

Born-Infeld-Einstein theory with matter

Dan N. Vollick

Irving K. Barber School of Arts and Sciences, University of British Columbia Okanagan, 3333 University Way, Kelowna, B.C. V1V 1V7, Canada

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The field equations associated with the Born-Infeld-Einstein action including matter are derived using a Palatini variational principle. Scalar, electromagnetic, and Dirac fields are considered. It is shown that an action can be chosen for the scalar field that produces field equations identical to the usual Einstein field equations minimally coupled to a scalar field. In the electromagnetic and Dirac cases the field equations reproduce the standard equations only to lowest order. The spherically symmetric electrovac equations are studied in detail. It is shown that the resulting Einstein equations correspond to gravity coupled to a modified Born-Infeld theory. It is also shown that the solution becomes singular on a spherical shell which implies that all charged particles must have a finite size. The space-time is therefore more singular than the Reissner-Nordström space-time, which is only singular at a point. Mass terms for the fields are also considered.

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I. INTRODUCTION

In the 1930's Born and Infeld [1] attempted to eliminate the divergent self-energy of the electron by modifying Maxwell's theory. Born-Infeld electrodynamics follows from the Lagrangian

$$L = -\frac{1}{4\pi b} \left\{ \sqrt{-\det(g_{\mu\nu} + bF_{\mu\nu})} - \sqrt{-\det(g_{\mu\nu})} \right\}, \quad (1)$$

where $g_{\mu\nu}$ is the metric tensor and $F_{\mu\nu}$ is the electromagnetic field tensor. In the weak field limit this Lagrangian reduces to the Maxwell Lagrangian plus small corrections. For strong fields the field equations deviate significantly from Maxwell's theory and the self-energy of the electron can be shown to be finite. The Born-Infeld action also appears in string theory. The action for a D-brane is of the Born-Infeld form with two fields, a gauge field on the brane and the projection of the Neveu-Schwarz B-field onto the brane [2].

In a recent paper [3] I considered using a Palatini variational approach to derive the field equations associated with the action

$$L = -\frac{1}{\kappa b} \left\{ \sqrt{-\det(g_{\mu\nu} + bR_{\mu\nu})} - \sqrt{-\det(g_{\mu\nu})} \right\}, \quad (2)$$

where $R_{\mu\nu}$ is the Ricci tensor, and $\kappa = 8\pi G$. This action has also been examined using a purely metric variation by Deser and Gibbons [4], Feigenbaum, Freund, and Pigli [5], and Feigenbaum [6].

In this paper I will consider adding matter to the theory by including it in the determinant in Eq. (2). The forms of matter that I investigate in this paper include scalar, electromagnetic, and Dirac fields. In all the cases except for the scalar field there are corrections to the standard equations. In the scalar field case it is shown that it is possible to choose a Lagrangian that gives the standard equations exactly. The fields produced by a spherically symmetric

charge distribution are also studied. It is shown that the resulting Einstein equations correspond to gravity coupled to a modified Born-Infeld theory and that point charges are not allowed in the theory. In fact, the solution is singular on a spherical shell in contrast to the Reissner-Nordström solution which is only singular at a point.

II. THE FIELD EQUATIONS

The field equations for the theory follow from the Born-Infeld-Einstein action

$$L = -\frac{1}{\kappa b} \left\{ \sqrt{-\det(g_{\mu\nu} + bR_{\mu\nu} + \kappa b M_{\mu\nu})} - \sqrt{-\det(g_{\mu\nu})} \right\}, \quad (3)$$

where $R_{\mu\nu}$ is the Ricci tensor, $\kappa = 8\pi G$, b is a constant, and $M_{\mu\nu}$ is the matter contribution. For a simple example of the matter contribution consider a massless scalar field ϕ . In this case $M_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi$. The Ricci tensor is given by

$$R_{\mu\nu} = \partial_\nu \Gamma_{\mu\alpha}^\alpha - \partial_\alpha \Gamma_{\mu\nu}^\alpha - \Gamma_{\beta\alpha}^\alpha \Gamma_{\mu\nu}^\beta + \Gamma_{\beta\mu}^\alpha \Gamma_{\alpha\nu}^\beta \quad (4)$$

and the connection is taken to be symmetric. Note that $R_{\mu\nu}$ is not symmetric in general.

If the curvature and matter terms are small an expansion of (3) gives

$$L \simeq -\sqrt{g} \left(\frac{1}{2\kappa} R + \frac{1}{2} M \right). \quad (5)$$

Thus, if M does not vanish, the lowest order contribution to the matter Lagrangian is

$$L_M \simeq -\frac{1}{2} \sqrt{g} M. \quad (6)$$

In the case of the electromagnetic field M vanishes, so higher order terms will be needed.

Varying the action with respect to $g_{\mu\nu}$ gives

$$\sqrt{P}(P^{-1})^{(\mu\nu)} + \kappa b \sqrt{P}(P^{-1})^{\alpha\beta} \frac{\partial M_{\beta\alpha}}{\partial g_{\mu\nu}} = \sqrt{g}g^{\mu\nu}, \quad (7)$$

where $P_{\mu\nu} = g_{\mu\nu} + bR_{\mu\nu} + \kappa bM_{\mu\nu}$, P^{-1} is the inverse of P , $(P^{-1})^{(\mu\nu)}$ is the symmetric part of P^{-1} , $P = -\det(P_{\mu\nu})$, and $g = -\det(g_{\mu\nu})$. If $M_{\mu\nu}$ is independent of the metric, as will be the case for massless scalar and electromagnetic fields, the second term on the left-hand side in Eq. (7) will vanish and the field equations will simplify significantly.

Varying with respect to $\Gamma_{\mu\nu}^\alpha$ gives

$$\nabla_\alpha[\sqrt{P}(P^{-1})^{(\mu\nu)}] - \frac{1}{2}\nabla_\beta\{\sqrt{P}[\delta_\alpha^\mu(P^{-1})^{\beta\nu} + \delta_\alpha^\nu(P^{-1})^{\beta\mu}]\} = 0. \quad (8)$$

Here I have assumed that $M_{\mu\nu}$ is independent of the connection. This will be true for scalar and electromagnetic fields, but not for the Dirac field. The Dirac field will be studied in detail later. Contracting over α and ν gives

$$\nabla_\alpha[\sqrt{P}(P^{-1})^{[\alpha\mu]}] = -\frac{3}{5}\nabla_\alpha[\sqrt{P}(P^{-1})^{(\alpha\mu)}], \quad (9)$$

where $(P^{-1})^{[\mu\nu]}$ is the antisymmetric part of P^{-1} .

From Eq. (9) we see that

$$\nabla_\alpha[\sqrt{P}(P^{-1})^{\alpha\mu}] = \frac{2}{5}\nabla_\alpha[\sqrt{P}(P^{-1})^{(\alpha\mu)}]. \quad (10)$$

Taking $M_{\mu\nu}$ to be independent of the metric and substituting Eqs. (7) and (10) into (8) gives

$$\nabla_\alpha[\sqrt{g}g^{\mu\nu}] - \frac{1}{5}\nabla_\beta\{\sqrt{g}[g^{\mu\beta}\delta_\alpha^\nu + g^{\nu\beta}\delta_\alpha^\mu]\} = 0. \quad (11)$$

Since the trace of the above system of equations vanishes there are four too few equations and the system is under determined. Thus, we expect four arbitrary functions in the solution. Such a solution is given by

$$\nabla_\alpha[\sqrt{g}g^{\mu\nu}] = \sqrt{g}[\delta_\alpha^\mu V^\nu + \delta_\alpha^\nu V^\mu], \quad (12)$$

where V^μ is an arbitrary vector. The connection that follows from this set of equations is given by [3]

$$\Gamma_{\mu\nu}^\alpha = \left\{ \begin{array}{c} \alpha \\ \mu\nu \end{array} \right\} + \frac{1}{2}[3g_{\mu\nu}V^\alpha - \delta_\mu^\alpha V_\nu - \delta_\nu^\alpha V_\mu], \quad (13)$$

where the first term on the right-hand side is the Christoffel symbol. This new ‘‘geometrical’’ vector field was discussed in detail in [3] and will not be considered here. I will therefore set $V_\mu = 0$ and the connection is the Christoffel symbol.

Thus, the gravitational field equations are given by (7) with the connection given by the Christoffel symbol if $M_{\mu\nu}$ is independent of the metric. The matter field equations are

given by varying the action with respect to the matter fields. I will study these equations in the subsequent sections, once a specific type of matter field has been chosen.

III. THE SCALAR FIELD

For the scalar field I will take $M_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi$. In this case $M_{\mu\nu}$ is symmetric and is independent of the metric. Note that $R_{\mu\nu}$ is symmetric since the connection is just the Christoffel symbol. This implies that $P_{\mu\nu}$ is symmetric.

Consider the general case in which $M_{\mu\nu}$ is symmetric and independent of the metric. Taking the determinant of Eq. (7) gives $P = g$ and after some simple algebra we find that

$$G_{\mu\nu} = -\kappa\left(M_{\mu\nu} - \frac{1}{2}Mg_{\mu\nu}\right). \quad (14)$$

Thus, for the scalar field we have

$$G_{\mu\nu} = -\kappa\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi\right). \quad (15)$$

Note that these are the standard Einstein field equations minimally coupled to a scalar field.

The scalar field equation can be found by varying the action with respect to ϕ . The resulting field equation is

$$\nabla_\mu[\sqrt{P}(P^{-1})^{\mu\nu}\nabla_\nu\phi] = 0. \quad (16)$$

Using (7) gives the usual field equations

$$\nabla^\alpha\nabla_\alpha\phi = 0. \quad (17)$$

Note that the lowest order matter Lagrangian given in Eq. (6) is the standard Lagrangian for a scalar field. Therefore, the higher order terms in both the gravitational and matter Lagrangians do not modify the field equations. This extends the universality of the Einstein vacuum equations discovered by Ferraris *et al.* [7,8] to Born-Infeld Einstein actions with a scalar field. In their papers Ferraris *et al.* showed that Lagrangians of the form $L = F(R)$ and $L = f(R^{\mu\nu}R_{\mu\nu})$ always give the Einstein vacuum equations with a cosmological constant under a Palatini variation.

Other choices for $M_{\mu\nu}$ are possible. The most general expression that is bilinear in $\nabla\phi$ is $M_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi + \alpha g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi$, where α is a constant. Such choices lead to more complicated theories in which the connection is no longer the Christoffel symbol.

IV. THE ELECTROMAGNETIC FIELD

For the electromagnetic field take $M_{\mu\nu} = \alpha F_{\mu\nu}$, where α is a constant. First consider the electromagnetic field equations. Varying the action with respect to A_μ gives

$$\nabla_\mu[\sqrt{P}(P^{-1})^{[\mu\nu]}] = 0. \quad (18)$$

To lowest order this reduces to the usual field equation

$$\nabla_{\mu} F^{\mu\nu} = 0. \quad (19)$$

The lowest order contribution to the electromagnetic field Lagrangian cannot be computed from (6) since $M = 0$. The Lagrangian to first order in the curvature and second order in the electromagnetic field is given by

$$L \simeq -\sqrt{g} \left[\frac{1}{2\kappa} R + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right]. \quad (20)$$

Thus, to this order, we obtain the standard Einstein-Maxwell Lagrangian.

It is interesting to note that the inclusion of the electromagnetic field $F_{\mu\nu}$ excludes the possibility of the ‘‘geometric’’ vector field discussed earlier. To see this note that (9) and (18) imply that

$$\nabla_{\alpha} [\sqrt{P} (P^{-1})^{(\alpha\mu)}] = 0. \quad (21)$$

This together with (7) and (11) implies that

$$\nabla_{\alpha} [\sqrt{g} g^{\mu\nu}] = 0, \quad (22)$$

which tells us that the connection is the Christoffel symbol and that there is no room for an extra geometric vector field.

Now consider the gravitational field equations. They are difficult to solve exactly, but if the fields are sufficiently weak they can be expanded to second order in the fields. The following expressions, computed to quadratic order in the field variables, will be useful in finding the field equations

$$(P^{-1})^{(\mu\nu)} = g^{\mu\nu} - bR^{\mu\nu} + \alpha^2 \kappa^2 b^2 F^{\mu}_{\alpha} F^{\alpha\nu} + b^2 R^{\mu}_{\alpha} R^{\alpha\nu} \quad (23)$$

and

$$\sqrt{P} = \sqrt{g} \left[1 + \frac{1}{2} bR + \frac{1}{8} b^2 R^2 + \frac{1}{4} \alpha^2 \kappa^2 b^2 F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{4} b^2 R_{\alpha\beta} R^{\beta\alpha} \right]. \quad (24)$$

The field equations up to and including quadratic terms in the fields are given by

$$G_{\mu\nu} = \frac{1}{8} b [R^2 g_{\mu\nu} - 4RR_{\mu\nu} - 2g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + 8R^{\mu}_{\alpha} R^{\alpha\nu}] - \alpha^2 \kappa^2 b \left[F_{\mu}^{\alpha} F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right]. \quad (25)$$

It is interesting to note that the nonlinear terms involving the Ricci tensor, in the first bracket, vanish due to the Rainich conditions [9]. Thus, to order b we obtain the usual Einstein equations if we take $\alpha^2 = 1/(\kappa b)$. Note that this implies that b is positive.

In general it seems that it is difficult to write the field equations in the form $G_{\mu\nu} = -\kappa T_{\mu\nu}$. However, it is possible to write them in this form for spherically symmetric fields. In this case we can take

$$A_{\mu} = (\phi, \vec{0}), \quad (26)$$

where ϕ depends only on r and the metric is diagonal. A short calculation shows that

$$P_{\mu\nu} = \begin{bmatrix} q_{00} & \Phi' & 0 & 0 \\ -\Phi' & q_{11} & 0 & 0 \\ 0 & 0 & q_{22} & 0 \\ 0 & 0 & 0 & q_{33} \end{bmatrix} \quad (27)$$

and

$$(P^{-1})^{\mu\nu} = \begin{bmatrix} \frac{q_{11}}{\Delta} & -\frac{\Phi'}{\Delta} & 0 & 0 \\ \frac{\Phi'}{\Delta} & \frac{q_{00}}{\Delta} & 0 & 0 \\ 0 & 0 & \frac{1}{q_{22}} & 0 \\ 0 & 0 & 0 & \frac{1}{q_{33}} \end{bmatrix} \quad (28)$$

where $q_{\mu\nu} = g_{\mu\nu} + bR_{\mu\nu}$, $\Delta = q_{00}q_{11} + (\Phi')^2$, and $\Phi = \sqrt{\kappa b} \phi$. Taking the determinant of both sides of (7) gives

$$g = -q_{00}q_{11}q_{22}q_{33}. \quad (29)$$

The field equations can then be written as

$$(P^{-1})^{(\mu\nu)} = \sqrt{\frac{q_{00}q_{11}}{\Delta}} g^{\mu\nu}. \quad (30)$$

The three independent equations are

$$q_{00} = -\sqrt{q_{00}q_{11}\Delta} g^{11}, \quad (31)$$

$$q_{11} = -\sqrt{q_{00}q_{11}\Delta} g^{00}, \quad (32)$$

and

$$q_{22} = \sqrt{\frac{\Delta}{q_{00}q_{11}}} g_{22}. \quad (33)$$

From (31) and (32) we find that

$$\Delta = g_{00}g_{11} \quad \text{and} \quad g_{00}R_{11} = g_{11}R_{00}. \quad (34)$$

Using these two equations gives

$$g_{11}b^2R_{00}^2 + 2g_{00}g_{11}bR_{00} + g_{00}(\Phi')^2 = 0. \quad (35)$$

The solution to this equation is

$$R_{00} = -\frac{g_{00}}{b} \left[1 - \sqrt{1 - g^{00}g^{11}(\Phi')^2} \right]. \quad (36)$$

Similarly we have

$$R_{11} = -\frac{g_{11}}{b} \left[1 - \sqrt{1 - g^{00}g^{11}(\Phi')^2} \right] \quad (37)$$

and

$$R_{22} = -\frac{g_{22}}{b} \left[1 - \sqrt{\frac{1}{1 - g^{00}g^{11}(\Phi')^2}} \right]. \quad (38)$$

The Einstein field equations can be written in covariant form as

$$G_{\mu\nu} = -\kappa \left\{ \frac{F_{\mu}^{\alpha} F_{\nu\alpha}}{\sqrt{1 - \frac{1}{2} a^2 F^2}} - \frac{g_{\mu\nu}}{a^2} \left[1 - \sqrt{1 - \frac{1}{2} a^2 F^2} \right] \right\}, \quad (39)$$

where I have taken $b = a^2/\kappa$ with a constant and $F^2 = F^{\alpha\beta} F_{\alpha\beta}$. Note that this equation is not general. I have only shown that it is valid if $P_{\mu\nu}$ is of the form given in (27). The energy-momentum tensor is identical to the Born-Infeld energy-momentum tensor, except for the sign in the square roots and the sign in front of the metric.

The electromagnetic field equations given in (18) can be written as

$$\nabla_{\mu} \left[\frac{F^{\mu\nu}}{\sqrt{1 - \frac{1}{2} a^2 F^2}} \right] = 0. \quad (40)$$

These field equations are the same as the Born-Infeld field equations with the sign in the square root changed.

In Born-Infeld theory the square root $\sqrt{1 - \frac{1}{2} a^2 F^2}$, that appears in the above equations, is replaced by $\sqrt{1 + \frac{1}{2} a^2 F^2}$. This difference is important in terms of the allowed magnitude of the electric field. In flat space-time the Born-Infeld square root is $\sqrt{1 - a^2 E^2}$, so that the maximum magnitude of the electric field is $1/a$. In the theory presented here the square root is $\sqrt{1 + a^2 E^2}$, so that the square root does not, by itself, constrain the magnitude of the electric field. For example consider the electromagnetic field equations in a vacuum outside a spherically symmetric charge distribution in flat space-time. The electromagnetic field equation is

$$\vec{\nabla} \cdot \left[\frac{\vec{E}}{\sqrt{1 + a^2 E^2}} \right] = 0 \quad (41)$$

and the solution is

$$\frac{\vec{E}}{\sqrt{1 + a^2 E^2}} = \frac{Q}{r^2} \hat{r}, \quad (42)$$

where Q is a constant. Note that although E can be arbitrarily large the left-hand side in the above equation has a maximum magnitude of $1/a$ as $E \rightarrow \infty$. Thus, we cannot let $r \rightarrow 0$ on the right-hand side and the particle has to have a finite size. In fact, any charged particle will have to have a radius that is greater than or equal to $\sqrt{a|Q|}$.

Now consider the solution in curved space-time. The general spherically symmetric solution for any theory with a Lagrangian of the form

$$-\frac{1}{2\kappa} R + L(F^2), \quad (43)$$

is given by [10–15]

$$ds^2 = - \left[1 - \frac{2m(r)}{r} \right] dt^2 + \left[1 - \frac{2m(r)}{r} \right]^{-1} dr^2 + r^2 d\Omega^2 \quad (44)$$

and

$$D = \frac{Q}{r^2} dt \wedge dr, \quad (45)$$

where

$$D^{\mu\nu} = -2 \frac{\partial L}{\partial F_{\mu\nu}}, \quad (46)$$

$$\frac{dm(r)}{dr} = 4\pi r^2 H(s), \quad (47)$$

$s = -\frac{1}{4} D^{\mu\nu} D_{\mu\nu} = Q^2/(2r^4)$ and H is the ‘‘Hamiltonian’’ given by

$$H = -\frac{1}{2} D^{\mu\nu} F_{\mu\nu} - L. \quad (48)$$

The electromagnetic field equations are

$$\nabla_{\mu} D^{\mu\nu} = 0 \quad (49)$$

and the energy-momentum tensor is

$$T_{\mu\nu} = D_{\mu}^{\alpha} F_{\nu\alpha} + g_{\mu\nu} L. \quad (50)$$

The field equations (39) and (40) can be obtained if we take

$$L = \frac{1}{a^2} \left[\sqrt{1 - \frac{1}{2} a^2 F^2} - 1 \right]. \quad (51)$$

For this Lagrangian

$$D^{\mu\nu} = \frac{F^{\mu\nu}}{\sqrt{1 - \frac{1}{2} a^2 F^2}}, \quad (52)$$

$$H(s) = \frac{1}{a^2} \left[1 - \sqrt{1 - 2a^2 s} \right], \quad (53)$$

and Eq. (47) can be written as

$$\frac{dm(r)}{dr} = \frac{4\pi}{a^2} \left[r^2 - \sqrt{r^4 - a^2 Q^2} \right]. \quad (54)$$

Thus, just as in the flat space-time case, the particle has to have a coordinate radius that is greater than or equal to $\sqrt{a|Q|}$.

To see if the space-time becomes singular as we approach $\sqrt{a|Q|}$ consider the Ricci tensor which is given by

$$R = -2 \left[\frac{rm'' + 2m'}{r^2} \right]. \quad (55)$$

From (54) we have

$$R = -\frac{32\pi}{a^2} \left[1 - \frac{r^2 - \frac{a^2 Q^2}{2r^2}}{\sqrt{r^4 - a^2 Q^2}} \right]. \quad (56)$$

Thus, there is a curvature singularity at $r = \sqrt{a|Q|}$ and the particle has to have a coordinate radius greater than $\sqrt{a|Q|}$ to avoid the singularity. It is interesting to note that the space-time is more singular than the Reissner-Nordström solution, which is only singular at a point.

V. THE DIRAC FIELD

In this section I will examine the Dirac field in Born-Infeld-Einstein theory. To deal with spinors it is convenient to introduce a tetrad $e^a{}_\mu$ satisfying

$$e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu} \quad \text{and} \quad e^a{}_\mu e^b{}_\nu g^{\mu\nu} = \eta^{ab}. \quad (57)$$

The Riemann tensor is defined to be

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac} \omega_\nu{}^b{}_c - \omega_\nu{}^{ac} \omega_\mu{}^b{}_c, \quad (58)$$

where $\omega_\mu{}^{ab}$ is the spin connection (see [16] for a discussion of spinors).

The action will be taken to be

$$L = -\frac{1}{\kappa b} \left\{ \sqrt{-\det(g_{\mu\nu} + bR_{\mu\nu} + \kappa b M_{\mu\nu})} - \sqrt{-\det(g_{\mu\nu})} \right\}, \quad (59)$$

where

$$R_{\mu\nu} = -R_{\mu\beta}{}^{ab} e_{a\nu} e_b{}^\beta, \quad (60)$$

$M_{\mu\nu} = -2i\bar{\Psi}\gamma_\mu D_\nu\Psi$, Ψ is the Dirac field, $\gamma^\mu = e_a{}^\mu\gamma^a$, $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}$,

$$D_\mu\Psi_e = \partial_\mu\Psi_e + \frac{1}{2}\omega_\mu{}^{ab}\Sigma_{ab}\Psi_e, \quad (61)$$

and $\Sigma_{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$ are the generators of the Dirac representation. Note that the Dirac Lagrangian depends on the spin connection. Of course, other choices for $M_{\mu\nu}$ are possible. For example one could symmetrize on the indices μ and ν . However, since $R_{\mu\nu}$ is not symmetric this does not simplify the field equations. The Lagrangian, to lowest order, corresponding to this choice for $M_{\mu\nu}$ is given by (6) and is the usual Dirac Lagrangian.

Varying the action with respect to the field $\bar{\Psi}$ gives

$$(P^{-1})^{\mu\nu} \gamma_\nu D_\mu\Psi = 0. \quad (62)$$

To lowest order we obtain the standard Dirac equation $\gamma^\alpha D_\alpha\Psi = 0$. Varying the action with respect to the tetrad gives, to lowest order, the field equations

$$G_{\mu\nu} = i\kappa\bar{\Psi}\gamma_\nu D_\mu\Psi, \quad (63)$$

where the field equation, to lowest order, for Ψ has been used.

The field equation for the connection is found by varying the action with respect to the connection. Note that the connection appears in both the Ricci tensor and in $M_{\mu\nu}$. The details are rather messy, but to lowest order we get the usual result (see [17] for details). This is not surprising since to lowest order the matter Lagrangian is the standard Dirac Lagrangian.

VI. MASS TERMS

In this section I will consider the inclusion of mass terms into the theory. Let $\tilde{M}_{\mu\nu}$ be the matter contribution without a mass term. A simple approach to include a mass term would be to let

$$M_{\mu\nu} = \tilde{M}_{\mu\nu} + \Omega g_{\mu\nu}, \quad (64)$$

where Ω depends on the field variable. For example, in the case of a scalar field $\Omega = \frac{1}{4}m^2\phi^2$ and for the Dirac field $\Omega = \frac{1}{4}m\bar{\Psi}\Psi$, where m is the mass of the field. For simplicity consider the case of the scalar field. Equation (7) gives the field equations

$$\sqrt{P}(P^{-1})^{(\mu\nu)} = \frac{\sqrt{g}g^{\mu\nu}}{1 + \frac{1}{4}\kappa b m^2 \phi^2}. \quad (65)$$

The field equations given by varying the action with respect to the connection are still given by (8). Thus the connection is no longer the Christoffel symbol with respect to the metric $g_{\mu\nu}$. In fact, the connection is the Christoffel symbol with respect to the metric $h_{\mu\nu} = (1 + \frac{1}{4}\kappa b m^2 \phi^2)^{-1} g_{\mu\nu}$. Thus, $R_{\mu\nu}$ and $P_{\mu\nu}$ will be symmetric. It is interesting to note that the field equations (65) imply that $h_{\mu\nu} = P_{\mu\nu}$.

One can also show that the Einstein equations are given by

$$R_{\mu\nu} = -\kappa \left[\nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} m^2 \phi^2 \left(\frac{1 + \frac{1}{8} \kappa b m^2 \phi^2}{1 + \frac{1}{4} \kappa b m^2 \phi^2} \right) g_{\mu\nu} \right]. \quad (66)$$

These are not the standard Einstein equations with a mass term. However, if $\kappa b m^2 \phi^2 \ll 1$ we get the standard equations to lowest order. There is nothing wrong with these field equations, but a simpler set of field equations can be found. These equations follow from the Lagrangian

$$L = -\frac{1}{\kappa b} \left\{ \sqrt{-\det \left[g_{\mu\nu} + bR_{\mu\nu} + \kappa b \left(\nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} m^2 \phi^2 g_{\mu\nu} \right) \right]} - \sqrt{-\det \left[\left(1 + \frac{1}{2} \kappa b m^2 \phi^2 \right)^{1/2} g_{\mu\nu} \right]} \right\}. \quad (67)$$

Varying this action with respect to the connection gives the same equations as before and varying it with respect to the metric gives

$$\sqrt{P}(P^{-1})^{(\mu\nu)} = \sqrt{g}g^{\mu\nu}. \quad (68)$$

The connection is therefore given by the Christoffel symbol and the Einstein equations are

$$R_{\mu\nu} = -\kappa \left[\nabla_{\mu} \phi \nabla_{\nu} \phi + \frac{1}{2} m^2 \phi^2 g_{\mu\nu} \right]. \quad (69)$$

Thus, the resulting field equations are exactly the Einstein equations coupled to a massive scalar field.

It is possible to obtain the standard equations with respect to the metric $h_{\mu\nu}$ from the action (3) if one takes

$$\kappa b \Omega = \sqrt{1 + \frac{1}{2} \kappa b m^2 \phi^2} - 1. \quad (70)$$

In this case $h_{\mu\nu} = (1 + \frac{1}{2} \kappa b m^2 \phi^2)^{-1/2} g_{\mu\nu}$ and the field equations are given by (69) with $g_{\mu\nu}$ replaced by $h_{\mu\nu}$.

A similar approach can be used for the Dirac and electromagnetic fields. For the Dirac field the mass terms will be of the form $m^2 \bar{\Psi} \Psi g_{\mu\nu}$ and for the electromagnetic field the mass terms will be of the form $m^2 A_{\mu} A_{\nu}$. The electromagnetic mass term, $m^2 A_{\mu} A_{\nu}$, is actually simpler to deal

with than the scalar field mass term since it is independent of the metric.

VII. CONCLUSION

The field equations for the Born-Infeld-Einstein action including matter were derived using the first order or Palatini formalism. For the scalar field it was shown that it is possible to choose a Lagrangian that gives the standard Einstein equations minimally coupled to a scalar field even in the massive case. For the electromagnetic and Dirac fields the standard equations are only reproduced to lowest order.

The spherically symmetric electrovac equations were studied in detail. It was shown that the resulting Einstein equations correspond to gravity coupled to a modified Born-Infeld theory. It was also shown that point charges are not allowed in the theory. The space-time is singular on a spherical shell and is therefore more singular than the Reissner-Nordström solution, which is only singular at a point.

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