Gravitational field of relativistic gyratons

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The metric ansatz (1) is used to describe the gravitational field of a beam pulse of spinning radiation (gyraton) in an arbitrary number of spacetime dimensions D. First we demonstrate that this metric belongs to the class of metrics for which all scalar invariants constructed from the curvature and its covariant derivatives vanish. Next, it is shown that the vacuum Einstein equations reduce to two linear problems in $(D-2)$-dimensional Euclidean space. The first is to find the static magnetic potential $\mathbf{A}$ created by a pointlike source. The second requires finding the electric potential Φ created by a pointlike source surrounded by given distribution of the electric charge. To obtain a generic gyraton-type solution of the vacuum Einstein equations it is sufficient to allow the coefficients in the corresponding harmonic decompositions of solutions of the linear problems to depend arbitrarily on retarded time u and substitute the obtained expressions in the metric ansatz. These solutions are generalizations of the gyraton metrics found in [V. P. Frolov and D. V. Fursaev, Phys. Rev. D **71**, 104034 (2005).]. We discuss properties of the solutions for relativistic gyratons and consider special examples.

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I. INTRODUCTION

Studies of the gravitational fields of beams and pulses of light have a long history. Tolman [1] found a solution in the linearized approximation. Peres [2,3] and Bonnor [4] obtained exact solutions of the Einstein equations for a pencil of light. These solutions belong to the class of pp waves. The generalization of these solutions to the case where the beam of radiation carries angular momentum has been found recently in [5]. Such a solution corresponds to a pulsed beam of radiation with negligible radius of cross section, finite duration in time, and which has both finite energy E, and angular momentum J. An ultrarelativistic source with these properties is called a gyraton.

The gyraton-type solutions are of general interest, since, for example, they allow one to address the question: What is the gravitational field of a photon or ultrarelativistic electron or proton? This question becomes important in the discussion of possible mini-black-hole production in future collider or cosmic ray experiments. In the absence of spin, one can use the Aichelburg-Sexl metric [6,7] to describe the gravitational field of each of the colliding particles. Such an approach allows one to estimate the cross section for mini-black-hole formation [8–11] (for a general review see [12]). The metric obtained in [5] makes it possible to consider the gravitational scattering and mini-black-hole formation in the interaction of particles with spin. The estimates show [5] that the spin-spin and spin-orbit interactions may be important at the threshold energies for mini-black-hole formation.

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In the present paper we study the gravitational field of gyratons. We start by discussing the general properties of the metric (1) describing the gravitational field of relativistic gyratons (Sec. II). First we show that this metric belongs to the class of metrics for which all the scalar invariants constructed from the curvature and its covariant derivatives vanish identically. For $\mathbf{A}=0$ this result was obtained by Amati and Klimcik [13] and Horowitz and Steif [14], who argued that such metrics are classical solutions to string theory. (For a general discussion of spacetimes with vanishing curvature invariants see [15–17].)

After this we show that the vacuum Einstein equations for metric (1) in a spacetime with arbitrary number D of dimensions reduce to the linear problems for the gravitoelectric (Φ) and gravitomagnetic ($\mathbf{A}$) potentials in the $(D-2)$-dimensional Euclidean space. These linearized problems can be easily solved. The solutions obtained in [5] are characterized by the property that only lowest harmonics are present in the harmonic decomposition of Φ and $\mathbf{A}$. For this reason one can consider the gyraton solutions presented in [5] as some ground state, while the more general solutions obtained in this paper are their excitations (or distortions). It should be emphasized that the vacuum solutions are valid only outside the region occupied by gyratons. In order to obtain a solution describing the total spacetime one needs to obtain a solution inside the gyraton. This solution depends on the gyraton structure. In the present paper we do not discuss concrete gyraton models. But since we obtain a general solution for the vacuum metric outside a gyraton, one can guarantee that for any model of the gyraton there exists a corresponding solution, so that the characteristics of the gyraton are "encoded" in the parameters of the exterior vacuum metric.

After discussing the asymptotic properties of the gyraton metrics (Sec. III), we consider general solutions for 4- (Sec. IV) and 5-dimensional (Sec. V) gyraton metrics. Section VI discusses the higher-dimensional gyraton metrics. In Sec. VII we summarize the obtained results and discuss open problems.

II. METRIC FOR RELATIVISTIC GYRATONS

A. Gyraton metric ansatz

Let us consider the Brinkmann [18] metric in D-dimensional spacetime of the form

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -2du dv + d\mathbf{x}^2 + \Phi du^2 + 2(\mathbf{A}, d\mathbf{x})du, \end{aligned} \tag{1}$$

$$\Phi = \Phi(u, \mathbf{x}), \qquad A_a = A_a(u, \mathbf{x}). \tag{2}$$

Evidently, $l^\mu \partial_\mu = \partial_v$ is the null Killing vector.

When $\Phi = \mathbf{A} = 0$, the coordinates $x^1 = v = (t + \xi)/\sqrt{2}$ and $x^2 = u = (t - \xi)/\sqrt{2}$ are null. The coordinate u remains null for the metric (1). The metric is generated by an object moving with the velocity of light in the ξ direction. The coordinates $(x^3, \ldots, x^D)$ are coordinates of an n-dimensional space ($n = D - 2$) transverse to the direction of motion. We use boldface symbols to denote vectors in this space. For example, $\mathbf{x}$ is a vector with components x^a ($a = 3, \ldots, D$). We denote by r the length of this vector, $r = |\mathbf{x}|$. We also denote

$$d\mathbf{x}^2 = \sum_{a=3}^{D} (dx_a)^2, \qquad (\mathbf{A}, d\mathbf{x}) = \sum_{a=3}^{D} A_a dx^a, \tag{3}$$

$$\triangle = \sum_{a=3}^{D} \partial_a^2, \qquad \mathrm{div}\mathbf{A} = \sum_{a=3}^{D} A^a_{,a}. \tag{4}$$

Later we assume that the sum is taken over the repeated indices and omit the summation symbol. Working in the Cartesian coordinates we shall not distinguish between upper and lower indices.

The form of the metric (1) is invariant under the following (gauge) transformation:

$$\begin{gathered} v \to v + \lambda(u, \mathbf{x}), \qquad A_a \to A_a - \lambda_{,a}, \\ \Phi \to \Phi - 2\lambda_{,u}. \end{gathered} \tag{5}$$

It is also invariant under rescaling

$$u \to au, \qquad v \to a^{-1}v, \qquad \Phi \to a^2\Phi, \qquad \mathbf{A} \to a\mathbf{A}. \tag{6}$$

It is easy to show that for the metric (1)

$$\sqrt{-g} = 1, \tag{7}$$

and the inverse metric is

$$g^{\mu\nu}\partial_\mu \partial_\nu = -(\Phi - \mathbf{A}^2)\partial_v^2 - 2\partial_u\partial_v + 2A_a\partial_a\partial_v + \partial_a^2. \tag{8}$$

The nonvanishing components of the Christoffel symbol $\Gamma_{\mu,\nu\lambda}$ are

$$\begin{gathered} \Gamma_{v,\mu\nu} = \Gamma_{\mu,\nu v} = \Gamma_{a,bc} = 0, \qquad \Gamma_{u,uu} = \frac{1}{2}\partial_u\Phi, \\ \Gamma_{a,uu} = \partial_u A_a - \frac{1}{2}\partial_a\Phi, \qquad \Gamma_{u,ua} = \frac{1}{2}\partial_a\Phi, \\ \Gamma_{u,ab} = \frac{1}{2}(\partial_a A_b + \partial_b A_a), \qquad \Gamma_{a,bu} = -\frac{1}{2}F_{ab}, \end{gathered} \tag{9}$$

where

$$F_{ab} = \partial_a A_b - \partial_b A_a. \tag{10}$$

We shall also need the Christoffel symbols $\Gamma^\mu_{\nu\lambda} = g^{\mu\alpha}\Gamma_{\alpha,\nu\lambda}$. Their nonvanishing components are

$$\begin{gathered} \Gamma^v_{uu} = -\frac{1}{2}(\Phi_{,u} + A^a\Phi_{,a}) + A^a A_{a,u}, \\ \Gamma^v_{ua} = -\frac{1}{2}(\Phi_{,a} - F_{ab}A^b), \qquad \Gamma^v_{ab} = -\frac{1}{2}(A_{a,b} + A_{b,a}), \\ \Gamma^a_{uu} = A^a{}_{,u} - \frac{1}{2}\Phi^{,a}, \qquad \Gamma^b_{ua} = \frac{1}{2}F_a{}^b. \end{gathered} \tag{11}$$

It is easy to check that

$$l_{\mu;\nu} = 0. \tag{12}$$

It means that the null Killing vector $\mathbf{l}$ is covariantly constant. In the 4-dimensional case, spacetimes admitting a (covariantly) constant null vector field are called plane-fronted gravitational waves with parallel rays, or briefly pp waves (see e.g. [19–21]). Similar terminology is often used for higher-dimensional metrics (see e.g. [22,23]).

B. Curvature invariants

In the next section we derive conditions under which metric (1) is Ricci flat and hence obeys the vacuum Einstein equations. But before this let us prove that the metric (1) belongs to the class of metrics with vanishing curvature invariants. Namely, all the local scalar invariants constructed from the Riemann tensor and its covariant derivatives for the metric (1) vanish. This statement is valid *off shell*, that is the metric need not be a solution of the vacuum Einstein equations. This property is well known for the 4-dimensional case, since pp-wave solutions are of Petrov type N. Generalization of this result to higher-dimensional metrics (1) with $\mathbf{A} = 0$ was given in [13,14]. (For a general discussion of spacetimes with vanishing curvature invariants see [15–17].)

To demonstrate that curvature invariants vanish for the metric (1), let us consider a covariant tensor $A_{\mu\cdots\nu}$. We shall call such a tensor *degenerate* if it has the following properties: It does not depend on v, and its components, which either contain at least one index v or do not contain index u, vanish. Since ∂_v is the Killing vector, the Riemann curvature tensor does not depend on v. Using the expressions for the Christoffel symbols (11) one can show that the only nonvanishing components of the Riemann tensor are $R_{[au][bu]}$, $R_{[ab][cu]}$, and $R_{[cu][ab]}$. Hence it is degenerate. Let us demonstrate now that the action of a covariant derivative ∇_μ on a degenerate tensor $A_{\mu\cdots\nu}$ gives a tensor which is also degenerate. Really, since $\Gamma^\alpha_{v\mu}=0$ one has

$$\nabla_v A_{\mu\cdots\nu} = \partial_v A_{\mu\cdots\nu} - \Gamma^\alpha_{v\mu}A_{\alpha\cdots\nu} - \cdots \Gamma^\alpha_{v\nu}A_{\mu\cdots\alpha} = 0. \tag{13}$$

Thus the covariant differentiation of the degenerate tensor cannot have a nonvanishing v component. Since $\Gamma^u_{\mu\nu}=0$, the covariant differentiation cannot also produce a nonvanishing component which does not contain index u.

Consider now a scalar invariant constructed from any set of degenerate tensors and $g^{\mu\nu}$. The only nonvanishing component of $g^{\mu\nu}$ which contains an index u is $g^{uv}=-1$. Hence a scalar invariant constructed from degenerate tensors and metrics always vanishes.

C. Calculation of the Ricci tensor

In order to calculate the Ricci tensor for the metric (1) let us introduce the following vectors:

$$V^\lambda = \frac{\partial x^\lambda}{\partial v}, \qquad U^\lambda = \frac{\partial x^\lambda}{\partial u}, \qquad e^\lambda_{(a)} = \frac{\partial x^\lambda}{\partial x^a}. \tag{14}$$

One has

$$V^\beta{}_{;\alpha} = \Gamma^\beta_{v\alpha} = 0, \qquad U^\beta{}_{;\alpha} = \Gamma^\beta_{u\alpha}, \qquad e^\lambda_{(a);\beta} = \Gamma^\lambda_{a\beta}, \tag{15}$$

$$U^\beta{}_{;\alpha} e^\alpha_{(c);\beta} = \Gamma^\beta_{u\alpha}\Gamma^\alpha_{c\beta} = \Gamma^b_{ua}\Gamma^a_{cb} + \Gamma^u_{ua}\Gamma^a_{cu} = 0. \tag{16}$$

The last equality holds because $\Gamma^a_{cb} = \Gamma^u_{ua} = 0$.

The Ricci identity implies

$$R_{\alpha\beta}Y^\alpha X^\beta = (X^\beta{}_{;\alpha}Y^\alpha)_{;\beta} - X^\beta{}_{;\alpha}Y^\alpha_\beta - X^\beta{}_{;\beta\alpha}Y^\alpha. \tag{17}$$

From the relation $V^\beta{}_{;\alpha}=0$ it follows that $R_{\alpha\beta}Y^\alpha V^\beta = 0$ and hence

$$R_{v\alpha} = 0. \tag{18}$$

Let us set $Y^\alpha = e^\alpha_{(a)}$ and $X^\beta = U^\beta$, then using (16) one has

$$R_{au} = (U^\beta{}_{;\alpha}e^\alpha_{(a)})_{;\beta} - U^\beta{}_{;\beta\alpha}e^\alpha_{(a)}. \tag{19}$$

Using (7) one obtains

$$U^\beta{}_{;\beta} = \frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g}\delta^\beta_u) = 0. \tag{20}$$

One also has

$$U^\beta{}_{;\alpha}e^\alpha_{(a)} = \Gamma^\beta_{ua} = \frac{1}{2}\delta^\beta_b F_a{}^b - \frac{1}{2}\delta^\beta_v(\Phi_{,a} - F_{ab}A^b). \tag{21}$$

Since Φ and A_a do not depend on v, and $\Gamma^\alpha_{v\beta}=0$, one has

$$R_{au} = \frac{1}{2}\partial_b F_a{}^b. \tag{22}$$

Similarly

$$R_{uu} = (U^\beta{}_{;\alpha}U^\alpha)_{;\beta} - U^\beta{}_{;\alpha}U^\alpha{}_{;\beta} - U^\beta{}_{;\beta\alpha}U^\alpha. \tag{23}$$

Relation (20) implies that the last term on the right-hand side vanishes. Since $U^\beta{}_{;\alpha}U^\alpha = \Gamma^\beta_{uu}$ using (11) one obtains

$$(U^\beta{}_{;\alpha}U^\alpha)_{;\beta} = \partial_b(A^b{}_{,u} - \Phi^{,b}). \tag{24}$$

One also has

$$U^\beta{}_{;\alpha}U^\alpha{}_{;\beta} = \Gamma^\beta_{u\alpha}\Gamma^\alpha_{u\beta} = \Gamma^b_{ua}\Gamma^a_{ub} = -\frac{1}{4}F_{ab}F^{ab}. \tag{25}$$

Combining these results one obtains

$$R_{uu} = \partial_u\,\mathrm{div}\mathbf{A} - \frac{1}{2}\triangle\Phi + \frac{1}{4}\mathbf{F}^2, \tag{26}$$

where

$$\mathrm{div}\,\mathbf{A} = \partial_a A^a, \qquad \mathbf{F}^2 = F_{ab}F^{ab}, \qquad \triangle\Phi = \partial_a\partial^a\Phi. \tag{27}$$

Finally, let us substitute $X^\alpha = e^\alpha_{(a)}$ and $Y^\beta = e^\beta_{(b)}$ into (17), then one has

$$R_{ab} = (e^\beta_{(b);\alpha}e^\alpha_{(a)})_{;\beta} - e^\beta_{(b);\alpha}e^\alpha_{(a);\beta} - e^\beta_{(b);\beta\alpha}e^\alpha_{(a)}. \tag{28}$$

Using (11) it is easy to show that

$$e^\beta_{(b);\alpha}e^\alpha_{(a)} = \Gamma^\beta_{ab} = -\frac{1}{2}\delta^\beta_v(A_{a,b} + A_{b,a}), \tag{29}$$

$$e^\beta_{(b);\alpha}e^\alpha_{(a);\beta} = \Gamma^\beta_{b\alpha}\Gamma^\alpha_{a\beta} = 0, \qquad e^\beta_{(b);\beta} = \partial_\beta\delta^\beta_b = 0. \tag{30}$$

Hence $R_{ab}=0$.

D. Vacuum equations for gravitational field of a gyraton

To summarize, the metric (1) is a solution of vacuum Einstein equations if and only if the following equations are satisfied:

$$\partial_b F_a{}^b = 0, \tag{31}$$

$$\triangle\Phi - 2\partial_u\,\mathrm{div}\mathbf{A} = \frac{1}{2}\mathbf{F}^2. \tag{32}$$

In the next section it will be shown that for solutions describing a gyraton with finite energy and angular momentum the quantities $\Phi_{,a}$ and F_{ab} are vanishing at transverse space infinity. We assume that the homogeneous equations are valid everywhere outside the point $\mathbf{x}=0$

where a pointlike source is located. It is easy to see that the left-hand side of (32) is gauge invariant, that is invariant under the transformations (5). In the "Lorentz" gauge $A^a_{,a} = 0$ Eq. (32) takes the form

$$\triangle\Phi = \frac{1}{2}\mathbf{F}^2. \tag{33}$$

Here $\mathbf{F}^2 = F_{ab}F^{ab}$. Using the analogy of gravity with electromagnetism, one can say that the problem of solving the D-dimensional vacuum Einstein equations for the gyraton metric is reduced to finding an electric potential φ and magnetic field F_{ab} created by a local source in the $(D-2)$-dimensional Euclidean space. For these solutions the retarded time u plays a role of an external parameter which enters through the dependence of pointlike sources on u.

As we already mentioned in the Introduction, in physical applications there always exists a source of the gravitational field which generates the metric (1). We called this source a gyraton [5]. In order to obtain a solution describing the total system, one must obtain a solution inside the region occupied by the gyraton and to glue it together with a vacuum solution (1) outside it. In the present paper we study only solutions outside the gyraton. We shall obtain a *general* solution of the magnetostatic equation (31) for pointlike currents localized at the point $\mathbf{x} = 0$. Since this equation is linear this current can be written as a linear combination of $\delta(\mathbf{x})$ and its derivatives. Similarly, one can write a general solution of the equation

$$\triangle\varphi = 0, \tag{34}$$

with a charge density, localized at $\mathbf{x} = 0$, or, what is equivalent, with the charge density proportional to $\delta(\mathbf{x})$ and its derivatives. It is convenient to write $\Phi = \varphi + \psi$, where

$$\triangle\psi = \frac{1}{2}\mathbf{F}^2. \tag{35}$$

After finding $A_a(u, \mathbf{x})$ and $\varphi(u, \mathbf{x})$, one needs only to find the "induced" potential $\psi(u, \mathbf{x})$ determined by Eq. (35). A formal solution of this problem can be obtained as follows. The Green function of the n-dimensional Laplace operator

$$\triangle\mathcal{G}_n(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}') \tag{36}$$

is

$$\mathcal{G}_2(\mathbf{x}, \mathbf{x}') = -\frac{1}{2\pi}\ln|\mathbf{x} - \mathbf{x}'|, \quad \text{if } n = 2, \tag{37}$$

$$\mathcal{G}_n(\mathbf{x}, \mathbf{x}') = \frac{g_n}{|\mathbf{x} - \mathbf{x}'|^{n-2}}, \quad \text{if } n > 2, \tag{38}$$

$$g_n = \frac{\Gamma(\frac{n-2}{2})}{4\pi^{n/2}}. \tag{39}$$

Using these Green functions one can present the solution for ψ in the form

$$\psi(u, \mathbf{x}) = -\frac{1}{2}\int d\mathbf{x}'\mathcal{G}_n(\mathbf{x}, \mathbf{x}')\mathbf{F}^2(u, \mathbf{x}'). \tag{40}$$

Let us emphasize that in the general case the solution (40) is only formal and may not have a well-defined sense. The reason is that for a pointlike current, $\mathbf{F}$ has a singularity at $\mathbf{x} = 0$. If one considers this singular function as a distribution, one needs to define what is the meaning of $\mathbf{F}^2$ in (40). This problem does not exist for a distributed source (gyraton). If we do not want to input an explicit form of the matter distribution within the gyraton, we can proceed as follows [24].

Suppose $\mathbf{F}$ and φ are solutions with localized sources. Let us surround a point $\mathbf{x} = 0$ by a $(D-3)$-dimensional surface σ. For example, one may choose σ to be a round $(D-3)$-dimensional sphere of small radius ϵ. Denote by $F^{(\sigma)}_{ab} = F_{ab}\vartheta(\sigma)$, where $\vartheta(\sigma)$ is equal to 1 outside σ and vanishes inside σ. The "magnetic" field $F^{(\sigma)}_{ab}$ obeys the equation

$$\partial_b F^{(\sigma)}_a = -n_b F_a{}^b\delta(\sigma), \tag{41}$$

where $\mathbf{n}$ is the unit normal to the σ vector directed to the exterior of σ. In other words, the field $F^{(\sigma)}_{ab}$ corresponds to the special case of an extended gyraton for which its angular momentum density is localized on σ. The value $\psi^{(\sigma)}$ obtained for $F^{(\sigma)}_{ab}$ by using (40) is well defined. Certainly this function $\psi^{(\sigma)}$ depends on the choice of σ. Suppose σ' is another surface, surrounding $\mathbf{x} = 0$, and lying inside σ. It is easy to see that outside σ one has

$$\triangle(\psi^{(\sigma)} - \psi^{(\sigma)}) = 0. \tag{42}$$

That is outside σ these two solutions $\psi^{(\sigma)}$ and $\psi^{(\sigma)}$ differ by a term which can be absorbed into the solution φ.

For a distributed source (a gyraton) one can use a similar procedure. If one is interested in the gravitational field of the gyraton outside a surface σ surrounding the matter distribution one can calculate $\psi^{(\sigma)}$ and choose φ correspondingly. For a given distribution of the gyraton matter, the parameters of the vacuum solution outside the gyraton are uniquely specified. In Secs. IV and V we shall give explicit examples of the vacuum gyraton solutions.

III. ENERGY AND ANGULAR MOMENTUM OF A GYRATON

A. Weak field approximation

The asymptotics of functions Φ and A_a at the transverse-spatial infinity are related to the energy and angular momentum of a gyraton. In order to find these relations let us consider a linearized problem. Let us write the Minkowski metric in the form

$$ds_0^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -2dudv + d\mathbf{x}^2. \tag{43}$$

Then its perturbation $h_{\mu\nu}$ generated by the stress-energy tensor $T_{\mu\nu}$ obeys the equation

$$\Box h_{\mu\nu} = -\kappa \bar{T}_{\mu\nu}, \qquad \bar{T}_{\mu\nu} = \left(T_{\mu\nu} - \frac{1}{n}\eta_{\mu\nu}T\right), \tag{44}$$

where $\kappa = 16\pi G$.

In the general case, if the metric (1) is a solution of the Einstein equations, then the stress-energy tensor which generates this solution possesses the following properties: (1) Its nonvanishing components are T_{uu} and T_{ua}; (2) these components do not depend on v; and (3) it obeys the conservation law $T_{\mu\nu}{}^{;\nu} = 0$. The latter condition in the linear approximations reduces to the relation

$$T_{ua,a} = 0. \tag{45}$$

The metric perturbation $h_{\mu\nu}$ generated by such a stress-energy tensor also does not depend on v. Thus instead of D-dimensional $\Box$ operator in (44) one can substitute the n-dimensional flat Laplace operator $\triangle$

$$\triangle h_{\mu\nu} = -\kappa T_{\mu\nu}. \tag{46}$$

We omit the bar over $T_{\mu\nu}$ since its trace vanishes.

Using the Green function (36) one can write the following expression for $h_{\mu\nu}(\mathbf{x})$:

$$h_{\mu\nu}(u, \mathbf{x}) = \kappa \int d\mathbf{x}' \mathcal{G}_n(|\mathbf{x}_\perp - \mathbf{x}'|)\bar{T}_{\mu\nu}(u, \mathbf{x}'). \tag{47}$$

B. Metric asymptotics

Consider $T_{\mu\nu}(u, \mathbf{x})$ as a function of $\mathbf{x}$ and suppose that it vanishes outside of some compact region. We denote by l the size (in the transverse direction) of this region. To determine the field at far distance $r \gg l$ we use the following relation:

$$|\mathbf{x} - \mathbf{x}'| \sim r - \frac{(\mathbf{x}, \mathbf{x}')}{r}. \tag{48}$$

Thus if one point, $\mathbf{x}$, is at a far distance from the source, while the other is close to it one has the following asymptotics for the Green functions:

$$\mathcal{G}_2(\mathbf{x}, \mathbf{x}') = -\frac{1}{2\pi}\ln r + \frac{(\mathbf{x}, \mathbf{x}')}{2\pi r^2} + \cdots, \quad \text{if } n = 2, \tag{49}$$

$$\mathcal{G}_n(\mathbf{x}, \mathbf{x}') = \frac{g_n}{r^{n-2}} + \frac{g_n(n-2)(\mathbf{x}, \mathbf{x}')}{r^n} + \cdots, \quad \text{if } n > 2, \tag{50}$$

where $\cdots$ denote the terms of higher order in $1/r$. Similarly, one has

$$h_{\mu\nu} = -\frac{\kappa}{2\pi}\ln r \mathcal{T}_{\mu\nu} + \frac{\kappa}{2\pi r^2} x^a \mathcal{J}_{a\mu\nu} + \cdots, \quad \text{if } n = 2, \tag{51}$$

$$h_{\mu\nu} = \frac{\kappa g_n}{r^{n-2}}\mathcal{T}_{\mu\nu} + \frac{\kappa g_n(n-2)}{r^n} x^a \mathcal{J}_{a\mu\nu} + \cdots, \quad \text{if } n > 2, \tag{52}$$

where

$$\mathcal{T}_{\mu\nu} = \int d\mathbf{x} T_{\mu\nu}, \qquad \mathcal{J}_{a\mu\nu} = \int d\mathbf{x} x_a T_{\mu\nu}. \tag{53}$$

The structure of the stress-energy tensor implies that only components h_{uu} and h_{ua} do not vanish. In order to relate the coefficients, which enter the asymptotic expressions for these components, to physical quantities such as energy and angular momentum we use the following relations:

$$\int d\mathbf{x} T_{ua} = -\int d\mathbf{x} T_{uc}{}^{,c} x_a = 0, \tag{54}$$

$$\int d\mathbf{x}(T_u^a x^b + T_u^b x^a) = -\int d\mathbf{x} T_u^c x^a x^b = 0. \tag{55}$$

We used here the conservation law (45).

The energy E and the angular momentum J_{ab} in the flat spacetime are defined by the relation

$$E = \int d\xi d\mathbf{x} T_{tt}, \qquad J_{ab} = \int d\xi d\mathbf{x}(T_{ta}x_b - T_{tb}x_a). \tag{56}$$

The integration is performed over a surface $t = \text{const}$. At this surface $d\xi = -\sqrt{2}du$, thus

$$\int_{-\infty}^{\infty} d\xi(\cdots) = \sqrt{2}\int_{-\infty}^{\infty} du(\cdots). \tag{57}$$

One also has

$$T_{tt} = \frac{1}{2}T_{uu}, \qquad T_{ta} = \frac{1}{\sqrt{2}}T_{ua}. \tag{58}$$

By combining these relations one obtains

$$E = \int du \varepsilon(u), \qquad J_{ab} = \int du j_{ab}(u), \tag{59}$$

$$\varepsilon(u) = \frac{1}{\sqrt{2}}\int d\mathbf{x} T_{uu}, \qquad j_{ab}(u) = 2\int d\mathbf{x} T_{ua}x_b. \tag{60}$$

Relation (55) shows that $j_{ab} = -j_{ba}$. The function $\varepsilon(u)$ describes the energy-density profile of the gyraton as a function of the retarded time u, while $j_{ab}(u)$ are similar profile functions for the components of the density of the angular momentum.

Using these results one obtains

$$\Phi \sim h_{uu} = \kappa\sqrt{2}\varepsilon\begin{cases} -\frac{1}{2\pi}\ln r & \text{if } n = 2, \\ \frac{g_n}{r^{n-2}} & \text{if } n > 2, \end{cases} \tag{61}$$

$$A_a \sim h_{ua} = \frac{\kappa g_n(n-2) j_{ab} x^b}{r^n}. \tag{62}$$

The latter relation is valid in the 4-dimensional spacetime (for $n = 2$) if one substitutes $1/(2\pi)$ for $g_n(n-2)$.

C. Canonical form

If j_{ab} were a time independent antisymmetric matrix then by making rotations

$$x^a = O^a{}_b \tilde{x}^b, \qquad x_a = \tilde{x}_c O^c{}_a \tag{63}$$

one would be able to bring h_{ua} into a form where instead of j_{ab} stands its block canonical form [25]

$$\tilde{j}_{ab} = \begin{pmatrix} 0 & j_1 & 0 & 0 & \cdots \\ -j_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & j_2 & \cdots \\ 0 & 0 & -j_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{64}$$

In the presence of time dependence the situation is slightly more complicated. Let us consider transformations (63) with time dependent orthogonal matrix $O^a{}_b(u)$. Then

$$\begin{aligned} dx^a &= O^a{}_b d\tilde{x}^b + \dot{O}^a{}_b \tilde{x}^b du, \\ dx_a &= O^c{}_a d\tilde{x}_c + \dot{O}^c{}_a \tilde{x}_c du, \end{aligned} \tag{65}$$

where $\dot{B} = \partial_u B$. Under these transformations the metric (1) preserves its form with

$$\tilde{A}_a = A_b O^b{}_a + B_{ab}\tilde{x}^b, \tag{66}$$

$$\tilde{\Phi} = A_a \dot{O}^a{}_b \tilde{x}^b + C_{ab}\tilde{x}^a\tilde{x}^b, \tag{67}$$

where

$$B_{ab} = O_{ac}\dot{O}^c{}_b = -\dot{O}_{ac}O^c{}_b, \tag{68}$$

$$C_{ab} = \dot{O}_{ac}\dot{O}^c{}_b = -B_{ac}B^c{}_b. \tag{69}$$

It is easy to see that $B_{ab}\tilde{x}^b$ is itself a solution of the magnetostatic equations and corresponds to a constant magnetic field with $F_{ab} = -B_{ab}$. It means that the (linearly growing at infinity) terms generated by time dependent rigid rotations can be compensated by adding to A_a a new solution corresponding to a constant magnetic field. This is a direct analog of the Larmor theorem in gravitomagnetism [26].

To summarize, we demonstrated that by making a time dependent rotation and adding to A_a a vector potential for a homogeneous time dependent magnetic field it is always possible to transform a solution (1) into the form where Φ and A_a have the asymptotics (61) and (62), where j_{ab} is an antisymmetric matrix in its canonical block form (64).

IV. 4-DIMENSIONAL GYRATONS

A. General solution

Before analyzing general gyratonlike solutions in an arbitrary number of spacetime dimensions, we consider special lower-dimensional cases.

Let us derive a gyraton metric in a 4-dimensional spacetime. In this case the number of transverse dimensions $n = 2$ and our problem reduces to 2D electro- and magnetostatics.

Let us consider Eq. (31) for the magnetic field. Any antisymmetric tensor of the second order in a 2-dimensional space can be written as $F_{ab} = Fe_{ab}$, where e_{ab} is the totally antisymmetric tensor. Substituting this representation into (31) one obtains that $F = \text{const}$. It is easy to see that the corresponding vector potential A_a can be written as

$$A_3 = \alpha x^4, \qquad A_4 = \beta x^3, \qquad F = \beta - \alpha. \tag{70}$$

The gauge transformation (5) with $\lambda = \gamma x^3 x^4$ changes the coefficients $\alpha \to \alpha - \gamma$ and $\beta \to \beta - \gamma$ but preserves the value F.

Equation (35) takes the form

$$\triangle\psi = \frac{1}{2}F^2. \tag{71}$$

If $F \neq 0$, the solution ψ does not vanish at infinity. We exclude this case. Thus we put $F = 0$.

Let us choose a 2-dimensional contour surrounding the source at $\mathbf{x} = 0$. When $F_{ab} = 0$, the value of the integral

$$j(u) = \frac{2}{\kappa}\oint_C A_a dx^a, \qquad j(u) = \frac{1}{2}\epsilon^{ab} j_{ab}, \tag{72}$$

does not depend on the choice of the contour C. This quantity which enters the solution (1) has the meaning of the angular momentum of the gyraton. In polar coordinates (r, ϕ)

$$x^3 + ix^4 = re^{i\phi}, \tag{73}$$

the corresponding potential A_a can be written as

$$A_r = 0, \qquad A_\phi = \frac{\kappa}{4\pi} j(u). \tag{74}$$

Let us consider now Eq. (34) for the 2-dimensional "electric" potential φ. A solution corresponding to a pointlike charge is

$$\varphi_0 = -\frac{\kappa\sqrt{2}}{2\pi}\varepsilon(u)\ln r. \tag{75}$$

Any other solution of this equation decreasing at infinity can be written as

$$\varphi = \varphi_0 + \sum_{n=-\infty}^{\infty}{}' \frac{b_n}{r^{|n|}} e^{in\phi}, \qquad \bar{b}_n = b_{-n}. \tag{76}$$

$\sum'$ indicates that the term $n = 0$ is excluded. In the electromagnetic analogy, the harmonics with $n \geq 1$ describe the field created by an electric n pole. Since $F = 0$, $\Phi = \varphi$ and the solution for a distorted gyraton in 4-dimensional spacetime is

$$ds^2 = -2dudv + dr^2 + r^2 d\phi^2 + \frac{\kappa}{2\pi} j(u) du d\phi + \left[-\frac{\kappa\sqrt{2}}{2\pi} \varepsilon(u) \ln r + \varphi \right] du^2, \tag{77}$$

where $\varphi = \varphi(u, r, \phi)$ is given by (76) with $b_n = b_n(u)$.

It should be emphasized that the solution (77), which is a special case of (1), is the pp-wave metric. The properties of pp-wave metrics in 4-dimensional spacetime are well known (see e.g. [19–21]). In particular, a vacuum pp-wave metric in a simply connected region can be written in the form where $\mathbf{A} = 0$. In the case of a gyraton, because of the presence of a singularity at $\mathbf{x} = 0$, the gauge transformations (5) cannot be used to banish the potential $\mathbf{A}$ *globally*. The situation here is similar to the well-known Aharonov-Bohm effect [27,28]. The topological invariant $j(u)$, which has the meaning of the density of the angular momentum of the gyraton, is similar to the magnetic flux in the Aharonov-Bohm case.

In the study of the Aharonov-Bohm effect it is usually helpful to consider at first a tube of finite radius where the magnetic field is localized. Similarly, in order to obtain a well-defined and finite expression for the gyraton metric, one must consider a spreaded source of finite size. In the 4-dimensional case the procedure proposed in Sec. II does not work. We describe now a simple model of an extended gyraton in the 4-dimensional case. Let us modify the expression for (74) as follows:

$$A_r = 0, \qquad A_\phi = \frac{\kappa}{4\pi} j(u) \left[\frac{r^2}{r_0^2} \vartheta(r_0 - r) + \vartheta(r - r_0) \right]. \tag{78}$$

For this modified vector potential the field strength $F_{ab} = e_{ab} F$ is

$$F = \frac{\kappa}{2\pi} \frac{j(u)}{r_0^2} \vartheta(r_0 - r). \tag{79}$$

In other words, the field strength F is constant inside a disk of radius r_0. Outside this disk the field vanishes, while the contour integral (72) is $j(u)$ as earlier.

The function ψ for such an extended gyraton can be found by using (37) and (40). In polar coordinates one has

$$\psi = \frac{F^2}{4\pi} \int_0^{r_0} dr' r' Q, \tag{80}$$

$$Q = \int_0^{2\pi} d\phi \ln(r^2 + r'^2 - 2rr' \cos\phi). \tag{81}$$

For $r > r_0$ one has $Q = 4\pi \ln r$. Thus

$$\psi = \frac{1}{2} F^2 r_0^2 \ln r. \tag{82}$$

It means that outside the gyraton ψ has the same form as φ_0 and can be absorbed into the latter by renormalizing the function $\varepsilon(u)$.

B. Boosted 4D NUT metric

As an aside, it is worth mentioning that the Aichelburg-Sexl boost [6] of the Newman-Unti-Tamburino (NUT) stationary vacuum geometry [29] is a particular member of the class (77).

A convenient symmetric form of the NUT metric is

$$ds^2 = \frac{dr^2}{f(r)} + (r^2 + a^2)(d^2\theta + \sin^2\theta d\phi^2) - f(r)(dt - 2a\cos\theta d\phi)^2, \tag{83}$$

where

$$f(r) = \frac{r^2 - 2mr - a^2}{r^2 + a^2}. \tag{84}$$

It is known [30,31] that the (singular) source of this geometry consists of a pair of semi-infinite line sources along the axis ($\theta = 0$ and $\theta = \pi$ respectively), endowed with equal and opposite average angular momenta $\pm a/2$ per unit length, and joined to a massive particle at the origin. The two line sources are massless to linear order in a, i.e., to within terms of the order of the gravitational potential energy, which cannot be localized unambiguously.

To boost the metric (83), it is sufficient to consider its linearized form

$$ds^2 = \left(1 + \frac{2m}{r}\right)(d\rho^2 + \rho^2 d\phi^2 + d\bar{z}^2) + 4a\frac{\bar{z}}{r} d\phi d\bar{t} - \left(1 - \frac{2m}{r}\right) d\bar{t}^2, \tag{85}$$

where $r^2 = \rho^2 + \bar{z}^2$. We apply a Lorentz transformation

$$\bar{z} = \frac{1}{2}(ve^{-\chi} - ue^{\chi}), \qquad \bar{t} = \frac{1}{2}(ve^{-\chi} + ue^{\chi}), \tag{86}$$

where $u = t - z$ and $v = t + z$. In the limit $\chi \to \infty$ this sends the originally static source moving along the path $z = t$ in the new frame. Noting that $\lim_{\chi\to\infty} \bar{z}/r = \epsilon(u) = \pm 1$, rescaling mass and angular momentum according to

$$m = \mu e^{\chi}, \qquad a = \alpha e^{\chi}, \tag{87}$$

and using the distributional identity

$$\lim_{\chi\to\infty} \frac{e^{\chi}}{\sqrt{\rho^2 + u^2 e^{2\chi}}} = \frac{1}{|u|} - \delta(u) \ln|\rho/l|, \tag{88}$$

where l is an arbitrary length scale, we readily obtain the limiting form

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 - dudv - 2\mu\delta(u)\ln|\rho/l|du^2 - 2\alpha\epsilon(u)dud\phi. \tag{89}$$

[In (89), we have absorbed the term $1/|u|$ from (88) by a transformation of v.] This is a special case of (77). It represents a pair of semi-infinite gyratons with equal and opposite angular momentum densities $\alpha\epsilon(u)$, joined to the Aichelburg-Sexl boosted particle of energy μ. Classification of the 4-dimensional pp waves with an impulsive profile based on their symmetries can be found in [32].

V. 5-DIMENSIONAL GYRATONS

A. General solution

In order to obtain a solution for the gravitational field of a 5-dimensional gyraton one needs to analyze electro- and magnetostatics in a flat 3-dimensional space.

Let us consider first the magnetic equation (31). Using the standard 3-dimensional notations one can write these equations in the form

$$\mathbf{B} = \text{curl}\,\mathbf{A}, \qquad \text{curl}\,\mathbf{B} = 0. \tag{90}$$

The second equation implies that there exists a function Υ, the magnetic scalar potential, such that the magnetic field B is

$$\mathbf{B} = -\nabla\Upsilon. \tag{91}$$

The first of the equations (90) implies that the magnetic potential obeys the following equation:

$$\triangle\Upsilon = 0. \tag{92}$$

Let (r, θ, ϕ) be the spherical coordinates

$$x^3 + ix^4 = r\sin\theta e^{i\phi}, \qquad x^5 = r\cos\theta. \tag{93}$$

Then the general solution of (92) decreasing at infinity can be written as follows:

$$\Upsilon = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}\frac{Y_{lm}(\theta, \phi)}{r^{l+1}}, \tag{94}$$

where the complex coefficients a_{lm} obey the conditions $\bar{a}_{lm} = a_{l-m}$. Here $Y_{lm}(\theta, \phi)$ are spherical harmonics

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\theta)e^{im\phi}. \tag{95}$$

The magnetic induction vector $\mathbf{B}$ is

$$\mathbf{B} = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}\nabla\left(\frac{Y_{lm}}{r^{l+1}}\right). \tag{96}$$

In order to find the corresponding vector potential $\mathbf{A}$ one needs to solve the following equation:

$$\text{curl}\;\mathbf{A} = -\nabla\Upsilon. \tag{97}$$

It can be done by using the properties of vector spherical harmonics. Let us denote

$$\Psi_{lm}(\theta, \phi) = r\nabla Y_{lm}(\theta, \phi), \tag{98}$$

$$\Phi_{lm}(\theta, \phi) = \mathbf{r}\times\nabla Y_{lm}(\theta, \phi). \tag{99}$$

The vector spherical harmonics obey the following relations [33]:

$$\nabla\times\left(\frac{\Phi_{lm}}{r^{l+1}}\right) = \nabla\times\left(\frac{\mathbf{r}\times\Psi_{lm}}{r^{l+2}}\right). \tag{100}$$

$$\nabla\cdot\left(\frac{\Phi_{lm}}{r^{l+1}}\right) = 0. \tag{101}$$

Using the first of these relations one finds

$$\mathbf{A} = \mathbf{A}_0 - \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{a_{lm}}{l}\frac{\Phi_{lm}}{r^{l+1}}. \tag{102}$$

The relation (101) shows that the solution (102) obeys the following gauge condition:

$$\text{div}\,\mathbf{A} = 0. \tag{103}$$

We denote by $\mathbf{A}_0$ a vector potential for the $l = 0$ case which requires a special treatment, since in this case $\Phi_0 = 0$ and ratio Φ_{lm}/l are not determined.

A general solution of the equation (34) for φ can be written as

$$\varphi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} b_{lm}\frac{Y_{lm}(\theta, \phi)}{r^{l+1}}, \tag{104}$$

where the coefficients b_{lm} obey the conditions $\bar{b}_{lm} = b_{l-m}$. For a gyraton solution coefficients a_{lm} and b_{lm} are arbitrary functions of the retarded time u. To obtain ψ one can use (40) with the

$$\mathcal{G}(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}. \tag{105}$$

B. Monopole solution

As we mentioned, the case of a magnetic monopole ($l = 0$) is special. Let us consider it in more detail. The magnetic potential Υ for the magnetic monopole is

$$\Upsilon = -\frac{\mu}{r}, \tag{106}$$

where μ is an arbitrary function of u. The magnetic induction vector has components

$$B_r = \frac{\mu}{r^2}, \qquad B_\theta = B_\phi = 0. \tag{107}$$

The corresponding vector potential is of the form

$$A_r = A_\theta = 0, \qquad A_\phi = -\mu\cos\theta. \tag{108}$$

The potential obeys the condition $\text{div}\mathbf{A} = 0$ and the potential ψ is

$$\psi = \frac{\mu^2}{4r^2}. \tag{109}$$

The corresponding monopole solution for the gyraton is

$$ds^2 = -2dudv + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + \left(\varphi + \frac{\mu^2(u)}{4r^2}\right)du^2 - 2\mu(u)\cos\theta du d\phi. \tag{110}$$

Here $\varphi = \varphi(u, r, \theta, \phi)$ is a solution of the equation (34).

Similarly to the 4D case this metric is related to the boosted NUT-like metric. Consider a metric ($R^2 = r^2 + w^2$)

$$ds^2 = -\left(1 - \frac{2m}{R^2}\right)dt^2 + 4a\cos\theta d\phi dt + \left(1 + \frac{2m}{R^2}\right)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + dw^2). \tag{111}$$

It is Ricci flat to linear order and it is a linearized version of 5D NUT spacetime. (An exact counterpart of this linearized metric does not seem to be known.) Applying the boost

$$w = \frac{1}{2}(ue^{\chi} - ve^{-\chi}), \qquad t = \frac{1}{2}(ue^{\chi} + ve^{-\chi}) \tag{112}$$

rescaling $me^{2\chi} = \frac{1}{2}\mu^2$, $ae^{\chi} = -\mu$, and noting

$$\lim_{\chi\to\infty} R^{-2} = r^{-2}, \tag{113}$$

we recover (110) with $\varphi = 1$ in the limit $\chi \to \infty$.

C. Dipole solution

The spherical harmonics for the $l = 1$ case are

$$Y_{10} = \alpha\frac{x^5}{r}, \qquad \alpha = \sqrt{\frac{3}{4\pi}}, \tag{114}$$

$$Y_{11} = \bar{Y}_{1-1} = -\frac{\alpha}{\sqrt{2}}\frac{x^3 + ix^4}{r}. \tag{115}$$

Notice that $\mathbf{r} \times \nabla F(r) = 0$. Using this property we obtain

$$\Phi_{10} = \frac{\alpha}{r}(x^4, -x^3, 0), \tag{116}$$

$$\Phi_{11} = -i\frac{\alpha}{\sqrt{2}r}(x^5, -ix^5, x^3 + ix^4). \tag{117}$$

Let us denote

$$\omega^{ab} = x^a dx^b - x^b dx^a. \tag{118}$$

Then the expressions for $(\mathbf{A}_{1m}, d\mathbf{x})$ take the form

$$(\mathbf{A}_{10}, d\mathbf{x}) = a_{10}\frac{\alpha}{r^3}\omega^{34}, \tag{119}$$

$$(\mathbf{A}_{11}, d\mathbf{x}) = a_{11}\frac{\alpha}{r^3}(\omega^{45} - i\omega^{35}). \tag{120}$$

The vector potential for a general dipole solution can be written as follows:

$$(\mathbf{A}, d\mathbf{x}) = \frac{\kappa}{8\pi}\frac{j_{ab}x^b dx^a}{r^3}, \tag{121}$$

where

$$j_{ab}x^b dx^a = \frac{8\pi\alpha}{\kappa}[a_{10}\omega^{34} + \sqrt{2}\mathrm{Re}(a_{11})\omega^{45} + \sqrt{2}\mathrm{Im}(a_{11})\omega^{35}], \tag{122}$$

and a_{10}, a_{11} are arbitrary functions of u.

VI. HIGHER-DIMENSIONAL CASE

Let us discuss first the scalar (electrostatic) equation (34) in the n-dimensional Euclidean space R^n

$$\triangle\varphi = 0. \tag{123}$$

To solve this equation it is convenient to decompose the potential φ into the scalar spherical harmonics [34]

$$Y^l = r^{-l}\mathcal{Y}^l, \qquad \mathcal{Y}^l = C_{c_1\cdots c_l}x^{c_1}\cdots x^{c_l}, \tag{124}$$

where $C_{c_1\cdots c_l}$ is a symmetric traceless rank-l tensor. It is easy to see that the number of linearly independent components of coefficients $C_{c_1\cdots c_l}$ is

$$d_0(n, l) = \frac{(l + n - 3)!(2l + n - 2)}{l!(n - 2)!}. \tag{125}$$

These harmonics are eigenfunctions of the invariant Laplace operator on a unit sphere S^{n-1} with eigenvalues $-l(n + l - 2)$. For each l there exists $d_0(n, l)$ linearly independent harmonics. We shall use an index q to enumerate the independent harmonics. The functions Y^{lq} form a complete set, so that any smooth function F on S^{n-1} can be decomposed as

$$F = \sum_{l=0}^{\infty}\sum_{q} F_{lq}Y^{lq}. \tag{126}$$

Consider now a special mode $F_{lq}(r)Y^{lq}$. It is a decreasing-at-infinity solution of (123) if $F_{lq} \sim r^{-(n+l-2)}$. This can be proved by using the properties of the scalar spherical harmonics. We demonstrate this directly by using the relations (124).

First, it is easy to check that

$$\triangle\mathcal{Y}^l = 0, \qquad x^d\partial_d\mathcal{Y}^l = l\mathcal{Y}^l. \tag{127}$$

Using these relations one obtains

$$\triangle(f(r)\mathcal{Y}^l) = \left(f'' + \frac{(n + 2l - 1)}{r}f'\right)\mathcal{Y}^l. \tag{128}$$

Thus for $f = 1/r^{n+2l-2}$ the mode functions $f(r)\mathcal{Y}^l$ obey the Eq. (123). To summarize, a general solution of the electrostatic equation (123) can be written in the form

$$\phi = \sum_{l=0}^{\infty} \sum_{q} \frac{\mathcal{Y}^{lq}}{r^{n+2l-2}}. \tag{129}$$

In the gyraton solution (1) $d_0(n, l)$ independent components of $C_{c_1 \cdots c_{l-1}}$ are arbitrary functions of u.

In a similar way, one can obtain solutions of the equations of magnetostatics in n-dimensional Euclidean space by using the vector spherical harmonics [34]. Let us denote

$$A_a^l = f(r) \mathcal{Y}_a^l, \tag{130}$$

$$\mathcal{Y}_a^l = C_{abc_1 \cdots c_{l-1}} x^b x^{c_1} \cdots x^{c_{l-1}}. \tag{131}$$

Here $C_{abc_1 \cdots c_{l-1}}$ is a $(l+1)$th-rank constant tensor which possesses the following properties: it is antisymmetric under the interchange of a and b, and it is traceless under the contraction of any pair of indices [34].

First, let us demonstrate that A_a^l obeys the gauge condition

$$\partial^a A_a^l = 0. \tag{132}$$

Notice that

$$\partial_a f(r) = f'(r) \frac{x^a}{r}. \tag{133}$$

Thus

$$\partial^a A_a^l = f \partial^a \mathcal{Y}_a^l = 0. \tag{134}$$

The latter equality follows from the fact that when ∂_a is acting on one of x it effectively produces a contraction of two indices in C which vanishes.

In the gauge (132) the magnetostatic field equation (31) reduces to the following equation:

$$\triangle A_a^l = 0. \tag{135}$$

It is easy to get

$$\triangle \mathcal{Y}_a^l = 0, \qquad x^b \partial_b \mathcal{Y}_a^l = l \mathcal{Y}_a^l. \tag{136}$$

Using these relations one obtains

$$\triangle (f \mathcal{Y}_a^l) = \left(f'' + \frac{n+2l-1}{r} f' \right) \mathcal{Y}_a^l. \tag{137}$$

Hence $\mathcal{Y}_a^l$ is a solution of (135) if

$$f'' + \frac{n+2l-1}{r} f' = 0. \tag{138}$$

Solving this equation we get $f = 1/r^{n+2l-2}$. Hence a general decreasing-at-infinity solution of the magnetostatic equations in the n-dimensional space (31) can be written as

$$A_a = \sum_{l=1}^{\infty} \sum_{q} \frac{\mathcal{Y}_a^{lq}}{r^{n+2l-2}}. \tag{139}$$

Again, we use an index q to enumerate different linearly independent vector spherical harmonics. The total number of these harmonics for given l is [34]

$$d_1(n, l) = \frac{l(n+l-2)(n+2l-2)(n+l-3)!}{(n-3)!(l+1)!}. \tag{140}$$

In the gyraton solution (1) the coefficients $C_{abc_1 \cdots c_{l-1}}$ in the decomposition (139) are arbitrary functions of the retarded time u. For a given solution $\mathbf{A}$ relation (40) allows one to find ψ.

VII. SUMMARY AND DISCUSSIONS

The main result of this paper is that the vacuum Einstein equations for the gyraton-type metric (1) in an arbitrary number of spacetime dimensions D can be reduced to linear problems in the Euclidean $(D-2)$-dimensional space. These problems are (1) to find a static electric field φ created by a pointlike source; (2) to find a magnetic field $\mathbf{A}$ created by a pointlike source. The retarded time u plays the role of an external parameter. One can include u dependence by making the coefficients in the harmonic decomposition for φ and $\mathbf{A}$ to be arbitrary functions of u. After choosing the solutions of these two problems one can define ψ by means of Eq. (40). By substituting $\Phi = \varphi + \psi$ and $\mathbf{A}$ into the metric ansatz one obtains a vacuum solution of the Einstein equations.

Such a gyratonlike solution has a singularity located at the spatial point $\mathbf{x} = 0$ during some interval of the retarded time u. It means that the corresponding pointlike source is moving with the velocity of light. Energy E and angular momentum J_{ab} are finite. It was demonstrated that for given energy and angular momentum the gyraton can also have other characteristics, describing the deviation of Φ from spherical symmetry (in the transverse space R^n) and the presence of higher than dipole terms in the multipole expansion of $\mathbf{A}$. One can interpret such solutions as excitations or distortions of the gyraton solutions obtained earlier in [5].

It should be emphasized that the pointlike sources are certainly an idealization. In [5] it was shown that gyraton solutions can describe the gravitational field of beam-pulse spinning radiation. In such a description one uses the geometric optics approximation. For its validity the size of the cross section of the beam must be much larger than the wavelength of the radiation. In the presence of spin J one can expect additional restrictions on the minimal size of both, the cross-section size and the duration of the pulse. As usual in physics, one must have in mind that in the possible physical applications the obtained solution is valid only outside some region surrounding the immediate neighborhood of the singularity.

The gyraton solutions might be used for studying the gravitational interaction of ultrarelativistic particles with spin. The gyraton metrics might be also interesting as possible exact solutions in the string theory.

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