PHYSICAL REVIEW D **72,** 084028 (2005)

# Bulk-boundary thermodynamic equivalence, and the Bekenstein and cosmic-censorship bounds for rotating charged AdS black holes

G. W. Gibbons,[1] M. J. Perry,[1] and C. N. Pope[2]

[1]*DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 OWA, United Kingdom*
[2]*George P. & Cynthia W. Mitchell Institute for Fundamental Physics, Texas A&M University, College Station, Texas 77843-4242, USA*
(Received 9 August 2005; published 27 October 2005)

We show that one may pass from bulk to boundary thermodynamic quantities for rotating anti-de Sitter (AdS) black holes in arbitrary dimensions so that if the bulk quantities satisfy the first law of thermodynamics then so do the boundary conformal field theory (CFT) quantities. This corrects recent claims that boundary CFT quantities satisfying the first law may only be obtained using bulk quantities measured with respect to a certain frame rotating at infinity, and which therefore do not satisfy the first law. We show that the bulk black-hole thermodynamic variables, or equivalently therefore the boundary CFT variables, do not always satisfy a Cardy-Verlinde type formula, but they do always satisfy an AdS-Bekenstein bound. The universal validity of the Bekenstein bound is a consequence of the more fundamental cosmic-censorship bound, which we find to hold in all cases examined. We also find that at fixed entropy, the temperature of a rotating black hole is bounded above by that of a nonrotating black hole, in four and five dimensions, but not in six or more dimensions. We find evidence for universal upper bounds for the area of cosmological event horizons and black-hole horizons in rotating black-hole spacetimes with a positive cosmological constant.

DOI: 10.1103/PhysRevD.72.084028 PACS numbers: 04.70.Dy, 04.50.+h

## I. INTRODUCTION

There has been much progress recently in constructing solutions of the supergravity equations describing rotating and charged black holes in $n$-dimensional anti-de Sitter (AdS) backgrounds [1–9]. A primary motivation for this work was the elucidation of the thermodynamics of these black holes, with a view to comparing it with that of the dual conformal field theory (CFT) [10–12] on the boundary of the spacetime, which approaches AdS with radius of curvature $l$.[1] In particular the correct energies and angular momenta for Kerr-AdS black holes, measured with respect to a frame that is nonrotating at infinity, were calculated in all dimensions in [13], where it was also demonstrated that these quantities satisfy the first law of thermodynamics,

$$dE = TdS + \Omega^i dJ_i. \tag{1.1}$$

This resolved some of the apparent ambiguities in earlier work, that had focused on energies and angular velocities measured with respect to a particular frame rotating at infinity, which we shall denote with primes throughout this paper.[2] As shown in [13], these do not satisfy the first law of thermodynamics:

$$dE' \neq T' dS' + \Omega^{i\prime} dJ_i', \tag{1.2}$$

since the asymptotic rotation rate in this frame depends on the black-hole rotation parameters.

In a recent paper, Cai *et al.* [14] noticed that by passing from the bulk quantities $(E', J_i')$ to the dual CFT quantities $(e', j_i', \ldots)$ on the boundary, and by including an additional pressure term $p'$ and suitably defined volume term $v'$, the equation

$$de' = t' ds' + \omega^{i\prime} dj_i' - p' dv' \tag{1.3}$$

holds. They interpreted this equation as the first law for the dual CFT, and furthermore they made the surprising claim that no such analogous CFT thermodynamic variables can be introduced that are dual to the unprimed bulk quantities, and which satisfy the first law,

$$de = tds + \omega^i dj_i - pdv. \tag{1.4}$$

One purpose of this paper is to refute this surprising claim, and on the contrary to demonstrate that following the perfectly standard transcription rules relating bulk and boundary quantities, the first law (1.4) does indeed hold. We also raise questions as to whether the interpretation of the primed quantities given in [14] is physically correct.

A remarkable feature of [14], following earlier work in [15,16], is the observation that the primed bulk quantities satisfy a Cardy-Verlinde [17] type formula,

$$S' = \frac{2\pi l}{n-2}\sqrt{E_c'(2E' - E_c')}, \qquad E_c' = (n-1)E' - (n-2)(T'S' + \Omega^{i\prime} J_i'), \tag{1.5}$$

whereas the unprimed quantities measured with respect to a nonrotating frame at infinity do not. This has motivated us to reexamine the old question of whether or not AdS

[1]In $n = 5$ dimensions, the dual boundary conformal field theory is $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory, where $N^2 = \pi c^3 l^3/(2\hbar G_5)$.

[2]Not all quantities measured in the rotating coordinate system are changed from their values in the asymptotically static frame, but for clarity we denote all quantities measured in the rotating frame with primes in the present discussion.

 

black holes satisfy some sort of possibly modified Cardy-Verlinde formula, and a Bekenstein-type bound of the form[3]

$$E \geq \frac{(n-2)S}{2\pi l}. \tag{1.6}$$

The existence of the Bekenstein bound is a necessary but not sufficient condition for the validity of a Cardy-Verlinde formula. In fact, we find that while there appears to be no universal formulation of a modified Cardy-Verlinde formula that will cover all of the charged and rotating AdS black holes that we know of, we do find that a Bekenstein bound holds in all cases.

In fact, the Bekenstein bound follows from a stronger and more fundamental inequality, the cosmic-censorship bound, which takes the form

$$E \geq \frac{(n-2)A}{16\pi l}\left[l\left(\frac{A}{\mathcal{A}_{n-2}}\right)^{-1/(n-2)} + \frac{1}{l}\left(\frac{A}{\mathcal{A}_{n-2}}\right)^{1/(n-2)}\right], \tag{1.7}$$

where $A$ is the area of the event horizon, or more generally, in time-dependent cases, the area of the outermost apparent horizon, and $\mathcal{A}_{n-2}$ is the volume of the unit $(n-2)$-sphere. We show that the recently constructed exact solutions for rotating and charged AdS black holes give strong support for the conjectured cosmic-censorship bound.

As well as lower bounds for the energy in terms of the entropy, it is well known that there are interesting upper bounds for the temperature as a function of entropy, for black holes in asymptotically flat spacetimes. It turns out that these may be generalized to the static anti-de Sitter case, taking the form

$$4\pi T \leq (n-3)\left(\frac{A}{\mathcal{A}_{n-2}}\right)^{-1/(n-2)} + (n-1)l^{-2}\left(\frac{A}{\mathcal{A}_{n-2}}\right)^{1/(n-2)}, \tag{1.8}$$

in $n$ dimensions. That is, the temperature is never greater than the value it would have in the Schwarzschild-AdS solution with the same entropy. We investigate whether the bound extends to stationary black holes, finding that it is obeyed by all Kerr-AdS black holes in four and five dimensions, but not in dimensions six or higher.

The uncharged rotating black-hole solutions are of course valid also if the cosmological constant is taken to be positive, corresponding to sending $l^2 \to -l^2$. In this case, an additional, cosmological, horizon is present. We verify that these solutions support the general conjecture that the area $A_C$ of the cosmological horizon satisfies

$$A_C \leq \mathcal{A}_{n-2} l^2, \tag{1.9}$$

and the black-hole horizon satisfies the inequality

$$A_H \leq \mathcal{A}_{n-2} l^{n-2}\left(\frac{n-3}{n-1}\right)^{(n-2)/2}. \tag{1.10}$$

The plan of the paper is as follows. In Section II we establish a general equivalence between bulk and boundary thermodynamics, and explain our disagreement with some of the work in [14]. In Section III, we review the Cardy-Verlinde formula and its consequence, the AdS-Bekenstein bound, explaining why it is saturated at the Hawking-Page phase transition. We show that a simple modification holds for Reissner-Nordström-AdS black holes, and gives rise to a strengthened form of the Bekenstein bound. We also show that in all dimensions rotating black holes without charge satisfy the AdS-Bekenstein bound. In Section IV, we examine a large number of examples of rotating and/or charged black holes, finding that despite the failure in general of the Cardy-Verlinde formula, the AdS-Bekenstein bound, or its strengthened electrostatic form, holds. In Section V we discuss how the AdS-Bekenstein bound may be regarded as a consequence of the AdS cosmic-censorship bound, and demonstrate in all the examples we have checked that the AdS cosmic-censorship bound does indeed hold, in some cases strengthened by an electrostatic contribution. Section VI discusses upper bounds for the temperature of AdS black holes, in terms of their entropy. We find that rotating black holes in four and five dimensions always have a temperature that is less than that of the Schwarzschild-AdS solution with the same area. However, rotating black holes of dimension six or higher do not satisfy such a bound. Section VII contains a brief discussion of the areas of the cosmological and black-hole horizons for rotating black holes with positive cosmological constant. We collect for the reader's convenience, in an appendix, the pertinent formulae for Kerr-AdS black holes in arbitrary dimensions. Our conclusions are contained in Section VIII.

## II. BULK AND BOUNDARY THERMODYNAMICS

The purpose of this section is to show that there is a universal rule allowing one to pass between bulk and boundary quantities in such a way that if one set of quantities satisfies the first law of thermodynamics, then so will the other.

### A. Tolman or UV/IR scaling transformations

It is quite generally true that if an arbitrary thermodynamic system satisfies the first law of thermodynamics without a pressure term,

$$dE = TdS + \Omega^i dJ_i + \Phi_i dQ_i, \tag{2.1}$$

then associated with it is a second system, with pressure equal to the energy density divided by the spatial dimen-

[3]Note that the Bekenstein bound does not contain Newton's constant, and so it makes sense for any thermodynamic quantity in an AdS background. However, in this paper our concern is solely with black holes.

sion, which satisfies the first law with pressure term:

$$de = tds + \omega^i dj_i + \phi_i dq_i - pdv. \qquad (2.2)$$

Actually, since the first system need have no natural dimension associated to it, the spatial dimension $(n-2)$ of the second system can be arbitrary. The thermodynamic quantities of the second system, denoted by lowercase letters, are related to those of the first system by

$$e = \frac{l}{y}E, \qquad \omega^i = \frac{l}{y}\Omega^i, \qquad \phi_i = \frac{l}{y}\Phi_i, \qquad t = \frac{l}{y}T,$$
$$s = S, \qquad j_i = J_i, \qquad q_i = Q_i, \qquad (2.3)$$

with

$$v = \mathcal{A}_{n-2}y^{n-2}, \qquad p = \frac{e}{(n-2)v}, \qquad (2.4)$$

where $\mathcal{A}_{n-2}$ is the volume of the unit $(n-2)$-sphere. Here, $y$ is to be thought of as the radius of the second system, and $l$ is an arbitrary constant, which in our application is related to the cosmological constant, so that $R_{\mu\nu} = -(n-1)l^{-2}g_{\mu\nu}$. Note that in (2.3) the intensive quantities $(T, \Omega^i, \Phi_i)$ are scaled, as is the energy $E$, while the extensive quantities $(S, J_i, Q_i)$ are not scaled.

As it stands, the above result is a mathematical triviality. However, in the case we are considering, where the first system is a rotating charged black hole in an $\mathrm{AdS}_n$ background, it allows us to relate the bulk thermodynamic quantities associated with the black hole to the boundary quantities associated with the dual conformal field theory. The quantities $(E, T, S, \Omega^i, J_i, \Phi_i, Q_i)$ are all evaluated with respect to a coordinate frame $(t, y, \hat{\mu}_i, \varphi_i)$ that is nonrotating and asymptotically spherical at infinity.[4] In these coordinates, the metric of the rotating black hole approaches the AdS metric

$$d\bar{s}^2 = -(1 + y^2 l^{-2})dt^2 + \frac{dt^2}{1 + y^2 l^{-2}} + y^2 \sum_{k=1}^{N+\epsilon}(d\hat{\mu}_k^2 + \hat{\mu}_k^2 d\varphi_k^2), \qquad (2.5)$$

in $n = 2N + \epsilon + 1$ dimensions, where $\epsilon = (n-1) \bmod 2$ (see [13], and the appendix, for a more detailed discussion). Note that the radial coordinate $y$ is related to the Fefferman-Graham coordinate $z \sim l^2/y$ for which the metric asymptotes to $-l^2dt^2/z^2 + l^2dz^2/z^2 + l^4/z^2 d\Omega^2_{n-2}$.

The Killing vector $\partial/\partial t$ is normalized so that near infinity

$$g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) \to -\frac{y^2}{l^2}, \qquad (2.6)$$

and this fixes the normalization of the quantities $(E, T, \Omega^i, \Phi_i)$. A boundary conformal field theory living on a surface $y$ = constant will thus occupy the volume $v$ given in (2.4). The intensive quantities $(t, \omega^i, \phi_i)$ of the CFT are then given by the standard Tolman redshifting factor, or, in the language of the AdS/CFT correspondence, the UV/IR connection, which coincides with our formulae in (2.3). The pressure $p$ is that expected of a conformally invariant system, the trace of whose energy-momentum tensor should vanish. One reason why the extensive quantities $S$, $J_i$, and $Q_i$ cannot scale under the UV/IR connection is that they are subject to quantization conditions, and are given by integers.

The upshot of the above discussion is that the introduction of the pressure term is a triviality, which ensures that if the first law of thermodynamics holds in the bulk, then it holds also in the boundary CFT.

### B. Relation to earlier work

Although for us the introduction of the pressure term is, as we have explained above, a triviality, because our bulk quantities satisfy the first law of thermodynamics, it is less transparent if bulk thermodynamic variables are chosen that do not satisfy the first law. As we showed in [13], the way to obtain bulk thermodynamic quantities for black holes that satisfy the first law is by calculating them with respect to a frame that is nonrotating at infinity. The energy measured in this frame can be derived [13] using the Ashtekar-Magnon-Das conformal definition of mass in AdS backgrounds [18,19]. It has also been shown [20] that the same expression can be derived from the superpotential of Katz, Bičák, and Lynden-Bell [21]. A further calculation leading to the same expression for the energy was given recently in [22].

There are, of course, infinitely many frames one could choose that do rotate, with different rotation rates, at infinity. One popular choice is the asymptotically rotating coordinate system in which Carter first wrote the Kerr-AdS black hole in four dimensions [23]. Analogous rotating frames were introduced in five dimensions by Hawking, Hunter, and Taylor-Robinson [1], and in all higher dimensions in [2]. In these papers, the metrics are given in a coordinate system which is rotating with angular velocity

$$\Omega^i_\infty = \frac{a_i}{l^2} \qquad (2.7)$$

with respect to an asymptotically static frame, where $a_i$ are the rotation parameters. (See the appendix for a summary of the salient details of the Kerr-AdS metrics. In the appendix, the metrics are given in an asymptotically static coordinate system.)

The geometrical significance of this particular rotating frame is that with respect to it the Kerr-Schild congruence, which was used to construct the solution, is nonrotating at infinity. However, this in itself does not appear to endow it

[4]The time coordinate $t$ should not be confused with the CFT temperature $t$—it should be clear from the context which is which.

with any privileged dynamical significance. Nevertheless one can certainly, as has been done in some of the literature, associate with it energies and angular velocities, which we shall denote by primes, that are given in terms of the unprimed nonrotating thermodynamic quantities by [13]

$$E' = E - \frac{a_i}{l^2} J_i, \qquad \Omega^{i\prime} = \Omega^i - \frac{a_i}{l^2}, \tag{2.8}$$

with all the other quantities being the same in the primed and the unprimed frame.[5] Note that

$$E' - \Omega^{i\prime} J_i' = E - \Omega^i J_i = E' - \Omega^{i\prime} J_i. \tag{2.9}$$

Throughout the rest of this paper, we shall use the symbols $E'$ and $\Omega^{i\prime}$ to denote energies and angular velocities measured with respect to the asymptotically rotating frames for which (2.7) holds.

Although $E'$ appears to have no special physical significance, it turns out that it provides a useful bound for the true energy $E$, in other words

$$E \geq E', \tag{2.10}$$

with equality if and only if the black hole is nonrotating.

As we noted in [13],

$$dE' \neq TdS + \Omega^{i\prime} dJ_i. \tag{2.11}$$

However, Klemm *et al.* [15,16] discussed the thermodynamics of rotating AdS black holes with a single nonvanishing rotation parameter, and obtained an extended system involving a chemical potential $\mu$ and number $N$, satisfying the first law. More recently, Cai *et al.* [14] have introduced thermodynamic quantities which in our notation we shall write as $e'$, $t'$, $s'$, $\omega^{i\prime}$, $j_i'$, $p'$, $v'$, given by

$$v' = \frac{\mathcal{A}_{n-2} r^{n-2}}{\prod_j \Xi_j}, \qquad p' = \frac{e'}{(n-2)v'}, \tag{2.12}$$

$$e' = \frac{l}{r} E', \qquad \omega^{i\prime} = \frac{l}{r} \Omega^{i\prime}, \qquad t' = \frac{l}{r} T, \qquad s' = S, \qquad j_i' = J_i, \tag{2.13}$$

and they have shown that these satisfy the first law

$$de' = t' ds' + \omega^{i\prime} dj_i' - p' dv'. \tag{2.14}$$

[5] The reason why $J_i = J_i'$ is that "passing to the rotating frame" means in effect choosing a new timelike Killing field from which $E'$ is constructed, but retaining the same angular Killing fields from which the $J_i$ are constructed. In other words, one introduces the new rotating coordinates $(t', \varphi_i')$, related to the asymptotically static coordinates $(t, \varphi_i)$ by $t' = t$, $\varphi_i' = \varphi_i - a_i l^{-2} t$, and associates the energy $E'$ with the Killing vector $\partial/\partial t' = \partial/\partial t + a_i l^2 \partial/\partial \varphi_i$, as opposed to the energy $E$ associated with $\partial/\partial t$. Thus passing to a rotating frame is *not* the same as performing an asymptotic $SO(n-1,2)$ transformation; it is merely picking a new basis for the Lie algebra $\mathfrak{so}(n-1,2)$.

Note that the formula (2.12) for $v'$ is not as we defined in (2.4), but rather has the additional factor $\prod_i \Xi_i$ in the denominator. This is needed in order to get the first law (2.14) in the primed variables to work out, to compensate for the failure of the bulk primed quantities to satisfy the first law. It is also not difficult to see that if one chooses a different rotation rate at infinity, replacing the right-hand side of (2.7) with some general functions of the rotations $a_i$, then one cannot in general find a formula for $v'$ of the form (2.12) with the factor $\prod_i \Xi_i$ replaced by a suitable function of the $a_i$. To that extent, the fact that the primed CFT quantities satisfy the first law (2.14) is not entirely fortuitous. Nevertheless, it seems to us that it is the unprimed CFT quantities given by (2.3) and (2.4) that most closely correspond to the physical situation that motivated the work in [1]. In other words, the relevant CFT should rotate, relative to a frame nonrotating at infinity, with the *same* angular velocity as that of the black hole in the bulk theory.

One could pass to a frame that is corotating with the black hole, i.e. one whose angular velocity with respect to the frame that is nonrotating at infinity is equal to $\Omega^i$, given by (A7). The associated Killing vector

$$K = \frac{\partial}{\partial t} + \Omega^i \frac{\partial}{\partial \varphi^i} \tag{2.15}$$

(expressed using the asymptotically nonrotating coordinates in (A2)) coincides on the horizon with its null generator, and is, provided $|a_i| < l$, everywhere timelike outside the horizon. This has the desirable feature that local energy densities measured with respect to this Killing vector are everywhere positive [24]. However, it has the distinct disadvantage that when considering any energy exchange between the rotating black hole and its environment, one must change to a new rotating frame because in general $\Omega^i$ changes. It is for this reason that the first law of thermodynamics does not hold with respect to the primed quantities. More generally, one could consider a Killing vector of the form

$$\tilde{K} = \frac{\partial}{\partial t} + \tilde{\Omega}^i \frac{\partial}{\partial \varphi^i}. \tag{2.16}$$

A simple calculation shows that on the horizon,

$$g(\tilde{K}, \tilde{K}) = g_{ij}(\tilde{\Omega}^i - \Omega^i)(\tilde{\Omega}^j - \Omega^j), \tag{2.17}$$

where $g_{ij} = g(\partial/\partial\varphi_i, \partial/\partial\varphi_j)$, and thus we see that for any angular velocity $\tilde{\Omega}^i$ that differs from $\Omega^i$, the associated Killing vector $\tilde{K}$ is spacelike on (and therefore in the neighborhood of) the horizon. In particular, this applies to the choice $\tilde{\Omega}^i = \Omega^i_\infty = a_i/l^2$. Thus to use the primed frame would neither achieve positivity of the local energy density nor a simple form for the first law. It seems, therefore, that neither it, nor any other frame that is rotating at infinity (other than, possibly, the frame that is rotating with the angular velocity of the black hole) has any particular merit or advantage over the frame that is nonrotating at infinity. Of course physical results cannot depend upon an arbitrary choice of frame.[6] It is clear that we can describe a rigidly rotating gas either as being at rest in a rotating frame, or moving in a nonrotating frame. The choice which seems to us the most straightforward and simple is the latter. Similarly, the calculations in [25] need not have been done using the primed quantities, and we disagree with the assertion in [25] that it is necessary to use the primed quantities in order "to extract data useful to a dual CFT." As we have seen above, this is a trivial matter using the nonrotating frame.

Using the primed energy $E'$ is precisely analogous to using the kinetic energy of a particle with respect to a rotating frame, such as that of the earth. It can be done, but it is then necessary to consider the contributions to the energy and the equations of motion due to the centrifugal and Coriolis forces. If the particle is freely moving on the rotating platform, and the rotation rate is changed, the kinetic energy with respect to the rotating frame will obviously change, while it will clearly be constant with respect to an inertial frame. There would seem to be no merit in introducing an artificially time-dependent rotating frame merely to describe straightline motion in inertial coordinates. If instead of free particles we considered a gas in a state of rigid rotation, we would have a situation rather more analogous to that of the CFT. The gas would exert a pressure on its container, which in principle could be measured in any rotating frame, but the two that are most relevant are surely the rigid rotating frame corotating the gas, or the one that is nonrotating with respect to an inertial coordinate system. As we have explained earlier, if the rotation rate changes with time, it is the latter which is more convenient. Choosing to use the energy $E'$ in the rotating black-hole problem is the equivalent of using a frame that is neither nonrotating at infinity nor is it rotating at the angular velocity of the black-hole horizon. Furthermore, in previous work where the energy $E'$ has nevertheless been used, the necessary corrections to compensate for the changes in the rotation rate of the primed frame have been omitted.

In [14], the CFT is assumed to be on a surface of large $r$ in Boyer-Lindquist coordinates, and the spatial volume is supposed to be the volume of that surface. It should be noted that although this spatial surface has the topology of an $(n-2)$-sphere it does not have an $SO(n-1)$ isometry group even asymptotically at large $r$. If one nevertheless chooses this $r=$ constant boundary one must face up to the fact that the temperature will be space dependent and there will be no conventional thermodynamic interpretation where a global temperature is well defined.

For example, in four dimensions the Kerr-AdS metric at large $r$ approaches the form

$$ds_4^2 = \frac{r^2\Delta_\theta}{l^2\Xi}\left[-dt^2 + l^2\left\{\frac{\Xi d\theta^2}{\Delta_\theta^2} + \frac{\sin^2\theta}{\Delta_\theta}(d\phi + al^{-2}dt)^2\right\}\right], \tag{2.18}$$

where

$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \qquad \Xi = 1 - \frac{a^2}{l^2}. \tag{2.19}$$

Defining a new coordinate $\hat{\theta}$ by $\tan\hat{\theta} = (\tan\theta)/\sqrt{\Xi}$, it can be seen that the 2-metric enclosed in braces is nothing but the standard unit 2-sphere, whose volume is of course $4\pi$. The metric in the square brackets is that of a three-dimensional rotating Einstein universe. The CFT metric is in fact conformal to this, and with respect to this metric the spatial volume is

$$\frac{2\pi r^2}{\Xi}\int_0^\pi d\hat{\theta}\,\sin\hat{\theta}\Delta_\theta = \frac{4\pi r^2 l}{a\sqrt{\Xi}}\arcsin(a/l). \tag{2.20}$$

This is not equal to $4\pi r^2/\Xi$, which is the value given in [14]. We are thus unsure as to precisely which spatial volume the quantity $4\pi r^2/\Xi$ in [14] is supposed to be. Similar remarks apply to all the higher-dimensional expressions for $v'$ given in [14] and reproduced in (2.12).

A striking feature of the work in [14] is the finding that the quantities $e'$, $t'$, $s'$, $\omega^{i\prime}$, $j_i'$, $p'$, and $v'$ satisfy a suggested formula of E. Verlinde [17], which itself was based on an attempt to incorporate Bekenstein's conjecture [26] of some sort of bound relating entropy, energy, and radius. This has motivated us to look in more detail at the general question of such formulae and bounds, which we shall do in the next section.

## III. THE CARDY-VERLINDE FORMULA AND THE BEKENSTEIN BOUND

### A. The ideal Cardy-Verlinde gas

According to a proposal of E. Verlinde, a CFT living on an $(n-1)$-dimensional Einstein Static Universe (ESU) of radius $y$ and hence metric

$$ds^2 = -dt^2 + y^2 d\Omega_{n-2}^2, \tag{3.1}$$

where $d\Omega_{n-2}^2$ is the canonical round metric on $S^{n-2}$ should have

(a) A pressure, energy, and volume related by

$$p = \frac{1}{n-2}\frac{e}{v}, \qquad v = \mathcal{A}_{n-2}y^{n-2}, \tag{3.2}$$

(b) An entropy $s$ given by

$$s = \frac{2\pi y}{n-2}\sqrt{e_c(2e - e_c)}, \tag{3.3}$$

[6] It should be emphasized that the different expressions for the energy and angular velocity in different frames is not the result of the coordinate transformation *per se*, but of using the transformed time coordinate when defining the energy and angular velocity.

where $e_c$ is a measure of the extent to which the energy $e$ is nonextensive and given by

$$e_c := (n-2)(e - ts - \omega j - \phi q + pv). \tag{3.4}$$

Subsequent work [27] showed that for free field theory, the Cardy-Verlinde formula does not hold exactly, but it agrees, up to a constant factor, in the high-temperature limit. That is, at large $T$ one has $E_c \propto T^{n-3}$, and $E \propto T^{n-1}$, and hence $S \propto T^{n-2}$. However the factor of proportionality is wrong. Nevertheless, it is still possible that it, or some modified form, may hold in the strongly interacting case that is relevant for the AdS/CFT correspondence.

In this limit, it is more convenient to discuss bulk, rather than boundary, quantities. As we have emphasized in Section II A, one can freely translate back and forth between the two descriptions. Using the Tolman redshifting formula, or the UV/IR relation between lowercase and capital letter quantities, one sees that (3.3) may be rewritten in terms of bulk quantities as

$$S = \frac{2\pi l}{n-2}\sqrt{E_c(2E - E_c)}, \tag{3.5}$$

where $E_c$ is given by

$$E_c := (n-2)\bigg[\bigg(1 + \frac{1}{n-2}\bigg)E - TS - \Omega J - \Phi Q\bigg] \tag{3.6}$$

and we identify $E$ as the conformal generator of $J_{0n} \in \mathfrak{so}(n-1, 2)$ associated to the asymptotically static Killing field $\frac{\partial}{\partial t}$,

$$E = l^{n-3} J_{0n}. \tag{3.7}$$

For later purposes, we rewrite (3.6) as[7]

$$E_c = (n-1)E - (n-2)[TS + \Omega J - \Phi Q]. \tag{3.8}$$

From now on we shall primarily be concerned with the Cardy-Verlinde formula in terms of "bulk," that is black-hole, quantities. However, it is worth remarking that in some papers the anti-de Sitter radius $l$ is replaced by $r_+$, where typically $r_+$ is the radius of the horizon in Schwarzschild or, in the rotating case, Boyer-Lindquist coordinates. In effect this amounts to setting $y = r_+$ in the redshifted CFT form. However, unless $r_+ \gg l$, this cannot be regarded as a legitimate application of the UV/IR relation, since if $r_+ \sim l$ then $-g_{tt}$ is not well approximated by $y/l$.

[7] The symbol $:=$ means that the quantity on the left-hand side is defined by the expression on the right-hand side. In particular we shall, for the sake of clarity, *always* stick with this primary definition of $e_c$, and the analogous blue shifted quantities $E_c$. If we need to modify the definition we will indicate any modified quantity by a circumflex, thus $\hat{e}_c$. The importance of this cannot be overemphasized. If one does not stick to the primary definition (3.8), then the question of the existence or nonexistence of such a formula becomes completely meaningless, since one could always define $e_c$ in such a way that the Cardy-Verlinde formula became a trivial identity.

### B. Anti-de Sitter Bekenstein bound and Hawking-Page transition

The (strict) Cardy-Verlinde formula (3.5) may be regarded as a formula for the energy $E$ in terms of the entropy $S$ and nonextensive energy $E_c$:

$$E = \frac{1}{2}E_c + \frac{1}{2E_c}\bigg[\frac{(n-2)S}{2\pi l}\bigg]^2. \tag{3.9}$$

Minimization with respect to $E_c$ leads to the lower bound

$$2\pi l E \geq (n-2)S, \tag{3.10}$$

which we shall refer to as the *Anti-de Sitter Bekenstein Bound*, regardless of whether it arises from a Cardy-Verlinde formula. Clearly the anti-de Sitter Bekenstein bound is a necessary, *but not sufficient* condition for the existence of a Cardy-Verlinde formula.

This lower bound for the energy in terms of the entropy, or alternatively upper bound for the entropy in terms of the energy, is attained when

$$E = E_c = \frac{(n-2)S}{2\pi l}. \tag{3.11}$$

Note that for $E > \frac{(n-2)S}{2\pi l}$ there are two values of $E_c$ satisfying the Cardy-Verlinde formula (3.9), while if $E < \frac{(n-2)S}{2\pi l}$, there are none.

The calculation above may be reorganized as follows. The thermodynamic potential $\Psi$ of the bulk black hole is given by

$$\Psi = E - TS - \Omega J - \Phi Q. \tag{3.12}$$

Thus

$$E_c = E + (n-2)\Psi, \qquad 2E - E_c = E - (n-2)\Psi. \tag{3.13}$$

The Cardy-Verlinde formula can be cast in the Pythagorean form

$$\frac{E^2}{(n-2)^2} = \frac{S^2}{4\pi^2 l^2} + \Psi^2. \tag{3.14}$$

We see from (3.14) that the Bekenstein bound is attained if and only if the thermodynamic potential vanishes,

$$\Psi = 0. \tag{3.15}$$

If one accepts the quantum statistical relation between thermodynamic potential and Euclidean action $I$,

$$\Psi = TI, \tag{3.16}$$

then the Bekenstein bound is attained when the Euclidean action vanishes. In the black-hole case, this indicates that the Euclidean black-hole solution with large energy $E$ no longer has smaller action than that of flat space, and a type of phase transition is indicated, as was first discussed by Hawking and Page [28] in the case of $AdS_4$ black holes, and by Witten [29] in the case of $AdS_5$ black holes.

The bound (3.10) resembles the controversial universal bound suggested by Bekenstein for systems in flat Minkowski spacetime, except that Bekenstein's putative universal Minkowski bound contains an undefined radius. In the AdS-Bekenstein Bound (3.10), this radius is taken to be that of $AdS_n$. However, for large radius we can use the Tolman redshifting formulae (which are of course valid only for large radius since we are using an approximate form for the metric near the boundary), and we may just as well write

$$2\pi y e \geq (n-2)s, \tag{3.17}$$

where now the radius is that of $S^{n-2}$. For clarity, we shall call this latter bound the *Spherical Bekenstein Bound* or the $S^{n-2}$-Bekenstein bound.

Note that neither the $AdS_n$ Bekenstein bound nor the $S^{n-2}$ Bekenstein bound, like that in Bekenstein's original and rather imprecise Minkowski-spacetime bound, contain Newton's constant or make any specific reference to gravity. Moreover, the $AdS_n$ Bekenstein bound (3.10) reduces, in the Minkowski limit $l \to \infty$, to the undemanding requirement that the energy be non-negative.

### C. Nonrotating Reissner-Nordström black holes

Remarkably, the Cardy-Verlinde formula is satisfied by Schwarzschild-AdS black holes in arbitrary dimensions. However, it is violated by Reissner-Nordström-AdS black holes. Nevertheless, a simple minimal modification does hold, namely

$$\frac{(n-2)S}{2\pi l} = \sqrt{E_c(2E - E_c - \Phi Q)}, \tag{3.18}$$

or in Pythagorean form,

$$\left(E - \frac{1}{2}\Phi Q\right)^2 = \left(\frac{(n-2)S}{2\pi l}\right)^2 + \left(\frac{\Psi}{(n-2} + \frac{1}{2}\Phi Q\right)^2. \tag{3.19}$$

The minimally modified Bekenstein bound becomes

$$E \geq \frac{1}{2}\Phi Q + \frac{(n-2)S}{2\pi l}, \tag{3.20}$$

with equality if and only if

$$(n-2)\Psi + \frac{1}{2}\Phi Q = 0, \tag{3.21}$$

or

$$(n-2)TI + \frac{1}{2}\Phi Q = 0. \tag{3.22}$$

Because $\Phi Q \geq 0$, we see that the $AdS_n$ Bekenstein bound holds for Reissner-Nordström-AdS black holes.

### D. Rotating black holes without charge

As observed in [14], the Cardy-Verlinde formula does *not* hold for rotating black holes in any dimension $n \geq 4$, if one uses the thermodynamic quantities defined with respect to a nonrotating frame at infinity. Remarkably, however, it was found in [14] (see [15,16] for earlier discussions) that in all dimensions it does hold if one uses the quantities defined with respect to a frame that rotates with angular velocities $-a_i/l^2$ at infinity. Because $E \geq E'$, one obtains an inequality,

$$E \geq E' = \frac{1}{2}E_c + \frac{1}{2E_c}\left[\frac{(n-2)S}{2\pi l}\right]^2, \tag{3.23}$$

whence

$$E \geq \frac{(n-2)S}{2\pi l}. \tag{3.24}$$

In other words, although the conserved quantities measured with respect to a frame nonrotating at infinity do not satisfy the Cardy-Verlinde formula, they do satisfy the Bekenstein bound.

In fact, one does not need to pass to the quantities $E'$ and $\Omega^{i\prime}$ to establish that rotating AdS black holes satisfy the $AdS_n$ Bekenstein bound. From (A12), we have

$$S = \frac{\mathcal{A}_{n-2} m l}{4(\prod_j \Xi_j)} \frac{r_+/l}{1 + r_+^2/l^2} \tag{3.25}$$

and hence, since $x/(1+x^2) \leq \frac{1}{2}$, we have

$$S \leq \frac{\mathcal{A}_{n-2} m l}{4(\prod_j \Xi_j)}, \tag{3.26}$$

with equality if and only if $r_+ = l$. From results in [13], the Euclidean action for the $n$-dimensional Kerr-AdS black hole is given by

$$I = \frac{\beta \mathcal{A}_{n-2} m}{8\pi(\prod_i \Xi_i)} \frac{l^2 - r_+^2}{l^2 + r_+^2}, \tag{3.27}$$

where $\beta$ is the inverse Hawking temperature. Thus we see that $r_+ = l$ corresponds to the Hawking-Page transition, where the Euclidean action vanishes.

From (A10) and (A11) we have

$$E \geq \frac{\mathcal{A}_{n-2} m(n-2)}{8\pi(\prod_j \Xi_j)}, \tag{3.28}$$

since $\Xi_i \leq 1$, with equality if and only if the black hole is nonrotating, i.e. if and only if all $a_i = 0$. Combining (3.26) and (3.28) gives the $AdS_n$ Bekenstein bound (3.24), with equality if and only if the black hole is nonrotating, and at the Hawking-Page transition.

## IV. FURTHER EXAMPLES

From the previous work, it is natural to wonder whether a simple modification of the Cardy-Verlinde formula, involving the use of $E'$, and the electrostatic term $\Phi Q$, continues to hold for more complicated black holes with, for example, more than one charge, or the recently constructed solutions representing rotating black holes with one or more charge, and one or more rotation parameters. In this section we shall study the various cases, and find that while no universal simple modified Cardy-Verlinde formula appears to exist that covers all these cases, we do find in many cases we have studied that

$$E'_c(2E' - E'_c - \Phi_i Q_i) \geq \left(\frac{(n-2)S}{2\pi l}\right)^2, \qquad (4.1)$$

which implies the electrostatic form of the $\mathrm{AdS}_n$ Bekenstein bound,

$$E \geq \frac{1}{2}\Phi_i Q_i + \frac{(n-2)S}{2\pi l}. \qquad (4.2)$$

### A. Nonrotating black holes with multiple charges

Modifications of the Cardy-Verlinde formula for multicharge nonrotating black holes in gauged supergravities have been discussed in [16,30]. Here, we give a related discussion, focussing, in particular, on the electrostatic AdS-Bekenstein bound, which we show to be satisfied in all the four, five, and seven-dimensional examples that we consider.

#### *1. Four-dimensional multicharge black holes*

The general four-charge solutions in four-dimensional gauged supergravity are given by

$$\begin{aligned}
ds_4^2 &= -\Big(\prod_i H_i\Big)^{-1/2} f dt^2 + \Big(\prod_i H_i\Big)^{1/2}[f^{-1}d\rho^2 + \rho^2 d\Omega_2^2],\\
A_i &= (1 - H_i^{-1})\sqrt{\frac{q_i+\mu}{q_i}}dt,\\
X_i &= \Big(\prod_j H_j\Big)^{1/4} H_i^{-1},\\
H_i &= 1 + \frac{q_i}{\rho},\\
f &= 1 - \frac{\mu}{\rho} + g^2\rho^2\prod_i H_i,
\end{aligned} \qquad (4.3)$$

where $X_i = e^{(1/2)\vec{a}_i\cdot\vec{\varphi}}$, where $\vec{\varphi}$ denotes the three canonically normalized scalar fields, the $\vec{a}_i$ are constant vectors satisfying $\vec{a}_i \cdot \vec{a}_j = 4\delta_{ij} - 1$, and the Lagrangian is given by

$$e^{-1}\mathcal{L} = \frac{1}{16\pi}R - \frac{1}{8\pi}(\partial\varphi)^2 - \frac{1}{16\pi}\sum_i X_i^{-2}F_i^2 + \frac{g^2}{16\pi}\sum_{i<j}X_iX_j. \qquad (4.4)$$

(We use units where $G = 1$. See [31] for a more extensive discussion of the notation and conventions that we are using.) Note that in order for the solution to be real, and free of naked singularities, we must have $q_i \geq 0$.

Straightforward calculations give

$$\begin{aligned}
E &= \frac{1}{2}\mu + \frac{1}{4}\sum_i q_i,\\
Q_i &= \sqrt{q_i(\mu + q_i)},\\
E_c &= \frac{1}{4}\sum_i(\rho_+ + q_i),\\
2E - E_c - \sum_i \Phi_i Q_i &= \frac{1}{4}\rho_+(\mu - \rho_+)\sum_i\frac{1}{(\rho_+ + q_i)},\\
S &= \frac{\pi}{g}\sqrt{\rho_+(\mu - \rho_+)},
\end{aligned} \qquad (4.5)$$

where $\rho_+$ is the radius of the horizon, i.e. the largest root of $f(\rho) = 0$.

One observes that unless the charges are equal, $q_i = q$, the minimally modified Cardy-Verlinde formula (3.18) fails. However, using the fact that the arithmetic mean is never less than the harmonic mean, one finds that

$$E - \frac{1}{2}\sum_i Q_i\Phi_i \geq \frac{1}{2}E_c + \frac{g^2S^2}{2\pi^2E_c}. \qquad (4.6)$$

Thus although the Cardy-Verlinde formula is violated,

$$S \neq \frac{\pi}{g}\sqrt{E_c(2E - E_c - \sum_i\Phi_iQ_i)}, \qquad (4.7)$$

nevertheless, the electrostatic AdS-Bekenstein bound still holds,

$$E \geq \frac{1}{2}\sum_i Q_i\Phi_i + \frac{gS}{\pi}, \qquad (4.8)$$

despite the fact that the scalar fields, and hence also the potential $-g^2\sum_{i<j}X_iX_j$, are space dependent. In these cases $1/g$ is only the *asymptotic* value of the anti-de Sitter radius.

#### *2. Five-dimensional multicharge black holes*

The general three-charge solutions in five-dimensional gauged supergravity are given by

$$ds_5^2 = -\Big(\prod_i H_i\Big)^{-2/3} f\,dt^2 + \Big(\prod_i H_i\Big)^{1/3}[f^{-1}d\rho^2 + \rho^2 d\Omega_3^2],$$
$$A_i = (1 - H_i^{-1})\sqrt{\frac{q_i + \mu}{q_i}}dt,$$
$$X_i = \Big(\prod_j H_j\Big)^{1/3} H_i^{-1}, \tag{4.9}$$
$$H_i = 1 + \frac{q_i}{\rho^2},$$
$$f = 1 - \frac{\mu}{\rho^2} + g^2\rho^2\prod_i H_i,$$

where $X_i = e^{(1/2)\vec{a}_i\cdot\vec{\varphi}}$, where $\vec{\varphi}$ denotes the two canonically normalized scalar fields, and the relevant Lagrangian is given by

$$e^{-1}\mathcal{L} = \frac{1}{16\pi}R - \frac{1}{8\pi}(\partial\varphi)^2 - \frac{1}{16\pi}\sum_i X_i^{-2}F_i^2 + \frac{g^2}{4\pi}\sum_i X_i^{-1}. \tag{4.10}$$

(We again use units where $G = 1$.) Again we must have $q_i \geq 0$ in order to have a real solution with no naked singularities.

After straightforward calculation, we find that

$$E = \frac{3}{8}\pi\mu + \frac{1}{4}\pi\sum_i q_i,$$
$$Q_i = \frac{1}{4}\pi\sqrt{q_i(\mu + q_i)},$$
$$E_c = \frac{1}{4}\pi\sum_i(\rho_+^2 + q_i), \tag{4.11}$$
$$2E - E_c - \sum_i \Phi_i Q_i = \frac{1}{4}\pi\rho_+^2(\mu - \rho_+^2)\sum_i\frac{1}{\rho_+^2 + q_i},$$
$$S = \frac{\pi^2}{2g}\rho_+(\mu - \rho_+^2)^{1/2},$$

where $\rho_+$ is the largest root of $f(\rho) = 0$. Again we see that the minimally modified Cardy-Verlinde formula (3.18) fails unless the $q_i$ are all equal. Again, using the fact that the arithmetic mean is never less than the harmonic mean, we obtain the inequality

$$E_c(2E - E_c - \sum_i \Phi_i Q_i) \geq \Big(\frac{3gS}{2\pi}\Big)^2, \tag{4.12}$$

and hence we derive the minimally modified AdS-Bekenstein bound

$$E - \frac{1}{2}\sum_i \Phi_i Q_i \geq \frac{3gS}{2\pi}. \tag{4.13}$$

Note that again, $1/g$ is only the asymptotic AdS radius.

### 3. Seven-dimensional multicharge black holes

The nonrotating multicharge black-hole solutions of maximal gauged supergravity in seven dimensions take the form

$$ds_7^2 = -(H_1H_2)^{-4/5}f\,dt^2 + (H_1H_2)^{1/5}[f^{-1}d\rho^2 + \rho^2 d\Omega_5^2],$$
$$A_i = (1 - H_i^{-1})\sqrt{\frac{q_i + \mu}{q_i}}dt,$$
$$X_i = (H_1H_2)^{2/5}H_i^{-1}, \tag{4.14}$$
$$H_i = 1 + \frac{q_i}{\rho^4},$$
$$f = 1 - \frac{\mu}{\rho^2} + g^2\rho^2 H_1H_2,$$

with the charges carried by the $U(1) \times U(1)$ gauge fields in the Abelian subgroup of $SO(5)$. Straightforward calculations show that

$$E = \frac{5}{16}\pi^2\mu + \frac{1}{4}\pi^2\sum_i q_i,$$
$$Q_i = \frac{1}{4}\pi^2\sqrt{q_i(\mu + q_i)},$$
$$E_c = \frac{1}{8}\pi^2\Big(5\rho_+^4 + \sum_i q_i\Big),$$
$$2E - E_c - \sum_i \Phi_i Q_i = \frac{1}{8}\pi^2\Big[5(\mu - \rho_+^4) + 3\sum_i q_i - 2\sum_i\frac{q_i(\mu + q_i)}{\rho_+^4 + q_i}\Big], \tag{4.15}$$
$$S = \frac{\pi^3}{4g}\rho_+^2(\mu - \rho_+^4)^{1/2}.$$

The verification that these solutions satisfy the electrostatic AdS-Bekenstein bound is slightly more complicated than for the four-dimensional and five-dimensional cases. This is presumably related to the fact that unlike in $n = 4$ and $n = 5$, in these seven-dimensional solutions the scalar fields are nonconstant even when the charges are set equal. In fact the easiest way to verify the Bekenstein bound is by performing a direct calculation of

$$X \equiv E - \frac{1}{2}\sum_i \Phi_i Q_i - \frac{5Sg}{2\pi}, \tag{4.16}$$

and verifying that $X$ is non-negative. We find that

$$X = \frac{1}{\rho_+^2}[5\rho_+^6(g\rho_+ - 1)^2 + q_1q_2g^2 + 3(q_1 + q_2)\rho_+^2(1 + g^2\rho_+^2) - 10\sqrt{(\rho_+^4 + q_1)(\rho_+^4 + q2)}]. \tag{4.17}$$

Using the Maclaurin-Cauchy inequality

$$\prod_i b_i^{s_i} \le \sum_i b_i s_i, \quad \text{where} \quad \sum_i s_i = 1, \tag{4.18}$$

we may deduce that

$$\sqrt{(\rho_+^4 + q_1)(\rho_+^4 + q2)} \le \frac{1}{2}(2\rho_+^4 + q_1 + q_2), \tag{4.19}$$

and hence we see that

$$X \ge \frac{1}{\rho_+^2}\bigg[5\rho_+^6(g\rho_+ - 1)^2 + q_1 q_2 g^2 + (q_1 + q_2)\rho_+^2\bigg[3\Big(g\rho_+ - \frac{5}{6}\Big)^2 + \frac{11}{12}\bigg]\bigg], \tag{4.20}$$

which proves that $X \ge 0$ and hence the Bekenstein bound is satisfied.

#### *4. Relation to previous work*

A more complicated modification of the Cardy-Verlinde formula has been proposed, in terms of the parameters $q_i$ and $\mu$ which appear in the multicharge metrics [30]. This modification incorporates the idea that the pressure of the associated conformal field theory should be reduced by its electrostatic self-repulsion. However, when reexpressed in terms of the fundamental thermodynamic variables $E$, $\Phi_i$, $Q_i$, $T$, and $S$, the modification takes on a rather complicated form, which appears to be a somewhat *ad hoc* construction designed to ensure the continued validity of the Cardy-Verlinde formula in these particular examples.

The modified Cardy-Verlinde formula in [30] is given by

$$S = \frac{2\pi}{(n-2)g}\sqrt{\hat{E}_c(2E - 2E_q - \hat{E}_c)}, \tag{4.21}$$

where

$$\hat{E}_c = E_c - E_q, \qquad E_q = \frac{\mathcal{A}_{n-2}(n-3)}{16\pi}\sum_i q_i. \tag{4.22}$$

This implies

$$E = \frac{1}{2}\hat{E}_c + \frac{1}{2\hat{E}_c}\Big(\frac{(n-2)gS}{2\pi}\Big)^2 + E_q, \tag{4.23}$$

and hence, minimizing with respect to $\hat{E}_c$, one again obtains the AdS-Bekenstein bound

$$E \ge E_q + \frac{(n-2)gS}{2\pi} \ge \frac{(n-2)gS}{2\pi}. \tag{4.24}$$

It is interesting to compare $E_q$, given in (4.22), with $\sum_i \Phi_i Q_i$, which is given by

$$\frac{\mathcal{A}_{n-2}(n-3)}{16\pi}\sum_i \frac{q_i(\mu + q_i)}{\rho_+^{n-3} + q_i}, \tag{4.25}$$

from which it follows that $\sum \Phi_i Q_i \ge E_q$, since $\rho_+^{n-3} \le \mu$.

### B. Rotating charged black holes

Various solutions for charged rotating black holes in gauged supergravities have been obtained. In this section, we study the generalized Bekenstein bounds for these cases.

#### *1. Four-dimensional Kerr-Newman-AdS black holes*

One might have hoped that, at least for the four-dimensional Kerr-Newman-AdS solution, a simple modification of the Cardy-Verlinde formula would work. However, one finds that the minimally modified Cardy-Verlinde formula, even in terms of quantities measured with respect to the canonically rotating frame, fails. Explicitly

$$E'_c(2E' - E'_c - \Phi Q) = \Big(\frac{S}{\pi l}\Big)^2 + \frac{16a^2Q^2}{l^2}, \tag{4.26}$$

where $a$ is given in terms of the extensive quantities by

$$a = \sqrt{\frac{E'^2 l^4}{4J^2} + l^2} - \frac{E' l^2}{2|J|}. \tag{4.27}$$

Note that

$$E'_c(2E' - E'_c - \Phi Q) \ge \Big(\frac{S}{\pi l}\Big)^2 \tag{4.28}$$

whence

$$E' \ge \frac{1}{2}\Phi Q + \frac{1}{2}E'_c + \frac{S^2}{2\pi^2 l^2 E'_c} \ge \frac{1}{2}\Phi Q + \frac{S}{\pi l}. \tag{4.29}$$

But since $E \ge E'$, we have

$$E \ge \frac{1}{2}\Phi Q + \frac{S}{\pi l}. \tag{4.30}$$

In other words, once again, despite the fact that the Cardy-Verlinde formula fails to hold, the electrostatic $AdS_4$-Bekenstein bound still holds.

#### *2. Four-dimensional rotating black holes with pairwise equal charges*

The solution for rotating black holes in four-dimensional gauged supergravity with four charges that are set pairwise equal was given in [6]. The thermodynamic quantities were evaluated in [32], where it was shown that the conserved energy, angular momentum, and charges are given by

$$E = \frac{2m + q_1 + q_2}{2\Xi^2},$$
$$J = \frac{a(2m + q_1 + q_2)}{2\Xi^2},$$
$$Q_1 = Q_2 = \frac{\sqrt{q_1(2m + q_1)}}{4\Xi}, \tag{4.31}$$
$$Q_3 = Q_4 = \frac{\sqrt{q_2(2m + q_2)}}{4\Xi},$$

where the parameters $q_i$ are related to the boost parameters $\delta_i$ in [6,32] by $q_i = 2m\sinh^2\delta_i$.

We find that

$$E_c'(2E' - E_c' - \sum_i \Phi_i Q_i) = \left(\frac{S}{\pi l}\right)^2 + \frac{Xg^2}{4\Xi r_+}, \tag{4.32}$$

where

$$X = 2a^2(q_1 + q_2)(a^2 + a^2g^2q_1q_2 + g^2q_1^2q_2^2) + [q_1q_2(q_1 - q_2)^2 + a^4g^2(q_1^2 + 6q_1q_2 + q_2^2) + a^2((q_1 - q_2)^2 + 2q_1^2 + 2q_2^2 + 3g^2q_1^3q_2 + 3g^2q_2^3q_1 + 10g^2q_1^2q_2^2)]r_+ + (q_1 + q_2)(2a^2 + 2a^4g^2 + (q_1 - q_2)^2 + a^2g^2(q_1^2 + q_2^2) + 10a^2g^2q_1q_2]r_+^2 + [(q_1 - q_2)^2 + 3a^2g^2(q_1^2 + q_2^2) + 10a^2g^2q_1q_2]r_+^3 + 2a^2g^2(q_1 + q_2)r_+^4. \tag{4.33}$$

Since the parameters $q_1$ and $q_2$ must be positive, the positivity of $X$ is manifest. Hence we obtain the Bekenstein bound

$$E \geq \frac{1}{2}\sum_i \Phi_i Q_i + \frac{Sg}{\pi}. \tag{4.34}$$

### 3. Five-dimensional rotating black holes with charges

The solution for a rotating black hole in five-dimensional minimal gauged supergravity, with independent rotation parameters in the two orthogonal planes in the transverse space, was obtained recently [9]. The solution can equivalently be viewed as a solution of $SO(6)$-gauged supergravity, with three equal charges carried by the $U(1)^3$ Abelian subgroup. The thermodynamic quantities were also evaluated in [9], and it was shown that the energy, angular momenta, and charge are given in terms of the parameters $m$, $a$, and $b$ in the metric by

$$E = \frac{\pi m(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b) + 2\pi qabg^2(\Xi_a + \Xi_b)}{4\Xi_a^2\Xi_b^2},$$
$$J_a = \frac{2\pi ma + \pi qb(1 + a^2g^2)}{4\Xi_a^2\Xi_b},$$
$$J_b = \frac{2\pi mb + \pi qa(1 + b^2g^2)}{4\Xi_b^2\Xi_a}, \qquad Q = \frac{\pi\sqrt{3}q}{4\Xi_a\Xi_b}, \tag{4.35}$$

where $g = 1/l$. We find that

$$E_c'(2E' - E_c' - \Phi Q) = \left(\frac{3Sg}{2\pi}\right)^2 + \frac{\pi^2g^2qX}{16\Xi_a^2\Xi_b^2[(r_+^2 + a^2)(r_+^2 + b^2) + abq]r_+}, \tag{4.36}$$

where $X$ is given by

$$X = q^3a^2b^2 + q^2ab(18a^2b^2 + 15(a^2 + b^2)r_+^2 + 4a^2b^2r^2 + 9r^4) + q(r_+^2 + a^2)(r_+^2 + b^2)(18a^2b^2 + 9(a^2 + b^2)r_+^2 + 10a^2b^2g^2r_+^2) + 6ab(r_+^2 + a^2)(r_+^2 + b^2)^2(1 + g^2r_+^2). \tag{4.37}$$

Since $X$ is manifestly positive, at least when $q$, $a$, and $b$ are positive, we therefore obtain the Bekenstein bound

$$E \geq \frac{1}{2}\Phi Q + \frac{3Sg}{2\pi}. \tag{4.38}$$

The bound is saturated if $a = b = q = 0$.

A further solution for charged rotating five-dimensional black holes was obtained in [8], which corresponds to a case where the three charges carried by the $U(1)^3 \in SO(6)$ gauge fields are still all nonzero, but with two of them being equal, and the third related to the first two in a specific way. Thus the solutions in [8] have four independent parameters, namely, the mass, the two rotations, and a parameter characterizing the charges. For these solutions we find

$$E_c(2E' - E_c - \sum_i \Phi_i Q_i) = \left(\frac{3Sg}{2\pi}\right)^2 + \frac{\pi^2 g^2 q X}{16\Xi_a^2 \Xi_b^2 r_+^2[(r_+^2 + a^2)(r_+^2 + b^2) + q r_+^2]}, \quad (4.39)$$

where $X$ is given by

$$X = 3(a^2 + b^2)(r_+^2 + a^2)(r_+^2 + b^2)(1 + g^2 r_+^2) + q^3 r_+^2(3a^2b^2g^2 + 2r_+^2 + (a^2 + b^2)g^2 r_+^2) + q(r_+^2 + a^2)(r_+^2 + b^2) \times [3a^2b^2 + (a^2 + b^2)r_+^2(8 + 7g^2 r_+^2) + g^2 r_+^2(a^4 + b^4 + 11a^2b^2) + 2r_+^4] + q^2 r_+^2[5a^2b^2 + (a^2 + b^2) \times (7r_+^2 + 4a^2b^2g^2 + 5g^2 r_+^4) + 2g^2 r_+^2(a^4 + b^4 + 8a^2b^2)]. \quad (4.40)$$

This is manifestly positive when $q$ is positive, and so again the AdS-Bekenstein bound (4.38) is satisfied.

## V. THE COSMIC-CENSORSHIP BOUND

So far, we have established that for many of the stationary black-hole solutions we have examined, the electrostatic $\mathrm{AdS}_n$ Bekenstein bound (4.2) is satisfied. In this section, we shall propose that this is a consequence of a more basic and more general lower bound for the energy $E$ of any initial data set for the Einstein equations with negative cosmological constant, coupled to a matter system that satisfies the dominant energy condition. This *Cosmic-Censorship Bound* is expressed in terms of the area $A$ of the outermost apparent horizon of that initial data set. In the case that there is no charge, the postulated lower bound reads

$$E \geq \frac{(n-2)A}{16\pi l}\left[l\left(\frac{A}{\mathcal{A}_{n-2}}\right)^{-1/(n-2)} + \frac{1}{l}\left(\frac{A}{\mathcal{A}_{n-2}}\right)^{1/(n-2)}\right]. \quad (5.1)$$

Some consequences of this bound, which is a more global extension of Hawking's variational principle for black holes [33], are:

(1) If $l \rightarrow \infty$, then (5.1) reduces to a bound first proposed in $n = 4$ dimensions by Penrose, who observed that it is a necessary condition for the cosmic-censorship hypothesis [34]. (See [35,36].)

(2) The proposed bound (5.1) implies a generalization of the $\mathrm{AdS}_n$ Bekenstein bound, to the nonstationary case. Noting that the quantity in square brackets in (5.1) must be greater than or equal to 2, we have a generalization of Bekenstein's bound to the time-dependent case when one may no longer equate entropy with $1/4$ of the area of an apparent horizon:

$$E \geq \frac{(n-2)A}{8\pi l}. \quad (5.2)$$

(3) The cosmic-censorship bound (5.1) is attained for the case of a Schwarzschild-anti-de Sitter black hole, and we propose that this is the only case for which it is saturated.

The strongest physical argument in favor of (5.1) is as follows. Consider an initial data set with total energy $E_{\mathrm{initial}}$, a single outermost apparent horizon of area $A_{\mathrm{initial}}$, and vanishing total angular momentum and charge. According to standard lore, this should settle down to a stationary state described by a Schwarzschild-de Sitter black hole with total energy $E_{\mathrm{final}}$ and event-horizon area $A_{\mathrm{final}}$, where

$$E_{\mathrm{final}} = \frac{(n-2)A_{\mathrm{final}}}{16\pi l}\left[l\left(\frac{A_{\mathrm{final}}}{\mathcal{A}_{n-2}}\right)^{-1/(n-2)} + \frac{1}{l}\left(\frac{A_{\mathrm{final}}}{\mathcal{A}_{n-2}}\right)^{1/(n-2)}\right]. \quad (5.3)$$

Assuming cosmic censorship, the apparent horizon lies inside the event horizon, and since, in the time-symmetric case, the apparent horizon is a minimal surface, its area gives a lower bound for the area of the event horizon. Now applying Hawking's theorem stating that the area of the horizon is nondecreasing, we obtain

$$A_{\mathrm{final}} \geq A_{\mathrm{initial}}. \quad (5.4)$$

In anti-de Sitter spacetime, unlike in asymptotically flat spacetimes, the total energy is constant, and therefore

$$E_{\mathrm{final}} = E_{\mathrm{initial}}. \quad (5.5)$$

It follows that

$$E_{\mathrm{initial}} \geq \frac{(n-2)A_{\mathrm{initial}}}{16\pi l}\left[l\left(\frac{A_{\mathrm{initial}}}{\mathcal{A}_{n-2}}\right)^{-1/(n-2)} + \frac{1}{l}\left(\frac{A_{\mathrm{initial}}}{\mathcal{A}_{n-2}}\right)^{1/(n-2)}\right]. \quad (5.6)$$

If the initial-value set had nonvanishing charge or angular momentum, it would be expected to settle down to the relevant stationary solution carrying those charges or angular momenta. However, the energy due to the charge or angular momentum could be extracted by dropping particles carrying charge or angular momentum into the black hole. In this process, the area of the event horizon cannot decrease, but the energy may. Thus we expect the energy of the black hole with charge or angular momentum to be less than that of a Schwarzschild-AdS black hole with the same event-horizon area.

We shall now review some of the additional evidence for this form of the cosmic-censorship bound, and provide some further support for it.

In the case of $n = 4$ dimensions and time-symmetric initial data, Jang and Wald's extension [37] of Geroch's [38] suggested method of proof of the positive mass theorem for asymptotically flat metrics using the inverse mean-curvature flow may be extended [39,40] to cover the case of asymptotically anti-de Sitter metrics. Furthermore, if one does so one obtains precisely the proposed lower bound (5.1). The Geroch-Jang-Wald proposed method of proof has been made into a rigorous theorem by Huisken and Ilmanen [41,42]. It seems plausible, but there is as yet no rigorous proof, that their methods will extend to the anti-de Sitter case.

There is no general proof of the original asymptotically flat cosmic-censorship inequality in higher dimensions. In [43], it is shown to hold in the case of a collapsing shell, using the obvious generalization of the four-dimensional calculations in [44].

There exists a natural generalization of the inverse mean-curvature flow to higher-dimensional time-symmetric initial-value sets [40]. This might yield a proof of the higher-dimensional inequality if on each level surface

$$\int [\bar{R} - (n-2)(n-3)] dA \geq 0, \tag{5.7}$$

where $\bar{R}$ is the Ricci scalar of the $(n-2)$-dimensional metric on the level surface, and the integration is over this surface.

In the static spherically symmetric case, the inequality (5.1) may be proved as follows. We write the metric as

$$ds^2 = -e^{2\nu(r)}\left(1 - \frac{2\mathfrak{m}(r)}{r^{n-3}}\right)dt^2 + \left(1 - \frac{2\mathfrak{m}(r)}{r^{n-3}}\right)^{-1} dr^2 + r^2 d\Omega_{n-2}^2. \tag{5.8}$$

There is a horizon of area $A = \mathcal{A}_{n-2} r_+^{n-2}$ at $r = r_+$, where

$$r_+^{n-3} = 2\mathfrak{m}(r_+). \tag{5.9}$$

The Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{5.10}$$

(where we include the contribution of the cosmological constant in $T_{\mu\nu}$) imply

$$\frac{d\mathfrak{m}}{dr} = \frac{8\pi}{n-2} r^{n-2} T_{\hat{t}\hat{t}}, \tag{5.11}$$

where the hats indicate components in an orthonormal frame. We have

$$T_{\mu\nu} = T_{\mu\nu}^{\text{cosmic}} + T_{\mu\nu}^{\text{matter}}, \tag{5.12}$$

where $T_{\mu\nu}^{\text{cosmic}}$ is the contribution from the cosmological term. As $r$ tends to infinity,

$$\mathfrak{m}(r) \longrightarrow m - \frac{r^{n-1}}{2l^2}, \tag{5.13}$$

where $l$ is the asymptotic de Sitter radius. We may integrate (5.11) from the horizon to infinity, to obtain

$$m = \frac{1}{2} r_+^{n-3} + \frac{r_+^{n-1}}{2l^2} + \frac{8\pi}{n-2} \int_{r_+}^{\infty} dr r^{n-2} T_{\hat{0}\hat{0}}^{\text{matter}} + \int_{r_+}^{\infty} dr r^{n-2} \left[\frac{n-1}{2l^2} + \frac{8\pi}{n-2} T_{\hat{t}\hat{t}}^{\text{cosmic}}\right]. \tag{5.14}$$

If $T_{\hat{t}\hat{t}}^{\text{cosmic}}$ is constant and $T_{\hat{t}\hat{t}}^{\text{matter}}$ satisfies the positive-energy condition, we obtain the cosmic-censorship bound, which will be saturated if and only if $T_{\hat{t}\hat{t}}^{\text{matter}} = 0$, i.e. for the Schwarzschild-de Sitter metric.

If $T_{\hat{t}\hat{t}}^{\text{cosmic}}$ is not constant, because of the presence of varying scalar fields, we still obtain a lower bound for $m$ if the integrand is positive. Unfortunately, in the gauged supergravities we have considered the integrand is in fact negative, because the potential is in general more negative than its negative value at its vanishing-scalar stationary point. However, even in this case the cosmic-censorship bound may continue to hold, because kinetic energy term for the scalars is positive. Indeed, this is what happens in the examples we have examined.

### A. Cosmic censorship for Kerr-AdS black holes in arbitrary dimension

It is straightforward to show that the general Kerr-AdS black holes in arbitrary spacetime dimension $n$ satisfy the cosmic-censorship bound (5.1). First, we note that one can characterize the Kerr-AdS metrics by their rotation parameters $a_i$, together with the radius $r_+$ of the outer horizon. The mass parameter $m$ appearing in the metric (A2) is then solved for using $V(r_+) = m$. It is then helpful to introduce a new parametrization in terms of $y_i$ and $z$ instead of $a_i$ and $r_+$, where

$$y_i \equiv \frac{r_+^2 + a_i^2}{r_+^2 \Xi_i}, \qquad z = \frac{r_+}{l}. \tag{5.15}$$

Clearly we must have $y_i \geq 1$, $z \geq 0$, and

$$r_+ = zl, \qquad a_i^2 = \frac{(y_i - 1) z^2 l}{(1 + z^2 y_i)}, \qquad \Xi_i = \frac{1 + z^2}{1 + z^2 y_i}. \tag{5.16}$$

We begin by considering the case when $n = 2N + 1$ is odd. From (A4), (A10), and (A12), the energy $E$ is given by

$$E = \frac{r_+^{n-3} \mathcal{A}_{n-2} (\prod_j y_j)}{16\pi} \left(n - 2 + 2z^2 \sum_i y_i\right), \tag{5.17}$$

while the right-hand side of (5.1) is given by

$$\frac{(n-2)r_+^{n-2}(\prod_j y_j)}{16\pi l}\left[z\left(\prod_i y_i\right)^{1/(n-2)}+\frac{1}{z}\left(\prod_i y_i\right)^{-1/(n-2)}\right], \tag{5.18}$$

and so to show that the cosmic-censorship bound is satisfied, we must show that

$$(n-2)\left[1-\left(\prod_i y_i\right)^{-1/(n-2)}\right] + z^2\left[2\sum_i y_i - 1-(n-2)\left(\prod_i y_i\right)^{1/(n-2)}\right]\geq 0. \tag{5.19}$$

The first bracketed term in (5.19), i.e. the term independent of $z$, is manifestly positive since $y_i \geq 1$. For the terms at order $z^2$ we may use the Maclaurin-Cauchy inequality (4.18) to show that

$$\left(\prod_i y_i\right)^{1/(n-2)} \leq \frac{1}{n-2}\sum_i y_i + \frac{n-3}{2(n-2)}. \tag{5.20}$$

Substituting this into the second bracketed term in (5.19) shows that

$$2\sum_i y_i - 1-(n-2)\left(\prod_i y_i\right)^{1/(n-2)} \geq \sum_{i=1}^{N}(y_i-1)\geq 0, \tag{5.21}$$

and hence the cosmic-censorship bound is proved.

In the case of even dimensions $n = 2N + 2$, an analogous calculation shows that cosmic censorship is satisfied if

$$(n-2)\left[1-\left(\prod_i y_i\right)^{-1/(n-2)}\right] + z^2\left[2\sum_i y_i -(n-2)\left(\prod_i y_i\right)^{1/(n-2)}\right]\geq 0. \tag{5.22}$$

Again using (4.18), we can show that

$$\left(\prod_i y_i\right)^{1/(n-2)} \leq \frac{1}{n-2}\sum_i y_i + \frac{1}{2} \tag{5.23}$$

and so since $y_i \geq 1$, the inequality (5.22) can indeed be seen to hold.

### B. Cosmic censorship for four-dimensional charged rotating black holes

The cosmic-censorship bound (5.1) can be generalized in the case of charged black-hole solutions [40,45]. In four dimensions, it becomes

$$E \geq \frac{A}{8\pi l}\left[l\left(\frac{A}{4\pi}\right)^{-1/2}+\frac{1}{l}\left(\frac{A}{4\pi}\right)^{1/2}+lQ^2\left(\frac{A}{4\pi}\right)^{-3/2}\right], \tag{5.24}$$

with equality being attained for the Reissner-Nordström-AdS solution. Calculating $E^2$ minus the square of the right-hand side of (5.24) for the Kerr-Newman-AdS solution, we find

$$E^2-(\mathrm{RHS})^2 = \frac{4(1+\mathrm{g}^2\mathrm{r}_+^2)[\mathrm{a}^2+\mathrm{q}^2+\mathrm{r}_+^2+\mathrm{g}^2\mathrm{r}_+^2(\mathrm{r}_+^2+\mathrm{a}^2)]^2}{\Xi^4\mathrm{r}_+^2(\mathrm{r}_+^2+\mathrm{a}^2)}, \tag{5.25}$$

which is manifestly positive, thus demonstrating that the inequality (5.24) is obeyed in this case.

### C. Cosmic censorship for five-dimensional charged rotating black holes

For solutions of five-dimensional minimal gauged supergravity, the generalized cosmic-censorship bound can be written as

$$\frac{8E}{3\pi} \geq \left(\frac{A}{2\pi^2}\right)^{2/3} + g^2\left(\frac{A}{2\pi^2}\right)^{4/3} + \frac{16Q^2}{3\pi^2}\left(\frac{A}{2\pi^2}\right)^{-2/3}, \tag{5.26}$$

with equality being attained in the case of the nonrotating Reissner-Nordström-AdS black hole. From the results obtained in [9], we find that the inequality (5.26) translates into the requirement that

$$\frac{2}{3}h[3+z^2(y_1+y_2)] + \frac{1}{3}y_1y_2[3-z^2+2z^2(y_1+y_2)] + \frac{h^2(1+z^2)}{3(y_1-1)(y_2-1)}[3-z^2+2z^2(y_1+y_2)] - (y_1y_2+h)^{2/3} - z^2(y_1y_2+h)^{4/3} - \frac{h^2(y_1y_2+h)^{-2/3}}{(y_1-1)(y_2-1)}(1+z^2y_1)(1+z^2y_2) \geq 0, \tag{5.27}$$

where

$$y_1 = \frac{r_+^2 + a^2}{r_+^2 \Xi_a}, \qquad y_2 = \frac{r_+^2 + b^2}{r_+^2 \Xi_b}, \qquad z = g r_+, \qquad h = \frac{abq}{r_+^4 \Xi_a \Xi_b}, \tag{5.28}$$

with $y_i \geq 1$, $z > 0$, $h \geq 0$. We have studied (5.27) numerically and find that it appears to be satisfied for all allowed values of the parameters $(y_1, y_2, z, h)$, and thus it appears that the generalized cosmic-censorship bound is obeyed by the five-dimensional charged rotating AdS black holes obtained in [9].

## VI. AN UPPER BOUND FOR THE TEMPERATURE?

It is well known that the presence of matter with a positive-energy density tends to reduce the temperature of a black hole, because of the redshift produced by the gravitational field of the matter. It is also well known that charged or rotating black holes tend to have a smaller temperature for the same entropy than their neutral or nonrotating versions. A general explanation for this observation was provided by Visser in the static spherically symmetric case in four dimensions with no cosmological term [46]. In this section we shall generalize Visser's observation, and apply it to the Hawking-Page transition.

For the spherically symmetric static metric (5.8), the Einstein equations imply

$$\frac{d\mathfrak{m}}{dr} = \frac{8\pi r^{n-2} T_{\hat{t}\hat{t}}}{(n-2)}, \tag{6.1}$$

$$\frac{d\nu}{dr} = \frac{8\pi r^{n-2}(T_{\hat{t}\hat{t}} + T_{\hat{r}\hat{r}})}{(n-2)[r^{n-3} - 2\mathfrak{m}(r)]}, \tag{6.2}$$

where the hats indicate components in an orthonormal frame.

The surface gravity is

$$\kappa = 2\pi T = \frac{1}{2r_+} e^{\nu(r_+)}\left(n - 3 - 2r^{4-n}\frac{d\mathfrak{m}}{dr}\right). \tag{6.3}$$

This becomes

$$\kappa = 2\pi T = \frac{1}{2r_+} e^{\nu(r_+)}\left[(n-3) - \frac{16\pi r_+^2}{(n-2)}(T^{\mathrm{cosmic}}_{\hat{t}\hat{t}} + T^{\mathrm{matter}}_{\hat{t}\hat{t}})\right]. \tag{6.4}$$

If $T^{\mathrm{cosmic}}_{\hat{t}\hat{t}}$ is constant, then (6.4) becomes

$$\kappa = 2\pi T = \frac{1}{2r_+} e^{\nu(r_+)}\left[(n-3) + (n-1)g^2 r_+^2 - \frac{16\pi r_+^2}{(n-2)} T^{\mathrm{matter}}_{\hat{t}\hat{t}}\right]. \tag{6.5}$$

If the matter satisfies the dominant energy condition then

$$T^{\mathrm{matter}}_{\hat{t}\hat{t}} \geq |T^{\mathrm{matter}}_{\hat{r}\hat{r}}| \geq 0. \tag{6.6}$$

Moreover

$$\nu(r) = -\int_r^\infty \frac{8\pi r'^{n-2}(T^{\mathrm{matter}}_{\hat{t}\hat{t}} + T^{\mathrm{matter}}_{\hat{r}\hat{r}})}{(n-2)[r'^{n-3} - 2m(r')]} dr'. \tag{6.7}$$

Thus $\nu(r)$ will be nonpositive and

$$4\pi T \leq \frac{(n-3)}{r_+} + (n-1)g^2 r_+. \tag{6.8}$$

If $T^{\mathrm{cosmic}}_{\hat{t}\hat{t}}$ is not constant, one might expect the kinetic term for the scalars to compensate for any extra positive contribution from $-T^{\mathrm{cosmic}}_{\hat{t}\hat{t}}$, and the inequality to continue to hold.

In the Schwarzschild-AdS case the inequality (6.8) becomes an equality. The minimum value of the right-hand side occurs at

$$r_+ = \frac{1}{g}\sqrt{\frac{n-3}{n-1}}, \tag{6.9}$$

at which

$$T = \frac{g\sqrt{(n-1)(n-3)}}{2\pi}. \tag{6.10}$$

This lower bound for the temperature is associated with the Hawking-Page phase transition. Below this temperature, there is no black-hole solution, while above it, there are two. The Hawking-Page transition itself occurs at $r_+ = 1/g$, for which $T = (n-2)g/(2\pi)$. For temperatures greater than (6.10) but smaller than $(n-2)g/(2\pi)$, both Schwarzschild-AdS solutions have larger Euclidean action than that of anti-de Sitter spacetime.

The general inequality (6.8) for spherically symmetric black holes may be recast in the form

$$4\pi T \leq (n-3)\left(\frac{A}{\mathcal{A}_{n-2}}\right)^{-1/(n-2)} + (n-1)g^2\left(\frac{A}{\mathcal{A}_{n-2}}\right)^{1/(n-2)}, \tag{6.11}$$

where $A$ is the area of the outer horizon. Equality is achieved in the case of Schwarzschild-AdS black holes.

One might think that when the inequality is expressed in the form (6.11), it would continue to hold for rotating as well as nonrotating black holes. In other words, one might conjecture that the minimum temperature, as a function of entropy, is always less than or equal to the minimum temperature of the Schwarzschild-AdS case. In fact, we find that the bound (6.11) is obeyed by all Kerr-AdS black holes in $n = 4$ and $n = 5$ dimensions. However, counterexamples can be found for Kerr-AdS black holes in all dimensions greater than or equal to 6.

To discuss the situation in arbitrary dimensions, it is again helpful to use the parametrization introduced in (5.15). In odd dimensions $n = 2N + 1$, showing that the inequality (6.11) is obeyed is equivalent to showing that

$$(n-3)\left(\prod_i y_i\right)^{-1/(n-2)} - 2\sum_i y_i^{-1} + 2 + (n-1)\left[\left(\prod_i y_i\right)^{1/(n-2)} - 1\right] z^2 \geq 0. \quad (6.12)$$

This must hold for the $z^0$ and $z^2$ terms independently. It is clearly true for the $z^2$ terms, since $y_i \geq 1$, and so checking the temperature bound for odd-dimensional Kerr-AdS black holes amounts to checking whether

$$(N-1)\left(\prod_{i=1}^N y_i\right)^{-1/(2N-1)} \geq \sum_{i=1}^N y_i^{-1} - 1 \quad (6.13)$$

for all $y_i \geq 1$. It is straightforward to see that in five dimensions, for which $N = 2$, the function

$$(y_1 y_2)^{-1/3} - \frac{1}{y_1} - \frac{1}{y_2} + 1 \quad (6.14)$$

is non-negative for all $y_i \geq 1$, since it can be written in the manifestly non-negative form

$$(y_1 y_2)^{-1}\{(y_1 - 1)(y_2 - 1) + [(y_1 y_2)^{2/3} - 1]\}. \quad (6.15)$$

This shows that all Kerr-AdS black holes in five dimensions obey the temperature bound (6.11). However, if $N \geq 3$ it is clear that the inequality in (6.13) can be violated for valid choices of the parameters $y_i$. For example, we can take $y_1 = y_2 = 1$, thus ensuring that the right-hand side of (6.13) is at least 1, and then choose the remaining $y_i$ large enough so that the left-hand side of (6.13) is less than 1. Clearly if $z$, which is independently specifiable and subject only to the restriction $z \geq 0$, is chosen to be sufficiently small, then the order $z^2$ terms in (6.12) will not be sufficiently positive to overwhelm the negative contribution from the terms at order $z^0$, and so (6.12) will be violated.

In even dimensions $n = 2N + 2$, the analogous calculation shows that for these Kerr-AdS black holes the temperature inequality (6.11) is equivalent to

$$(n-3)\left(\prod_i y_i\right)^{-1/(n-2)} - 2\sum_i y_i^{-1} + 1 + (n-1)\left[\left(\prod_i y_i\right)^{1/(n-2)} - 1\right] z^2 \geq 0. \quad (6.16)$$

Again, the terms at order $z^2$ are clearly positive, and so showing that (6.16) is satisfied is equivalent to showing that

$$\left(N - \frac{1}{2}\right)\left(\prod_{i=1}^N y_i\right)^{-1/2N} \geq \sum_{i=1}^N y_i^{-1} - \frac{1}{2}. \quad (6.17)$$

Clearly this inequality is always obeyed in four dimensions, corresponding to $N = 1$, since the function

$$\frac{1}{2} y_1^{-1/2} - \frac{1}{y_1} + \frac{1}{2} = \frac{1}{2} y_1^{-1}[(y_1 - 1) + (y_1^{1/2} - 1)] \quad (6.18)$$

is manifestly non-negative for all $y_1 \geq 1$. Thus all Kerr-AdS black holes in four dimensions obey the temperature bound (6.11). It is clear, however, that the inequality (6.17) can be violated for valid choices of the parameters, $y_i \geq 1$, if $N$ is greater than or equal to 2 (i.e. in even dimensions $n$ greater than or equal to 6). For example, we could take $y_1 = 1$, and then by taking the remaining $y_i$ large enough, the left-hand side of (6.17) can be made arbitrarily small, while the right-hand side exceeds $\frac{1}{2}$. By also taking $z$ sufficiently small, this means that (6.16) can be violated when $N \geq 2$.

More generally, one can see that in all dimensions $n \geq 6$, there exist regions in the $(y_i, z)$ parameter space for which the inequalities (6.12) or (6.16) are violated, and using (5.16) these can be translated back into regions in the parameter space for $(a_i, r_+)$ for which the temperature inequality (6.11) is not obeyed.

It is also worth remarking that similar conclusions are obtained if we consider asymptotically flat, rather than asymptotically AdS, rotating black holes. From (5.15) we see that the asymptotically flat case, which arises when $g = 0$, corresponds to taking $z$ to zero. We saw above that in the asymptotically AdS case there were terms in the inequality that were of order $z^2$, and terms of order $z^0$. The former were always consistent with the inequality, and it was the $z^0$ terms, which are the ones that survive in the $g \to 0$ limit, that had to be investigated in more detail. Thus the conclusions for asymptotically flat rotating black holes are the same as those for asymptoticallty AdS rotating black holes, namely, that violations of the temperature inequality (6.11) can occur in all dimensions 6 and higher, in cases where some of the rotations are small and some are large.

Finally, we should emphasize that our finding of violations of the inequality (6.11) does not contradict or threaten any cherished beliefs. The inequality was derived for static solutions, and, although commonly such considerations can lead to conjectured inequalities that have a wider range of applicability, as in the case of the cosmic-censorship bound (5.1), there is no *a priori* reason why it should do so in this case. The result could, perhaps, be viewed as a salutary reminder that a conjecture that holds up well in low dimensions may run into trouble in higher dimensions.

## VII. COSMOLOGICAL EVENT HORIZONS

In this section we take the cosmological constant to be positive, thus

$$R_{\mu\nu} = \frac{n-1}{l^2} g_{\mu\nu}. \quad (7.1)$$

In order to obtain the necessary formulae one makes the substitution $l^2 \to -l^2$.

In the case of pure de Sitter spacetime, $dS_n$, one has $m = a_i = 0$ and there is a cosmological horizon [47] at $r = l$. If $m > 0$, this is at

$$r = r_C \leq l. \tag{7.2}$$

Inside the cosmological horizon there will, in general, be a black-hole horizon, at $r = r_H$ say.

If the spacetime dimension $n$ is even, then the area of the cosmological horizon $A_C$ is easily seen to be bounded above by the value in pure $dS_n$,

$$A_C \leq \mathcal{A}_{n-2} l^{n-2}. \tag{7.3}$$

(For the four-dimensional case, see [39,48].) In some sense, $S_{\max} = \frac{1}{4}\mathcal{A}_{n-2} l^{n-2}$ represents the largest amount of information that can ever be lost through the cosmological horizon.

By manipulations similar to those in the case of a negative cosmological constant, one may convince oneself that

$$\left(\frac{A_C}{\mathcal{A}_{n-2}}\right)^{(n-3)/(n-2)}\left(1 - \frac{1}{l^2}\left(\frac{A_C}{\mathcal{A}_{n-2}}\right)^{2/(n-2)}\right) \leq \left(\frac{A_H}{\mathcal{A}_{n-2}}\right)^{(n-3)/(n-2)}\left(1 - \frac{1}{l^2}\left(\frac{A_H}{\mathcal{A}_{n-2}}\right)^{2/(n-2)}\right) \tag{7.4}$$

with equality only for the Kottler, i.e. Schwarzschild-de Sitter, solution. In the Reissner-Nordström-de Sitter case, one can do more, and obtain

$$\left(\frac{A_C}{\mathcal{A}_{n-2}}\right)^{(n-3)/(n-2)}\left(1 - \frac{1}{l^2}\left(\frac{A_C}{\mathcal{A}_{n-2}}\right)^{2/(n-2)}\right) - \Phi_C Q \leq \left(\frac{A_H}{\mathcal{A}_{n-2}}\right)^{(n-3)/(n-2)}\left(1 - \frac{1}{l^2}\left(\frac{A_H}{\mathcal{A}_{n-2}}\right)^{2/(n-2)}\right) - \Phi_H Q, \tag{7.5}$$

where $\Phi_C$ and $\Phi_H$ are the electrostatic potentials of the cosmological horizon and black-hole horizon. Actually, only the potential difference between the two horizons enters the inequality, as must be the case by gauge invariance.

An interesting question is whether there is an upper bound to the area of a black hole in a background de Sitter spacetime [49]. For the Schwarzschild-de Sitter solution, there is such an upper bound, which occurs when the two horizons coincide. This happens when

$$r_C = r_H = l\sqrt{\frac{n-3}{n-1}}. \tag{7.6}$$

It is natural therefore to conjecture that more generally,

$$A_H \leq \mathcal{A}_{n-2} l^{n-2}\left(\frac{n-3}{n-1}\right)^{(1/2)(n-2)}. \tag{7.7}$$

It is easy to check that this is true for the Reissner-Nordström-de Sitter solution in any dimension. The radius $r$ at which the two horizons coincide is easily seen to be less than $l\sqrt{\frac{n-3}{n-1}}$ and so the area of a charged black hole in a background de Sitter spacetime is indeed never greater than $\mathcal{A}_{n-1} l^{n-2}(\frac{n-3}{n-1})^{n-2}$.

In the rotating case the black hole and cosmological horizons coincide when $m$, considered as a function of $r$, has a vanishing derivative. One may check that this happens at

$$r_C = r_H < l\sqrt{\frac{n-3}{n-1}}. \tag{7.8}$$

We have verified that the inequality (7.7) is satisfied for rotating black holes in a variety of cases. These include the general Kerr-de Sitter metrics in four and six dimensions; the Kerr-de Sitter metrics with equal angular momenta in five and seven dimensions, and the Kerr-de Sitter metrics with equal angular momenta in all even dimensions.

Here, we shall just present the proof for the case of Kerr-de Sitter metrics with equal angular momenta in all even dimensions $n = 2N + 2$. From the formulae collected in the appendix, and setting $a_i = a$, we can show that the condition for double root $r_C = r_H$ can be expressed as

$$l^2 = \frac{r_H^2[(n-1)r_H^2 - a^2]}{(n-3)r_H^2 - a^2}, \tag{7.9}$$

while the area of the horizon is given by

$$A_H = \mathcal{A}_{n-2} l^{n-2}\left(\frac{r_H^2 + a^2}{l^2 + a^2}\right)^{(n-2)/2}. \tag{7.10}$$

It is straightforward to see that

$$\frac{r_H^2 + a^2}{l^2 + a^2} = \frac{n-3}{n-1} - \frac{2a^2}{(n-1)[(n-1)r_H^2 - a^2]} \leq \frac{n-3}{n-1}, \tag{7.11}$$

thus proving that these Kerr-AdS black holes indeed satisfy the bound (7.7).

## VIII. CONCLUSIONS

In this paper, we have studied the relation between the thermodynamics of the bulk variables describing rotating black holes in gauged supergravities, and the corresponding variables in the boundary CFT. We have shown that by using the standard UV/IR connection between the bulk and the boundary, bulk quantities that satisfy the first law of thermodynamics are mapped into boundary quantities that likewise satisfy the first law of thermodynamics. An important point when considering rotating AdS black holes is that the natural conformal boundary at large distance is defined with respect to a coordinate frame in which the metric is asymptotically static and asymptotically spherical.

Our results have clarified some previous puzzling claims in the literature, including, in particular, the assertion that to get boundary quantities that satisfy the first law one must start from bulk quantities that do not. This assertion was based on calculations performed in a specific frame that is rotating and nonspherical at infinity, with an angular velocity that depends on the rotation parameters of the black hole. In our opinion this is not a convenient or natural frame to use, and we believe that this is why it led to apparently puzzling conclusions.

In this context, it is perhaps worth remarking that in much of the literature on the subject of rotating AdS black holes, there is a tendency to refer to just two choices of frame, namely, the frame that is asymptotically static, and the frame with rotation rates given by (2.7) at infinity. In our opinion, the discussion of whether the thermodynamic quantities such as energy should be defined with respect to the former or the latter frame is misplaced. In reality there are infinitely many different frames that could be chosen, with arbitrary choices of asymptotic rotation rates. Asymptotically static frames enjoy a preferred status, and, as we showed in [13], the quantities defined in an asymptotically static frame satisfy the first law of thermodynamics. Frames whose asymptotic rotation rates depend upon the black-hole rotation parameters (such as the frame specified by (2.7)) seem to be particularly unnatural from the point of view of thermodynamic discussions, since one would need to include extra terms to compensate for the changing centrifugal and Coriolis contributions to the energy. Furthermore, if physical results (such as the energy) depend upon the choice of frame (in the sense that they depend upon the choice of timelike Killing vector used to define the energy, etc.), then a justification is called for as to why some specific frame, rather than one with some other rate of rotation, has been chosen. We have argued that for thermodynamic discussions, at least, the asymptotically static frame is the physically natural one.

Some of the results in [14,16] show that in a different context, namely, the discussion of the Cardy-Verlinde formula, there is a significant merit to considering the energy function $E'$ defined with respect to the frame with asymptotic angular velocity given by (2.7). It was shown in five dimensions in [16], and in higher dimensions in [14], that the Cardy-Verlinde formula (3.5) holds for rotating AdS black holes, provided that one uses energies and angular velocities measured with respect to the frame with angular velocities given by (2.7) at infinity. As far as we are aware, there is no *a priori* reason why a frame with this particular angular velocity should be singled out in this context, but the observation is certainly an interesting one.

Results had also been obtained for modifications to the Cardy-Verlinde formula when applied to nonrotating charged black holes [15,16,30]. The recent construction of black holes that have both rotation and charge has provided a wider spectrum of examples where the Cardy-Verlinde formula can be tested, and we have reported some results in the present paper. It seems that there is no natural and universal modification which encompasses all the cases. Nevertheless, as we have shown, there is a closely related and physically more significant result that does always hold for all the rotating charged black holes, namely, the existence of an AdS-Bekenstein bound (3.10), and its electrostatic generalization (4.2). The AdS-Bekenstein bound is itself a consequence of a more fundamental cosmic-censorship bound, and we have explicitly demonstrated for many of the rotating and charged black holes that this bound is indeed satisfied.

We have also examined the question of whether there is an upper bound for the temperature as a function of entropy for black holes in AdS backgrounds. In four and five dimensions, we found that the temperature of a rotating AdS black hole is always less than that of the Schwarzschild-AdS black hole of the same entropy. In six or more dimensions, by contrast, we find that for certain choices of the rotation parameters, the rotating AdS black hole can have a higher temperature than the Schwarzschild-AdS black-hole of the same entropy. We also discussed area inequalities for rotating black holes with a positive cosmological constant, for which there is a cosmological horizon as well as a black-hole horizon.

Finally, it is worth remarking that the whole question of how one defines, and in practice calculates, the energies of asymptotically AdS spacetimes is a subtle one, and many interesting open avenues for research remain. Many of the results for the energies of rotating AdS black holes were obtained first by the method of integrating the first law of thermodynamics, since this provides an unambiguous procedure that avoids the uncertainties associated with the regularization and subtraction procedures that are needed in some other definitions of the energy. However, it would be interesting to establish a more direct connection between the calculations based on the first law, and calculations based on the computation of conserved charges.

## ACKNOWLEDGMENTS

G. W. G. thanks the Centre for Mathematical Sciences, Zheijiang University, Hangzhou, and C. N. P. thanks the Relativity and Cosmology group, Cambridge, for hospitality during the course of this work. Research supported in part by DOE Grant No. DE-FG03-95ER40917 and NSF Grant No. INTO3-24081.

## APPENDIX: GENERAL KERR-ADS BLACK HOLE IN ARBITRARY DIMENSIONS

In this appendix, we collect some general results on rotating asymptotically AdS black holes in arbitrary dimension $n$. The solution was obtained in $n = 4$ in [23], in $n = 5$ in [1], and in $n \geq 6$ in [2,3]. Results on the thermodynamics of the arbitrary-dimension rotating AdS black holes were obtained in [13].

The metrics have $N \equiv [(n - 1)/2]$ independent rotation parameters $a_i$ in $N$ orthogonal 2-planes. We have $n = 2N + 1$ when $n$ is odd, and $n = 2N + 2$ when $n$ is even. Defining $\epsilon \equiv (n - 1) \bmod 2$, so that $n = 2N + 1 + \epsilon$, the metrics can be described by introducing $N$ azimuthal angles $\phi_i$, and $(N+\epsilon)$ "direction cosines" $\mu_i$ obeying the constraint

$$\sum_{i=1}^{N+\epsilon} \mu_i^2 = 1. \tag{A1}$$

In Boyer-Lindquist type coordinates that are asymptotically nonrotating, the metrics are given by [2,3]

$$\begin{aligned} ds^2 = {}& -W(1+r^2l^{-2})dt^2 + \frac{2m}{U}\left(W\,dt - \sum_{i=1}^{N}\frac{a_i\mu_i^2 d\varphi_i}{\Xi_i}\right)^2 \\ & + \sum_{i=1}^{N}\frac{r^2+a_i^2}{\Xi_i}\mu_i^2 d\varphi_i^2 + \frac{U\,dr^2}{V-2m} \\ & + \sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{\Xi_i}d\mu_i^2 - \frac{l^{-2}}{W(1+r^2l^{-2})} \\ & \times\left(\sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{\Xi_i}\mu_i d\mu_i\right)^2, \end{aligned} \tag{A2}$$

where

$$W \equiv \sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{\Xi_i}, \qquad U \equiv r^\epsilon \sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{r^2+a_i^2}\prod_{j=1}^{N}(r^2+a_j^2), \tag{A3}$$

$$V \equiv r^{\epsilon-2}(1+r^2l^{-2})\prod_{i=1}^{N}(r^2+a_i^2), \qquad \Xi_i \equiv 1 - a_i^2 l^{-2}. \tag{A4}$$

They satisfy $R_{\mu\nu} = -(n-1)l^{-2}g_{\mu\nu}$.

The constant-$r$ spatial surfaces at large distance are inhomogeneously distorted $(n-2)$-spheres. Making the coordinate transformations

$$\Xi_i y^2 \hat{\mu}_i^2 = (r^2+a_i^2)\mu_i^2, \tag{A5}$$

where $\sum_i \hat{\mu}_i^2 = 1$, the metrics at large $y$ approach the standard AdS form

$$\begin{aligned} d\bar{s}^2 = {}& -(1+y^2l^{-2})dt^2 + \frac{dt^2}{1+y^2l^{-2}} \\ & + y^2\sum_{k=1}^{N+\epsilon}(d\hat{\mu}_k^2 + \hat{\mu}_k^2 d\varphi_k^2), \end{aligned} \tag{A6}$$

with round $(n-2)$-spheres of volume $\mathcal{A}_{n-2}y^{n-2}$ at radius $y$, where $\mathcal{A}_{n-2}$ is the volume of the unit $(n-2)$-sphere.

The angular velocities of the horizon, measured relative to the frame that is nonrotating at infinity, are given by

$$\Omega^i = \frac{(1+r_+^2 l^{-2})a_i}{r_+^2+a_i^2}, \tag{A7}$$

and the angular momenta are

$$J_i = \frac{m a_i \mathcal{A}_{n-2}}{4\pi\Xi_i(\prod_j \Xi_j)}, \tag{A8}$$

where

$$\mathcal{A}_{n-2} = \frac{2\pi^{(n-1)/2}}{\Gamma[(n-1)/2]} \tag{A9}$$

is the volume of the unit $(n-2)$-sphere. As shown in [13], the energy of the black hole, again measured in the asymptotically static frame, is given by

$$\text{n} = \text{odd}: \quad E = \frac{m\mathcal{A}_{n-2}}{4\pi(\prod_j \Xi_j)}\left(\sum_{i=1}^{N}\frac{1}{\Xi_i} - \frac{1}{2}\right), \tag{A10}$$

$$\text{n} = \text{even}: \quad E = \frac{m\mathcal{A}_{n-2}}{4\pi(\prod_j \Xi_j)}\sum_{i=1}^{N}\frac{1}{\Xi_i}. \tag{A11}$$

The area of the event horizon is given by

$$A = \mathcal{A}_{n-2} r_+^{\epsilon-1}\prod_i \frac{r_+^2+a_i^2}{\Xi_i}. \tag{A12}$$

The Euclidean action was also calculated in [13], and found to be given by

$$I = \frac{\beta\mathcal{A}_{n-2}}{8\pi\prod_i \Xi_i}\left(m - r_+^\epsilon l^{-2}\prod_j (r_+^2+a_j^2)\right), \tag{A13}$$

where $\beta$ is the inverse of the Hawking temperature, which is given by

$$\text{n} = \text{odd}: \quad 2\pi T = r_+(1+r_+^2l^{-2})\sum_i \frac{1}{r_+^2+a_i^2} - \frac{1}{r_+}, \tag{A14}$$

n = even :

$$2\pi T = r_+(1+r_+^2l^{-2})\sum_i \frac{1}{r_+^2+a_i^2} - \frac{1-r_+^2l^{-2}}{2r_+}. \tag{A15}$$

The traditional asymptotically rotating Boyer-Lindquist coordinate system, where the angular velocities at infinity are given by (2.7), is related to the coordinates in (A2) by defining

$$\varphi_i' = \varphi_i - a_i l^{-2} t, \qquad t' = t. \tag{A16}$$

The energy $E'$ calculated in the asymptotically rotating frame, i.e. using the timelike Killing vector $\partial/\partial t'$, is given by [13]

$$E' = E - \frac{1}{l^2}\sum_i a_i J_i = \frac{(n-2)m\mathcal{A}_{n-2}}{8\pi(\prod_j \Xi_j)}. \tag{A17}$$

[1] S. W. Hawking, C. J. Hunter, and M. M. Taylor-Robinson, Phys. Rev. D **59**, 064005 (1999).
[2] G. W. Gibbons, H. Lü, D. N. Page, and C. N. Pope, J. Geom. Phys. **53**, 49 (2005).
[3] G. W. Gibbons, H. Lü, D. N. Page, and C. N. Pope, Phys. Rev. Lett. **93**, 171102 (2004).
[4] M. Cvetič, H. Lü, and C. N. Pope, Phys. Lett. B **598**, 273 (2004).
[5] M. Cvetič, H. Lü, and C. N. Pope, Phys. Rev. D **70**, 081502 (2004).
[6] Z. W. Chong, M. Cvetič, H. Lü, and C. N. Pope, Nucl. Phys. **B717**, 246 (2005).
[7] Z. W. Chong, M. Cvetič, H. Lü, and C. N. Pope, hep-th/0412094.
[8] Z. W. Chong, M. Cvetič, H. Lü, and C. N. Pope, Phys. Rev. D **72**, 041901 (2005).
[9] Z. W. Chong, M. Cvetič, H. Lü, and C. N. Pope, hep-th/0506029 [to appear in Phys. Rev. Lett.].
[10] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[11] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[12] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[13] G. W. Gibbons, M. J. Perry, and C. N. Pope, Classical Quantum Gravity **22**, 1503 (2005).
[14] R. G. Cai, L. M. Cao, and D. W. Pang, Phys. Rev. D **72**, 044009 (2005).
[15] D. Klemm, A. C. Petkou, and G. Siopsis, Nucl. Phys. **B601**, 380 (2001).
[16] D. Klemm, A. C. Petkou, G. Siopsis, and D. Zanon, Nucl. Phys. **B620**, 519 (2002).
[17] E. Verlinde, hep-th/0008140.
[18] A. Ashtekar and A. Magnon, Classical Quantum Gravity **1**, L39 (1984).
[19] A. Ashtekar and S. Das, Classical Quantum Gravity **17**, L17 (2000).
[20] N. Deruelle and J. Katz, Classical Quantum Gravity **22**, 421 (2005).
[21] J. Katz, Classical Quantum Gravity **2**, 423 (1985); J. Katz, J. Bičák, and D. Lynden-Bell, Phys. Rev. D **55**, 5957 (1997).
[22] S. Deser, I. Kanik, and B. Tekin, Classical Quantum Gravity **22**, 3383 (2005).
[23] B. Carter, Commun. Math. Phys. **10**, 280 (1968).
[24] S. W. Hawking and H. S. Reall, Phys. Rev. D **61**, 024014 (2000).
[25] B. M. N. Carter and I. P. Neupane, Phys. Rev. D **72**, 043534 (2005).
[26] J. D. Bekenstein, Phys. Rev. D **23**, 287 (1981).
[27] D. Kutasov and F. Larsen, J. High Energy Phys. 01 (2001) 001.
[28] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983).
[29] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998).
[30] R. G. Cai, Phys. Rev. D **63**, 124018 (2001).
[31] M. Cvetič, M. J. Duff, P. Hoxa, J. T. Liu, H. Lü, J. X. Lu, R. Martinez-Acosta, C. N. Pope, H. Sati, and T. A. Tran, Nucl. Phys. **B558**, 96 (1999).
[32] M. Cvetič, G. W. Gibbons, H. Lü, and C. N. Pope, hep-th/0504080.
[33] S. W. Hawking, Commun. Math. Phys. **33**, 323 (1973).
[34] R. Penrose, Ann. N.Y. Acad. Sci. **224**, 125 (1973).
[35] H. L. Bray and P. T. Chrusciel, *The Einstein Equations and the Large Scale Behavior of Gravitational Fields*, edited by H. Friedrich and P. T. Chrusciel (Birkhaeuser, Basel, 2004), p. 39.
[36] H. L. Bray, Not. Am. Math. Soc. **49**, 1372 (2002).
[37] P. S. Jang and R. M. Wald, J. Math. Phys. (N.Y.) **18**, 41 (1977).
[38] R. Geroch, Ann. N.Y. Acad. Sci. **224**, 108 (1973).
[39] W. Boucher, G. W. Gibbons, and G. T. Horowitz, Phys. Rev. D **30**, 2447 (1984).
[40] G. W. Gibbons, Classical Quantum Gravity **16**, 1677 (1999).
[41] G. Huisken and T. Ilmanen, Int. Math. Res. Not. **20**, 1045 (1997).
[42] G. Huisken and T. Ilmanen, J. Diff. Geom. **59**, 353 (2001).
[43] C. Barrabes, V. P. Frolov, and E. Lesigne, Phys. Rev. D **69**, 101501 (2004).
[44] G. W. Gibbons, Classical Quantum Gravity **14**, 2905 (1997).
[45] P. S. Jang, Phys. Rev. D **20**, 834 (1979).
[46] M. Visser, Phys. Rev. D **46**, 2445 (1992).
[47] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).
[48] T. Shiromizu, K. Nakao, H. Kodama, and K. I. Maeda, Phys. Rev. D **47**, R3099 (1993).
[49] S. A. Hayward, T. Shiromizu, and K. I. Nakao, Phys. Rev. D **49**, 5080 (1994).