Topological solitons in the noncommutative plane and quantum Hall Skyrmions

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We analyze topological solitons in the noncommutative plane by taking a concrete instance of the quantum Hall system with the $SU(N)$ symmetry, where a soliton is identified with a Skyrmion. It is shown that a topological soliton induces an excitation of the electron number density from the ground-state value around it. When a judicious choice of the topological charge density $J_0(\mathbf{x})$ is made, it acquires a physical reality as the electron density excitation $\Delta\rho^{\rm cl}(\mathbf{x})$ around a topological soliton, $\Delta\rho^{\rm cl}(\mathbf{x}) = -J_0(\mathbf{x})$. Hence a noncommutative soliton carries necessarily the electric charge proportional to its topological charge. A field-theoretical state is constructed for a soliton state irrespectively of the Hamiltonian. In general, it involves an infinitely many parameters. They are fixed by minimizing its energy once the Hamiltonian is chosen. We study explicitly the cases where the system is governed by the hard-core interaction and by the noncommutative CP^{N-1} model, where all these parameters are determined analytically and the soliton excitation energy is obtained.

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I. INTRODUCTION

Noncommutative geometry [1] has attracted much attention in a recent development of string theory, as triggered in part by the invention of noncommutative instantons [2] and solitons [3]. Since then, noncommutative solitons have been studied in various noncommutative field theories such as the Abelian Higgs model [4] and the CP^{N-1} model [5].

The standard way of constructing a noncommutative field theory is to replace the ordinary product by the Groenewold-Moyal [6,7] product ($\star$ product) in the Lagrangian density of the corresponding commutative theory. The replacement leads to interesting features reflecting the peculiarities of noncommutativity. In this light, the noncommutative field theory appears as a theory of classical fields and comprises no underlying microscopic (second quantized) origin.

On the other hand, it has long been known that the quantum Hall (QH) system [8,9] has its essence in the noncommutativity [10,11] of the coordinate of an electron in the lowest Landau level (LLL) and that its algebraic structure [12–15] is W_∞. A charged excitation is a topological soliton identified with a Skyrmion [16], whose existence has been confirmed experimentally [17–19] by measuring the number of flipped spins per excitation. Accordingly there must be a microscopic soliton state in a consistent quantum field theory on the noncommutative plane. Indeed, a candidate of the microscopic Skyrmion state was proposed [20–22] to carry out a Hartree-Fock approximation to estimate its excitation energy. Thus the QH system is an ideal laboratory to play with noncommutative geometry and noncommutative solitons. However, no indication of noncommutativity has so far been reproduced in the classical picture corresponding to these microscopic approaches.

The aim of this paper is to fill out such a lack of interplay between the noncommutativity in the microscopic theory and the noncommutativity in the classical theory.

Taking a concrete instance of the QH system, we investigate topological solitons in the noncommutative plane with the coordinate $\mathbf{x} = (x, y)$. The noncommutativity is represented in terms of the Moyal bracket as

$$[x, y]_\star = -i\theta \tag{1.1}$$

with $\theta > 0$. Employing the picture of planar electrons performing cyclotron motion in strong magnetic field, we demonstrate that the noncommutativity encoded at *microscopic* level develops into the noncommutative *kinematical* properties of the corresponding *macroscopic* objects, which are the expectation values of quantum operators. Such an observation is quite reasonable, since the noncommutativity appears as a property of the plane itself, while the *dynamics* is the subsequent structure built over the plane. This approach is particularly useful to explore topological solitons since the topological charge is essentially a geometrical property.

In the noncommutative field theory the basic object is the Weyl-ordered operator [23] together with its symbol. We denote the holomorphic basis by $\{|n\rangle; n = 0, 1, \cdots\}$, in which the Weyl-ordered operator acts. We consider the planar electron field $\psi_\mu(\mathbf{x})$ carrying the $SU(N)$ isospin index μ in the lowest Landau level, where the physical variable is the $U(N)$ density $D_{\mu\nu}(\mathbf{x}) \equiv \psi_\nu^\dagger(\mathbf{x})\psi_\mu(\mathbf{x})$ comprising the electron density $\rho(\mathbf{x})$ and the $SU(N)$ isospin density $S_a(\mathbf{x})$. We define the associated bare density $\hat{D}_{\mu\nu}(\mathbf{x})$, subject to the so-called $W_\infty(N)$ algebra[24], and its classical field $\hat{D}^{\rm cl}_{\mu\nu}(\mathbf{x})$ that is the expectation value by a Fock state. We explore the kinematical and dynamical properties of the classical field $\hat{D}^{\rm cl}_{\mu\nu}(\mathbf{x})$.

We summarize our new results. First of all, we show that the classical field configuration, in particular, the Skyrmion configuration, satisfies the noncommutative constraint

$$\sum_{\sigma=1}^{N} \hat{D}^{\rm cl}_{\mu\sigma}(\mathbf{x}) \star \hat{D}^{\rm cl}_{\sigma\nu}(\mathbf{x}) = \frac{1}{2\pi\theta}\hat{D}^{\rm cl}_{\mu\nu}(\mathbf{x}). \tag{1.2}$$

Furthermore, we resolve the above noncommutative constraint by introducing the noncommutative CP^{N-1} field $n_\mu(\mathbf{x})$ with its complex conjugate $\bar{n}_\mu(\mathbf{x})$,

$$\hat{D}^{\rm cl}_{\mu\nu}(\mathbf{x}) = \frac{1}{2\pi\theta} n_\mu(\mathbf{x}) \star \bar{n}_\nu(\mathbf{x}). \tag{1.3}$$

Recall that the noncommutative CP field has so far been introduced by hand as a mere generalization of the ordinary (commutative) CP field [5].

We then define the topological charge density by the formula

$$J_0(\mathbf{x}) = \frac{1}{2\pi\theta}\sum_\mu [\bar{n}_\mu(\mathbf{x}), n_\mu(\mathbf{x})]_\star. \tag{1.4}$$

It follows from (1.3) and (1.4) that the density excitation $\Delta\rho^{\rm cl}(\mathbf{x})$ is essentially the topological charge density,

$$\Delta\rho^{\rm cl}(\mathbf{x}) = -J_0(\mathbf{x}). \tag{1.5}$$

It is a novel property of a noncommutative soliton that it induces an excitation of the electron number density around it as dictated by this formula. There are many different but equivalent ways of defining the topological charge, but only this definition has such a special property.

An immediate consequence is that a topological soliton carries necessarily the electron number $\Delta N_{\rm e}^{\rm cl} = -Q$, where $\Delta N_{\rm e}^{\rm cl} = \int d^2x \Delta\rho^{\rm cl}(\mathbf{x})$ and $Q = \int d^2x J_0(\mathbf{x})$. We should mention that this property has been known [14,16,25] since the first proposal of Skyrmions in QH systems. Nevertheless, there has been no observation that the relation (1.5) holds rigorously with the choice of the topological charge density (1.4).

Another remarkable result is that the Skyrmion carrying $Q = 1$ is constructed as a $W_\infty(N)$-rotated state of a hole state. The corresponding CP^{N-1} field is given by

$$\mathfrak{n}_\mu = \sum_{n=0}^{\infty}[u_\mu(n)|n\rangle\langle n| + v_\mu(n)|n+1\rangle\langle n|], \tag{1.6}$$

where $\mathfrak{n}_\mu$ is the Weyl-ordered operator whose symbol is $n_\mu(\mathbf{x})$, while $u_\mu(n)$ and $v_\mu(n)$ are infinitely many parameters characterizing the Skyrmion. These parameters are fixed once the Hamiltonian is given.

As an example we study the case where the Hamiltonian is given by the hard-core four-fermion interaction [26]. Determining those parameters explicitly we obtain the Skyrmion state as an eigenstate of the Hamiltonian. The solution is found to possess a factorizable property, $S_a^{\rm cl}(\mathbf{x}) = \rho^{\rm cl}(\mathbf{x}) S_a(\mathbf{x})$, where $S_a(\mathbf{x})$ is the solution in the ordinary CP^{N-1} model. We call such an isospin texture the factorizable Skyrmion. For the sake of completeness we analyze the Skyrmion solution in the noncommutative CP^{N-1} model [5]. It is intriguing that Skyrmions found in the hard-core model [26] and the noncommutative CP^{N-1} model[5] are the same, though these two models are very different. Finally we point out that a factorizable Skyrmion cannot be a physical excitation in the Hamiltonian system consisting of the Coulomb and Zeeman interactions.

This paper is organized as follows. In Sec. II we recapture the basic moments of noncommutative geometry. In Sec. III we review the properties of the density operator for electrons in the lowest Landau level. In Sec. IV we discuss the topological charge in the noncommutative plane. In Sec. V we consider the four-fermion repulsive interaction governing the dynamics of electrons in the lowest Landau level. In Sec. VI we analyze the Skyrmion as a $W_\infty(N)$-rotated state of a hole state. In Sec. VII we construct the microscopic Skyrmion state in the hard-core model. In Sec. VIII we analyze the noncommutative CP^{N-1} model. In Sec. IX we derive the effective theory for the classical density $\hat{D}^{\rm cl}_{\mu\nu}(\mathbf{x})$ and study a generic structure of classical equations of motion. We also make the derivative expansion of the energy and derive the noncommutative CP^{N-1} model as the lowest-order term. Section X is devoted to discussions. In particular, we briefly summarize the property of Skyrmions in the realistic Coulomb interaction model.

II. NONCOMMUTATIVE GEOMETRY

The essence of noncommutative geometry becomes clearer when formulated in algebraic terms [27]. Commutativity of a plane implies the algebra of smooth functions over the plane, with the algebraic operation to be the ordinary multiplication. The plane with the coordinate $\mathbf{x} = (x, y)$ is said to be noncommutative if the algebraic operation is defined by

$$f(\mathbf{x}) \star h(\mathbf{x}) = e^{-(i/2)\theta\nabla_x\wedge\nabla_y} f(\mathbf{x})h(\mathbf{y})|_{\mathbf{y}=\mathbf{x}}, \tag{2.1}$$

where θ is the parameter of noncommutativity. The derivative is

$$\partial_i f(\mathbf{x}) = -\frac{i}{\theta}\epsilon_{ij}[x_j, f(\mathbf{x})]_\star, \tag{2.2}$$

where $[f, g]_\star \equiv f \star g - g \star f$ is the Moyal bracket. The $\star$ product (2.1) is known to be the only possible deformation of the ordinary product provided the associativity is required.

From (2.1) we get

$$x \star y - y \star x = -i\theta, \tag{2.3}$$

implying that the coordinates of a plane are noncommutative with respect to the algebraic multiplication law.

We introduce operators X and Y forming the oscillator algebra,

$$XY - YX = -i\theta. \tag{2.4}$$

The mapping $(x, y) \mapsto (X, Y)$ generates that of a function $f(\mathbf{x})$ into the corresponding operator $O[f]$, which is the Weyl-ordered operator[23],

$$O[f] = \frac{1}{2\pi}\int d^2k e^{i\mathbf{k}\mathbf{X}} f(\mathbf{k}), \tag{2.5}$$

where $f(\mathbf{k})$ is the Fourier transformation of $f(\mathbf{x})$,

$$f(\mathbf{k}) = \frac{1}{2\pi}\int d^2x e^{-i\mathbf{k}\mathbf{x}} f(\mathbf{x}). \tag{2.6}$$

The function $f(\mathbf{x})$ is referred to as the symbol of $O[f]$.

The inverse of (2.5) is given by

$$f(\mathbf{k}) = \theta\,\mathrm{Tr}(e^{-i\mathbf{k}\mathbf{X}} O[f]), \tag{2.7}$$

from which we find

$$\int d^2x f(\mathbf{x}) = 2\pi\theta\,\mathrm{Tr}(O[f]). \tag{2.8}$$

It follows from (2.5) that

$$O[f]O[g] = O[f \star g], \tag{2.9}$$

which generalizes the correspondence between (2.3) and (2.4).

The creation and annihilation operators are constructed from the algebra (2.4),

$$b = \frac{X - iY}{\sqrt{2\theta}}, \qquad b^\dagger = \frac{X + iY}{\sqrt{2\theta}}, \tag{2.10}$$

generating the holomorphic basis,

$$|n\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}}|0\rangle, \qquad b|0\rangle = 0, \tag{2.11}$$

in the Fock space. The Weyl-ordered operator $O[f]$ is presented as

$$O[f] = \sum_{mn} O_{mn}[f]|m\rangle\langle n|, \tag{2.12}$$

where the matrix $O_{mn}[f]$ is given by

$$O_{mn}[f] = \langle m|O[f]|n\rangle. \tag{2.13}$$

The inversion formula (2.7) reads

$$f(\mathbf{k}) = \theta\sum_{mn}\langle n|e^{-i\mathbf{k}\mathbf{X}}|m\rangle O_{mn}[f]. \tag{2.14}$$

We have the chain of one-to-one mappings

$$f \quad \leftrightarrow \quad O[f] \quad \leftrightarrow \quad O_{mn}[f], \tag{2.15}$$

accompanied by the ones among the multiplication laws,

$$f \star h \quad \leftrightarrow \quad O[f]O[h] \quad \leftrightarrow \quad \sum_j O_{mj}[f]O_{jn}[h]. \tag{2.16}$$

In the subsequent sections we will reproduce the above constructions in terms of expectation values of field operators in the QH system.

III. ELECTRONS IN THE LOWEST LANDAU LEVEL

The system of electrons in the lowest Landau level provides us with the simplest and concrete example where the principle of noncommutativity acquires a natural realization. In this section we recollect the basic moments and introduce the bare operators which play the key role in the consequent development. We assume the electron to carry the $U(N)$ isospin index. We have $N = 2$ for the monolayer QH system with the spin degree of freedom, and $N = 4$ for the bilayer QH system with the spin and layer degrees of freedom [8].

A. Quantum mechanics

A planar electron performs cyclotron motion in a homogeneous perpendicular magnetic field ($A_i^{\text{ext}} = \frac{1}{2}B_\perp\epsilon_{ij}x_j$). The electron coordinate $\mathbf{x} = (x, y)$ is decomposed into the guiding center $\mathbf{X} = (X, Y)$ and the relative coordinate $\mathbf{R} = (R_x, R_y)$, $\mathbf{x} = \mathbf{X} + \mathbf{R}$, where $R_i = -\epsilon_{ij}P_j/eB_\perp$ with $\mathbf{P} = (P_x, P_y)$ the covariant momentum.

In the first quantized picture the operators $\mathbf{X}$ and $\mathbf{R}$ are

$$X_i = \frac{1}{2}x_i - i\ell_B^2\epsilon_{ij}\partial_j, \tag{3.1a}$$

$$R_i = \frac{1}{2}x_i + i\ell_B^2\epsilon_{ij}\partial_j. \tag{3.1b}$$

The canonical commutation relation implies

$$[X_i, X_j] = -i\ell_B^2\epsilon_{ij}, \tag{3.2a}$$

$$[P_i, P_j] = i\frac{\hbar^2}{\ell_B^2}\epsilon_{ij}, \tag{3.2b}$$

$$[X_i, P_j] = 0, \tag{3.2c}$$

where ℓ_B is the magnetic length defined by $\ell_B^2 = \hbar/eB_\perp$.

The kinetic Hamiltonian with the electron mass M,

$$H_{\mathrm{K}} = \frac{\mathbf{P}^2}{2M} = \frac{1}{2M}(P_x - iP_y)(P_x + iP_y) + \frac{1}{2}\hbar\omega_c, \tag{3.3}$$

creates the equidistant Landau levels with gap energy $\hbar\omega_c = \hbar eB_\perp/M$.

Because of the noncommutative relation (3.2a) an electron cannot be localized to a point and occupies an area $2\pi\ell_B^2$ in each Landau level. It is highly degenerate. The degree of degeneracy is given by the maximal affordable density of identical electrons given by $\rho_\Phi = (2\pi\ell_B^2)^{-1}$. It is equal to the magnetic flux density, $\rho_\Phi = B_\perp/\Phi_D$ with $\Phi_D = 2\pi\hbar/e$ the Dirac flux quantum. The maximal possible density is given by $N\rho_\Phi$ for electrons carrying the isospin index $\mu = 1, \ldots, N$.

The algebra (3.2a) acts independently within each level. Provided the magnetic field is strong enough, the gap energy $\hbar\omega_c$ becomes large compared with other characteristic energies associated with thermal fluctuations or electrostatic interactions. Then, excitations across Landau levels are practically suppressed, and electrons turn out to be confined to the lowest Landau level. Namely, the degree of freedom associated with the algebra (3.2b) is frozen, and the kinematic Hamiltonian is quenched. This is the LLL projection [10,11].

Consequently, the kinematics of the system becomes governed solely by the algebra (3.2a), which is identical to (2.4). The coordinates x and y standing in (2.3) appear in (3.1a) as coordinates in the space of representation of (3.2a). The parameter of noncommutativity is $\theta = \ell_B^2$ and is carried through all the subsequent accounts.

The wave functions of electrons in the lowest Landau level are given by

$$\langle \mathbf{x}|n\rangle = \frac{z^n}{\sqrt{2^{1+n}\pi\theta n!}}e^{-|z|^2/4} \tag{3.4}$$

with $z = (x+iy)/\sqrt{\theta}$, where n labels the degenerate states with respect to the orbital momentum eigenvalue. We call the state $|n\rangle$ the Landau site. We denote the number of Landau sites by N_Φ, which is equal to the number of flux quanta passing through the system. The set (3.4) agrees with the $\mathbf{x}$ representation of the states (2.11). Though $\langle \mathbf{x}|n\rangle$ are orthonormal, their set is not complete, leading to the nonlocalizability of an electron to a point when it is confined to the lowest Landau level.

B. Second quantization and bare densities

In constructing the second quantized picture we introduce the fermion field operator

$$\psi_\mu(\mathbf{x}) = \sum_{n=0}^{\infty}\langle \mathbf{x}|n\rangle c_\mu(n), \tag{3.5}$$

where $\mu = 1, \ldots, N$ is the isospin index associated with the algebra $U(N)$. The fermion operators satisfy the standard anticommutation relations,

$$\{c_\mu(m), c_\nu^\dagger(n)\} = \delta_{\mu\nu}\delta_{mn}. \tag{3.6}$$

The operator $c_\mu^\dagger(n)$ creates an electron with the isospin μ in the Landau site n.

The physical variables are the number density $\rho(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{x})$ and the isospin density $S_a(\mathbf{x}) = \frac{1}{2}\psi^\dagger(\mathbf{x})\lambda_a\psi(\mathbf{x})$ with the Gell-Mann matrix λ_a. It is convenient to introduce the density operator,

$$D_{\mu\nu}(\mathbf{x}) \equiv \psi_\nu^\dagger(\mathbf{x})\psi_\mu(\mathbf{x}), \tag{3.7}$$

comprising the number and isospin densities as

$$D_{\mu\nu}(\mathbf{x}) = \frac{1}{N}\delta_{\mu\nu}\rho(\mathbf{x}) + (\lambda_a)_{\mu\nu}S_a(\mathbf{x}). \tag{3.8}$$

Substituting (3.5) into (3.7), we obtain

$$D_{\mu\nu}(\mathbf{x}) = \sum_{mn}\langle n|\mathbf{x}\rangle\langle \mathbf{x}|m\rangle D_{\mu\nu}(m,n), \tag{3.9}$$

where

$$D_{\mu\nu}(m,n) \equiv c_\nu^\dagger(n)c_\mu(m). \tag{3.10}$$

Using the relation

$$\int d^2x\langle n|\mathbf{x}\rangle\langle \mathbf{x}|m\rangle e^{-i\mathbf{k}\mathbf{x}} = e^{-(1/4)\theta k^2}\langle n|e^{-i\mathbf{k}\mathbf{X}}|m\rangle, \tag{3.11}$$

we get

$$D_{\mu\nu}(\mathbf{x}) = \frac{1}{2\pi}\int d^2k e^{-(1/4)\theta k^2}\hat{D}_{\mu\nu}(\mathbf{k})e^{i\mathbf{k}\mathbf{x}}, \tag{3.12}$$

where

$$\hat{D}_{\mu\nu}(\mathbf{k}) = \frac{1}{2\pi}\sum_{mn}\langle n|e^{-i\mathbf{k}\mathbf{X}}|m\rangle D_{\mu\nu}(m,n). \tag{3.13}$$

Its Fourier transform reads

$$\hat{D}_{\mu\nu}(\mathbf{x}) = \frac{1}{2\pi}\int d^2k\hat{D}_{\mu\nu}(\mathbf{k})e^{i\mathbf{k}\mathbf{x}}, \tag{3.14}$$

which is related to $D_{\mu\nu}(\mathbf{x})$ as

$$D_{\mu\nu}(\mathbf{x}) = \frac{1}{\pi\theta}\int d^2y e^{-|\mathbf{x}-\mathbf{y}|^2/\theta}\hat{D}_{\mu\nu}(\mathbf{y}). \tag{3.15}$$

We call $\hat{D}_{\mu\nu}$ the bare density. This relation implies that the physical density $D_{\mu\nu}(\mathbf{x})$ is not localizable to a point.

Integrating both sides of (3.15) we get

$$\int d^2x D_{\mu\nu}(\mathbf{x}) = \int d^2x\hat{D}_{\mu\nu}(\mathbf{x}), \tag{3.16}$$

which indicates that (3.15) is just a smearing of $\hat{D}_{\mu\nu}(\mathbf{x})$ over some area of order of θ. The operator (3.14) may be regarded to describe some kind of cores located inside the physical objects associated with $D_{\mu\nu}(\mathbf{x})$.

C. $W_\infty(N)$ algebra

The algebraic relation

$$\begin{aligned}[][D_{\mu\nu}(m,n), D_{\sigma\tau}(s,t)] &= \delta_{\mu\tau}\delta_{mt}D_{\sigma\nu}(s,n)\\ &\quad - \delta_{\sigma\nu}\delta_{sn}D_{\mu\tau}(m,t)\end{aligned} \tag{3.17}$$

holds between the density matrix operator, as is easily derived from the anticommutation relation (3.6). We combine this with the magnetic-translation group property,

$$e^{i\mathbf{k}\mathbf{X}}e^{i\mathbf{k}'\mathbf{X}} = e^{i(\mathbf{k}+\mathbf{k}')\mathbf{X}}\exp\left[\frac{i}{2}\theta\mathbf{k}\wedge\mathbf{k}'\right], \tag{3.18}$$

which summarizes the noncommutativity of the plane. In this way we obtain

$$2\pi[\hat{D}_{\mu\nu}(\mathbf{k}), \hat{D}_{\sigma\tau}(\mathbf{k}')] = \delta_{\mu\tau}e^{+(i/2)\theta\mathbf{k}\wedge\mathbf{k}'}\hat{D}_{\sigma\nu}(\mathbf{k}+\mathbf{k}') - \delta_{\sigma\nu}e^{-(i/2)\theta\mathbf{k}\wedge\mathbf{k}'}\hat{D}_{\mu\tau}(\mathbf{k}+\mathbf{k}') \tag{3.19}$$

in terms of the bare operator (3.13). We may rewrite it as

$$[\hat{\rho}(\mathbf{k}), \hat{\rho}(\mathbf{k}')] = \frac{i}{\pi}\hat{\rho}(\mathbf{k}+\mathbf{k}')\sin\left(\theta\frac{\mathbf{k}\wedge\mathbf{k}'}{2}\right), \tag{3.20a}$$

$$[\hat{S}_a(\mathbf{k}), \hat{\rho}(\mathbf{k}')] = \frac{i}{\pi}\hat{S}_a(\mathbf{k}+\mathbf{k}')\sin\left(\theta\frac{\mathbf{k}\wedge\mathbf{k}'}{2}\right), \tag{3.20b}$$

$$[\hat{S}_a(\mathbf{k}), \hat{S}_b(\mathbf{k}')] = \frac{i}{2\pi}f_{abc}\hat{S}_c(\mathbf{k}+\mathbf{k}')\cos\left(\theta\frac{\mathbf{k}\wedge\mathbf{k}'}{2}\right) + \frac{i}{2\pi}d_{abc}\hat{S}_c(\mathbf{k}+\mathbf{k}')\sin\left(\theta\frac{\mathbf{k}\wedge\mathbf{k}'}{2}\right) + \frac{i}{2\pi N}\delta_{ab}\hat{\rho}(\mathbf{k}+\mathbf{k}')\sin\left(\theta\frac{\mathbf{k}\wedge\mathbf{k}'}{2}\right), \tag{3.20c}$$

where the summation over the repeated isospin index is understood. We have named this the $W_\infty(N)$ algebra [24] since it is the $SU(N)$ extension of W_∞. Its physical implication is an intrinsic entanglement between the electron density and the isospin density. The entanglement is removed in the commutative limit, $\theta \to 0$, where it is reduced to the U(1) ⊗ SU(N) algebra.

IV. NONCOMMUTATIVE KINEMATICS

In this section we study how the noncommutativity presented in microscopic theory generates a constraint on classical objects. We first define the class of Fock states we work with and then derive a noncommutative relation satisfied by the expectation value of the bare density. We subsequently introduce the noncommutative CP^{N-1} field by resolving the noncommutative constraint, and discuss topological aspects with the use of the noncommutative CP^{N-1} field.

A. Fock states

We consider the class of Fock states which can be written as

$$|\mathfrak{S}\rangle = e^{iW}|\mathfrak{S}_0\rangle, \tag{4.1}$$

where W is an arbitrary element of the algebra $W_\infty(N)$ which represents a general linear combination of the operators (3.10). The state $|\mathfrak{S}_0\rangle$ is assumed to be of the form

$$|\mathfrak{S}_0\rangle = \prod_{\mu,n}[c^\dagger_\mu(n)]^{\nu_\mu(n)}|0\rangle, \tag{4.2}$$

where $\nu_\mu(n)$ takes the value either 0 or 1 specifying whether the isospin state μ at the Landau site n is occupied or not, respectively.

The amount of electrons at the Landau site n is given by

$$\nu(n) = \sum_{\mu=1}^{N}\nu_\mu(n), \tag{4.3}$$

and may take a value from 0 up to N. The filling factor is defined by

$$\nu = \frac{1}{N_\Phi}\sum_{m=0}^{N_\Phi-1}\nu(m), \tag{4.4}$$

where the thermodynamical limit $N_\Phi \to \infty$ is implied. The electron density is homogeneous in the ground state, $\nu(n) = \text{constant}$.

The electron number of the $W_\infty(N)$-rotated state (4.1) is easily calculable,

$$\langle\mathfrak{S}|N_e|\mathfrak{S}\rangle = \langle\mathfrak{S}_0|e^{-iW}N_e e^{+iW}|\mathfrak{S}_0\rangle = \langle\mathfrak{S}_0|N_e|\mathfrak{S}_0\rangle, \tag{4.5}$$

since the total electron number

$$N_e = \sum_n\sum_\mu c^\dagger_\mu(n)c_\mu(n) \tag{4.6}$$

is a Casimir operator. The electron number of the state $|\mathfrak{S}\rangle$ is the same as that of the state $|\mathfrak{S}_0\rangle$. We set

$$\Delta N_e^{cl} = \langle\mathfrak{S}|N_e|\mathfrak{S}\rangle - \langle g|N_e|g\rangle = \int d^2x\Delta\hat{\rho}^{cl}(\mathbf{x}), \tag{4.7}$$

where $\langle g|N_e|g\rangle$ is the total electron number in the ground state $|g\rangle$, and ΔN_e^{cl} is the number of extra electrons carried by the excitation described by the state $|\mathfrak{S}\rangle$. It is an integer since $\langle\mathfrak{S}_0|N_e|\mathfrak{S}_0\rangle$ is an integer as well as $\langle g|N_e|g\rangle$.

The class of states of the form (4.1) together with (4.2) does not embrace the whole Fock space. Nevertheless, it is general enough to cover all physically relevant cases in our analysis where the filling factor ν is taken to be an integer. Indeed, as far as we know, perturbative excitations are isospin waves and nonperturbative excitations are Skyrmions in QH systems. The corresponding states belong surely to this category.

B. Noncommutative constraints

The classical field is the expectation value of a second quantized operator such as

$$\hat{D}^{cl}_{\mu\nu}(\mathbf{x}) \equiv \langle\mathfrak{S}|\hat{D}_{\mu\nu}(\mathbf{x})|\mathfrak{S}\rangle. \tag{4.8}$$

From (3.13) we have

$$\hat{D}^{cl}_{\mu\nu}(\mathbf{k}) = \frac{1}{2\pi}\sum_{mn}\langle n|e^{-i\mathbf{k}\mathbf{X}}|m\rangle D^{cl}_{\mu\nu}(m,n), \tag{4.9}$$

where

$$D^{cl}_{\mu\nu}(m,n) = \langle\mathfrak{S}|c^\dagger_\nu(n)c_\mu(m)|\mathfrak{S}\rangle. \tag{4.10}$$

It is identical to (2.14) when we set $f(\mathbf{k}) = \hat{D}^{cl}_{\mu\nu}(\mathbf{k})$ and

$O_{mn}[f] = D^{\rm cl}_{\mu\nu}(m,n)/2\pi\theta$. Hence, from the matrix element $D^{\rm cl}_{\mu\nu}(m,n)$ we may construct a Weyl-ordered operator whose symbol is $\hat{D}^{\rm cl}_{\mu\nu}(\mathbf{x})$.

As we derive in the Appendix the classical density satisfies the relation

$$\sum_{\sigma,s} D^{\rm cl}_{\mu\sigma}(m,s)D^{\rm cl}_{\sigma\nu}(s,n) = D^{\rm cl}_{\mu\nu}(m,n). \tag{4.11}$$

In terms of symbols it reads

$$\sum_{\sigma=1}^{N} \hat{D}^{\rm cl}_{\mu\sigma}(\mathbf{x}) \star \hat{D}^{\rm cl}_{\sigma\nu}(\mathbf{x}) = \frac{1}{2\pi\theta}\hat{D}^{\rm cl}_{\mu\nu}(\mathbf{x}). \tag{4.12}$$

The noncommutativity encoded in (3.2a) has become perceptible at the classical level in terms of the bare quantity $\hat{D}^{\rm cl}_{\mu\nu}(\mathbf{x})$. This is the noncommutative constraint on the bare density.

We may rewrite (4.12) as the constraints on the bare densities $\hat{\rho}^{\rm cl}(\mathbf{x})$ and $\hat{S}^{\rm cl}_a(\mathbf{x})$ given by

$$\hat{D}^{\rm cl}_{\mu\nu}(\mathbf{x}) = \frac{1}{N}\delta_{\mu\nu}\hat{\rho}^{\rm cl}(\mathbf{x}) + (\lambda_a)_{\mu\nu}\hat{S}^{\rm cl}_a(\mathbf{x}), \tag{4.13}$$

where $\hat{\rho}^{\rm cl}(\mathbf{x}) \equiv \langle\mathfrak{S}|\hat{\rho}(\mathbf{x})|\mathfrak{S}\rangle$ and $\hat{S}^{\rm cl}_a(\mathbf{x}) \equiv \langle\mathfrak{S}|\hat{S}_a(\mathbf{x})|\mathfrak{S}\rangle$. Substituting this into (4.12) and using the properties of Gell-Mann matrices we come to

$$\hat{S}^{\rm cl}_a(\mathbf{x}) \star \hat{S}^{\rm cl}_a(\mathbf{x}) = \frac{1}{4\pi\theta}\hat{\rho}^{\rm cl}(\mathbf{x}) - \frac{1}{2N}\hat{\rho}^{\rm cl}(\mathbf{x}) \star \hat{\rho}^{\rm cl}(\mathbf{x}), \tag{4.14a}$$

$$\frac{1}{2}(if_{abc} + d_{abc})\hat{S}^{\rm cl}_b(\mathbf{x}) \star \hat{S}^{\rm cl}_c(\mathbf{x}) = \frac{1}{4\pi\theta}\hat{S}^{\rm cl}_a(\mathbf{x}) - \frac{1}{2N}\hat{\rho}^{\rm cl}(\mathbf{x}) \star \hat{S}^{\rm cl}_a(\mathbf{x}) - \frac{1}{2N}\hat{S}^{\rm cl}_a(\mathbf{x}) \star \hat{\rho}^{\rm cl}(\mathbf{x}). \tag{4.14b}$$

These two relations are the manifestation of the microscopic noncommutativity at the level of the classical fields.

C. Noncommutative CP^{N-1} field

Because of the involution of the $\star$ product in (4.14), the structure of the fields $\hat{\rho}^{\rm cl}(\mathbf{x})$ and $\hat{S}^{\rm cl}_a(\mathbf{x})$ is very complicated, and a comprehensive analysis of these equations is quite far from being trivial. We investigate relatively simple cases of integer filling factors. In the physics of QH systems the integer value of ν is related to the integer quantum Hall effect, which is much simpler in structure than the fractional one realized at a certain fractional value of ν. In this case we are able to present a systematic way of resolving the noncommutative relations (4.12) or (4.14).

We introduce the ν amount ($r = 1, \ldots, \nu$) of fields $n^r(\mathbf{x})$ each representing a column with N complex components. They are required to be orthonormal

$$\bar{n}^{\,r}_{\mu}(\mathbf{x}) \star n^s_\mu(\mathbf{x}) = \delta^{rs}, \tag{4.15}$$

where $\bar{n}$ denotes the complex conjugate of n. Here and hereafter the summation over the repeated isospin index μ is understood. Then it is trivial to see that the relation (4.12) is resolved by

$$\hat{D}^{\rm cl}_{\mu\nu}(\mathbf{x}) = \frac{1}{2\pi\theta}\sum_{s=1}^{\nu} n^s_\mu(\mathbf{x}) \star \bar{n}^s_\nu(\mathbf{x}), \tag{4.16}$$

or

$$\hat{\rho}^{\rm cl}(\mathbf{x}) = \frac{1}{2\pi\theta}\sum_s n^s_\mu(\mathbf{x}) \star \bar{n}^s_\mu(\mathbf{x}), \tag{4.17a}$$

$$\hat{S}^{\rm cl}_a(\mathbf{x}) = \frac{1}{4\pi\theta}\sum_s (\lambda_a)_{\mu\nu} n^s_\nu(\mathbf{x}) \star \bar{n}^s_\mu(\mathbf{x}). \tag{4.17b}$$

Changing the order of multiplicatives in (4.17a) and using (4.15) we obtain

$$\hat{\rho}^{\,\rm cl}(\mathbf{x}) = \frac{\nu}{2\pi\theta} + \frac{1}{2\pi\theta}\sum_s [n^s_\mu(\mathbf{x}), \bar{n}^s_\mu(\mathbf{x})]_\star. \tag{4.18}$$

The first term represents the homogeneous part of the electron density. Note that $\rho_0 \equiv \nu\rho_\Phi = \nu/2\pi\theta$ is the electron density in the ground state. Comparing this with (4.7) we find

$$\Delta\hat{\rho}^{\rm cl}(\mathbf{x}) = \frac{1}{2\pi\theta}\sum_s [n^s_\mu(\mathbf{x}), \bar{n}^s_\mu(\mathbf{x})]_\star, \tag{4.19}$$

which is the density excitation from the ground state realized in the state (4.1).

At the filling factor $\nu = 1$, which we study mostly in what follows, we have a single N-component column $n(\mathbf{x})$ satisfying

$$\bar{n}_\mu(\mathbf{x}) \star n_\mu(\mathbf{x}) = 1, \tag{4.20}$$

and the bare density reads

$$\hat{D}^{\rm cl}_{\mu\nu}(\mathbf{x}) = \frac{1}{2\pi\theta} n_\mu(\mathbf{x}) \star \bar{n}_\nu(\mathbf{x}), \tag{4.21}$$

or

$$\hat{\rho}^{\rm cl}(\mathbf{x}) = \frac{1}{2\pi\theta} n_\mu(\mathbf{x}) \star \bar{n}_\mu(\mathbf{x}), \tag{4.22a}$$

$$\hat{S}^{\rm cl}_a(\mathbf{x}) = \frac{1}{4\pi\theta}(\lambda_a)_{\mu\nu} n_\nu(\mathbf{x}) \star \bar{n}_\mu(\mathbf{x}). \tag{4.22b}$$

They are represented as

$$\mathfrak{n}^\dagger_\mu \mathfrak{n}_\mu = 1, \tag{4.23}$$

and

$$O[\hat{D}^{\rm cl}_{\mu\nu}] = \frac{1}{2\pi\theta}\mathfrak{n}_\mu \mathfrak{n}^\dagger_\nu \tag{4.24}$$

in terms of the Weyl-ordered operator $\mathfrak{n}_\mu$ whose symbol is the CP^{N-1} field $n_\mu(\mathbf{x})$, $\mathfrak{n}_\mu = O[n_\mu]$.

We study a local noncommutative phase transformation of $n_\mu(\mathbf{x})$. Let $U(\mathbf{x})$ be a complex valued function satisfying

$$\bar{U}(\mathbf{x}) \star U(\mathbf{x}) = U(\mathbf{x}) \star \bar{U}(\mathbf{x}) = 1, \tag{4.25}$$

which is constructed as

$$U = e_\star^{i\xi} \equiv 1 + i\xi + \frac{i^2}{2!}\xi \star \xi + \cdots \tag{4.26}$$

with $\xi(\mathbf{x})$ being an arbitrary function. Then the local $U(1)$ transformation of $n_\mu(\mathbf{x})$ is given by

$$n_\mu(\mathbf{x}) \to n'_\mu(\mathbf{x}) \equiv n_\mu(\mathbf{x}) \star U(\mathbf{x}), \tag{4.27a}$$

$$\bar{n}_\mu(\mathbf{x}) \to \bar{n}'_\mu(\mathbf{x}) \equiv \bar{U}(\mathbf{x}) \star \bar{n}_\mu(\mathbf{x}), \tag{4.27b}$$

which are consistent as a matter of $\overline{(f \star h)} = \bar{h} \star \bar{f}$. The classical density (4.22) as well as the noncommutative normalization condition (4.20) are invariant under the transformation.

In general, at the integer filling factor ν, we introduce the Grassmannian $G_{N,\nu}$ field [28] as a set of the ν amount of the CP^{N-1} fields,

$$Z(\mathbf{x}) = (n_\mu^1, n_\mu^2, \cdots, n_\mu^\nu). \tag{4.28}$$

Under a local $U(\mathbf{x})$ transformation,

$$Z(\mathbf{x}) \to Z'(\mathbf{x}) \equiv Z(\mathbf{x}) \star U(\mathbf{x}), \tag{4.29}$$

the classical density (4.17) as well as the noncommutative normalization condition (4.15) are invariant.

D. Topological charge

There exist several equivalent ways of defining the topological charge associated with the CP^{N-1} field in the ordinary theory on the commutative plane. One of them reads

$$Q = \frac{1}{2\pi i}\epsilon_{kl}\sum_s \int d^2x (\partial_k \bar{n}_\mu^s)(\partial_l n_\mu^s). \tag{4.30}$$

We generalize it to the noncommutative theory as

$$Q = \frac{1}{2\pi\theta}\sum_s \int d^2x [\bar{n}_\mu^s(\mathbf{x}), n_\mu^s(\mathbf{x})]_\star, \tag{4.31}$$

since the Moyal bracket is expanded as

$$[\bar{n}_\mu^s(\mathbf{x}), n_\mu^s(\mathbf{x})]_\star = -i\theta\epsilon_{jk}\partial_j \bar{n}_\mu^s(\mathbf{x})\partial_k n_\mu^s(\mathbf{x}) + O(\theta^2), \tag{4.32}$$

and the quantity (4.31) is reduced to the topological charge (4.30) in the commutative limit ($\theta \to 0$).

We now show that the quantity (4.31) satisfies all standard requirements that the topological charge should meet. The topological charge density is

$$J_0(\mathbf{x}) = \frac{1}{2\pi\theta}\sum_{s,\mu}[\bar{n}_\mu^s(\mathbf{x}), n_\mu^s(\mathbf{x})]_\star. \tag{4.33}$$

First, due to an important property of the star product such that

$$f(\mathbf{x}) \star g(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x}) + \partial_k \Lambda_k(\mathbf{x}), \tag{4.34}$$

the density is expressed as $J_0(\mathbf{x}) = \partial_k \Lambda_k(\mathbf{x})$ with a certain function $\Lambda_k(\mathbf{x})$. Thus (4.31) reads

$$Q = \int d^2x J_0(\mathbf{x}) = \oint dx_k \Lambda_k(\mathbf{x}), \tag{4.35}$$

where the contour integration is taken along a circle of an infinite radius. At the spatial infinity, the normalization (4.20) turns into a commutative condition (since the derivatives must vanish), and the asymptotic behavior is given by $n_\mu(\mathbf{x}) \sim r^0$, yielding $\Lambda_k \sim r^{-1}$. Thus, $Q \sim \oint d\vartheta$ with ϑ the azimuthal angle. The integration over the angle gives the number of windings. This is the standard way how the number of windings appear.

We next examine the conservation law. We obtain

$$\begin{aligned}\partial_0 J_0 &= \frac{1}{2\pi\theta}\sum_{s,\mu}[\partial_0 \bar{n}_\mu^s(\mathbf{x}), n_\mu^s(\mathbf{x})]_\star \\ &\quad + \frac{1}{2\pi\theta}\sum_{s,\mu}[\bar{n}_\mu^s(\mathbf{x}), \partial_0 n_\mu^s(\mathbf{x})]_\star \\ &= \partial_k J_k,\end{aligned} \tag{4.36}$$

where the existence of J_k is guaranteed by the property (4.34). Based on dimensional considerations, the quantity J_k behaves as $J_k \sim r^{-2}$ asymptotically, implying the vanishing of the corresponding surface integral, and the conservation of the topological charge, $\partial_0 Q = 0$, holds.

As we have stated, there are several ways of defining the topological charge Q. Needless to say, they are all equivalent. Nevertheless the topological charge densities are different, and, in general, they have no physical meaning. Among them the topological charge density (4.33) has a privileged role that it is essentially the electron density excitation induced by the topological soliton,

$$J_0(\mathbf{x}) = -\Delta\hat{\rho}^{\text{cl}}(\mathbf{x}), \tag{4.37}$$

as follows from the comparison of (4.33) with (4.19). Hence the topological charge is given by

$$Q = -\int d^2x \Delta\hat{\rho}^{\text{cl}}(\mathbf{x}) = -\Delta N_{\text{e}}^{\text{cl}}, \tag{4.38}$$

which is an integer. Consequently the noncommutative soliton carries necessarily the electric charge proportional to its topological charge. This is a very peculiar phenomenon that occurs in the noncommutative theory.

V. HAMILTONIAN

We have so far been concerned with the kinematical properties of the noncommutative electron system. Recall that the LLL projection quenches the kinetic Hamiltonian (3.3) and brings about the noncommutativity into the system. We now investigate interactions between electrons which organize the system dynamically.

A. Four-fermion interactions

We take a four-fermion interaction term,

$$H_{\mathrm{V}} = \frac{1}{2}\int d^2x d^2y V(\mathbf{x}-\mathbf{y})\Delta\rho(\mathbf{x})\Delta\rho(\mathbf{y}), \tag{5.1}$$

where $\Delta\rho(\mathbf{x})$ is the density excitation operator,

$$\Delta\rho(\mathbf{x}) = \rho(\mathbf{x}) - \rho_0, \tag{5.2}$$

and ρ_0 denotes the homogeneous density of charges providing the electric neutrality in the ground state. In the real QH system $V(\mathbf{x})$ is given by the Coulomb potential, but here we only assume that it represents a repulsive interaction.

As we have already mentioned, we consider the system at an integer filling factor ν. Each Landau site can accommodate up to N electrons of different isospins. The cases of $\nu = 0$ and $\nu = N$ are trivial since the isospin polarization is zero, where Landau sites are either all empty or all occupied. Nontrivial filling factors are $\nu = 1, \ldots, N-1$. The fillings ν and $N-\nu$ are equivalent in the sense of particle-hole symmetry.

Substituting the density operators (3.5) into (5.1) we obtain

$$H = \sum_{mnij} V_{mnij} D_{\mu\mu}(n,m) D_{\nu\nu}(j,i) - \nu(N_{\mathrm{e}} + \Delta N_{\mathrm{e}})\epsilon_{\mathrm{D}}, \tag{5.3}$$

where $N_{\mathrm{e}} = \nu N_\Phi + \Delta N_{\mathrm{e}}$, and

$$V_{mnij} = \frac{1}{4\pi}\int d^2k V(\mathbf{k}) e^{-\theta\mathbf{k}^2/2}\langle m|e^{i\mathbf{k}\mathbf{X}}|n\rangle\langle i|e^{-i\mathbf{k}\mathbf{X}}|j\rangle, \tag{5.4}$$

with $V(\mathbf{k})$ the Fourier transformation of the potential $V(\mathbf{x})$. We have used the notation

$$\Delta N_{\mathrm{e}} = \int d^2x \Delta\rho(\mathbf{x}), \tag{5.5}$$

$$\epsilon_{\mathrm{D}} = \sum_j V_{nnjj} = \frac{\rho_\Phi}{2}\int d^2x V(\mathbf{x}). \tag{5.6}$$

For later convenience we also define

$$\epsilon_{\mathrm{X}} = \sum_j V_{njjn} = \frac{\rho_\Phi}{2}\theta\int d^2k V(\mathbf{k}) e^{-\theta\mathbf{k}^2/2}. \tag{5.7}$$

Here, ϵ_{D} and ϵ_{X} are the direct and exchange energy parameters, respectively. Note that $\sum_j V_{nnjj}$ and $\sum_j V_{njjn}$ are independent of n, and the summation is taken over j with n arbitrarily fixed.

B. Spontaneous symmetry breaking

Using the four-fermion interaction Hamiltonian (5.1) we evaluate the energy of the state $|\mathfrak{S}\rangle$ in the class (4.1). In the Appendix we derive the decomposition formula,

$$E_{\mathrm{V}} \equiv \langle\mathfrak{S}|H_{\mathrm{V}}|\mathfrak{S}\rangle = E_{\mathrm{D}} + E_{\mathrm{X}}, \tag{5.8}$$

where

$$E_{\mathrm{D}} = V_{mnij} D^{\mathrm{cl}}_{\mu\mu}(n,m) D^{\mathrm{cl}}_{\nu\nu}(j,i) - \nu(N^{\mathrm{cl}}_{\mathrm{e}} + \Delta N^{\mathrm{cl}}_{\mathrm{e}})\epsilon_{\mathrm{D}}, \tag{5.9a}$$

$$E_{\mathrm{X}} = -V_{mnij} D^{\mathrm{cl}}_{\mu\nu}(j,m) D^{\mathrm{cl}}_{\nu\mu}(n,i) + N^{\mathrm{cl}}_{\mathrm{e}}\epsilon_{\mathrm{X}}, \tag{5.9b}$$

with $N^{\mathrm{cl}}_{\mathrm{e}} = \nu N_\Phi + \Delta N^{\mathrm{cl}}_{\mathrm{e}}$. Here E_{D} and E_{X} represent the direct and exchange energies.

It is straightforward to represent them in the momentum space by using (4.9) and (5.4),

$$E_{\mathrm{D}} = \pi\int d^2k V(\mathbf{k}) e^{-(1/2)\theta k^2}|\Delta\hat{\rho}^{\mathrm{cl}}(\mathbf{k})|^2, \tag{5.10a}$$

$$E_{\mathrm{X}} = \pi\int d^2k \Delta V_{\mathrm{X}}(\mathbf{k})\left[|\hat{S}^{\mathrm{cl}}_a(\mathbf{k})|^2 + \frac{1}{2N}|\hat{\rho}^{\mathrm{cl}}(\mathbf{k})|^2\right], \tag{5.10b}$$

where $\Delta V_{\mathrm{X}}(\mathbf{k}) = V_{\mathrm{X}}(0) - V_{\mathrm{X}}(\mathbf{k})$ with

$$V_{\mathrm{X}}(\mathbf{k}) \equiv \frac{\theta}{\pi}\int d^2k' e^{-i\theta\mathbf{k}\wedge\mathbf{k}'} e^{-(1/2)\theta k'^2} V(\mathbf{k}'). \tag{5.11}$$

Note that both the energies are positive semidefinite since $V(\mathbf{k}) > 0$ and $\Delta V_{\mathrm{X}}(\mathbf{k}) > 0$ for a repulsive interaction. Here, $\hat{\rho}^{\mathrm{cl}}(\mathbf{k}) = \langle\mathfrak{S}|\hat{\rho}(\mathbf{k})|\mathfrak{S}\rangle$ and $\hat{S}^{\mathrm{cl}}_a(\mathbf{k}) = \langle\mathfrak{S}|\hat{S}_a(\mathbf{k})|\mathfrak{S}\rangle$. It is remarkable that, though the Hamiltonian (5.1) involves no isospin variables, the energy (5.8) of a state does. The direct energy E_{D} is insensitive to isospin orientations, and it vanishes for the homogeneous electron distribution since $\Delta\hat{\rho}^{\mathrm{cl}}(\mathbf{k}) = 0$. The exchange energy E_{X} depends on isospin orientations. The isospin texture is homogeneous when the isospin is completely polarized, where $\hat{S}^{\mathrm{cl}}_a(\mathbf{k}) \propto \delta(\mathbf{k})$. Furthermore, $\hat{\rho}^{\mathrm{cl}}(\mathbf{k}) \propto \delta(\mathbf{k})$ due to the homogeneous electron distribution. For such an isospin orientation the exchange energy also vanishes since $\Delta V_{\mathrm{X}}(0) = 0$ in (5.10b). On the other hand, $E_{\mathrm{X}} > 0$ if the isospin is not polarized completely since $\hat{S}^{\mathrm{cl}}_a(\mathbf{k})$ contains nonzero momentum components. Consequently the isospin-polarized state has the lowest energy, which is zero. Namely, all isospins are spontaneously polarized along an arbitrarily chosen direction. It implies a spontaneous symmetry breaking due to the exchange interaction implicit in the Hamiltonian (5.1).

C. Ground state and Goldstone modes

The ground state is a spontaneously symmetry broken one, where the electron density is homogenous, $\rho^{\mathrm{cl}}(\mathbf{x}) = \rho_0 \equiv \nu/2\pi\theta$, and the isospin field is additionally a constant, $\hat{S}_a^{\rm cl}(\mathbf{x}) = S_a^0/2\pi\theta$. Substituting these into the noncommutative constraint (4.14) we find

$$S_a^0 S_a^0 = \frac{\nu(N-\nu)}{2N}, \tag{5.12a}$$

$$d_{abc} S_b^0 S_c^0 = \frac{N-2\nu}{N} S_a^0. \tag{5.12b}$$

At the filling factor ν the isospin is normalized on the ground state in this way. The asymptotic behavior of the topological soliton is determined by these equations.

We study small perturbative fluctuations of the isospin on the ground state. We may keep the electron density unperturbed, $\rho^{\rm cl}(\mathbf{x}) = \rho_0$, to minimize the direct energy (5.10a). Setting

$$\hat{S}_a^{\rm cl}(\mathbf{x}) = S_a(\mathbf{x})/2\pi\theta, \tag{5.13}$$

and substituting these into (4.14), we find

$$S_a(\mathbf{x}) \star S_a(\mathbf{x}) = \frac{\nu(N-\nu)}{2N}, \tag{5.14a}$$

$$(if_{abc} + d_{abc}) S_b(\mathbf{x}) \star S_c(\mathbf{x}) = \frac{N-2\nu}{N} S_a(\mathbf{x}). \tag{5.14b}$$

The Goldstone modes, which are small fluctuations of the isospins, are subject to these constraints. The exchange energy (5.10b) may be used as the effective Hamiltonian for the Goldstone modes. However, there are $N^2 - 1$ real components in $S_a(\mathbf{x})$, but not all of them are independent. Because of this fact, except for the case of $SU(2)$, it is quite difficult to analyze Goldstone modes based on (5.10b). Later we shall derive the effective Hamiltonian in terms of the CP^{N-1} field: See (9.19).

VI. QUANTUM HALL SKYRMIONS

We proceed to analyze topological excitations. As we have shown, a topological soliton necessarily carries the electron number. The simplest charge excitation is the hole excitation. It is curious but, as we shall soon see, a hole is a kind of Skyrmion in the noncommutative plane. A generic Skyrmion state is constructed as a $W_\infty(N)$-rotated state of the hole state. We start with a Skyrmion with $Q = 1$, and subsequently pass to a multi-Skyrmion with $Q = k$. It is to be remarked that the Hamiltonian is not necessary to develop a theory of Skyrmions. What we use is only the fact that the ground state is a spontaneously symmetry broken one.

A. Microscopic Skyrmions

The ground state is expressed as

$$|{\rm g}\rangle = \prod_{n=0}^{\infty} g_\mu c_\mu^\dagger(n)|0\rangle, \tag{6.1}$$

where a constant vector g_μ stands for the isospin component spontaneously chosen, obeying the normalization condition

$$\bar{g}_\mu g_\mu = 1. \tag{6.2}$$

The ground state satisfies

$$\rho(m,n)|{\rm g}\rangle = \delta_{mn}|{\rm g}\rangle, \tag{6.3a}$$

$$\langle {\rm g}|S_a(m,n)|{\rm g}\rangle = \frac{1}{2}\delta_{mn}\bar{g}_\mu(\lambda_a)_{\mu\nu}g_\nu. \tag{6.3b}$$

The simplest charge excitation is a hole excitation given by

$$|{\rm h}\rangle = \bar{g}_\mu c_\mu(0)|{\rm g}\rangle, \tag{6.4}$$

with its energy $\langle {\rm h}|H_{\rm V}|{\rm h}\rangle = \epsilon_{\rm X}$.

We consider a $W_\infty(N)$-rotated state of the hole state $|{\rm h}\rangle$,

$$|\mathfrak{S}_{\rm sky}\rangle = e^{iW}|{\rm h}\rangle = e^{iW}\prod_{n=0}^{\infty} g_\mu c_\mu^\dagger(n+1)|0\rangle, \tag{6.5}$$

where W is an element of the $W_\infty(N)$ algebra (3.19). As we have noticed in (4.7), the electron number of this state is the same as that of the hole state, or $\Delta N_{\rm e}^{\rm cl} = -1$.

The simplest $W_\infty(N)$ rotation mixes only the nearest neighboring sites, and is given by the choice of $W = \sum_{n=0}^{\infty} W_n$ with

$$iW_n = \alpha_n h_\mu g_\nu^* c_\mu^\dagger(n) c_\nu(n+1) - \alpha_n h_\mu^* g_\nu c_\nu^\dagger(n+1) c_\mu(n), \tag{6.6}$$

where α_n is a real parameter, and h_μ is a constant vector orthogonal to g_μ,

$$\bar{h}_\mu h_\mu = 1, \qquad \bar{g}_\mu h_\mu = \bar{h}_\mu g_\mu = 0. \tag{6.7}$$

Note that W_n is a Hermitian operator belonging to the $W_\infty(N)$ algebra. It is to be remarked that $[W_n, W_m] = 0$, as follows from (6.7) in the case of $m = n+1$, and trivially in all other cases. We define

$$\xi^\dagger(n) = e^{+iW} g_\mu c_\mu^\dagger(n+1) e^{-iW}. \tag{6.8}$$

It is easy to show that

$$\xi^\dagger(n) = u_\mu(n) c_\mu^\dagger(n) + v_\mu(n) c_\mu^\dagger(n+1), \tag{6.9}$$

where we have set

$$u_\mu(n) = h_\mu \sin\alpha_n, \qquad v_\mu(n) = g_\mu \cos\alpha_n. \tag{6.10}$$

The conditions

$$\bar{u}_\mu(n) u_\mu(n) + \bar{v}_\mu(n) v_\mu(n) = 1, \tag{6.11a}$$

$$\bar{v}_\mu(n) u_\mu(n+1) = 0, \tag{6.11b}$$

follow on $u_\mu(n)$ and $v_\mu(n)$. It is necessary that

$$\lim_{n\to\infty} u_\mu(n) = 0, \qquad \lim_{n\to\infty} v_\mu(n) = g_\mu, \tag{6.12}$$

since the Skyrmion state should approach the ground state asymptotically.

The operator $\xi(m)$ satisfies the standard canonical anticommutation relation,

$$\{\xi(m), \xi^\dagger(n)\} = \delta_{mn}, \qquad \{\xi(m), \xi(n)\} = 0, \tag{6.13}$$

as is verified with the use of the condition (6.11). The hole state is a special limit of the Skyrmion state with $u_\mu(n) = 0$ and $v_\mu(n) = g_\mu$ for all n.

It is now easy to see that the $W_\infty(N)$-rotated state (6.5) is rewritten as

$$|\mathfrak{S}_{\rm sky}\rangle = \prod_{n=0}^{\infty} \xi^\dagger(n)|0\rangle \tag{6.14}$$

by the use of (6.8) and $e^{-iW_n}|0\rangle = 0$. This agrees with the Skyrmion state [20] proposed to carry out a Hartree-Fock approximation.

Multi-Skyrmion states are given by (6.14) with

$$\xi^\dagger(n) = u_\mu(n)c_\mu^\dagger(n) + v_\mu(n)c_\mu^\dagger(n+k), \tag{6.15}$$

where k indicates the amount of Skyrmions. The normalization condition is still given by (6.11a) but (6.11b) replaced by $\bar{v}_\mu(n)u_\mu(n+k) = 0$. In a single Skyrmion case the nearest neighboring sites are mixed in operators $\xi(n)$. For $k = 2$ the mixing appears among the sites with even n and separately among the ones with odd n, while no mixing appears among even and odd sites. In this sense there arise two separate chains of mixed sites, where the above steps can be carried out separately. In the case of general k there are k separate chains, so nothing essentially new occurs, and the analog of (6.5) appears as

$$|\mathfrak{S}_{\rm sky}\rangle = e^{iW}\prod_{n=0}^{\infty} g_\mu c_\mu^\dagger(n+k)|0\rangle = e^{iW}\prod_{n=0}^{k-1} \bar{g}_\mu c_\mu(n)|{\rm g}\rangle, \tag{6.16}$$

where $|{\rm g}\rangle$ is the ground state given by (6.1). The electron number is evidently given by $\Delta N_{\rm e}^{\rm cl} = -k$.

B. Noncommutative CP^{N-1} Skyrmions

We construct the noncommutative CP^{N-1} field describing the multi-Skyrmion state (6.14) with (6.15). For this purpose we calculate the quantities

$$D_{\mu\nu}^{\rm cl}(m,n) = \langle\mathfrak{S}_{\rm sky}|c_\nu^\dagger(n)c_\mu(m)|\mathfrak{S}_{\rm sky}\rangle \tag{6.17}$$

for the multi-Skyrmion state. It is straightforward to show

$$c_\mu(m)|\mathfrak{S}_{\rm sky}\rangle = [u_\mu(m)\xi(m) + v_\mu(m-k)\xi(m-k)]|\mathfrak{S}_{\rm sky}\rangle, \tag{6.18}$$

where the convention $v_\mu(m) = 0$ for $m < 0$ is introduced. We then use

$$\langle\mathfrak{S}_{\rm sky}|\xi^\dagger(m)\xi(n)|\mathfrak{S}_{\rm sky}\rangle = \delta_{mn} \tag{6.19}$$

together with (6.18), to find the only nonvanishing values of $D_{\mu\nu}^{\rm cl}(m,n)$ to be

$$\begin{aligned} D_{\mu\nu}^{\rm cl}(n,n) &= u_\mu(n)\bar{u}_\nu(n) + v_\mu(n-k)\bar{v}_\nu(n-k), \\ D_{\mu\nu}^{\rm cl}(n,n+k) &= u_\mu(n)\bar{v}_\nu(n), \\ D_{\mu\nu}^{\rm cl}(n+k,n) &= v_\mu(n)\bar{u}_\nu(n). \end{aligned} \tag{6.20}$$

This gives the Weyl-ordered operator as

$$\begin{aligned} 2\pi\theta O[D_{\mu\nu}^{\rm cl}] = {}& [u_\mu(n)\bar{u}_\nu(n) + v_\mu(n-k)\bar{v}_\nu(n-k)]|n\rangle \\ &\times \langle n| + u_\mu(n)\bar{v}_\nu(n)|n\rangle\langle n+k| \\ &+ v_\mu(n)\bar{u}_\nu(n)|n+k\rangle\langle n|. \end{aligned} \tag{6.21}$$

Comparing this with (4.24) we uniquely come to

$$\mathfrak{n}_\mu = \sum_{n=0}^{\infty}[u_\mu(n)|n\rangle\langle n| + v_\mu(n)|n+k\rangle\langle n|], \tag{6.22}$$

whose symbol is [27]

$$\begin{aligned} n_\mu(\mathbf{x}) = {}& 2e^{-\bar{z}z}\sum_{n=0}^{\infty}(-1)^n u_\mu(n)L_n(2\bar{z}z) \\ &+ 2^{(k/2)+1}z^k e^{-\bar{z}z}\sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{n!}}{\sqrt{(n+k)!}}v_\mu(n)L_n^k(2\bar{z}z). \end{aligned} \tag{6.23}$$

This is the noncommutative CP^{N-1} field describing the multi-Skyrmion state.

C. Topological charge of Skyrmions

We have already shown that the topological charge is given by $Q = -\Delta N_{\rm e}^{\rm cl} = k$ for the multi-Skyrmion state via the charge-number relation (4.38). We confirm this by an explicit calculation.

In the Weyl form the topological charge (4.31) is expressed as

$$Q = {\rm Tr}([\mathfrak{n}_\mu^\dagger, \mathfrak{n}_\mu]). \tag{6.24}$$

Here, we cannot use ${\rm Tr}([A,B]) = {\rm Tr}(AB) - {\rm Tr}(BA) = 0$, which is valid only if ${\rm Tr}(AB)$ and ${\rm Tr}(BA)$ are separately well defined. This is not the case here, and the trace operation must be carried out after the commutator is calculated.

Using (6.22) we obtain

$$[\mathfrak{n}_\mu^\dagger, \mathfrak{n}_\mu] = \sum_{n=0}^{\infty}\bar{v}_\mu(n)v_\mu(n)[|n\rangle\langle n| - |n+k\rangle\langle n+k|]. \tag{6.25}$$

This is still inappropriate for calculating the trace, since the separate pieces are divergent due to the behavior $|v_\mu(n)| \to 1$ as $n \to \infty$, as implied by the boundary condition (6.12). We rewrite the formula in terms of $u_\mu(n)$ instead of $v_\mu(n)$, and obtain

$$[\mathfrak{n}_\mu^\dagger, \mathfrak{n}_\mu] = \sum_{n=0}^{k-1} |n\rangle\langle n| - \sum_{n=0}^{\infty} \bar{u}_\mu(n) u_\mu(n) |n\rangle\langle n| + \sum_{n=0}^{\infty} \bar{u}_\mu(n) u_\mu(n) |n+k\rangle\langle n+k|. \quad (6.26)$$

Each piece is well defined since $u_\mu(n) \to 0$ as $n \to \infty$, and the traces can be calculated separately, which eventually leads to $Q = k$.

D. Hole state

In order to understand the difference of the noncommutative theory from the commutative one, it is instructive to analyze the hole state (6.4) in more detail. To make the argument as simple as possible, we consider the $SU(2)$ spin system with the ground state being the up-spin filled one,

$$|\mathrm{g}\rangle = \prod_{n=0}^{\infty} c_\uparrow^\dagger(n)|0\rangle. \quad (6.27)$$

The hole state is

$$|\mathrm{h}\rangle = c_\uparrow(0)|\mathrm{g}\rangle, \quad (6.28)$$

with its bare density being

$$\hat{\rho}^{\mathrm{cl}}(\mathbf{x}) = \rho_0(1 - 2e^{-r^2/\theta}), \quad (6.29a)$$

$$\hat{S}_x^{\mathrm{cl}}(\mathbf{x}) = \hat{S}_y^{\mathrm{cl}}(\mathbf{x}) = 0, \qquad \hat{S}_z^{\mathrm{cl}}(\mathbf{x}) = \frac{1}{2}\hat{\rho}^{\mathrm{cl}}(\mathbf{x}). \quad (6.29b)$$

It represents literally a hole of the size $\sqrt{\theta}$. The spin texture is trivial.

In the commutative theory, the topological number is given by (4.30), or

$$Q = \frac{1}{2\pi i}\epsilon_{kl}\sum_s \int d^2x (\partial_k \bar{n}_\mu)(\partial_l n_\mu), \quad (6.30)$$

which is equivalent to the Pontryagin number,

$$Q_{\mathrm{P}} = \frac{1}{\pi}\int d^2x \varepsilon_{abc}\varepsilon_{ij} S_a(\partial_i S_b)(\partial_j S_c). \quad (6.31)$$

Here S_a is the normalized spin density, $S_a \equiv \frac{1}{2}\bar{n}_\mu(\sigma_a)_{\mu\nu}n_\nu$, with σ_a the Pauli matrix. We would obviously conclude $Q_{\mathrm{P}} = 0$ for the trivial spin texture such as (6.29b).

However, this is not the case in the noncommutative theory. According to the argument presented in the previous subsection, its topological charge is given by

$$Q = \mathrm{Tr}([\mathfrak{n}_\mu^\dagger, \mathfrak{n}_\mu]) = \mathrm{Tr}(|0\rangle\langle 0|) = 1, \quad (6.32)$$

since $u_\mu(n) = 0$ for all n in (6.26). There is no mystery here. Let us explain this by calculating the topological number explicitly in the coordinate space.

The CP^1 field which gives the trivial spin texture (6.29b) via (4.17) is highly nontrivial,

$$n_\uparrow(\mathbf{x}) = 2^{3/2} z e^{-\bar{z}z} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} L_n^1(2\bar{z}z), \quad (6.33a)$$

$$n_\downarrow(\mathbf{x}) = 0. \quad (6.33b)$$

It is well defined everywhere. Only the asymptotic behavior contributes to the topological number (4.31) in the noncommutative formulation, or equivalently to (6.30) in the commutative formulation. Any finite order terms in n vanish in the limit $\bar{z}z \to \infty$ due to the term $e^{-\bar{z}z}$ in (6.33a). Thus, in calculating the topological charge, we may replace $1/\sqrt{(n+1)}$ by $\Gamma(n+\frac{3}{2})/\Gamma(n+2)$, which is valid for $n \gg 1$. We then use [29]

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} t^n L_n^{(c-1)}(x) = \frac{1}{(1-t)^a} M\left(a; c; \frac{xt}{t-1}\right), \quad (6.34)$$

where $M(a; b; x)$ is the Kummer function. Taking $a = \frac{3}{2}$, $c = 2$, and $t = -1$, we obtain

$$n_\uparrow(\mathbf{x}) \to z e^{-\bar{z}z} \frac{\Gamma(\frac{3}{2})}{\Gamma(2)} M\left(\frac{3}{2}; 2; \bar{z}z\right) \to \frac{z}{|z|} = e^{i\vartheta}, \quad (6.35)$$

where ϑ is the azimuthal angle. It carries a proper winding number. The topological number of a hole is clearly $Q = 1$, as is consistent with (6.32).

Hence we conclude that a hole is a kind of Skyrmion in the noncommutative theory: Its spin texture is trivial but the associated CP^{N-1} field is highly nontrivial and carries a proper winding number.

It is easy to see what happens in the commutative limit for the hole state, where the CP^1 field (6.33a) is reduced to

$$n_\uparrow(\mathbf{x}) = e^{i\vartheta}, \qquad n_\downarrow(\mathbf{x}) = 0. \quad (6.36)$$

It is ill defined at the origin. Hence we cannot calculate the topological charge by the formula (6.30) for the hole state. It is the essential feature of the noncommutative theory that such a singularity is regulated over the region of area θ. This is the reason why the noncommutative CP^1 field for the hole state is well defined everywhere and carries a winding number.

VII. HARD-CORE INTERACTION

We have so far analyzed the kinematical structure of multi-Skyrmions in the noncommutative plane. The CP^{N-1} field given by (6.23) contains infinitely many parameters $u_\mu(n)$ and $v_\mu(n)$. These variables are to be fixed so as to minimize the energy of the state. Let us carry out this program by taking the hard-core interaction [26] for the potential $V(\mathbf{x})$ in the Hamiltonian (5.1),

$$V(\mathbf{x} - \mathbf{y}) = \delta^2(\mathbf{x} - \mathbf{y}). \quad (7.1)$$

The Hamiltonian reads

$$H_{\mathrm{hc}} = \frac{1}{2}\int d^2x \Delta\rho(\mathbf{x})\Delta\rho(\mathbf{x}). \quad (7.2)$$

For simplicity we restrict the isospin $SU(N)$ symmetry to the spin $SU(2)$ symmetry in what follows.

The matrix element (5.4) is easily calculable to give

$$V_{mnij} = \frac{1}{8\pi\theta}\frac{\sqrt{(m+i)!}}{\sqrt{m!i!}}\frac{\sqrt{(n+j)!}}{\sqrt{n!j!}}\frac{\delta_{m+i,n+j}}{\sqrt{2^{m+i+n+j}}}, \quad (7.3)$$

which possesses the additional symmetry separately with respect to $m \leftrightarrow i$ and $n \leftrightarrow j$. Because of this symmetry the Hamiltonian (7.2) is reduced to

$$H_{\rm hc} = 2\sum_{mnij} V_{mnij} c_\uparrow^\dagger(m) c_\downarrow^\dagger(i) c_\downarrow(j) c_\uparrow(n) - \frac{1}{4\pi\theta}\Delta N_{\rm e}, \quad (7.4)$$

or

$$H_{\rm hc} = \int d^2x \psi_\uparrow^\dagger(\mathbf{x})\psi_\downarrow^\dagger(\mathbf{x})\psi_\downarrow(\mathbf{x})\psi_\uparrow(\mathbf{x}) - \frac{1}{4\pi\theta}\Delta N_{\rm e}, \quad (7.5)$$

where we have used

$$\epsilon_{\rm D} = \epsilon_{\rm X} = \frac{1}{4\pi\theta}, \quad (7.6)$$

which follows from (5.6) and (5.7).

A. Factorizable Skyrmions

We consider the multi-Skyrmion state (6.14) with (6.15), or

$$\xi^\dagger(n) = u(n)c_\downarrow^\dagger(n) + v(n)c_\uparrow^\dagger(n+k). \quad (7.7)$$

Because of the anticommutation relation (6.13) we have

$$\langle \mathfrak{S}_{\rm sky}|\xi^\dagger(m)\xi^\dagger(i)\xi(j)\xi(n)|\mathfrak{S}_{\rm sky}\rangle = \delta_{mn}\delta_{ij} - \delta_{mj}\delta_{in}, \quad (7.8)$$

and using (7.3) we obtain

$$\langle \mathfrak{S}_{\rm sky}|H_{\rm hc}|\mathfrak{S}_{\rm sky}\rangle = \frac{k}{4\pi\theta} + \frac{1}{8\pi\theta}\sum_{mn}\frac{(m+n+k)!}{2^{m+n+k}m!n!} \times \left| \frac{\sqrt{m!}}{\sqrt{(m+k)!}}v(m)u(n) - \frac{\sqrt{n!}}{\sqrt{(n+k)!}}v(n)u(m)\right|^2, \quad (7.9)$$

where we have taken into account $\Delta N_{\rm e}^{\rm cl} = -k$. The second term, being positive semidefinite, takes the minimum for

$$\frac{\sqrt{n!}}{\sqrt{(n+k)!}}\frac{v(n)}{u(n)} = \frac{1}{\omega^k}, \quad (7.10)$$

where ω is an arbitrary complex constant. This gives

$$|u(n)|^2 = \frac{|\omega|^{2k}}{|\omega|^{2k} + [(n+1)\cdots(n+k)]}, \quad (7.11a)$$

$$|v(n)|^2 = \frac{(n+1)\cdots(n+k)}{|\omega|^{2k} + [(n+1)\cdots(n+k)]} \quad (7.11b)$$

with the aid of the condition $|u(n)|^2 + |v(n)|^2 = 1$.

Using (6.18) we can actually verify that

$$H_{\rm hc}|\mathfrak{S}_{\rm sky}\rangle = -\frac{1}{4\pi\theta}\Delta N_{\rm e}^{\rm cl}|\mathfrak{S}_{\rm sky}\rangle = \frac{k}{4\pi\theta}|\mathfrak{S}_{\rm sky}\rangle, \quad (7.12)$$

when parameters $u_\mu(n)$ and $v_\mu(n)$ are given by (7.11). Consequently, $|\mathfrak{S}_{\rm sky}\rangle$ is an eigenstate of the Hamiltonian (7.5) with $\Delta N_{\rm e}^{\rm cl} = -k$. The eigenvalue is independent of the scale parameter ω, and the Skyrmion state is degenerate with the hole state.

The CP^1 field (6.22) corresponding to the multi-Skyrmion state is $|\mathfrak{S}_{\rm sky}\rangle$ expressed as

$$\mathfrak{n}_\uparrow = \sum_n v(n)|n+k\rangle\langle n|, \quad (7.13a)$$

$$\mathfrak{n}_\downarrow = \sum_n u(n)|n\rangle\langle n| \quad (7.13b)$$

with (7.11). We calculate this explicitly with the aid of (7.11).

From (2.11) we have

$$(b^\dagger)^k|n\rangle = \left[\frac{(n+k)!}{n!}\right]^{1/2}|n+k\rangle, \quad (7.14)$$

which leads to

$$(b^\dagger)^k\mathfrak{n}_\downarrow = \sum_{n=0}^{\infty} u(n)\left[\frac{(n+k)!}{n!}\right]^{1/2}|n+k\rangle\langle n|. \quad (7.15)$$

Here we use (7.10) and (7.13a) to find

$$(b^\dagger)^k\mathfrak{n}_\downarrow = \omega^k\mathfrak{n}_\uparrow, \quad (7.16)$$

which in terms of symbols turns into

$$z^k \star n_\downarrow(\mathbf{x}) = (\sqrt{2}\omega)^k n_\uparrow(\mathbf{x}), \quad (7.17)$$

where $z \star z^k = z^{k+1}$ has been used.

Then the CP^1 field reads

$$n(\mathbf{x}) = \frac{1}{\lambda^k}\begin{bmatrix} z^k \\ \lambda^k \end{bmatrix} \star n_\downarrow(\mathbf{x}), \quad (7.18)$$

with $\lambda \equiv \sqrt{2}\omega$, where $n_\downarrow(\mathbf{x})$ is the function of $r = |\mathbf{x}|$ only

$$n_\downarrow(\mathbf{x}) = 2e^{-\bar{z}z}\sum_{n=0}^{\infty}(-1)^n u(n) L_n(2\bar{z}z). \quad (7.19)$$

It is to be noticed that the spin part is factorized as in (7.18). It is reminiscent of the Skyrmion CP^1 field

$$n(\mathbf{x}) = \frac{1}{\sqrt{|z|^{2k} + |\lambda|^{2k}}}\begin{bmatrix} z^k \\ \lambda^k \end{bmatrix} \quad (7.20)$$

in the ordinary CP^1 model (9.22).

Finally we calculate the physical quantities $\rho^{\rm cl}(\mathbf{x})$ and $S_a^{\rm cl}(\mathbf{x})$ for the multi-Skyrmion state,

$$\rho^{\rm cl}(\mathbf{x}) = \langle \mathfrak{S}_{\rm sky}|\psi^\dagger(\mathbf{x})\psi(\mathbf{x})|\mathfrak{S}_{\rm sky}\rangle, \tag{7.21a}$$

$$S_a^{\rm cl}(\mathbf{x}) = \frac{1}{2}\langle \mathfrak{S}_{\rm sky}|\psi^\dagger(\mathbf{x})\sigma_a\psi(\mathbf{x})|\mathfrak{S}_{\rm sky}\rangle. \tag{7.21b}$$

Substituting (6.20) with (7.11) into (4.9), we obtain $\hat{D}^{\rm cl}_{\mu\nu}(\mathbf{k})$. Then, we construct $D^{\rm cl}_{\mu\nu}(\mathbf{x})$ from (3.12), which may be decomposed into $\rho^{\rm cl}(\mathbf{x})$ and $S_a^{\rm cl}(\mathbf{x})$.

In this way we come to

$$\rho^{\rm cl}(\mathbf{x}) = \frac{1}{2\pi\theta}\left[\left(\frac{r^2}{2\theta}\right)^k + |\omega|^{2k}\right] f\left(\frac{r^2}{2\theta}\right), \tag{7.22a}$$

$$S_x^{\rm cl}(\mathbf{x}) = \frac{+1}{2\pi\theta}\left(\frac{|\omega|r}{\sqrt{2\theta}}\right)^k f\left(\frac{r^2}{2\theta}\right)\cos k\vartheta, \tag{7.22b}$$

$$S_y^{\rm cl}(\mathbf{x}) = \frac{-1}{2\pi\theta}\left(\frac{|\omega|r}{\sqrt{2\theta}}\right)^k f\left(\frac{r^2}{2\theta}\right)\sin k\vartheta, \tag{7.22c}$$

$$S_z^{\rm cl}(\mathbf{x}) = \frac{1}{4\pi\theta}\left[\left(\frac{r^2}{2\theta}\right)^k - |\omega|^{2k}\right] f\left(\frac{r^2}{2\theta}\right), \tag{7.22d}$$

where $f(x)$ is given by

$$f(x) = e^{-x}\sum_{n=0}^{\infty}\frac{1}{|\omega|^{2k} + [(n+1)\cdots(n+k)]}\frac{x^n}{n!} \tag{7.23}$$

and ϑ is the azimuthal angle. It follows from (7.22) that

$$S_a^{\rm cl}(\mathbf{x}) = \rho^{\rm cl}(\mathbf{x})S_a(\mathbf{x}), \tag{7.24}$$

where

$$S_a(\mathbf{x}) = \frac{1}{2}\bar{n}(\mathbf{x})\sigma_a n(\mathbf{x}) \tag{7.25}$$

with (7.20). It is the nonlinear spin field normalized as $S_a(\mathbf{x})S_a(\mathbf{x}) = 1/4$, and given by

$$S_x(\mathbf{x}) = \frac{(|\lambda|r\sqrt{\theta})^k}{(r^2)^k + (|\lambda|^2\theta)^k}\cos k\vartheta, \tag{7.26a}$$

$$S_y(\mathbf{x}) = \frac{-(|\lambda|r\sqrt{\theta})^k}{(r^2)^k + (|\lambda|^2\theta)^k}\sin k\vartheta, \tag{7.26b}$$

$$S_z(\mathbf{x}) = \frac{1}{2}\frac{(r^2)^k - (|\lambda|^2\theta)^k}{(r^2)^k + (|\lambda|^2\theta)^k}. \tag{7.26c}$$

It agrees with the Skyrmion spin field in the ordinary $O(3)$ nonlinear sigma model. The factorizability (7.24) together with (7.26) is a peculiar property. We call such a Skyrmion a factorizable Skyrmion.

B. Zeeman interaction

QH effects occur in the external magnetic field. The electron couples with the magnetic field via the Zeeman term in the realistic system,

$$H_{\rm Z} = -\Delta_{\rm Z}\int d^2x S_z(\mathbf{x}), \tag{7.27}$$

where $\Delta_Z = |g|\mu_B B_\perp$ is the Zeeman gap with μ_B the Bohr magneton and g the magnetic g factor. It is interesting to calculate the Zeeman energy of the factorizable Skyrmion (7.24).

For simplicity we study the simplest Skyrmion with $Q = k = 1$. We may rewrite (7.23) as

$$f(x) = \frac{1}{|\omega|^2 + 1}e^{-x}M(|\omega|^2 + 1; |\omega|^2 + 2; x). \tag{7.28}$$

It behaves as

$$\lim_{x\to\infty} f(x) = \frac{1}{x} - \frac{|\omega|^2}{x^2} + \frac{|\omega|^2(|\omega|^2 - 1)}{x^3}. \tag{7.29}$$

Hence we obtain

$$\lim_{r\to\infty}\rho^{\rm cl}(\mathbf{x}) = \frac{1}{2\pi\theta}\left(1 - \frac{2\theta^2|\lambda|^2}{r^4}\right), \tag{7.30a}$$

$$\lim_{r\to\infty} S_z^{\rm cl}(\mathbf{x}) = \frac{1}{4\pi\theta}\left(1 - \frac{2\theta|\lambda|^2}{r^2}\right). \tag{7.30b}$$

The number of spins flipped around a Skyrmion is given by

$$N_{\rm spin} = \int d^2x\left\{S_z^{\rm cl}(\mathbf{x}) - \frac{1}{4\pi\theta}\right\}, \tag{7.31}$$

which is identified with the Skyrmion spin. We find $N_{\rm spin}$ to diverge logarithmically, unless $\lambda = 0$, due to the asymptotic behavior (7.30b). The Zeeman energy $\langle H_{\rm Z}\rangle = -\Delta_{\rm Z}N_{\rm spin}$ is divergent, except for the hole, from the infrared contribution however small the Zeeman effect is.

The factorizable Skyrmion (7.24) is no longer valid in the hard-core model with the Zeeman term. There exists surely a Skyrmion state which has a finite Zeeman energy once the factorizability is abandoned: See (10.9) for an example. Nevertheless, we can show that the hole state has the lowest energy. The reason reads as follows. The factorizable Skyrmion is an eigenstate of the hard-core Hamiltonian, $H_{\rm hc}|\mathfrak{S}_{\rm sky}\rangle = E_{\rm hc}|\mathfrak{S}_{\rm sky}\rangle$ with $E_{\rm hc} = |\Delta N_{\rm e}^{\rm cl}|/4\pi\theta$, as in (7.12). Accordingly any spin texture $|\mathfrak{S}\rangle$ possessing the same electron number $\Delta N^{\rm cl}$ has a higher energy, $\langle\mathfrak{S}|H_{\rm hc}|\mathfrak{S}\rangle \geq E_{\rm hc}$. Furthermore its Zeeman energy is larger than that of the hole, $\langle\mathfrak{S}|H_{\rm Z}|\mathfrak{S}\rangle \geq \frac{1}{2}\Delta_{\rm Z}$. Hence,

$$\langle\mathfrak{S}|(H_{\rm hc} + H_{\rm Z})|\mathfrak{S}\rangle \geq E_{\rm hc} + \frac{1}{2}\Delta_{\rm Z}, \tag{7.32}$$

where the equality holds for the hole state. Consequently there are no Skyrmions in the presence of the Zeeman interaction in the system with the hard-core interaction.

VIII. NONCOMMUTATIVE CP^{N-1} MODEL

In the previous section we have studied the Skyrmion state (6.22) in the hard-core model (7.1) and determined parameters $u_\mu(n)$ and $v_\mu(n)$ explicitly. We have emphasized that the Skyrmion itself is independent of the Hamiltonian. For the sake of completeness we analyze the Skyrmion state (6.22) in the noncommutative CP^{N-1} model.

The noncommutative CP^{N-1} model is defined by the Hamiltonian [5]

$$H_{CP} = \kappa\theta^2 \,\mathrm{Tr}[(\partial_m \mathfrak{n}_\mu^\dagger)(\partial_m \mathfrak{n}_\mu) + (\mathfrak{n}_\mu^\dagger \partial_m \mathfrak{n}_\mu)(\mathfrak{n}_\nu^\dagger \partial_m \mathfrak{n}_\nu)], \tag{8.1}$$

where the derivative is given by (2.2) or

$$\partial_i \mathfrak{n}_\mu = -\frac{i}{\theta}\epsilon_{ij}[X_j, \mathfrak{n}_\mu]. \tag{8.2}$$

It is convenient to introduce the notation

$$\mathcal{D}_z \mathfrak{n}_\mu \equiv \mathcal{D}_x \mathfrak{n}_\mu - i\mathcal{D}_y \mathfrak{n}_\mu, \quad \mathcal{D}_{\bar{z}} \mathfrak{n}_\mu \equiv \mathcal{D}_x n_\mu + i\mathcal{D}_y \mathfrak{n}_\mu, \tag{8.3}$$

where

$$\mathcal{D}_i \mathfrak{n}_\mu \equiv -i\theta^{-1}\epsilon_{ij}([X_j, \mathfrak{n}_\mu] - \mathfrak{n}_\mu \mathfrak{n}_\gamma [X_j, \mathfrak{n}_\gamma]), \tag{8.4}$$

and rewrite the noncommutative CP^{N-1} model (9.19) as

$$E_{\mathrm{X}} = \frac{1}{2}\kappa\theta^2 \,\mathrm{Tr}[(\mathcal{D}_z \mathfrak{n}_\mu)^\dagger(\mathcal{D}_z \mathfrak{n}_\mu) + (\mathcal{D}_{\bar{z}} \mathfrak{n}_\mu)^\dagger(\mathcal{D}_{\bar{z}} \mathfrak{n}_\mu)]. \tag{8.5}$$

We have already defined the topological charge Q by the formula (4.31). However, there are other equivalent ways of defining it, and here we use the representation

$$Q = \frac{1}{2}\theta \,\mathrm{Tr}[(\mathcal{D}_z \mathfrak{n}_\mu)^\dagger(\mathcal{D}_z \mathfrak{n}_\mu) - (\mathcal{D}_{\bar{z}} \mathfrak{n}_\mu)^\dagger(\mathcal{D}_{\bar{z}} \mathfrak{n}_\mu)]. \tag{8.6}$$

The equivalence is readily proved because the topological charge densities associated with (4.31) and (8.6) are different only by a total derivative term that does not contribute to the topological charge.

The Bogomol'nyi-Prasad-Sommerfield energy bound follows from (8.5) and (8.6),

$$E_{\mathrm{X}} \geq \kappa\theta|Q|. \tag{8.7}$$

This bound becomes saturated for

$$\mathcal{D}_{\bar{z}} \mathfrak{n}_\mu = 0 \quad \text{(Skyrmions)}, \tag{8.8}$$

or

$$\mathcal{D}_z \mathfrak{n}_\mu = 0 \quad \text{(anti-Skyrmions)}. \tag{8.9}$$

We consider the case of Skyrmions in what follows. The self-dual equation (8.8) takes the form

$$[b^\dagger, \mathfrak{n}_\mu] - \mathfrak{n}_\mu \mathfrak{n}_\gamma [b^\dagger, \mathfrak{n}_\gamma] = 0. \tag{8.10}$$

We consider explicitly the case of $SU(2)$ and look for the solution to (8.10) in the form (6.22), or

$$\mathfrak{n}_\uparrow = \sum_{m=0}^{\infty} v(m)|m+k\rangle\langle m|, \qquad \mathfrak{n}_\downarrow = \sum_{m=0}^{\infty} u(m)|m\rangle\langle m|, \tag{8.11}$$

where $|u(k)|^2 + |v(k)|^2 = 1$. This describes a multi-Skyrmion carrying the topological charge $Q = k$. We get

$$[b^\dagger, \mathfrak{n}_\uparrow] - \mathfrak{n}_\uparrow \mathfrak{n}_\mu [b^\dagger, \mathfrak{n}_\mu] = \sum_{m=0}^{\infty} f_\uparrow(m)|m+k+1\rangle\langle m|,$$
$$[b^\dagger, \mathfrak{n}_\downarrow] - \mathfrak{n}_\downarrow \mathfrak{n}_\mu [b^\dagger, \mathfrak{n}_\mu] = \sum_{m=0}^{\infty} f_\downarrow(m)|m+1\rangle\langle m|, \tag{8.12}$$

with

$$f_\uparrow(m) = v(m)\bar{u}(m+1)u(m+1)\sqrt{m+k+1} - v(m+1)\bar{u}(m+1)u(m)\sqrt{m+1},$$
$$f_\downarrow(m) = u(m)\bar{v}(m+1)v(m+1)\sqrt{m+1} - u(m+1)\bar{v}(m+1)v(m)\sqrt{m+k+1}. \tag{8.13}$$

Equation (8.10) holds if $f_\uparrow(m) = 0$ and $f_\downarrow(m) = 0$. These two equations are summarized into a single equation as

$$\frac{v(m)}{u(m)} = \frac{\sqrt{m+k}}{\sqrt{m}}\frac{v(m-1)}{u(m-1)}. \tag{8.14}$$

This recurrence relation is solved as

$$\frac{v(m)}{u(m)} = \frac{\sqrt{(m+k)!}}{\sqrt{m!}}\frac{1}{\omega^k}, \tag{8.15}$$

where we have introduced the complex parameter ω by

$$\omega^k \equiv \frac{u(0)}{v(0)}k!. \tag{8.16}$$

The parameters (8.15) are found to be the same as (7.10).

Consequently the Skyrmions are identical both in the hard-core model and the noncommutative CP^{N-1} model, though these two models are very different. We discuss the relation between them in Sec. X.

IX. EFFECTIVE THEORY

We have explored kinematical properties of the classical density matrix $\hat{D}^{\mathrm{cl}}_{\mu\nu}$. It is interesting to discuss the dynamical property of the classical density, especially, what the equation of motion for $\hat{D}^{\mathrm{cl}}_{\mu\nu}$ is.

A. Classical equation of motion

We start with the quantum equation of motion for the density matrix $D_{\mu\nu}(k, l)$,

$$i\hbar\frac{d}{dt}D_{\mu\nu}(k, l) = [D_{\mu\nu}(k, l), H_{\mathrm{V}}], \tag{9.1}$$

where H_{V} is the four-fermion interaction Hamiltonian (5.1). By using (3.17) it is explicitly calculated as

$$i\hbar \frac{d}{dt} D_{\gamma\alpha}(k, l) = V_{mnkj} D_{\mu\mu}(n, m) D_{\gamma\alpha}(j, l) - V_{mnil} D_{\mu\mu}(n, m) D_{\gamma\alpha}(k, i) + V_{knij} D_{\gamma\alpha}(n, l) D_{\mu\mu}(j, i) - V_{mlij} D_{\gamma\alpha}(k, m) D_{\mu\mu}(j, i). \quad (9.2)$$

We take the average of this equation with respect to the Fock state (4.1). Using

$$\langle \mathfrak{S} | D_{\nu\mu}(n, m) D_{\tau\sigma}(j, i) | \mathfrak{S} \rangle = D^{\text{cl}}_{\nu\mu}(n, m) D^{\text{cl}}_{\tau\sigma}(j, i) - D^{\text{cl}}_{\tau\mu}(j, m) D^{\text{cl}}_{\nu\sigma}(n, i) + \delta_{\nu\sigma}\delta_{ni} D^{\text{cl}}_{\tau\mu}(j, m), \quad (9.3)$$

where $D^{\text{cl}}_{\nu\mu}(n, m) = \langle \mathfrak{S} | D_{\nu\mu}(n, m) | \mathfrak{S} \rangle$, we obtain

$$i\hbar \frac{d}{dt} D^{\text{cl}}_{\gamma\alpha}(l, k) = 2V_{mnlj} D^{\text{cl}}_{\mu\mu}(n, m) D^{\text{cl}}_{\gamma\alpha}(j, k) - 2V_{mnik} D^{\text{cl}}_{\mu\mu}(n, m) D^{\text{cl}}_{\gamma\alpha}(l, i) - 2V_{mnlj} D^{\text{cl}}_{\gamma\mu}(j, m) D^{\text{cl}}_{\mu\alpha}(n, k) + 2V_{mnik} D^{\text{cl}}_{\gamma\mu}(l, m) D^{\text{cl}}_{\mu\alpha}(n, i). \quad (9.4)$$

Provided the classical density $D^{\text{cl}}_{\nu\mu}(n, m)$ is endowed with the Poisson structure

$$i\hbar [D^{\text{cl}}_{\mu\nu}(m, n), D^{\text{cl}}_{\sigma\tau}(i, j)]_{\text{PB}} = \delta_{\mu\tau}\delta_{mj} D^{\text{cl}}_{\sigma\nu}(i, n) - \delta_{\sigma\nu}\delta_{in} D^{\text{cl}}_{\mu\tau}(m, j), \quad (9.5)$$

it is straightforward to show that (9.4) is summarized into

$$\frac{d}{dt} D^{\text{cl}}_{\mu\nu}(m, n) = [D^{\text{cl}}_{\mu\nu}(m, n), E_{\text{V}}]_{\text{PB}}, \quad (9.6)$$

where $E_{\text{V}} \equiv \langle \mathfrak{S} | H_{\text{V}} | \mathfrak{S} \rangle$ is the average of the Hamiltonian. As we have noticed in (5.8), it consists of the direct energy (5.9a) and the exchange energy (5.9b).

It is remarkable that the Poisson structure (9.5) is precisely the same as the algebra (3.17). To see its significance more in detail we combine it with the magnetic-translation group property (3.18),

$$2\pi i\hbar [\hat{D}^{\text{cl}}_{\mu\nu}(\mathbf{k}), \hat{D}^{\text{cl}}_{\sigma\tau}(\mathbf{k}')]_{\text{PB}} = \delta_{\mu\tau} e^{+(i/2)\theta \mathbf{k}\wedge\mathbf{k}'} \hat{D}^{\text{cl}}_{\sigma\nu}(\mathbf{k}+\mathbf{k}') - \delta_{\sigma\nu} e^{-(i/2)\theta \mathbf{k}\wedge\mathbf{k}'} \times \hat{D}^{\text{cl}}_{\mu\tau}(\mathbf{k}+\mathbf{k}'). \quad (9.7)$$

It corresponds to the $W_\infty(N)$ algebra (3.19), indicating that the classical density $\hat{D}^{\text{cl}}_{\mu\nu}$ should obey the $W_\infty(N)$ algebra as well.

It is important that the Poisson structure (9.7) is resolved in terms of the noncommutative CP^{N-1} field. The relation (4.21) reads

$$\hat{D}^{\text{cl}}_{\mu\nu}(\mathbf{k}) = \frac{1}{4\pi^2\theta} \int d^2k' e^{-(i/2)\theta \mathbf{k}\wedge\mathbf{k}'} n_\mu(\mathbf{k}') \bar{n}_\nu(\mathbf{k}'-\mathbf{k}) \quad (9.8)$$

in the momentum space, where $\bar{n}_\mu(\mathbf{k})$ is the complex conjugate of $n_\mu(\mathbf{k})$. Substituting this into (9.7), provided

$$i\hbar [n_\mu(\mathbf{k}), \bar{n}_\nu(\mathbf{k}')]_{\text{PB}} = 2\pi\theta \delta_{\mu\nu}(\mathbf{k}'-\mathbf{k}), \quad (9.9)$$

or

$$i\hbar [n_\mu(\mathbf{x}), \bar{n}_\nu(\mathbf{y})]_{\text{PB}} = 2\pi\theta \delta_{\mu\nu}(\mathbf{x}-\mathbf{y}), \quad (9.10)$$

we can easily show that (9.7) holds as an identity.

B. Derivative expansion

We proceed to derive the low energy effective theory of the Hamiltonian system (5.1) in terms of the CP^{N-1} field. The corresponding energy consists of the direct and exchange energies E_{D} and E_{X} as given by (5.9a) and (5.9b), respectively. The direct energy is not interesting since it represents simply the classical energy of a charge distribution $-e\Delta\rho^{\text{cl}}(\mathbf{x})$.

We analyze the exchange energy E_{X}. Corresponding to (4.24) we find

$$D^{\text{cl}}_{\mu\nu}(m, n) = \langle m | \mathfrak{n}_\mu \mathfrak{n}^\dagger_\nu | n \rangle. \quad (9.11)$$

Substituting this into (5.9b) we come to

$$E_{\text{X}} = -\int \frac{d^2k}{4\pi} e^{-(1/2)\theta k^2} V(\mathbf{k}) \text{Tr}(\mathfrak{n}_\mu \mathfrak{n}^\dagger_\nu e^{i\mathbf{k}\mathbf{X}} \mathfrak{n}_\nu \mathfrak{n}^\dagger_\mu e^{-i\mathbf{k}\mathbf{X}}) + N^{\text{cl}}_{\text{e}} \epsilon_{\text{X}}, \quad (9.12)$$

where we have taken (5.4) into account. To bring it to a more reasonable form, we use

$$N^{\text{cl}}_{\text{e}} = \text{Tr}(\mathfrak{n}_\mu \mathfrak{n}^\dagger_\mu), \quad (9.13)$$

and rewrite the exchange energy as

$$E_{\text{X}} = \int \frac{d^2k}{4\pi} e^{-(1/2)\theta k^2} V(\mathbf{k}) \times \text{Tr}(\mathfrak{n}_\mu \mathfrak{n}^\dagger_\mu - \mathfrak{n}_\mu \mathfrak{n}^\dagger_\nu e^{i\mathbf{k}\mathbf{X}} \mathfrak{n}_\nu \mathfrak{n}^\dagger_\mu e^{-i\mathbf{k}\mathbf{X}}), \quad (9.14)$$

where the cancellation of divergences occurs at $\mathbf{k} = 0$. We may change the order of operators under the trace simultaneously in both terms so that the cancellation at $\mathbf{k} = 0$ is maintained. We come to

$$E_{\text{X}} = \int \frac{d^2k}{4\pi} e^{-(1/2)\theta k^2} V(\mathbf{k}) \times \text{Tr}(\mathbb{I} - \mathfrak{n}^\dagger_\mu e^{i\mathbf{k}\mathbf{X}} \mathfrak{n}_\mu \cdot \mathfrak{n}^\dagger_\nu e^{-i\mathbf{k}\mathbf{X}} \mathfrak{n}_\nu) \quad (9.15)$$

where we have used the normalization $\mathfrak{n}^\dagger_\mu \mathfrak{n}_\mu = \mathbb{I}$.

As far as we are concerned about sufficient smooth field configurations we may make the derivative expansion and take the lowest-order term. Here we note that

$$\mathbb{I} - \mathfrak{n}^\dagger_\mu e^{i\mathbf{k}\mathbf{X}} \mathfrak{n}_\mu \cdot \mathfrak{n}^\dagger_\nu e^{-i\mathbf{k}\mathbf{X}} \mathfrak{n}_\nu = [\mathfrak{n}^\dagger_\mu, e^{i\mathbf{k}\mathbf{X}}][e^{-i\mathbf{k}\mathbf{X}}, \mathfrak{n}_\mu] - \mathfrak{n}^\dagger_\mu [e^{i\mathbf{k}\mathbf{X}}, \mathfrak{n}_\mu] \times \mathfrak{n}^\dagger_\nu [e^{-i\mathbf{k}\mathbf{X}}, \mathfrak{n}_\nu]. \quad (9.16)$$

Because the derivative is given by (2.2) or

$$\partial_i \mathfrak{n}_\mu = -\frac{i}{\theta}\epsilon_{ij}[X_j, \mathfrak{n}_\mu], \tag{9.17}$$

the derivative expansion corresponds to the expansion in **X**,

$$e^{i\mathbf{kX}} = \mathbb{I} + ik_iX_i - \frac{1}{2}k_ik_jX_iX_j + O(X^3). \tag{9.18}$$

Substituting this into (9.15) and assuming the rotational invariance of $V(\mathbf{k})$ we integrate over the angle in the momentum space. In such a way, up to the lowest-order term of the derivative expansion, we come to

$$\begin{aligned} E_{\mathrm{X}} &= \kappa\,\mathrm{Tr}(\mathfrak{n}_\mu^\dagger X_iX_i\mathfrak{n}_\mu - \mathfrak{n}_\mu^\dagger X_i\mathfrak{n}_\mu\mathfrak{n}_\nu^\dagger X_i\mathfrak{n}_\nu) \\ &= \kappa\,\mathrm{Tr}([\mathfrak{n}_\mu^\dagger, X_i][X_i, \mathfrak{n}_\mu] - \mathfrak{n}_\mu^\dagger[X_i, \mathfrak{n}_\mu]\mathfrak{n}_\nu^\dagger[X_i, \mathfrak{n}_\nu]) \\ &= \kappa\theta^2\,\mathrm{Tr}[(\partial_m\mathfrak{n}_\mu^\dagger)(\partial_m\mathfrak{n}_\mu) + (\mathfrak{n}_\mu^\dagger\partial_m\mathfrak{n}_\mu)(\mathfrak{n}_\nu^\dagger\partial_m\mathfrak{n}_\nu)], \end{aligned} \tag{9.19}$$

where the constant κ is given by

$$\kappa \equiv \frac{1}{4}\int dk k^3 e^{-(1/2)\theta k^2}V(k). \tag{9.20}$$

It agrees with the noncommutative CP^{N-1} model [5]. It is interesting that the noncommutative CP^{N-1} model is derived from the four-fermion interaction Hamiltonian irrespective of the explicit form of the potential $V(\mathbf{x})$ as the lowest-order term in the derivative expansion.

We can use the noncommutative CP^{N-1} model (9.19) as the effective Hamiltonian for the Goldstone modes, which are small fluctuations of the CP^{N-1} field around the ground state. Such fluctuations occur without the density excitation ($\Delta\rho_{\mathrm{e}}^{\mathrm{cl}}(\mathbf{x}) = 0$), thus minimizing the direct energy as $E_{\mathrm{D}} = 0$. The condition reads

$$n_\mu(\mathbf{x}) \star \bar{n}_\mu(\mathbf{x}) = \bar{n}_\mu(\mathbf{x}) \star n_\mu(\mathbf{x}) = 1. \tag{9.21}$$

Furthermore, sufficiently smooth isospin waves are described in its commutative limit,

$$E_{\mathrm{X}} = \kappa\theta^2\int d^2x[(\partial_m n_\mu^\dagger)(\partial_m n_\mu) + (n_\mu^\dagger\partial_m n_\mu)(n_\nu^\dagger\partial_m n_\nu)], \tag{9.22}$$

which is the ordinary CP^{N-1} model. There are $N-1$ complex Goldstone modes.

X. DISCUSSION

We have analyzed the kinematical and dynamical properties of the classical density matrix $\hat{D}_{\mu\nu}^{\mathrm{cl}}(\mathbf{x})$ in the noncommutative plane. In particular, we have elucidated noncommutative solitons by taking a concrete instance of the QH system with the $SU(N)$ symmetry.

A microscopic Skyrmion state with $Q = 1$ is a $W_\infty(N)$-rotated state of a hole state. The Skyrmion texture is given by (1.6), or

$$\mathfrak{n}_\mu = \sum_{n=0}^{\infty}[u_\mu(n)|n\rangle\langle n| + v_\mu(n)|n+1\rangle\langle n|] \tag{10.1}$$

with $|u(n)|^2 + |v(n)|^2 = 1$. It contains infinitely many parameters. They are fixed when the Hamiltonian is given. We have studied explicitly the hard-core model and the noncommutative CP^{N-1} model. In these models all those parameters are determined analytically except for the scale parameter ω,

$$u^2(n) = \frac{\omega^2}{n+1+\omega^2}. \tag{10.2}$$

The Skyrmion energy is independent of the parameter ω.

It is intriguing that the Skyrmions are identical both in the hard-core model and the noncommutative CP^{N-1} model. Let us examine this equivalence in more detail. On one hand, in the noncommutative CP^{N-1} model the Skyrmion energy is given by

$$E_{CP} = \kappa\theta k, \tag{10.3}$$

as follows from (8.7). On the other hand, in the hard-core model it is given by (7.12), or

$$E_{\mathrm{hc}} = \frac{k}{4\pi\theta}. \tag{10.4}$$

The parameter κ is given by (9.20), which reads

$$\kappa = \frac{1}{4\pi\theta^2}. \tag{10.5}$$

Hence, these two energies are identical, $E_{CP} = E_{\mathrm{hc}}$. Namely, in the hard-core model the Skyrmion energy coming from the lowest-order term saturates the total energy. It implies that the direct energy and the higher order exchange energy have precisely canceled out each other. Such a cancellation is possible since both of them are short range in the hard-core model.

The Skyrmion solution obtained in these models satisfies the factorizability property

$$S_a^{\mathrm{cl}}(\mathbf{x}) = \rho^{\mathrm{cl}}(\mathbf{x})S_a(\mathbf{x}), \tag{10.6}$$

where $S_a(\mathbf{x}) = \frac{1}{2}\bar{n}(\mathbf{x})\lambda_a n(\mathbf{x})$ with the Gell-Mann matrix λ_a. We have called them factorizable Skyrmions. It is this class of Skyrmions that has been studied in almost all previous works on QH systems. For instance, we have studied Skyrmions [24,25] in the semiclassical approximation based on the composite-boson picture, but it allows us only to consider such a limited class of excitations.

However, as we have shown, the energy of the factorizable Skyrmion is infinite once the Zeeman effect is taken into account. Consequently realistic Skyrmions in the QH system are not factorizable Skyrmions. The factorizability follows from the particular form (10.2) of parameters $u(n)$. Any other choice produces an unfactorizable Skyrmion.

The electron-electron interaction is given by the Coulomb potential $V(\mathbf{x}) = e^2/4\pi\varepsilon|\mathbf{x}|$ in the realistic QH system. Since the interaction decreases continuously as the distance increases, the direct Coulomb energy E_D decreases as the Skyrmion size increases. However, a large Skyrmion requires a large Zeeman energy. Hence, when the Zeeman effect is very large, what is excited is the hole state (6.4). Thus the total Coulomb energy is

$$E_C = \epsilon_X = \sqrt{\frac{\pi}{2}} \frac{e^2}{8\pi\varepsilon\ell_B} \quad \text{(hole)}. \tag{10.7}$$

On the other hand, when the Zeeman effect is infinitesimal, an infinitely large Skyrmion is excited. It is well approximated by the factorizable Skyrmion, where the total Coulomb energy is

$$E_C = \frac{1}{2}\epsilon_X \quad \text{(large Skyrmion)}. \tag{10.8}$$

It is one half of the excitation energy of one hole.

To determine a Skyrmion excitation in general, it is necessary to carry out a numerical calculation by making an appropriate ansatz on parameters $u(n)$. For instance, it is reasonable to choose

$$u^2(n) = \frac{\omega^2 t^{2n+2}}{n+1+\omega^2}, \tag{10.9}$$

since the parameter t smoothly interpolates the hole ($t = 0$) and the factorizable Skyrmion ($t = 1$). As is reported elsewhere [30], it is possible to explain the experimental data [19] based on this ansatz in all range of the Zeeman gap Δ_Z.

One of our main results is the rigorous relation between the electron number density and the topological charge density,

$$\Delta\rho^{\text{cl}}(\mathbf{x}) = -J_0(\mathbf{x}) = -\frac{1}{2\pi\theta}\sum_\mu [\bar{n}_\mu(\mathbf{x}), n_\mu(\mathbf{x})]_\star. \tag{10.10}$$

The topological charge density thus defined enjoys a privileged role that it is a physical quantity essentially equal to the electron density excitation induced by a topological soliton. Accordingly the noncommutative soliton carries necessarily the electric charge. The topological charge $Q = \int d^2x J_0(\mathbf{x})$ becomes an integer for a wide class of states (4.1). This class of states contains all physically relevant states at integer filling factors. However, it does not contain states relevant at fractional filling factors. In such a system the topological charge Q would become fractional since it carries a fractional electric charge [31]. We wish to generalize the present scheme to fractional QH systems in a future work.

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APPENDIX: DECOMPOSITION FORMULA

The basic element in our analysis is the density matrix

$$D^{\text{cl}}_{\mu\nu}(m,n) = \langle\mathfrak{S}|c^\dagger_\nu(n)c_\mu(m)|\mathfrak{S}\rangle, \tag{A1}$$

obeying the relation (4.11), or

$$\sum_s D^{\text{cl}}_{\mu\sigma}(m,s)D^{\text{cl}}_{\sigma\nu}(s,n) = D^{\text{cl}}_{\mu\nu}(m,n). \tag{A2}$$

The decomposition formula (5.8) on the classical energy, or

$$E_V \equiv \langle\mathfrak{S}|H_V|\mathfrak{S}\rangle = E_D + E_X \tag{A3}$$

with (5.9), has played an important role. We prove these relations.

We consider a system with a finite number of Landau sites ($m = 0, 1, \ldots, N_\Phi - 1$) assuming the limit $N_\Phi \to \infty$ in final expressions. It is important that the $W_\infty(N)$ algebra (3.17) is not violated in such a finite system. For the sake of convenience we combine the isospin and site indices into a multi-index $M \equiv (\mu, m)$, which runs over the values $M = 1, 2, \ldots, NN_\Phi$. In terms of multi-indices, the $W_\infty(N)$ algebra turns out to be an algebra $U(NN_\Phi)$, and the transformation rules for fermion operators are

$$e^{-iW}c_I e^{+iW} = \sum_{I'}(U)_{II'}c_{I'}, \tag{A4a}$$

$$e^{-iW}c^\dagger_I e^{+iW} = \sum_{I'}c^\dagger_{I'}(U^\dagger)_{I'I}, \tag{A4b}$$

where U is an $(NN_\Phi)\times(NN_\Phi)$ unitary matrix

$$UU^\dagger = U^\dagger U = \mathbb{I}_{(NN_\Phi)\times(NN_\Phi)}. \tag{A5}$$

The state $|\mathfrak{S}_0\rangle$ given by (4.2) is expressed as

$$|\mathfrak{S}_0\rangle = \prod_{K=1}^{NN_\Phi}[c^\dagger_K]^{\nu_K}|0\rangle. \tag{A6}$$

Now it is easy to see

$$\langle\mathfrak{S}_0|c^\dagger_I c_J|\mathfrak{S}_0\rangle = \nu_J\delta_{IJ}, \tag{A7}$$

where $\delta_{MN} \equiv \delta_{\mu\nu}\delta_{mn}$.

The density matrix (A1) appears as

$$D^{\text{cl}}_{IJ} = \langle\mathfrak{S}|c^\dagger_J c_I|\mathfrak{S}\rangle. \tag{A8}$$

Employing $|\mathfrak{S}\rangle = e^{iW}|\mathfrak{S}_0\rangle$ together with (A4) and (A7) we come to

$$D^{\text{cl}}_{IJ} = \sum_K \nu_K(U)_{IK}(U^\dagger)_{KJ}. \tag{A9}$$

From this we get

$$\sum_J D^{\rm cl}_{IJ} D^{\rm cl}_{JK} = \sum_J (\nu_J)^2 (U)_{IJ} (U^\dagger)_{JK}, \quad \text{(A10)}$$

where we have used (A5).

Since $\nu_J = 0$ or 1, we have $(\nu_J)^2 = \nu_J$ and come to

$$\sum_J D^{\rm cl}_{IJ} D^{\rm cl}_{JK} = \sum_J \nu_J (U)_{IJ} (U^\dagger)_{JK} = D^{\rm cl}_{IK}, \quad \text{(A11)}$$

which is nothing but (A2), or (4.11).

We next calculate the quantity $\langle \mathfrak{S} | c_I^\dagger c_J^\dagger c_K c_L | \mathfrak{S} \rangle$. For this purpose we use the same technique and also

$$\langle \mathfrak{S}_0 | c_I^\dagger c_J^\dagger c_K c_L | \mathfrak{S}_0 \rangle = \nu_K \nu_L (\delta_{JK}\delta_{IL} - \delta_{IK}\delta_{JL}), \quad \text{(A12)}$$

which holds due to the given particular structure of (A6).

Collecting everything together we come to

$$\langle \mathfrak{S} | c_I^\dagger c_J^\dagger c_K c_L | \mathfrak{S} \rangle = D^{\rm cl}_{KJ} D^{\rm cl}_{LI} - D^{\rm cl}_{KI} D^{\rm cl}_{LJ}, \quad \text{(A13)}$$

which turns out to yield the decomposition formula (A3), or (5.8).

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