly at

$$\omega_M\sim m\sqrt{\rho_L}=\sqrt{eE_L}, \tag{5.23}$$

corresponding to the maximum harmonic order $q_M\sim\sqrt{\xi_L m/\omega_L}$, with $\xi_L=eE_L/m\omega_L$ the so-called adiabaticity parameter. This maximum value is independent of the electron mass and this is because we are working in the strong field regime $E_L\gg E_{\mathrm{cr}}$.

As mentioned at the end of Sec. III, it is quite natural to interpret the photon production process as the creation of a virtual electron-positron pair which, after absorbing a large number of laser photons, annihilates by emitting only two very energetic photons. The difference between the VHHG cutoff and that via AHHG is that in AHHG the electron "excursion" in the laser field is real, while in VHHG it is virtual, confined by the uncertainty relation $t\sim 1/\omega$. While the cutoff energy may be estimated by the maximally attained energy to be released as radiation, its actual evaluation differs from that for AHHG. The cutoff formula for VHHG (5.23) can be roughly estimated by equating the typical energy emitted with the typical energy that the laser field can supply to the electron (positron): $\omega_M\sim eE_L v^* t^*$, with v^* and t^* the velocity and the annihilation time, respectively, and by using the typical values $v^*=1$ and $t^*=1/\omega_M$. The previous expression $\omega_M\sim\sqrt{eE_L}$ gives the order of magnitude of the cutoff position. To check it better, we show in Fig. 3 a VHHG spectrum with $\rho_L=10$, corresponding to $E_L=1.3\times 10^{17}$ V/cm, and $\beta_L=0.025$, corresponding to $\omega_L=12.5$ KeV. From a numerical point of view, it is a very difficult task to evaluate the general expression of the spectrum (3.17) in the whole frequency range because of the integral on the variable s in the coefficients $c_q(\vartheta;\beta_L,\rho_L)$, $c_{\perp,q}(\vartheta;\beta_L,\rho_L)$, and $c_{\parallel,q}(\vartheta;\beta_L,\rho_L)$ [see Eqs. (2.18), (2.19), and (3.24)]. For this reason, we decided to use our analytic estimates Eqs. (5.5) and (5.22). Actually, to plot homogeneous quantities we obtained from Eq. (5.5) the number of photons created per unit time, unit volume, and unit angle ϑ

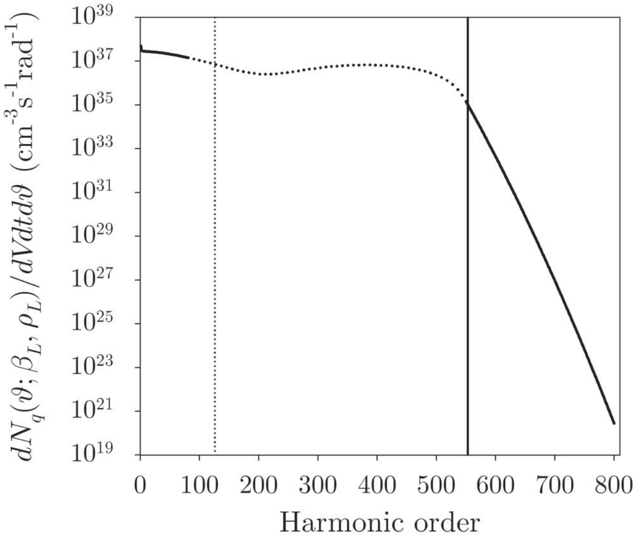

FIG. 3. VHHG spectrum with $\rho_L = 10$ corresponding to $E_L = 1.3 \times 10^{17}$ V/cm, $\beta_L = 0.025$ corresponding to $\omega_L = 12.5$ KeV and $\vartheta = \pi/4$. We used Eq. (5.24) for the first 80 harmonics, Eq. (5.22) for the harmonics 550–800, and a 5-order polynomial interpolating function in the remaining part (dotted part of the curve). The dotted vertical line corresponds to the cutoff formula (5.23) and the solid one to the corrected value $q_M \sim 553$ obtained from Eq. (5.25).

$$\frac{dN_q(\vartheta;\beta_L,\rho_L)}{dVdtd\vartheta} \simeq \frac{\alpha^2}{16\pi^5}(m\beta_L)^4\rho_L^2\frac{q^4}{(4q^2-1)^2} \times \sin^5\vartheta I^2(q\beta_L\sin\vartheta/2), \tag{5.24}$$

and we evaluated by means of this expression the first 80 harmonics. The value of the polar angle ϑ was put equal to $\pi/4$ [see discussion below Eq. (5.21)]. At this value, we can safely put $B(\vartheta) = B_\perp(\vartheta) = B_\parallel(\vartheta) = 1$ in Eq. (5.22) and use it to evaluate the harmonics from 550 to 800. The remaining part of the spectrum was obtained by interpolation (we used a 5-order polynomial function). The dotted vertical line in Fig. 3 shows the harmonic order corresponding to the cutoff rule (5.23). The order of magnitude is correct, but a better estimate can be obtained by equating Eq. (5.24) with Eq. (5.22). Of course, this is not rigorous because the two expressions hold in different frequency regions, but it is enough to obtain an estimate. A numerical solution of the resulting nonlinear equation

$$q_M^2 e^{-\pi q_M^2\beta_L^2/2\rho_L} = \frac{1}{2\pi^4}\frac{\beta_L^2}{\rho_L(\log\rho_L+1)^2} \tag{5.25}$$

gives the better value $q_M \sim 553$, represented by the solid vertical line in Fig. 3.

We have considered so far the laser field $E_L(t)$ as a given external field. We want to estimate here how large the backreaction of VHHG can be to distort the applied approximation [see the papers [37] for a discussion about backreaction effects in electron-positron pair production]. The laser energy density ε transformed per unit time into the emitted vacuum high-order harmonic spectrum can be estimated from Eq. (4.52) by using the expression (5.23) of the cutoff

$$\frac{d\varepsilon}{dt} \sim \sum_{q=1}^{q_M}\omega_q\frac{dN_q^{(\mathrm{E-H})}(\beta_L,\rho_L)}{dVdt} \simeq \frac{\alpha^2}{135}\frac{m^5}{8\pi^5}(\rho_L\beta_L)^3. \tag{5.26}$$

If τ is the laser pulse duration, then the total energy density transformed in VHHG is $\sim(d\varepsilon/dt)\tau$. Finally, the ratio η between this quantity and the initial mean laser energy density $E_L^2/4$ is given by $\eta = 2\alpha^3\rho_L\beta_L^2\omega_L\tau/(135\pi^4)$. By using the same data as those used in Fig. 3 and a typical pulse duration $\tau = 100$ fs, we obtain $\eta \sim 10^{-7}$ and we can conclude that the backreaction of VHHG on the external laser field is negligible at the considered conditions.

VI. COMPETING MECHANISM OF PHOTON PRODUCTION

Here we want to quote another mechanism of photon production from vacuum in the presence of a strong (also constant) electric field and to make a qualitative comparison with that discussed in the paper. In fact, also in a constant and uniform electric field E_L, the process represented by the Feynman diagram in Fig. 4, i.e. the photon plus electron-positron pair production from vacuum, takes place [15]. The total probability $dP(\mathbf{k};\rho_L)/d\mathbf{k}$ of photon creation per unit of photon momentum $\mathbf{k}$ obtained by summing on all the electron and positron states can be evaluated by using the optical theorem [15,26]. By adapting the notation of Ref. [15] to ours, the number of photons created per unit of photon momentum can be written as

$$\frac{dN(\mathbf{k};\rho_L)}{d\mathbf{k}} = 2\,\mathrm{Im}(L_{in}(\mathbf{k};\rho_L)), \tag{6.1}$$

where

$$L_{in}(\mathbf{k};\rho_L) = \int_{\Gamma^c-\Gamma^a} ds_1 \times \int_{\Gamma^c-\Gamma_2-\Gamma_3} ds_2\,\mathrm{sgn}(b)\mathcal{L}(s_1,s_2,\mathbf{k};\rho_L), \tag{6.2}$$

with

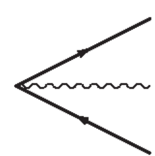

FIG. 4. Feynman diagram of the photon plus electron-positron pair production in the presence of a constant and uniform electric field. The thick fermion lines indicate that the electron and positron states have to be calculated in the presence of the external field.

$$\mathcal{L}(s_1, s_2, \mathbf{k}; \rho_L) = \frac{\alpha\sqrt{\pi}}{2(2\pi)^5}\frac{m^2VT}{\omega}\frac{e^{-i(s_1+s_2)/\rho_L}}{s_1 s_2 \sinh s_1 \sinh s_2} \times \left[2\cosh s_1 \cosh s_2 + i\frac{\rho_L}{\sinh s_1 \sinh s_2}\frac{\partial}{\partial b} + i\frac{\rho_L}{\cosh(s_1 - s_2)}\frac{\partial}{\partial a}\right]\mathcal{I}(s_1, s_2, \mathbf{k}; \rho_L), \tag{6.3}$$

$$a = \frac{s_1 + s_2}{s_1 s_2}, \tag{6.4}$$

$$b = \coth s_1 + \coth s_2, \tag{6.5}$$

$$\mathcal{I}(s_1, s_2, \mathbf{k}; \rho_L) = \frac{1}{ab}e^{-i(\beta^2/\rho_L)\sin^2\vartheta(1/a-1/b)}\Gamma\left(\frac{1}{2}, i\frac{\beta^2}{\rho_L b}\right), \tag{6.6}$$

and $\beta = \omega/m = |\mathbf{k}|/m$, and $\Gamma(x, z)$ is the incomplete Γ function [41]. Also, the complex paths where the integrals in Eq. (6.2) are performed are

$$\Gamma^c = [0, +\infty[, \tag{6.7}$$

$$\Gamma^a = [0 - i\pi, +\infty - i\pi[, \tag{6.8}$$

$$\Gamma_2 = \left] -\infty - i\frac{\pi}{2}, +\infty - i\frac{\pi}{2}\right[, \tag{6.9}$$

$$\Gamma_3 = [0 - i\pi, -\infty - i\pi[, \tag{6.10}$$

and along them the variable b results are always real [see Eqs. (6.2) and (6.5)]. In general, the result obtained by integrating Eq. (6.1) on $\mathbf{k}$ is plagued by different kinds of divergences [39] and to cure them is not a trivial task [15]. For this reason, we content ourselves with a qualitative comparison between our previous results and the number of photons produced per unit volume and unit time in a given frequency range corresponding to Eq. (6.1). Since our spectrum is discrete, it is sensible to compare the amplitude of the qth harmonic with the quantity

$$\frac{dN_q(\mathbf{k}; \rho_L/2)}{dVdT} = \frac{2}{VT}\omega_q^2\Delta\omega\int d\Omega\,\mathrm{Im}(L_{in}(\mathbf{k}_q; \rho_L/2)), \tag{6.11}$$

with Ω the solid angle, $\omega_q = |\mathbf{k}_q| = q\omega_L$ and $\Delta\omega = \omega_L$ (we have considered the previous quantity as calculated at $\rho_L/2$ to compensate for the fact that in our case the electric field is oscillating).

Now, if $(q\beta_L)^2 \gg \rho_L$, the main contribution to the double integral in $L_{in}(\mathbf{k}; \rho_L)$ comes from that along the path Γ^c. In fact, from Eqs. (6.4) and (6.6) one sees that the integrals along Γ^a, Γ_2, and Γ_3 contain damping exponentials. Also, we perform in the remaining double integral along Γ^c the derivatives with respect to a and b, and we make the change of variables

$$s_1 = s\frac{1-\nu}{2}, \tag{6.12}$$

$$s_2 = s\frac{1+\nu}{2}. \tag{6.13}$$

By observing from Eqs. (6.4) and (6.5) that

$$\frac{1}{a} = s\frac{1-\nu^2}{4}, \tag{6.14}$$

$$\frac{1}{b} = \frac{1}{2}\frac{\cosh s - \cosh\nu s}{\sinh s}, \tag{6.15}$$

and by looking at Eq. (6.3), one realizes that the relevant functions for an order-of-magnitude evaluation of the quantity in Eq. (6.11) are the following ones:

$$\mathcal{F}_1(s, \nu, \vartheta, q\beta_L; \rho_L) \equiv e^{-is/\rho_L}e^{-i[(q\beta_L)^2/\rho_L]\sin^2\vartheta[s(1-\nu^2)/4-(\cosh s-\cosh\nu s)/2\sinh s]}e^{-i[(q\beta_L)^2/\rho_L](\cosh s-\cosh\nu s)/2\sinh s} = e^{-is/\rho_L+[(q\beta_L)^2/\rho_L]h(s,\nu,\vartheta)}, \tag{6.16}$$

$$\mathcal{F}_2(s, \nu, \vartheta, q\beta_L; \rho_L) \equiv e^{-is/\rho_L}e^{-i[(q\beta_L)^2/\rho_L]\sin^2\vartheta[s(1-\nu^2)/4-(\cosh s-\cosh\nu s)/2\sinh s]}\Gamma\left(\frac{1}{2}, i\frac{(q\beta_L)^2}{\rho_L}\frac{1}{2}\frac{\cosh s - \cosh\nu s}{\sinh s}\right) = \mathcal{F}_1(s, \nu, \vartheta, q\beta_L; \rho_L)\Psi\left(\frac{1}{2}, \frac{1}{2}; i\frac{(q\beta_L)^2}{\rho_L}\frac{1}{2}\frac{\cosh s - \cosh\nu s}{\sinh s}\right), \tag{6.17}$$

where the function $h(s, \nu, \vartheta)$ is defined in Eq. (3.21) and $\Psi(x_1, x_2; z)$ is the confluent hypergeometric function [33]. In this way, we can conclude that the frequency region where the photons are emitted through the two mechanisms at hand are expected qualitatively to be not so different. In particular, in the cutoff region $(q\beta_L)^2 \gg \rho_L$, the function $\mathcal{F}_2(s, \nu, \vartheta, q\beta_L; \rho_L)$ gives a contribution negligible with respect to that of $\mathcal{F}_1(s, \nu, \vartheta, q\beta_L; \rho_L)$, because in this limit [41]

$$\Psi\left(\frac{1}{2},\frac{1}{2};i\frac{(q\beta_L)^2}{\rho_L}\frac{1}{2}\frac{\cosh s-\cosh\nu s}{\sinh s}\right)\sim\frac{\sqrt{\rho_L}}{q\beta_L}\ll 1. \tag{6.18}$$

In the weak field regime $\rho_L \ll 1$, the photon yield from this process is completely negligible with respect to that of the process treated before, because here an electron-positron pair must be also created and, if $\rho_L \ll 1$, this process is very unlikely. Instead, the order of magnitude of the quantity in Eq. (6.11) in the strong field regime and for $q\beta_L \ll 1$ is given by

$$\frac{dN_q(\mathbf{k};\rho_L/2)}{dVdT}\sim\frac{\alpha}{(2\pi)^5}\rho_L^2 q\beta_L^2 m^4. \tag{6.19}$$

This expression, of course, holds when the external field is rigorously constant, but at zero order we can say that a slowly varying external field gives the same probability but modulated in time. This is not true in the process we have discussed previously, because at zero order in the laser frequency the process is not primed. We can compare the number of photons given by Eq. (6.19) with the amplitude of the qth harmonic as given by Eq. (4.52). In this way, we see that the number of photons given by Eq. (4.52) is $\alpha\beta_L^2/q \ll 1$ less also because this second process involves only one QED vertex (see Fig. 4) and then $dN_q(\mathbf{k};\rho_L/2)/dVdT$ is proportional to α. Despite this, two observations are in order. First, this second process gives rise in the presence of an oscillating electric field to a spectrum also "oscillating" in amplitude that does not always overcome the spectrum due to the first process. Second, the two processes are very different. On the one hand, in this second process the spectrum is continuous, because together with the photon two other particles (an electron-positron pair) are created. On the other hand, in the first process two correlated photons are always created in the opposite direction, while here only one photon is created. In conclusion, at least in principle, the two processes are distinguishable also from an experimental point of view.

VII. SUMMARY AND CONCLUSIONS

We have studied the possibility of high-order harmonic generation in the field of two equally strong counterpropagating laser beams. The mechanism responsible for this vacuum high-order harmonic generation is the creation of a virtual electron-positron pair that absorbs a certain number of laser photons and then annihilates by producing a smaller number of high-energy photons. In various limiting situations, a closed expression for the harmonic yield has been obtained analytically. Actually, the phenomenon of VHHG is primed only for very strong laser fields such that $E_L \gg E_{\rm cr}$. In this case, we have found that, for frequencies smaller than the electron mass, the spectrum shows a flat behavior (*plateau*) (see Sec. IV B). At energies of the order of the electron mass, the spectrum decreases slowly (see Sec. V A). This behavior continues for larger harmonics, but roughly at $\omega_M \sim \sqrt{eE_L}$ a strong reduction (cutoff) of the radiation takes place. For larger harmonic orders, an exponential decreasing of the harmonic yield is found (see Sec. V B). In general, the physical situation is quite different from the atomic high-order harmonic generation. In AHHG, obviously, the photons are emitted by real particles (electrons), while here they are emitted by a virtual electron-positron pair. This explains the difference in the two cutoff scaling laws. Also, atomic energy levels have in conventional conditions definite parity, and for this reason only odd harmonics of the laser frequency are predicted theoretically and observed experimentally in AHHG. Instead, here we found that both even and odd harmonics are generated [see Eq. (3.17)].

For laser electric fields much less than $E_{\rm cr}$, the harmonic spectrum decreases as the power law $(E_L/E_{\rm cr})^{4q}$ also for the first harmonics (see Sec. IV A). From a theoretical point of view, this case is less interesting than the strong field case. Nevertheless, we have shown that from an experimental point of view this is, at the moment, the most interesting case. In fact, because of the very high value of $E_{\rm cr} \sim 10^{16}$ V/cm, it is impossible today to go beyond the weak field limit $E_L \ll E_{\rm cr}$. In this regime, we have seen that from an experimental point of view, contrary to the pair creation process, photon-photon scattering requires a much less extreme laser focusing. Finally, we have also shown that laser-assisted photon-photon scattering could be obtained experimentally by using three optical lasers with characteristics today available.

APPENDIX A

In the present appendix, we want to give some details to derive the expression of the effective Lagrangian density

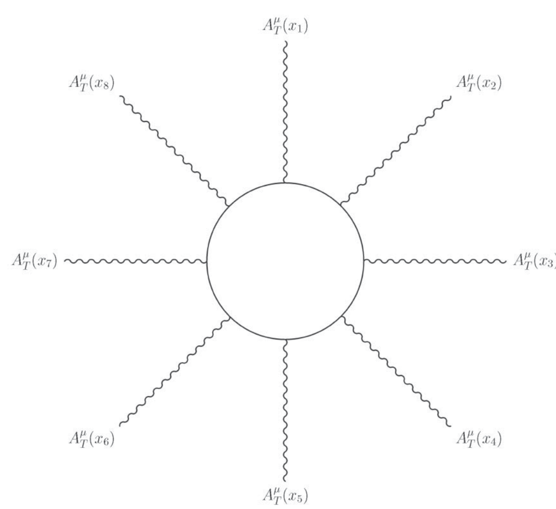

FIG. 5. Feynman diagram corresponding to the amputated one-loop n-point function (A3) with $n = 8$.

(2.5) [see also Eqs. (2.6), (2.7), and (2.8)]. We assume that the total electromagnetic field is described by the four-potential

$$A_T^\mu(x) = A^\mu(x) + \mathcal{A}^\mu(x), \tag{A1}$$

where x denotes here the four space-time coordinates, $A^\mu(x)$ represents a strong classical field and $\mathcal{A}^\mu(x)$ the radiation field. The space-time dependence of the classical field $A^\mu(x)$ is assumed to be assigned, while the effective action of the radiation field has to be determined. In general, the effective action corresponding to the total field $A_T^\mu(x)$ is given up to the one-loop approximation by [13]

$$\Gamma_1[A_T] = \Gamma_M[A_T] - i\sum_{n=1}^{\infty}\frac{1}{n!}\int dx_1\cdots dx_n \times\Gamma^{(n)}_{1;\mu_1\cdots\mu_n}(x_1,\ldots,x_n)A_T^{\mu_1}(x_1)\cdots A_T^{\mu_n}(x_n), \tag{A2}$$

where $\Gamma_M[A_T]$ is the classical Maxwell action and $\Gamma^{(n)}_{1;\mu_1\cdots\mu_n}(x_1,\ldots,x_n)$ are the amputated one-loop n-point functions (see Fig. 5)

$$\Gamma^{(n)}_{1;\mu_1\cdots\mu_n}(x_1,\ldots,x_n) = (-1)(n-1)!\,\mathrm{tr}[(e\gamma_{\mu_1})G(x_1,x_2)\cdots(e\gamma_{\mu_n})G(x_n,x_1)], \tag{A3}$$

with γ_{μ_i} the Dirac matrices, $G(x_a, x_b)$ the electron propagator in vacuum, and the symbol "tr" denoting the trace on the Dirac matrices [the factor $(n-1)!$ represents all the topologically inequivalent graphs that have to be taken into account]. Now, if we neglect the self-interactions of the radiation field with respect to its interactions with the strong external field, we look for *linear* equations of motion for the field $\mathcal{A}^\mu(x)$. In consequence, the corresponding action $\Gamma[\mathcal{A}]$ of the radiation field can be obtained by expanding Eq. (A2) and keeping only terms up to second order in $\mathcal{A}^\mu(x)$:

$$\begin{aligned}\Gamma[\mathcal{A}] = {}& i\,\mathrm{Im}(\Gamma_1[A]) + \Gamma_M[\mathcal{A}] - \frac{i}{2}\int dx_1dx_2\Gamma^{(2)}_{1;\mu_1\mu_2}(x_1,x_2)\mathcal{A}^{\mu_1}(x_1)\mathcal{A}^{\mu_2}(x_2)\\ &-\frac{i}{3!}\int dx_1dx_2dx_3\Gamma^{(3)}_{1;\mu_1\mu_2\mu_3}(x_1,x_2,x_3)\mathcal{A}^{\mu_1}(x_1)\mathcal{A}^{\mu_2}(x_2)A^{\mu_3}(x_3)\\ &-\frac{i}{3!}\int dx_1dx_2dx_3\Gamma^{(3)}_{1;\mu_1\mu_2\mu_3}(x_1,x_2,x_3)A^{\mu_1}(x_1)\mathcal{A}^{\mu_2}(x_2)\mathcal{A}^{\mu_3}(x_3)\\ &-\frac{i}{3!}\int dx_1dx_2dx_3\Gamma^{(3)}_{1;\mu_1\mu_2\mu_3}(x_1,x_2,x_3)\mathcal{A}^{\mu_1}(x_1)A^{\mu_2}(x_2)\mathcal{A}^{\mu_3}(x_3) + \ldots .\end{aligned} \tag{A4}$$

In the previous equation, we have dropped the *real* terms independent of $\mathcal{A}^\mu(x)$ that do not contribute to its equation of motion and the linear term in $\mathcal{A}^\mu(x)$ that after quantization does not give contribution, too. Instead, although it is independent of the radiation field, the imaginary part of the one-loop effective action of the field $A^\mu(x)$ has to be kept, because it takes into account that spontaneous electron-positron pair creation from vacuum can arise in the presence of the field $A^\mu(x)$ itself. Finally, the dots indicate the remaining terms quadratic in $\mathcal{A}^\mu(x)$ but containing all the powers of the external field $A^\mu(x)$ that is taken into account exactly. From Eq. (A3) and from the cyclic property of the trace, we see that

$$\begin{aligned}\Gamma^{(3)}_{1;\mu_1\mu_2\mu_3}(x_1,x_2,x_3) &= \Gamma^{(3)}_{1;\mu_2\mu_3\mu_1}(x_2,x_3,x_1)\\ &= \Gamma^{(3)}_{1;\mu_3\mu_1\mu_2}(x_3,x_1,x_2).\end{aligned} \tag{A5}$$

In this way, the effective action (A4) can be written as

$$\Gamma[\mathcal{A}] = i\,\mathrm{Im}(\Gamma_1[A]) + \Gamma_M[\mathcal{A}] - \frac{1}{2}\int dx_1dx_2\mathcal{A}^{\mu_1}(x_1)\Pi_{\mu_1\mu_2}(x_1,x_2;A)\mathcal{A}^{\mu_2}(x_2), \tag{A6}$$

with

$$\begin{aligned}\Pi_{\mu_1\mu_2}(x_1,x_2;A) = {}& i\Gamma^{(2)}_{\mu_1\mu_2}(x_1,x_2) + i\int dx_3\Gamma^{(3)}_{\mu_1\mu_2\mu_3}(x_1,x_2,x_3)A^{\mu_3}(x_3) + \ldots = -i\,\mathrm{tr}[(e\gamma_{\mu_1})G(x_1,x_2)(e\gamma_{\mu_2})G(x_2,x_1)]\\ &-2i\int dx_3\,\mathrm{tr}[(e\gamma_{\mu_1})G(x_1,x_2)(e\gamma_{\mu_2})G(x_2,x_3)(e\gamma_{\mu_3})G(x_3,x_1)]A^{\mu_3}(x_3) + \ldots\\ ={}& -i\,\mathrm{tr}[(e\gamma_{\mu_1})G(x_1,x_2;A)(e\gamma_{\mu_2})G(x_2,x_1;A)].\end{aligned} \tag{A7}$$

These two previous equations lead exactly to the expression (2.5) of the effective Lagrangian of the radiation field if the classical field is given by the constant and uniform electric field E_L. In fact, $G(x_1, x_2; A)$ is, in general, the exact electron propagator in the presence of the external field $A_\mu(x)$. The previous equation can be expressed by means of Feynman

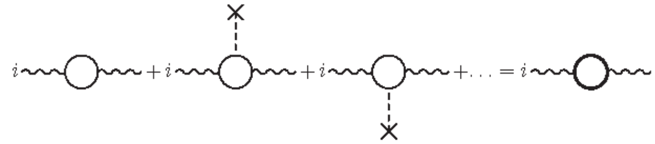

FIG. 6. Feynman diagram representation of Eq. (A7). The thick fermion line in the last Feynman diagram indicates that the corresponding propagators are calculated in the presence of the external field $A_\mu(x)$ [see Eq. (A7)].

diagrams as in Fig. 6 and the thick fermion line in the last Feynman diagram means that $G(x_1, x_2; A)$ is the exact electron propagator in the presence of the external field $A_\mu(x)$.

APPENDIX B

We devote the present appendix to obtaining explicitly the asymptotic expression of the coefficient $c_q(\vartheta; \beta_L, \rho_L)$ in the limit $(q\beta_L)^2 \gg \rho_L$ and $\rho_L \gg 1$. We start from the expression (3.24) of the coefficient $c(\vartheta, q\beta_L; \rho_L)$ which is valid for a constant field ρ_L, and we rewrite it for the sake of clarity:

$$c(\vartheta, q\beta_L; \rho_L) = -\frac{\alpha}{2\pi}\int_0^\infty ds\, \mathrm{Ei}\left(-\frac{is}{\rho_L}\right) \times \int_{-1}^{1}\frac{d\nu}{2} e^{[(q\beta_L)^2/\rho_L]h(s,\nu,\vartheta)}\left[\frac{(q\beta_L)^2}{\rho_L}F(s,\nu) \times \frac{\partial h(s,\nu,\vartheta)}{\partial s} + \frac{\partial F(s,\nu)}{\partial s}\right], \tag{B1}$$

with the functions $F(s, \nu)$ and $h(s, \nu, \vartheta)$ given by [see Eqs. (2.21) and (3.21)]

$$F(s, \nu) = \frac{s}{\sinh s}(\cosh \nu s - \nu \sinh \nu s \coth s), \tag{B2}$$

$$h(s, \nu, \vartheta) = -is\left(\sin^2\vartheta \frac{1-\nu^2}{4} + \frac{\cos^2\vartheta}{2}\frac{\cosh s - \cosh \nu s}{s \sinh s}\right). \tag{B3}$$

Since $(q\beta_L)^2 \gg \rho_L$, we can evaluate the double integral by using the saddle point method. At the moment we assume that $\vartheta \neq 0$ and $\vartheta \neq \pi/2$ [see the main text below Eqs. (5.20) and (5.21) for a discussion of these two cases]. Also, despite $(q\beta_L)^2 \gg \rho_L$, we also keep the second term proportional to $\partial F(s, \nu)/\partial s$ in the integral. In fact, since the first term is proportional to $\partial h(s, \nu, \vartheta)/\partial s$, they will give in the asymptotic limit a contribution of the same order of magnitude. Now the stationary points (s^*, ν^*) of the phase $h(s, \nu, \vartheta)$ are determined by the equations

$$\frac{\partial h(s^*, \nu^*, \vartheta)}{\partial s} = -i\sin^2\vartheta \frac{1-\nu^{*2}}{4} - i\frac{\cos^2\vartheta}{2} \frac{\cosh s^* \cosh \nu^* s^* - \nu^* \sinh \nu^* s^* \sinh s^* - 1}{\sinh^2 s^*} = 0, \tag{B4}$$

$$\frac{\partial h(s^*, \nu^*, \vartheta)}{\partial \nu} = is^* \frac{\sin^2\vartheta}{2}\nu^* + is^* \frac{\cos^2\vartheta}{2}\frac{\sinh \nu^* s^*}{\sinh s^*} = 0. \tag{B5}$$

A solution of the second equation would be $s^* = 0$, but, by inserting it in the first equation, then $|\nu^*| > 1$, while in the integral in Eq. (B1) $|\nu| \leq 1$. Then from the second equation we obtain $\nu^* = 0$. As a consequence, s^* must solve the equation

$$\cosh\frac{s^*}{2} = \pm i \cot\vartheta. \tag{B6}$$

This equation can be solved by putting $s_n^* = x_n^* + in\pi$ with x_n^* real and n integer and different from zero, then

$$\sinh\frac{x_n^*}{2} \equiv \sinh\frac{x^*}{2} = \pm \cot\vartheta \tag{B7}$$

independently of n. Since our integral in s goes from 0 to ∞, we choose $x^* > 0$. Also, as we will see, the only physically acceptable solutions are those with $n < 0$. In particular, the largest contribution to the integral comes from the stationary point

$$s^* \equiv s_{-1}^* = x^* - i\pi = 2\mathrm{arcsinh}(\cot\vartheta) - i\pi \tag{B8}$$

corresponding to $n = -1$ (for notational simplicity, we will not indicate the dependence of s^* and of x^* on ϑ). Also, by expanding the function $h(s, \nu, \vartheta)$ around the stationary point (s^*, ν^*), we obtain

$$h(s, \nu, \vartheta) \simeq h(s^*, \nu^*, \vartheta) + \frac{\partial^2 h(s^*, \nu^*, \vartheta)}{\partial s^2}\frac{(s-s^*)^2}{2} + \frac{\partial^2 h(s^*, \nu^*, \vartheta)}{\partial \nu^2}\frac{\nu^2}{2}, \tag{B9}$$

with

$$h(s^*, \nu^*, \vartheta) = -\frac{\pi}{2}\sin^2\vartheta - i\left(\cos\vartheta + \frac{x^*}{2}\sin^2\vartheta\right), \tag{B10}$$

$$\frac{\partial^2 h(s^*, \nu^*, \vartheta)}{\partial s^2} = -i \sin\vartheta \tan\vartheta, \tag{B11}$$

$$\frac{\partial^2 h(s^*, \nu^*, \vartheta)}{\partial \nu^2} = i\frac{s^*}{2}\left(\sin^2\vartheta + \cos^2\vartheta \frac{s^*}{\sinh s^*}\right). \tag{B12}$$

We point out that, by choosing a negative imaginary part of s^*, we have obtained a negative real part of $h(s^*, \nu^*, \vartheta)$, and this will ensure that for large frequencies the harmonic yield goes to zero (otherwise, it would diverge). From the previous equations, it is also easy to see that, near the stationary point (s^*, ν^*), the steepest descent passes through (s^*, ν^*), forming an angle $3\pi/4$ with the s real axis and an angle $(\pi - \chi(\vartheta))/2$ with

$$\chi(\vartheta) = \arctan\left[\frac{2x^* + (\pi^2 - x^{*2}\cos\vartheta)}{2\pi(1 - x^*\cos\vartheta)}\right] \tag{B13}$$

with the ν real axis. In this way, by performing a rotation in the s complex plane of an angle $\pi/4$ and a rotation in the ν complex plane of an angle $(\pi + \chi(\vartheta))/2$, we obtain the following asymptotic estimate of the integral in Eq. (B1):

$$\begin{aligned} c(\vartheta, q\beta_L; \rho_L) \sim &-\frac{\alpha}{2\pi} e^{[(q\beta_L)^2/\rho_L]h(s^*,\nu^*,\vartheta)} e^{-i\pi/4} e^{-i[\pi+\chi(\vartheta)]/2} \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \bigg\{ \frac{\partial F(s^*, \nu^*)}{\partial s} - \frac{(q\beta_L)^2}{\rho_L} \\ &\times \left[i \frac{\partial^2 G(s^*, \nu^*, \vartheta; \rho_L)}{\partial s^2} \frac{z_1^2}{2} + e^{-i\chi(\vartheta)} \frac{\partial^2 G(s^*, \nu^*, \vartheta; \rho_L)}{\partial \nu^2} \frac{z_2^2}{2} \right] \bigg\} e^{-[(q\beta_L)^2/\rho_L][\sin\vartheta\tan\vartheta z_1^2/2 + K(\vartheta) z_2^2/2]} \\ = &-\alpha \frac{\rho_L}{(q\beta_L)^2} \frac{e^{[(q\beta_L)^2/\rho_L]h(s^*,\nu^*,\vartheta)} e^{-i\pi/4} e^{-i[\pi+\chi(\vartheta)]/2}}{\sqrt{\sin\vartheta\tan\vartheta K(\vartheta)}} \bigg\{ \frac{\partial F(s^*, \nu^*)}{\partial s} - \frac{i}{2} \frac{\cos\vartheta}{\sin^2\vartheta} \frac{\partial^2 G(s^*, \nu^*, \vartheta; \rho_L)}{\partial s^2} \\ &- \frac{e^{-i\chi(\vartheta)}}{K(\vartheta)} \frac{\partial^2 G(s^*, \nu^*, \vartheta; \rho_L)}{\partial \nu^2} \bigg\}, \end{aligned} \tag{B14}$$

with

$$G(s, \nu, \vartheta; \rho_L) = \mathrm{Ei}\left(-\frac{is}{\rho_L}\right) F(s, \nu) \frac{\partial h(s, \nu, \vartheta)}{\partial s}, \tag{B15}$$

$$K(\vartheta) = \sqrt{\frac{x^{*2} + \pi^2}{16}} \left| \sin^2\vartheta + \cos^2\vartheta \frac{s^*}{\sinh s^*} \right|. \tag{B16}$$

As we have mentioned, even if the terms proportional to $(q\beta_L)^2/\rho_L$ seem to be much larger than the term proportional to $\partial F(s^*, \nu^*)/\partial s$, they give a contribution to the integral of the same order of magnitude.

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