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Chaos-order transition in Bianchi type I non-Abelian Born-Infeld cosmology

Vladimir V. Dyadichev* and Dmitri V. Gal'tsov†
Department of Theoretical Physics, Moscow State University, 119899, Moscow, Russia

Paulo Vargas Moniz‡,§
Astronomy Unit, School of Mathematical Sciences, University of London, Mile End Road, London E1 4NS, United Kingdom
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We investigate the Bianchi I cosmology with the homogeneous SU(2) Yang-Mills field governed by the non-Abelian Born-Infeld action. A similar system with the standard Einstein-Yang-Mills (EYM) action is known to exhibit chaotic behavior induced by the Yang-Mills field. When the action is replaced by the Born-Infeld–type non-Abelian action (NBI), the chaos-order transition is observed in the high-energy region. This is interpreted as a smothering effect due to (nonperturbative in α') string corrections to the classical EYM action. We give numerical evidence for the chaos-order transition and present an analytical proof of regularity of color oscillations in the limit of strong Born-Infeld nonlinearity. We also perform a general analysis of the Bianchi I NBI cosmology and derive an exact solution in the case of only the U(1) component excited. Our new exact solution generalizes the Rosen solution of the Bianchi I Einstein-Maxwell cosmology to the U(1) Einstein-Born-Infeld theory.

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I. INTRODUCTION

One of the key questions in theoretical cosmology is whether the space-time metric near the singularity is regular or chaotic. As was shown by Belinskii, Khalatnikov, and Lifshitz (BKL), the generic solution of the four-dimensional vacuum Einstein equations exhibits an oscillating behavior [1] which was later qualified as essentially chaotic (see [2] and references therein). Recently the issue of chaos in the early universe received renewed attention due to discovery that the multidimensional cosmologies with antisymmetric form fields are generically chaotic [3–7]. Namely, it was shown that the general solution near a spacelike singularity of the Einstein-dilaton-p-form field equations exhibits an oscillatory behavior of the BKL type. However the issue of chaos in superstring cosmology is still far from being clear. On the one hand, the previous considerations are based on the lowest in the α' level of the string theory. To go beyond this approximation in the closed string theories is difficult, since no exact in α' effective action is known. The lowest order corresponds to the supergravity approach employed in Refs. [3–7]. On the other hand, the presence of, e.g., quadratic curvature corrections obtained perturbatively can be of relevance. In particular, they may imply a damping of the BKL oscillations and corresponding chaotic behavior (see Refs. [8,9] for an analysis in a nonstring context). It is interesting to point out that similar effects were recently identified within a brane-world scenario [10].

An alternative route to investigate the issue of chaos in the string cosmology may, nevertheless, be considered. In more precise terms, we can employ the framework of open strings (in the bulk or ending on D-branes), where an exact in α' effective action is known: the Dirac-Born-Infeld action [11–13]. This allows one to study the role of nonperturbative string corrections beyond the lowest in α' level in string theory, with a view towards the analysis of the space-time singularity and possibly its corresponding chaotic behavior. We choose the cosmological model with a Yang-Mills (YM) matter source which exhibits chaotic behavior in the lowest α' approximation and explore whether chaos persists in the Born-Infeld case.

In fact, classical YM fields governed by the ordinary quadratic action exhibit chaotic behavior in various situations [14,15]. The simplest case is that of the homogeneous YM fields depending only on time in the flat space-time [16–18]: when only two YM components are excited, the problem is reduced to the well-known two-dimensional hyperbolic system $H = (p_x^2 + p_y^2 + x^2y^2)/2$ which is chaotic. Furthermore, in the lattice simulations of the inhomogeneous YM system, one observes the energy flow from the infrared to the ultraviolet region [19]. Therefore, it is believed that the chaotic behavior is typical for the purely classical YM equations, one of the arguments being the absence of solitons in this theory. In addition, it is known that adding the Higgs field to the YM theory leads to stabilization of chaos in the homogeneous systems. In this case the hyperbolic model is replaced by the system of coupled harmonic oscillators which is regular in the weak coupling regime.

When gravity is switched on, generically one observes a smoothing of the chaotic behavior. Still, the YM chaos

*Electronic address: vv.dyadichev@pisem.net
†Electronic address: galtsov@mail.phys.msu.ru
‡Electronic address: p.moniz@qmul.ac.uk
URL: http://webx.ubi.pt/~pmoniz
§On leave of absence from Departmento de Fisica, UBI, Covilha, Portugal.
Also at Centra-IST, Lisboa, Portugal.

unambiguously persists in some homogeneous cosmological models, such as an axisymmetric Bianchi I [20–22]. An interesting behavior of the Einstein-Yang-Mills (EYM) system was found inside the spherical black holes (which corresponds to the Kantowski-Sachs cosmology): the YM field has violent oscillating behavior near the singularity of the EYM black holes [23], though the oscillations are not chaotic. These oscillations are fully damped once the quadratic action is replaced by the Born-Infeld (BI) action [24].

A non-Abelian generalization of the Born-Infeld action was discussed phenomenologically long ago [25]; in the context of string theory it was suggested in Ref. [26]. Although to derive a closed form of the action from the string theory is problematical, there are several heuristic considerations which can be used to fix its form. Here we choose the simplest suggestion as a model which can be analyzed in some detail. In principle there could be two ways of introducing the Born-Infeld dynamics: in terms of the traditional open string theory in the bulk and its subsequent compactification, or within the brane-world scenario. Here we rather use the first, again aiming to construct the simplest model accounting for the Born-Infeld nonlinearity of the gauge field dynamics.

Some previous work on cosmological models with vector fields governed by the BI action includes the U(1) matter [27]. Such models are necessarily anisotropic or inhomogeneous since there is no homogeneous and isotropic configuration of the classical U(1) field. On the contrary, non-Abelian Born-Infeld cosmologies can be homogeneous and isotropic; these were recently investigated for the Friedmann-Robertson-Walker (FRW) models [28–31]. A brane-world generalization was also considered [32].

Here we study an anisotropic axially symmetric Bianchi I cosmology with the SU(2) Yang-Mills field governed by the non-Abelian Born-Infeld action. The Born-Infeld effect on the flat-space dynamics of the homogeneous axisymmetric YM field was shown to provide a chaos-order transition [33], so it can be expected that in the gravity coupled case this effect will be even more pronounced.

The paper is organized as follows. In Sec. II we give basic definitions and present the full set of the equations of motion. In Sec. III the scaling symmetries are identified which allow one to reduce the order of the system expressing it as two first order and two second order equations subject to a constraint. The low energy Yang-Mills limit is discussed in Sec. IV. Then (Sec. V) we consider the case of only the U(1) field component excited and derive an exact solution generalizing the Rosen solution of the Einstein-Maxwell system to the Born-Infeld dynamics. The following (Sec. VI) is devoted to the discussion of the structure of singularity in the general case. In Sec. VII we derive two additional constraint equations which hold in the high-energy limit and show that the generic solution is this regime is nonchaotic. In Sec. VIII the numerical results are presented followed by a discussion in Sec. IX.

II. GENERAL SETTING

As was discussed recently [13,25,26,34], the definition of the non-Abelian Born-Infeld (NBI) action is ambiguous. One can start with the U(1) BI action presented either in the determinant form,

$$S = \frac{1}{16\pi}\int\sqrt{-\det(g_{\mu\nu} - \beta^{-1}F_{\mu\nu})}d^4x, \tag{1}$$

or in the equivalent (in four space-time dimensions) "-square-root" form,

$$S = \frac{1}{16\pi}\int\sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{2\beta^2} - \frac{(\tilde{F}_{\mu\nu}F^{\mu\nu})^2}{16\beta^4}}\sqrt{-g}d^4x. \tag{2}$$

In the non-Abelian case $F_{\mu\nu}$ is matrix valued, and the trace over gauge matrices must be further specified. One particular definition—a symmetrized trace—is due to Tseytlin [26]. It prescribes a symmetrization of all products of $F_{\mu\nu}$ in the power expansion of the determinant (1) before the trace is taken. Inside the symmetrized series expansion the gauge generators effectively commute, so both the determinant (1) and the square-root (2) forms are equivalent. This property does not hold for other trace prescriptions, e.g., for an ordinary trace. In the latter case it is common to apply the trace to the square-root form (2). Note that string theory seems to require the symmetrized trace in the lower orders of the perturbation theory [13,26,35], while higher order corrections seem to violate this prescription [36–39]. Here we choose the "square-root/ordinary trace" Lagrangian just for its simplicity. It is worth noting that in the static case discussed recently both in the ordinary [40] and the symmetrized trace [24] versions, qualitative features of the solutions turn out to be similar. Thus we choose the action of the Einstein-NBI (ENBI) system in the following form:

$$S = -\frac{1}{4\pi}\int\left\{\frac{1}{4G}R + \beta^2(\mathcal{R} - 1)\right\}\sqrt{-g}d^4x, \tag{3}$$

where R is the scalar curvature, β is the BI critical field strength and $\mathcal{R}$ is the Born-Infeld square root,

$$\mathcal{R} = \sqrt{1 + \frac{1}{2\beta^2}F^a_{\mu\nu}F_a^{\mu\nu} - \frac{1}{16\beta^4}(\tilde{F}^a_{\mu\nu}F_a^{\mu\nu})^2}. \tag{4}$$

The limit $\beta \to \infty$ corresponds to the standard EYM theory with the action

$$S = -\frac{1}{4\pi}\int\left(\frac{1}{4G}R + \frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu}\right)\sqrt{-g}d^4x. \tag{5}$$

We consider an axially symmetric Bianchi I space-time described by the line element,

$$ds^2 = N^2dt^2 - b^2(dx^2 + dy^2) - c^2dz^2, \tag{6}$$

where functions N, b, and c depend on time t. In the YM case this problem was studied previously by Darian and Kunzle [20,21] and Barrow and Levin [22]. The gauge field compatible with the space-time symmetry is parametrized by two functions u, v of time

$$A = T_1 u dx + T_2 u dy + T_3 v dz, \tag{7}$$

where SU(2) generators are normalized according to $[T_1, T_2] = iT_3$. The corresponding field strength matrix-valued twoform is

$$F = \dot{u}(T_1 dt \wedge dx + T_2 dt \wedge dy) + \dot{v}T_3 dt \wedge dz + u^2 T_3 dx \wedge dy + uv(T_2 dz \wedge dx + T_1 dy \wedge dz). \tag{8}$$

Integrating over the 3-space, we obtain the following one-dimensional Lagrangian:

$$L = -\frac{1}{2}\frac{\dot{b}(\dot{b}c + 2\dot{c}b)}{GN} - \beta^2 N b^2 c(\mathcal{R} - 1), \tag{9}$$

where now

$$\mathcal{R} = \sqrt{1 - \frac{\mathcal{F}}{\beta^2} - \frac{\mathcal{G}^2}{\beta^4}}, \tag{10}$$

$$\mathcal{F} = \frac{2\dot{u}^2}{N^2b^2} + \frac{\dot{v}^2}{N^2c^2} - \frac{1}{b^2}\left(\frac{2u^2v^2}{c^2} + \frac{u^4}{b^2}\right), \tag{11}$$

$$\mathcal{G} = \frac{u(2\dot{u}v + \dot{v}u)}{Nb^2c}. \tag{12}$$

The quantity $\mathcal{F}$ is the YM Lagrangian, and it is convenient to present it as a difference of kinetic and potential terms

$$\mathcal{F} = T - U, \tag{13}$$

$$T = \frac{2\dot{u}^2}{N^2b^2} + \frac{\dot{v}^2}{N^2c^2}, \tag{14}$$

$$U = \left(\frac{2u^2v^2}{b^2c^2} + \frac{u^4}{b^4}\right). \tag{15}$$

Note that from two coupling parameters entering the action, G and β, one can be eliminated by an appropriate rescaling. In what follows we set $G = 1$.

The Einstein equations can be derived by variation of the one-dimensional action over N, b, c. Variation over N gives the Hamiltonian constraint

$$\mathcal{H} = \frac{\partial L}{\partial N} = 0, \tag{16}$$

where $\mathcal{H}$ reads in the synchronous gauge $N = 1$:

$$\mathcal{H} = \frac{1}{2}\dot{b}(\dot{b}c + 2\dot{c}b) + \frac{b^2c}{\mathcal{R}}[\beta^2(\mathcal{R} - 1) - U]. \tag{17}$$

Fixing this gauge from now on, we obtain the remaining Einstein equations:

$$\frac{\ddot{b}}{b} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} + \frac{\ddot{c}}{c} = 2\beta^2(\mathcal{R} - 1) + \frac{2}{\mathcal{R}}\left(\frac{\dot{u}^2}{b^2} - \frac{u^2v^2}{b^2c^2} - \frac{u^4}{b^4} + \frac{\mathcal{G}^2}{\beta^2}\right), \tag{18}$$

$$\frac{\ddot{b}}{b} + \frac{1}{2}\frac{\dot{b}^2}{b^2} = \beta^2(\mathcal{R} - 1) + \frac{1}{\mathcal{R}}\left(\frac{\dot{v}^2}{c^2} - 2\frac{u^2v^2}{b^2c^2} + \frac{\mathcal{G}^2}{\beta^2}\right). \tag{19}$$

The equations for the YM field can be presented in the following form:

$$\frac{\mathcal{R}}{c}\frac{d}{dt}\left[\frac{c}{\mathcal{R}}\left(\dot{u} + \frac{uv\mathcal{G}}{c\beta^2}\right)\right] = \frac{(\dot{u}v + \dot{v}u)\mathcal{G}}{c\beta^2} - \frac{u^3}{b^2} - \frac{uv^2}{c^2}, \tag{20}$$

$$c\mathcal{R}\frac{d}{dt}\left[\frac{b^2}{c\mathcal{R}}\left(\dot{v} + \frac{cuv\mathcal{G}}{\beta^2}\right)\right] = -2u^2v + \frac{2c\dot{u}u\mathcal{G}}{\beta^2}. \tag{21}$$

The energy-momentum tensor has the following non-vanishing components: the energy density

$$T_0^0 = \epsilon = \frac{\beta^2 + 2\Psi^2\Gamma^2 + \Psi^4}{4\pi\mathcal{R}} - \frac{\beta^2}{4\pi}, \tag{22}$$

the transversal pressure in the plane orthogonal to the symmetry axis

$$p_x = -T_x^x = -T_y^y = \frac{\Pi_\Gamma^2 + \Pi_\Psi^2 - \Gamma^2\Psi^2 - \beta^2}{4\pi\mathcal{R}} + \frac{\beta^2}{4\pi}, \tag{23}$$

and the longitudinal pressure

$$p_z = -T_z^z = \frac{2\Pi_\Psi - \Psi^4 - \beta^2}{4\pi\mathcal{R}} + \frac{\beta^2}{4\pi}. \tag{24}$$

Physical domain of the variables involved is bounded by the condition of reality of the square root $\mathcal{R}$ in the above formulas. Note that in Minkowski space this would mean the boundedness of the field strength from above by some value proportional to the critical field parameter β. In the curved space-time the corresponding boundary depends on the space-time metric variables and the condition of reality of $\mathcal{R}$ is more complicated. We will discuss the near-boundary behavior in detail in Sec. VII.

III. REDUCTION OF ORDER

The above system of equations looks like a dynamical system of the eight order in the presence of a constraint. However, it possesses additional scaling symmetries which can be used to reduce the system order by two (for the EYM action this possibility was noticed by Darian and Kunzle [20]). It is easy to check that under a scaling transformation

$$b \to \lambda b, \qquad c \to \lambda^{-2}c, \qquad u \to \lambda u, \qquad v \to \lambda^{-2}v,$$

the Lagrangian remains invariant. Moreover, under a separate rescaling in the b, u sector

$$b \to \lambda b, \qquad u \to \lambda u, \tag{25}$$

the Lagrangian scales as λ^2, and under the transformation

$$c \to \lambda c, \qquad v \to \lambda v, \tag{26}$$

as λ. The corresponding reduction of the EYM system is achieved by an introduction of new variables invariant under the above rescalings. Following Barrow and Levin [22], whose notation we will adopt in what follows (note that in Ref. [22] another convention $8\pi G = 1$ is used), we introduce the volume and shear variables

$$a = (b^2 c)^{1/3}, \qquad \chi = \left(\frac{b}{c}\right)^{1/3}, \tag{27}$$

together with the associated Hubble parameters

$$H_a = \frac{\dot{a}}{a}, \qquad H_\chi = \frac{\dot{\chi}}{\chi}, \tag{28}$$

as well as the scaled Yang-Mills variables

$$\Psi = \frac{u}{b}, \qquad \Gamma = \frac{v}{c}. \tag{29}$$

It is also convenient to use the scaled derivatives

$$\Pi_\Psi = \frac{\dot{u}}{b}, \qquad \Pi_\Gamma = \frac{\dot{v}}{c}, \tag{30}$$

which are related to $\dot{\Psi}, \dot{\Gamma}$ via

$$\Pi_\Psi = \dot{\Psi} + (H_a + H_\chi)\Psi, \tag{31}$$

$$\Pi_\Gamma = \dot{\Gamma} + (H_a - 2H_\chi)\Gamma. \tag{32}$$

Note that these are not the momenta conjugate to Ψ, Γ, the corresponding canonical momenta being

$$P_\Psi = \frac{2a^3}{\mathcal{R}}\left(\Pi_\Psi + \frac{\Psi\Gamma\mathcal{G}}{\beta^2}\right), \tag{33}$$

$$P_\Gamma = \frac{a^3}{\mathcal{R}}\left(\Pi_\Gamma + \frac{\Psi^2\mathcal{G}}{\beta^2}\right). \tag{34}$$

In terms of the new variables the Hamiltonian constraint reads

$$\frac{3}{2}(H_a^2 - H_\chi^2) + \beta^2 - [\beta^2 + \Psi^2(\Psi^2 + 2\Gamma^2)]\mathcal{R}^{-1} = 0. \tag{35}$$

The functions T, U and $\mathcal{G}$ entering $\mathcal{R}$ now take the form

$$T = 2\Pi_\Psi^2 + \Pi_\Gamma^2, \tag{36}$$

$$U = \Psi^4 + 2\Psi^2\Gamma^2, \tag{37}$$

$$\mathcal{G} = \Psi(2\Pi_\Psi\Gamma + \Pi_\Gamma\Psi). \tag{38}$$

From the Einstein equations, one can derive two *first order* equations for the Hubble parameters, which are linear in derivatives. Taking the sum of Eqs. (18) and (19) and the constraint equation (17), one obtains the following simple equation for $\dot{H}_a$:

$$\dot{H}_a + 3H_\chi^2 + \frac{2}{3\mathcal{R}}(T + U) = 0. \tag{39}$$

Similarly, taking twice the second Einstein equation (19) and subtracting (18) we get

$$\dot{H}_\chi + 3H_\chi H_a - \frac{2}{3\mathcal{R}}(\Pi_\Gamma^2 - \Pi_\Psi^2 + \Psi^4 - \Psi^2\Gamma^2) = 0. \tag{40}$$

Thus the Einstein equations reduce to two first order equations in the presence of a constraint.

Alternatively, one can introduce the Hubble factors with respect to b and c:

$$H_b = \frac{\dot{b}}{b}, \qquad H_c = \frac{\dot{c}}{c}, \tag{41}$$

and bring the Einstein equations into the form

$$\dot{H}_b + H_b(H_b - H_c) = -\frac{2}{\mathcal{R}}(\Pi_\Psi^2 + \Psi^2\Gamma^2), \tag{42}$$

$$\dot{H}_c + H_c(H_b - H_c) = -\frac{2}{\mathcal{R}}(\Pi_\Gamma^2 + \Psi^4), \tag{43}$$

with the Hamiltonian constraint

$$\frac{1}{2}H_b(H_b + 2H_c) - \frac{1}{\mathcal{R}}[\beta^2(1 - \mathcal{R}) + U] = 0. \tag{44}$$

In addition, we have two second order equations for the YM fields which read in terms of the new variables

$$\left(\frac{d}{dt} + H_b + H_c\right)\left[\frac{1}{\mathcal{R}}\left(\Pi_\Psi + \frac{\mathcal{G}\Psi\Gamma}{\beta^2}\right)\right] + \frac{1}{\mathcal{R}}\left[\Psi^3 + \Psi\Gamma^2 - \frac{\mathcal{G}(\Pi_\Psi\Gamma + \Pi_\Gamma\Psi)}{\beta^2}\right] = 0, \tag{45}$$

$$\left(\frac{d}{dt} + 2H_b\right)\left[\frac{1}{\mathcal{R}}\left(\Pi_\Gamma + \frac{\mathcal{G}\Psi^2}{\beta^2}\right)\right] + \frac{2}{\mathcal{R}}\left[\Psi^2\Gamma + \Psi\Gamma^2 - \frac{\mathcal{G}\Pi_\Psi\Psi}{\beta^2}\right] = 0. \tag{46}$$

IV. YM LIMIT

In the YM limit $\beta \to \infty$, the square-root factor in the above formulas should be replaced according to the relation

$$\lim_{\beta\to\infty}\beta^2(\mathcal{R} - 1) = -\frac{1}{2}\mathcal{F}. \tag{47}$$

The main qualitative difference between EYM and ENBI theories lies in the fact that the standard YM action is scale invariant (though not the EYM one) contrary to the NBI case. This leads to a partial decoupling of the YM dynamics from that of the space-time. Given Eq. (47), the constraint equation simplifies to

$$\frac{1}{2}[3(H_a^2 - H_\chi^2) - (T + U)] = 0. \tag{48}$$

Combining this with (39), one finds that one of the Einstein equations fully decouples and reduces to the vacuum form:

$$\dot{H}_a + H_\chi^2 + 2H_a^2 = 0. \tag{49}$$

However, the shear remains coupled to matter and obeys the equation

$$\dot{H}_\chi + 3H_\chi H_a + H_a^2 - H_\chi^2 = \Pi_\Psi^2 + \Psi^4. \tag{50}$$

Finally, the YM field equations become

$$\dot{\Pi}_\Psi + (H_b + H_c)\Pi_\Psi + \Psi(\Psi^2 + \Gamma^2) = 0, \tag{51}$$

$$\dot{\Pi}_\Gamma + 2H_b\Pi_\Gamma + 2\Psi^2\Gamma = 0, \tag{52}$$

where the definitions (31) and (32) have to be used.

The Hamiltonian form of the EYM equations can be further simplified using an exponential parametrization of the volume and shear variables

$$a = \mathrm{e}^\alpha, \qquad \chi = \mathrm{e}^\gamma. \tag{53}$$

The canonical momenta conjugate to α, γ are

$$P_\alpha = -3\mathrm{e}^{3\alpha}\dot{\alpha}, \qquad P_\gamma = 3\mathrm{e}^{3\alpha}\dot{\gamma}, \tag{54}$$

while the YM momenta (33) and (34) simplify to

$$P_\Psi = 2a^3\Pi_\Psi, \qquad P_\Gamma = a^3\Pi_\Gamma. \tag{55}$$

The Hamiltonian constraint (17) for the EYM system in terms of the momentum variables reads

$$\mathcal{H} = \mathrm{e}^{-3\alpha}\left[\frac{1}{6}(P_\alpha^2 - P_\gamma^2) - \frac{1}{4}(P_\Psi^2 + 2P_\Gamma^2)\right] - \frac{U}{2} = 0, \tag{56}$$

where the potential is given by the Eq. (37).

V. U(1) CASE

Consider the special case when only the v-component of the YM field is excited, corresponding to the U(1) subgroup of the gauge group. The Einstein equations (42) and (43) reduce to

$$\dot{H}_b + H_b(H_b - H_c) = 0, \tag{57}$$

$$\dot{H}_c + H_c(H_b - H_c) = -\frac{2}{\mathcal{R}}\Pi_\Gamma^2, \tag{58}$$

and the Hamiltonian constraint is

$$H_b(H_b + 2H_c) = 2\beta^2(\mathcal{R}^{-1} - 1). \tag{59}$$

Integrating the BI field equation

$$\frac{d}{dt}\left(\frac{b^2\Pi_\Gamma}{\mathcal{R}}\right) = 0, \tag{60}$$

one obtains

$$\frac{b^2\Pi_\Gamma}{\mathcal{R}} = 2b_0, \tag{61}$$

where b_0 is an integration constant, so that

$$\mathcal{R} = \sqrt{1 - \frac{\Pi_\Gamma^2}{\beta^2}} = \frac{1}{\sqrt{1 + x^2}}, \qquad x = \frac{2b_0}{\beta b^2}. \tag{62}$$

It is easy to see that the Einstein equation (57) is equivalent to

$$\frac{\ddot{b}}{\dot{b}} = \frac{\dot{c}}{c}, \tag{63}$$

which immediately gives a relation

$$\dot{b} = kc, \tag{64}$$

where k is a second integration constant. Now the constraint equation becomes the following separated equation for the function $b(t)$:

$$\dot{H}_b + \frac{3}{2}H_b^2 = \beta^2(\sqrt{1 + x^2} - 1), \tag{65}$$

while the second Einstein equation (58) is its time derivative. The right-hand side of this equation is positively definite. It follows that the system has no bounces. Indeed, if $H_b = 0$, from Eq. (58) it follows that $\dot{H}_b = 0$, which contradicts Eq. (65).

We can solve Eq. (65) considering instead of $b(t)$ an inverse function $t(b)$. Then

$$H_b = \frac{1}{bt'}, \tag{66}$$

where $t' = dt/db$. The equation for $t(b)$ following from (65) reads

$$\left(\frac{1}{t'}\right)^2\left(\frac{t''}{t'} - \frac{1}{2b}\right) = b\beta^2\left(1 - \sqrt{1 + \frac{4b_0^2}{\beta^2 b^4}}\right). \tag{67}$$

This is the linear first order equation for the function

$$z(b) = (1/t')^2, \tag{68}$$

namely,

$$z' + \frac{z}{b} + 2b\beta^2\left(1 - \sqrt{1 + \frac{4b_0^2}{\beta^2 b^4}}\right) = 0. \tag{69}$$

Its solution reads

$$z = \frac{2\beta^2}{b}\int\left(\sqrt{1 + \frac{4b_0^2}{\beta^2 b^4}} - 1\right)b^2 db + \frac{b_1}{b}, \tag{70}$$

where b_1 is a third integration constant. An integration can be done in terms of the hypergeometric function [41]:

$$z = \frac{2\beta^2}{3}\sqrt{b^4 + \frac{4b_0^2}{\beta^2}} - \frac{2\beta^2 b^2}{3} + \frac{8\beta b b_0}{3} F\left(\frac{1}{3}, \frac{3}{4}; \frac{5}{4}; \frac{1}{1+x^2}\right) + \frac{\tilde{b}_1}{b}, \qquad (71)$$

where $\tilde{b}_1 \neq b_1$ is another constant. Now, according to (68), the inverse function to the required solution is given by the integral

$$t(b) = \int \frac{db}{\sqrt{z(b)}} + t_0, \qquad (72)$$

where t_0 is the last integration constant in this process. Our solution generalizes the Rosen solution [42] to the Einstein-Born-Infeld theory.

Near the singularity $z \approx b_1/b$, so one has

$$H_b = \frac{\sqrt{b_1}}{b^{3/2}}. \qquad (73)$$

Integrating Eq. (72) one obtains

$$b = (b_1 t)^{2/3}, \qquad (74)$$

and then from Eq. (64)

$$c = \frac{2b_1^{2/3}}{3k} t^{-1/3}. \qquad (75)$$

Hence, we obtain a cigar singularity.

In the Maxwell case the situation is different. Indeed, in the limit $\beta \to \infty$ one has

$$z = -\frac{4b_0^2}{b^2} + \frac{b_1}{b}. \qquad (76)$$

Since z should remain positive, the region of b is limited by

$$b > b_{\min} = \frac{4b_0^2}{b_1}. \qquad (77)$$

Combining Eqs. (64) and (72) we obtain

$$b = b_{\min} + \frac{b_1 t^2}{4b_{\min}}, \qquad c = \frac{b_1 t}{2kb_{\min}}. \qquad (78)$$

This is a pancake singularity. Thus, the BI nonlinearity modifies the singularity from the pancake to the cigar type.

VI. SINGULARITY STRUCTURE

Consider now the general solution near the cosmological singularity. It turns out that, except for a special isotropic solution $b = c = a$, previously studied in [28–31], generic solutions have the same metric singularities as the vacuum Bianchi I solutions. Near the pancake singularity the solution is not analytic in terms of t, but it can be series expanded in terms of $t^{1/3}$. More precisely, one finds the following Laurent expansion containing four free parameters p, q, r, s:

$$H_a = \frac{1}{3t} - \frac{2rq + sp}{9pr} t^{-2/3} + O(t^{-1/3}), \qquad (79)$$

$$H_\chi = \frac{1}{3t} + \left(\frac{ps - qr}{9pr} - \frac{p}{\sqrt{2}}\right) t^{-2/3} + O(t^{-1/3}), \qquad (80)$$

$$\Psi = pt^{-2/3} + qt^{-1/3} + O(1), \qquad (81)$$

$$\Gamma = rt^{1/3} + st^{2/3} + O(t). \qquad (82)$$

The Γ-component of the YM field vanishes at $t = 0$, while Ψ is singular. The scale factor a and the shear χ near the pancake singularity both behave as $O(t^{1/3})$.

Near the cigar singularity the solution has a Laurent expansion in terms of t:

$$H_a = \frac{1}{3t} + \frac{4\bar{r}^2\bar{p}^2 - \bar{s}^2}{3\mathcal{R}_1} + O(t), \qquad (83)$$

$$H_\chi = -\frac{1}{3t} + \frac{2\bar{r}^2\bar{p}^2 + \bar{s}^2}{3\mathcal{R}_1} + O(t), \qquad (84)$$

$$\Psi = \bar{p} + \left(\bar{q} - \frac{2\bar{r}^2\bar{p}^3}{\mathcal{R}_1}\right) t + O(t^2), \qquad (85)$$

$$\Gamma = \bar{r}t^{-1} + \bar{s} + \frac{\bar{r}\bar{s}^2}{\mathcal{R}_1} + O(t), \qquad (86)$$

where the quantity $\mathcal{R}_1$ is the leading term in an expansion of the NBI square root:

$$\mathcal{R} = \mathcal{R}_1 t^{-1} + O(1), \quad \mathcal{R}_1 = \sqrt{\frac{2\bar{r}^2\bar{p}^2 - \bar{s}^2}{\beta^2} - \frac{\bar{p}^2(\bar{p}\,\bar{s} + 2\bar{r}\,\bar{q})^2}{\beta^4}}. \qquad (87)$$

The scale factor and the shear have the following expansions:

$$a = a_1\left(t^{1/3} + \frac{4\bar{r}^2\bar{p}^2 - \bar{s}^2}{3\mathcal{R}_1} t^{4/3} + O(t^{7/3})\right), \qquad (88)$$

$$\chi = \chi_1\left(t^{-1/3} + \frac{2\bar{r}^2\bar{p}^2 + \bar{s}^2}{3\mathcal{R}_1} t^{2/3} + O(t^{5/3})\right). \qquad (89)$$

The quantities p, q, r, s ($\bar{p}, \bar{q}, \bar{r}, \bar{s}$) are independent free parameters which, together with an arbitrariness associated with the time shift, provide five free constants needed to specify the generic solution for both singularity types.

VII. SOLUTION IN THE LIMIT $\beta = 0$

In order to better understand the effect of the BI nonlinearities on the gauge field dynamics, let us first study the strong field limit $F \gg \beta$, or, formally, $\beta \to 0$. The leading term in the square root (10) containing the pseudoscalar invariant $\mathcal{G}$ is negative definite. Therefore, the reality of the square root $\mathcal{R}$ in the limit $\beta \to 0$ may be ensured only if $\mathcal{G}$ tends to zero in this limit, in which case the limiting behavior must satisfy the constraint

$$\Psi\Pi_\Gamma + 2\Gamma\Pi_\Psi = 0. \tag{90}$$

One can show that this condition is compatible indeed with the equations of motion as $\beta \to 0$.

Given the condition (90), the square-root term will read

$$\mathcal{R} = \frac{\sqrt{\Psi^4 + 2\Gamma^2\Psi^2 - \Pi_\Gamma^2 - 2\Pi_\Psi^2}}{\beta}. \tag{91}$$

The matter terms in the Einstein equations (39) and (40) then tend to zero, so the gravitational degrees of freedom decouple

$$\dot{H}_a = -3H_\chi^2, \qquad \dot{H}_\chi = -3H_\chi H_a, \tag{92}$$

and the gravitational constraint tends to the vacuum form

$$H_a^2 - H_\chi^2 = 0. \tag{93}$$

The solutions of these equations describe the vacuum Kasner metrics either of the cigar type

$$H_a = H_\chi = \frac{1}{3t}, \tag{94}$$

or the pancake type

$$H_a = -H_\chi = \frac{1}{3t}, \tag{95}$$

where we set the singularity at $t = 0$.

Substituting the explicit expressions for the Hubble and shear parameters into the gauge field equations, one finds the second constraint involving the gauge variables,

$$\frac{\Psi^2\Gamma}{H_a} = C = \text{const}, \tag{96}$$

and thus in the remaining equations one can express the YM field either in terms of Γ, Π_Γ, or in terms of Ψ, Π_Ψ. One simple consequence of this constraint is that in the nontrivial case $C \neq 0$ the variables Ψ and Γ cannot have zeros outside the singularity, and thus should preserve their signs. From the NBI field equations, one then finds

$$\Pi_\Psi = \dot{\Psi}, \qquad \text{cigar}, \tag{97}$$

$$\Pi_\Psi = \dot{\Psi} - \frac{2\Psi}{3t}, \qquad \text{pancake}. \tag{98}$$

In both cases the dynamical equation for Ψ will be of the form

$$\ddot{\Psi} = f(\Psi, \dot{\Psi}, C, t) \tag{99}$$

with some function f. It describes oscillations with a decreasing amplitude. The second YM variable Γ is related to Ψ algebraically via constraints (90) and (96) and therefore oscillates with the same frequency exactly in a counterphase. Oscillations are fully *regular*, so no YM chaos can persist in the regime of the strong BI nonlinearity.

The general solution near the pancake singularity can be expanded with respect to the variable $\tau \equiv t^{1/3}$:

$$\Gamma = p_1^2\tau + \sqrt{6C}p_1\tau^2 + \frac{3C}{2}\tau^3 + q_1\tau^4 + O(x^5), \tag{100}$$

$$\Psi = \frac{\sqrt{C}}{\sqrt{3}p_1}\tau^{-2} + \frac{C}{\sqrt{2}p_1^2}\tau^{-1} + \frac{\sqrt{3}C^{3/2}}{2p_1^3} + O(\tau), \tag{101}$$

where p_1 and q_1 are free parameters.

Near the cigar singularity the solution can be expanded in terms of t:

$$\Gamma = p_1 t^{-1} + q_1 + \frac{3q_1^2 - p_1 C}{6p_1}t + O(t^2), \tag{102}$$

$$\Psi = \frac{\sqrt{C}}{\sqrt{3p_1}} - \frac{\sqrt{C}q_1}{2\sqrt{3}p_1^{3/2}}t + O(t^2). \tag{103}$$

VIII. CHAOS-ORDER TRANSITION

Now we address the problem numerically. Various methods were suggested to study a chaotic behavior in the context of gravity, where the absence of the canonical time variable prevents straightforward use of such convenient tools as the Lyapunov exponents (however, see [43]). In the case of the conformally invariant YM Lagrangian, one can use the approach of Ref. [22] to separate the dynamics of the YM field from the gravitational expansion and then apply the invariant technique of chaotic scattering showing that the set of all periodic orbits has fractal structure invariant under coordinate reparametrizations.

Here we did not intend to investigate the chaotic behavior in the Bianchi I EYM cosmology which was studied in detail in the past, but concentrated on the existence of the regular regime for the gravity coupled non-Abelian BI system in the high-energy limit. Note that such methods as Poincaré maps or fractal structure are not convenient tools to reveal the chaos-order transition in our case contrary to the flat space case [33]. The reason is simply that under the full time scale of the cosmological evolution any initial behavior ends up with the YM chaos at late time when the dynamics automatically enters the regime governed by the usual YM Lagrangian. For β of the order of unity or greater, the YM variables perform only a small number of oscillations during the epoch of high field intensity, the field then being diluted fast. However, if we set the parameter β sufficiently small, the time spent by the system in the highly nonlinear region will be large enough, and in this case the chaos-order transition manifests itself unambiguously. Therefore our strategy consisted first in establishing the regularity of the system in the high-energy limit formally equivalent to $\beta \to 0$; this was done in the previous section. Numerical calculations were intended to show that this limiting regime is realized indeed for a certain period of time after which the usual YM regime is inevitably reached with the corresponding chaotic behavior.

A numerical integration of the system for small values of β reveals the following. While the gauge field strength is considerably greater than the critical field β (being of course restricted by the reality of $\mathcal{R}$), both conditions (90) and (96) hold approximately, and the dynamics of the gauge field qualitatively coincides with that discussed in the previous section in the formal limit $\beta = 0$. Both variables Ψ and Γ perform nonlinear oscillations with decreasing amplitude in a counterphase with respect to each other and without crossing zero. In this region the YM oscillations are fully *regular*, while the evolution of the metric is governed mostly by curvature terms. The metric is close to the vacuum Kasner solution.

To test the validity of an approximate description of the system in terms of the limiting $\beta = 0$ solution, we checked the constraint equations (90) and (96) in the case of small, but finite β. The first constraint (90) is the condition of smallness of the pseudoscalar YM invariant $\mathcal{G}^2 \ll \beta^2 \mathcal{F}$. It was checked through the validity of the corresponding vacuum behavior of the metric. The second constraint has been directly checked numerically. Figure 1 illustrates the situation for the cigar solution with $\beta = 10^{-4}$. From this figure, one can see that the right-hand side of Eq. (96) evolves on time scales much larger than the period of oscillations of the gauge field variables. Therefore the approximate description given in the previous section corresponds well to numerical experiments for small but finite β. This regime is a regular evolution.

In the course of the overall volume expansion, the YM energy density falls down and contributions related to the BI nonlinearity relatively decrease and the system enters the standard YM regime. At the same time, the matter

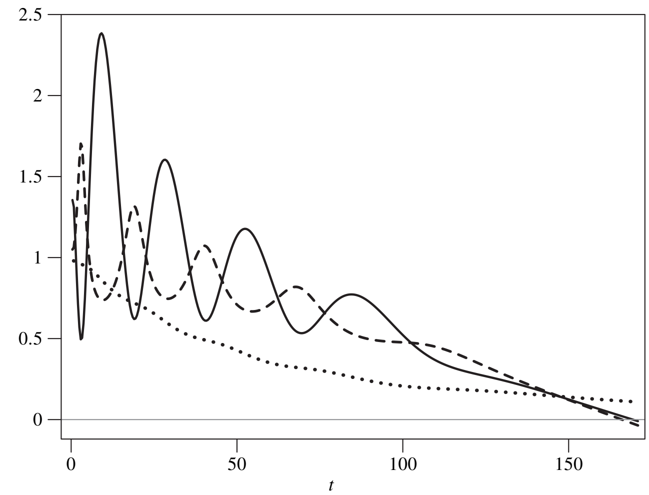

FIG. 2. The solution for the $\beta = 2 \times 10^{-3}$, regular phase, a cigar singularity. The solid line, $\Gamma t^{1/3}$; the dashed line, $\Psi t^{1/3}$; the dotted line, H_χ / H_a.

terms in the Einstein equation enter into play and tend, in particular, to reduce the shear anisotropy H_χ much faster than the Hubble parameter H_a. These features are illustrated in Figs. 2–4. Figure 2 shows the early regular evolution for $\beta = 2 \times 10^{-3}$ and the cigar-type singularity. Both variables Ψ and Γ oscillate in the positive region. The behavior of the shear anisotropy H_χ is smooth. Figure 3 demonstrates the same solution at later time. From this figure, one can notice the chaotic changes of the oscillation periods reflecting the transition to the chaotic phase. The function H_χ coupled to matter performs chaotic oscillations with decreasing (as compared to the Hubble parameter H_a) amplitude. The first zeros of the gauge functions Ψ and Γ serve as an approximate boundary separating the regular phase from the late chaotic region. Note that, for both types of singularity (pancake or cigar), the qualitative

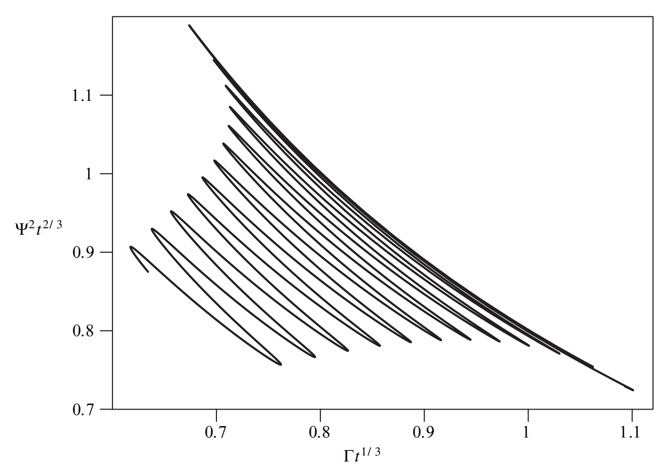

FIG. 1. The phase portrait $\Psi^2 t^{2/3}$ vs $\Gamma t^{1/3}$ for $\beta = 10^{-4}$ illustrating an approximate validity of the constraint 96.

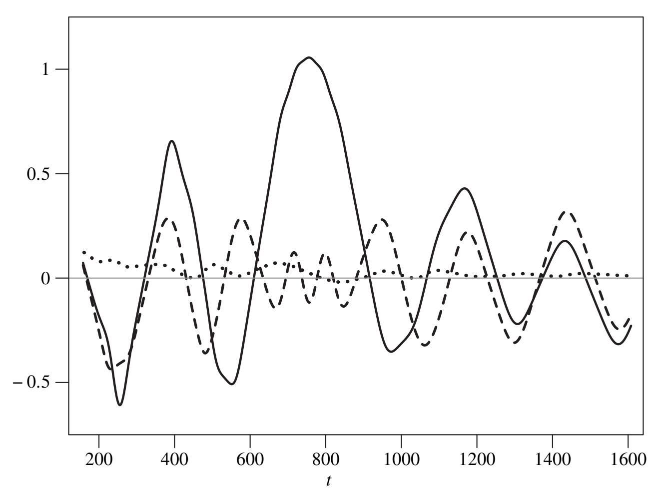

FIG. 3. Further development of the solution from Fig. 2 entering the chaotic regime.

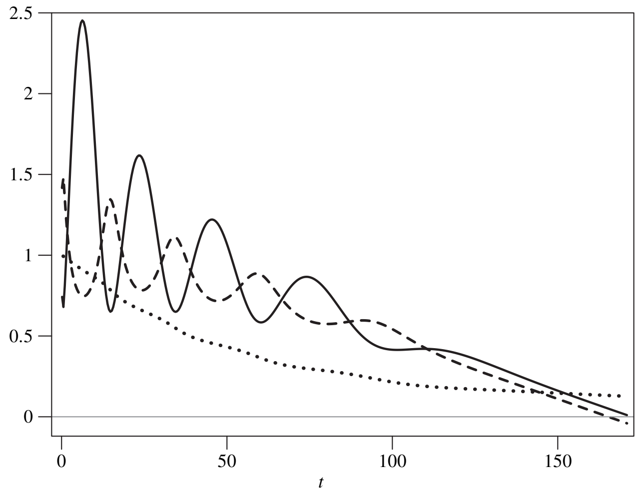

FIG. 4. Regular phase near the pancake singularity for $\beta = 2 \times 10^{-3}$. The solid line, $\Gamma t^{1/3}$; the dashed line, $\Psi t^{1/3}$; the dotted line, $-H_\chi/H_a$.

behavior of the solution is similar in both regular and chaotic phases except for the small vicinity of the singularity. This is illustrated in Fig. 4, which shows the solution with pancake singularity can be obtained from the solution shown in Fig. 2 by changing the sign of H_χ in the initial conditions (which were actually set at $t = 10$).

The Hubble parameter H_a does not exhibit chaotic behavior. It can be presented as $H_a = h(t)t^{-1}$, with some slowly varying smooth function $h(t)$. Numerical curves $h(t)$ are shown in Fig. 5 for various β. This function interpolates between the value $1/3$ at $t = 0$ (vacuum Kasner solution) and the value $1/2$ at $t = \infty$ corresponding to the isotropic "hot universe" cosmology. However, for small values of β, when the system stays in a highly nonlinear regime for a considerable time interval, there is a region where $h(t)$ is greater than $1/2$. This feature can be explained using the results of the FRW-BI model studied in [28–31]. Equation (35) implies that, once the contribution

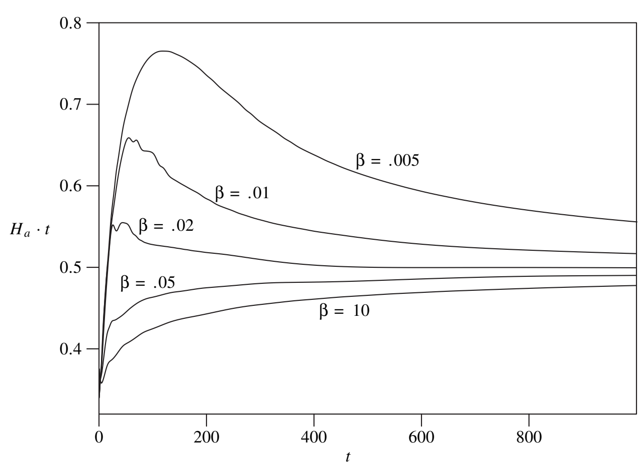

FIG. 5. Hubble function $h(t) = tH_a$ for various values of β.

of the anisotropy term H_χ decreases (i.e. the solution undergoes isotropization), the Hamiltonian constraint tends to that given by Friedmann equation. In the FRW case [28], one can derive an equation of state for the NBI matter which interpolates between that for conformal matter $\epsilon = 1/3p$ and the "string fluid" equation $\epsilon = -1/3p$ in the highly nonlinear regime. The latter corresponds to the value $h = 1$, which is, however, never achieved in the anisotropic case which we investigate here.

IX. DISCUSSION

The main goal of this paper was to test the nonperturbative effects of superstring theory on the issue of chaos in cosmology. It is worth noting that three different patterns of chaotic behavior in cosmology were identified. The first is the billiard-type behavior which is manifest in the Bianchi IX pure gravity and its supergravity (including multidimensional cases) generalizations. The second is chaos generated by the bouncing behavior of the metric as in the case of the FRW-scalar field cosmology. The third type is the matter-generated chaos which can also be observed in the corresponding flat space models. A typical example is given by the Bianchi I EYM cosmology (recently an interesting analysis was performed [44] of the YM field behavior in more general type A Bianchi spacetimes showing that basic features of the YM chaos persist there as well). From these three patterns the last one is the most appropriate for testing the superstring nonlocality effects accumulated in the BI non-Abelian action.

Our model does not pretend on playing a role in realistic cosmology, rather being a toy model for investigating the string nonlocality effects beyond the supergravity approximation. So we do not discuss the concrete origin of the YM fields and consider the simplest SU(2) configuration governed by the "square-root"-type non-Abelian Born-Infeld action. Our model does not consider the brane universe scenario either, but can be understood as a compactified open string cosmology of the traditional type. The corresponding brane-world generalization is simple to perform along the lines of [32] where the FRW case was studied. Furthermore, we do not pretend here to give an extensive discussion of the chaotic behavior of the corresponding EYM model (which can be found in the literature), but mostly concentrate on new features associated with the Born-Infeld type of dynamics. Therefore we do not present the Poincaré maps or the fractal structure patterns but demonstrate both analytically and numerically the existence of the region of regular motion in the high-energy limit.

Fortunately, in spite of the much more complicated nature of the Einstein-NBI dynamics as compared to the standard EYM dynamics, the system of equations still admits a reduction of order due to the presence of scaling symmetries similarly to the EYM case. Moreover, in the strong BI regime the axisymmetric Bianchi I NBI system can be reduced further due to existence of two additional asymptotic integrals of motion. This limit is characterized by the dynamical vanishing of the pseudoscalar quadratic invariant of the YM field. This simplifies dynamics considerably and leads to a decoupling of the gravitational degrees of freedom. Color oscillations are still governed by the BI nonlinearity and are reducible to the one-variable second order system predicting a perfectly regular behavior.

Numerical experiments show that the system behavior for sufficiently small β consists of a regular phase in the high-energy region near the singularity and the chaotic phase at later time. The regular phase is qualitatively similar to that described by the $\beta = 0$ approximate description. The chaos-order transition is observed when one is moving backward in time towards the singularity. The singularity itself is either of a cigar or a pancake type, as in the vacuum Bianchi I case, though the YM field does not tend to the vacuum configuration. Thus the nonperturbative in α' string corrections to the YM action suppress the YM chaos which takes place at lower energies where dynamics of the YM field is governed by the ordinary quadratic action.

Suppression of chaos in open string cosmology beyond the supergravity level gives rise to a further question: whether a purely gravitational dynamics, which is chaotic in the supergravity approximation [3], can be similarly modified by the nonperturbative string effects. The absence of the closed form effective actions for the closed strings makes this more difficult to investigate, though some partial results based on account for the higher curvature corrections indicate that this is likely to be so as well.

In the case of only an Abelian component excited, an exact analytic solution of the Einstein-BI system was found which generalizes the Rosen solution to the Einstein-Maxwell equations. It also exhibits a different behavior in the singularity as compared with the Einstein-Maxwell case.

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