

**Dynamics of a self-gravitating shell of matter**

Jerzy Kijowski

*Centrum Fizyki Teoretycznej PAN, Al. Lotników 32/46, 02-668 Warszawa, Poland*

Ewa Czuchry

*Katedra Metod Matematycznych Fizyki, ul. Hoża 74, 00-682 Warszawa, Poland*

(Received 5 July 2005; published 19 October 2005)

The dynamics of a self-gravitating shell of matter is derived from the Hilbert variational principle and then described as an (infinite-dimensional, constrained) Hamiltonian system. The method used here enables us to define a singular Riemann tensor of a noncontinuous connection via standard formulas of differential geometry, with derivatives understood in the sense of distributions. Bianchi identities for the singular curvature are proved. They match the conservation laws for the singular energy-momentum tensor of matter. The Rosenfeld-Belinfante and Noether theorems are proved to be valid still in the case of these singular objects. The assumption about the continuity of the four-dimensional space-time metric is widely discussed.

DOI: [10.1103/PhysRevD.72.084015](https://doi.org/10.1103/PhysRevD.72.084015)

PACS numbers: 04.20.Fy, 04.40.-b, 04.60.Ds

**I. INTRODUCTION**

In his seminal paper [1], Werner Israel considered the dynamics of a self-gravitating thin matter shell. The main purpose of his theory was to find a simple model describing gravitational collapse. In case of a realistic collapse, equations describing evolution of matter and gravity are extremely difficult to handle. Israel's idea was that many aspects of the collapse may be investigated within a toy model, which consists of a matter shell and the surrounding gravitational field. The dynamics of such a system reduces to a proper tailoring of the two different vacuum solutions describing the two sides of the shell.

We are going to present a systematic derivation of the Israel model from the variational principle and the construction of its canonical (Hamiltonian) structure. In our approach the space-time  $M$  consists of two parts, tailored together along a hypersurface  $S$ , which contains a moving matter shell. "Tailoring" means that the induced metric  $g_{ab}$  of  $S$  is continuous. On the other hand, the four-dimensional connection coefficients  $\Gamma_{\mu\nu}^\lambda$  may be discontinuous on  $S$ . It is proved that the singular part of the Einstein curvature tensor density of such a space-time contains derivatives of those discontinuities and may be defined in the sense of distributions as  $\mathcal{G}^a_b = \mathbf{G}^a_b \delta_S$ , where  $\delta_S$  is a Dirac-delta distribution concentrated on  $S$ . As we show in the sequel, the following relation holds:

$$\mathbf{G}^a_b = [Q^a_b], \quad (1.1)$$

where square brackets denote the jump of the extrinsic curvature across  $S$ , written in the ADM form. The singular Einstein tensor density matches the singular (concentrated on  $S$ ) energy-momentum tensor density of the matter shell. Distributional Gauss-Codazzi equations are then derived (and not *postulated*, as in the Israel approach). They imply that both the Einstein tensor density and the matter energy-momentum tensor density are conserved. We prove that the

conservation law  $\nabla_a \mathbf{G}^a_b = 0$  can be written in terms of the three-dimensional geometry of  $S$ .

The possibility of defining the singular Einstein tensor and calculating its divergence (in the sense of distributions) via the standard formulas of Riemannian geometry simplifies dramatically the calculational part of the theory. For this purpose we must assume that the four-dimensional metric is continuous across  $S$ . At this point our approach differs from the technics used by many authors, including W. Israel himself, who always stressed that no assumptions about the continuity of the four-metric, except for the continuity of the three-metric on the surface  $S$ , are necessary. This is, of course, true. We observe, however, that our "additional condition" on the continuity of the entire four-dimensional metric does not contain any geometrical or physical conditions imposed on configurations of the physical fields considered by us, but it is merely a gauge condition imposed on the coordinate systems used. Whenever an intrinsic three-dimensional metric of  $S$  is continuous, then also the remaining four components of the four-dimensional metric may become continuous after a suitable coordinate transformation. In this new coordinate system we can use our technics based on the distribution theory, but the dynamics derived this way is written in terms of intrinsic, geometric relations, having sense in an arbitrary system of reference. Hence, our derivation does not depend upon particular gauge conditions which we use. Note that even in a perfectly smooth, flat space-time, we can introduce a coordinate system in which only intrinsic three-dimensional metric on a fixed hypersurface  $S = \{x^3 = \text{const}\}$  is continuous, whereas the other four components of the four-dimensional metric  $g_{3\mu}$  may be discontinuous. Using these "singular coordinates" one can properly formulate e.g., the initial value problem for the Maxwell field, but nobody uses such a formulation for obvious reasons. Our additional assumption on the continuity of space-time metric is motivated by similar rea-

sons: not to complicate things which already are complicated enough. The technical flexibility we obtain this way, i.e., the possibility of defining the singular geometric objects living on  $S$  via standard geometric formulas, where the derivatives are understood in the sense of distribution, is the main profit of our approach.

The structure of the paper is as follows: At the beginning we derive Israel's model from the variational principle, first in the Lagrangian formulation, and then—after performing an appropriate Legendre transformation—in the Hamiltonian picture. Many technical elements of the construction presented here come from Refs. [2,3]. A significant simplification of the calculations is obtained using consequently tensor densities instead of tensors for such quantities like the matter energy-momentum and the Einstein curvature object (see also [4]). Part of the presented calculations was included in the Ph.D. thesis of one of us [5].

## II. PROPOSED DESCRIPTION OF ISRAEL'S MODEL

Consider a space-time consisting of two parts which are tailored together along a hypersurface  $S$ , whose nondegenerate metric has the signature  $(-, +, +)$ . Unlike in the original Israel approach (which also was used in [2]), here we restrict ourselves to coordinate systems for which all the components of the space-time metric are *continuous*. As was already mentioned, this condition does not limit the applicability of our formalism and may be treated as merely a “gauge condition” imposed on the coordinate system. It simplifies dramatically theoretical description of the model. Further simplification is obtained by using a coordinate system such that the hypersurface  $S$  is given by the equation  $\{x^3 = \text{const}\}$ . Metric derivatives along  $S$  (i.e.,  $\partial_a g_{\mu\nu}$ , where  $a = 1, 2, 3$ ) are continuous, whereas the transversal derivative (i.e.,  $\partial_3 g_{\mu\nu}$ ) may have jumps over  $S$ . We assume that the topology of  $S$  is of the type  $S^2 \times R^1$ , i.e., it describes a history of a matter concentrated on a two-dimensional surface with the topology of the sphere  $S^2$ .

The shell divides the space-time into the internal and external part with respect to the world tube  $S$ . In both parts vacuum Einstein equations may be derived in a standard way from the variational principle. Hence, the regular part of the Einstein tensor must vanish everywhere outside and inside of  $S$ . Only its singular part concentrated on  $S$  is left.

The singular part of the Riemann tensor is proportional to the (invariant) Dirac-delta distribution  $\delta_S$  concentrated on  $S$ , because first derivatives of the metric (and, whence, also the connection coefficients  $\Gamma_{\mu\nu}^\lambda$ ) may be discontinuous across  $S$ . In our particular coordinate system we have  $\delta_S \equiv \delta(x^3)$ . As will be seen in Sec. IV, the resulting singular part of the Einstein tensor will match the singular energy-momentum tensor describing matter concentrated on  $S$ . This singular part may be obtained from the standard formula for the Ricci tensor:

$$R_{\mu\nu} = \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_{(\mu} \Gamma_{\nu)\lambda}^\lambda + \Gamma_{\sigma\lambda}^\lambda \Gamma_{\mu\nu}^\sigma - \Gamma_{\mu\sigma}^\lambda \Gamma_{\nu\lambda}^\sigma. \quad (2.1)$$

A considerable simplification is obtained if we use the following combination of Christoffel symbols:

$$A_{\mu\nu}^\lambda := \Gamma_{\mu\nu}^\lambda - \delta_{(\mu}^\lambda \Gamma_{\nu)\kappa}^\kappa. \quad (2.2)$$

Then we have

$$R_{\mu\nu} = \partial_\lambda A_{\mu\nu}^\lambda - A_{\mu\sigma}^\lambda A_{\nu\lambda}^\sigma + \frac{1}{3} A_{\mu\lambda}^\lambda A_{\nu\sigma}^\sigma. \quad (2.3)$$

Because  $A$  may have only discontinuities of the “jump-type” across  $S$ , the derivatives of  $A$  along directions tangent to  $S$  are thus finite and belong to the regular part of the Ricci tensor. Hence, its singular part consists only of the transversal derivatives:

$$\text{sing}(R_{\mu\nu}) = \partial_3 A_{\mu\nu}^3 = \delta(x^3)[A_{\mu\nu}^3], \quad (2.4)$$

where the square brackets denote the jump of a specific quantity across  $S$ . Hence, we have the following formula for the singular part of the Einstein tensor density:

$$\text{sing}(\mathcal{G}_{\nu}^{\mu}) := \sqrt{|g|} \text{sing}\left(R^{\mu}_{\nu} - \frac{1}{2}R\right) = \delta(x^3)\mathbf{G}^{\mu}_{\nu}, \quad (2.5)$$

where

$$\mathbf{G}^{\mu}_{\nu} := \sqrt{|g|} \left( \delta^{\beta}_{\nu} g^{\mu\alpha} - \frac{1}{2} \delta^{\mu}_{\nu} g^{\alpha\beta} \right) [A^3_{\alpha\beta}] \quad (2.6)$$

is a quantity living on  $S$ . Now, we are going to show that it is actually a three-dimensional tensor density on  $S$ . For this purpose we first show that its components transversal to  $S$  vanish and, whence, its only nontrivial components are those tangential to  $S$ . To prove that it transforms like a tensor density on  $S$  let us observe that  $\mathcal{G}^{\mu}_{\nu}$  behaves like a four-dimensional density, which splits into the three-dimensional density on  $S$  and the one-dimensional density along  $x^3$ . But the Dirac-delta  $\delta(x^3)$  is already the density (and not a scalar) on the real axis  $x^3$  which proves that the remaining object  $\mathbf{G}$  behaves indeed like a 3-density. Hence, we have to prove the following:

*Lemma 1.*—

$$\mathbf{G}^{\perp}_{\nu} \equiv 0. \quad (2.7)$$

*Proof.*—On both sides of  $S$  consider the following combination of the connection coefficients:

$$\tilde{Q}^{\mu}_{\nu} := \sqrt{|g|} \left( g^{\mu\alpha} A^3_{\alpha\nu} - \frac{1}{2} \delta^{\mu}_{\nu} g^{\alpha\beta} A^3_{\alpha\beta} \right). \quad (2.8)$$

It is useful to encode the entire information about the metric in the following tensor density<sup>1</sup>

<sup>1</sup>In the so-called affine formulation of general relativity, proposed by one of us in 1978 (see Ref. [6]), this quantity plays role of the momentum canonically conjugate to connection  $\Gamma$ .

$$\pi^{\mu\nu} := \frac{1}{16\pi} \sqrt{|g|} g^{\mu\nu}. \quad (2.9)$$

Hence, we have

$$\frac{1}{16\pi} \tilde{Q}^\mu{}_\nu = \pi^{\mu\alpha} A^3_{\alpha\nu} - \frac{1}{2} \delta^\mu{}_\nu \pi^{\alpha\beta} A^3_{\alpha\beta}. \quad (2.10)$$

Then the Eq. (2.6) gives

$$\mathbf{G}^\mu{}_\nu := [\tilde{Q}^\mu{}_\nu]. \quad (2.11)$$

Now, we use the metricity condition for the connection  $\Gamma$ , which is fulfilled on both sides of  $S$  (indices  $a, b = 0, 1, 2$  label the coordinates on  $S$ ):

$$\begin{aligned} 0 &\equiv \nabla_a \pi^{33} = \partial_a \pi^{33} + 2\pi^{3\mu} \Gamma^3_{\mu a} - \pi^{33} \Gamma^{\mu}_{a\mu} \\ &= \partial_a \pi^{33} + 2\pi^{3\mu} A^3_{\mu a}, \end{aligned} \quad (2.12)$$

$$\begin{aligned} 0 &\equiv \nabla_a \pi^{3a} = \partial_a \pi^{3a} + \pi^{\mu a} \Gamma^3_{\mu a} + \pi^{3\mu} \Gamma^a_{\mu a} - \pi^{3a} \Gamma^{\mu}_{a\mu} \\ &= \partial_a \pi^{3a} + \pi^{ab} A^3_{ab} - \pi^{33} A^3_{33}, \end{aligned} \quad (2.13)$$

but the metric components  $\pi^{\mu\nu}$  and their derivatives along  $S$  are continuous across  $S$ . Hence, we obtain

$$\frac{1}{16\pi} \mathbf{G}^3{}_a = \pi^{3\mu} [A^3_{\mu a}] = -\frac{1}{2} [\partial_a \pi^{33}] = 0, \quad (2.14)$$

$$\frac{1}{16\pi} \mathbf{G}^3{}_3 = -\frac{1}{2} (\pi^{ab} [A^3_{ab}] - \pi^{33} [A^3_{33}]) = \frac{1}{2} [\partial_a \pi^{3a}] = 0, \quad (2.15)$$

which ends the proof because in our coordinate system we have  $\mathbf{G}^\perp{}_\nu = \mathbf{G}^3{}_\nu$ .  $\square$

### III. GEOMETRIC INTERPRETATION OF THE SINGULAR CURVATURE AND GENERALIZED BIANCHI IDENTITIES

Equation (2.11) expressing the singular part of Einstein tensor in terms of jumps of quantities  $\tilde{Q}^\mu{}_\nu$  across  $S$  is extremely useful in our derivation of the shell dynamics. However, is not satisfactory from the geometric viewpoint because quantities  $\tilde{Q}^\mu{}_\nu$  do not have any geometrical meaning on both sides of  $S$ , and only their jump (2.11) across  $S$  does. Now, we are going to prove that we obtain the same result replacing nongeometric object  $\tilde{Q}^\mu{}_\nu$  by a tensor density  $Q^\mu{}_\nu$  which is, by definition, orthogonal to  $S$  and whose restriction  $Q^a{}_b$  to  $S$  is equal to the external curvature of the hypersurface  $S$  in the ADM [7] representation. In our coordinate system  $S = \{x^3 = \text{const}\}$ , external curvature of  $S$  is given by

$$L_{ab} := -\frac{1}{\sqrt{g^{33}}} \Gamma^3_{ab} = -\frac{1}{\sqrt{g^{33}}} A^3_{ab}, \quad (3.1)$$

and its ADM version equals (cf. [3,8]):

$$Q^{ab} := \sqrt{|\det g_{cd}|} (L \hat{g}^{ab} - L^{ab}). \quad (3.2)$$

Here,  $\hat{g}^{ab}$  is a three-dimensional inverse of the induced metric  $g_{ab}$  on  $S$  and  $L^{ab} = \hat{g}^{ac} L_{cd} \hat{g}^{db}$ . It is easy to check that  $\hat{g}^{ab}$  may be calculated in terms of the four-dimensional inverse metric  $g^{\mu\nu}$  via the following formula (see [8]):

$$\hat{g}^{ab} = g^{ab} - \frac{g^{3a} g^{3b}}{g^{33}}. \quad (3.3)$$

In order to express coefficients  $\tilde{Q}^\mu{}_\nu$  in terms of the tensor density  $Q^\mu{}_\nu$ , observe that identities (2.12) and (2.13) can be solved algebraically with respect to  $A^3_{33}$  and  $A^3_{3a}$ . As a result, we have on both sides of the hypersurface  $S$  the following identities:

$$A^3_{33} = \frac{1}{\pi^{33}} (\partial_a \pi^{3a} + A^3_{ab} \pi^{ab}), \quad (3.4)$$

$$A^3_{3a} = -\frac{1}{2\pi^{33}} (\partial_a \pi^{33} + 2A^3_{ab} \pi^{3b}). \quad (3.5)$$

Hence, all the coefficients  $A^3_{\mu\nu}$  can be expressed in terms of  $A^3_{ab}$ , i.e., using (3.1) and (3.2), in terms of  $Q^{ab}$  and the metric. Using again formula (3.3) we obtain:

$$\begin{aligned} \tilde{Q}^a{}_b &= Q^a{}_b + 16\pi \left\{ -\frac{1}{2} \frac{\pi^{3a}}{\pi^{33}} \pi^{33}{}_{,b} \right. \\ &\quad \left. + \frac{1}{2} \delta^a_b \left( \frac{\pi^{3c}}{\pi^{33}} \pi^{33}{}_{,c} - \pi^{3c}{}_{,c} \right) \right\}. \end{aligned} \quad (3.6)$$

The last term is identical on both sides of  $S$ , because the metric components  $\pi^{\mu\nu}$  are continuous. Hence, their jump across  $S$  vanishes and, due to (2.11), we obtain:

$$\mathbf{G}^a{}_b := [Q^a{}_b], \quad (3.7)$$

whereas the transversal components  $\mathbf{G}^\perp{}_b$  vanish. Because the object  $Q^a{}_b$  is a well-defined tensor density on both sides of  $S$ , (its definition does not depend upon the coordinate system used), the tensorial character of the three-dimensional object  $\mathbf{G}^a{}_b$  has been proved.

Now, we are going to show that the total Einstein tensor  $\mathcal{G}^\mu{}_\nu$  of our space-time  $M$  fulfills Bianchi identities. Because regular part of  $\mathcal{G}^\mu{}_\nu$  is discontinuous across  $S$  and, moreover, it contains also a Dirac-delta-like singular part, these identities must be understood in a distributional sense. To prove them we shall use Gauss-Codazzi equations, relating transversal components  $\mathcal{G}^\perp{}_b$  of the Einstein tensor density  $\mathcal{G}^\mu{}_\nu$  to a divergence of external curvature  $Q$  on  $S$ :

$$\mathcal{G}^\perp{}_b + \bar{\nabla}_a Q^a{}_b \equiv 0, \quad (3.8)$$

where  $\bar{\nabla}$  denotes the intrinsic, three-dimensional covariant derivative on  $S$ . But the transversal component  $\mathcal{G}^\perp{}_b$  is a well-defined three-dimensional object on  $S$ . In our coordinate system, adapted to  $S$  in such a way that the coordinate  $x^3$  is constant on  $S$ , this quantity is simply equal to a ‘‘third’’ component:  $\mathcal{G}^\perp{}_b = \mathcal{G}^3{}_b$ . Taking a jump of this equation across  $S$  we obtain the following identity:

$$[\text{reg}(\mathcal{G})^\perp_b] + \bar{\nabla}_a \mathbf{G}^a_b \equiv 0. \quad (3.9)$$

Now, we are going to show that the left-hand side is exactly the singular part of the Bianchi divergence:  $\nabla_\mu \mathbf{G}^\mu_b \equiv 0$ . Indeed, the regular part of this quantity vanishes on both sides of  $S$  (a consequence of the standard Bianchi identities), whereas its singular part is proportional to  $\delta_S$ . To prove this statement observe that  $\mathcal{G}^3_b = 0$  and, whence, no derivative of the Dirac delta is produced when we apply the covariant derivative  $\nabla_3$  to the singular tensor density (2.5). Thus what remains are the ‘‘along  $S$ ’’ derivatives  $\nabla_a$ . As a result of this operation we obtain, therefore, the quantity  $\bar{\nabla}_a \mathbf{G}^a_b$  multiplied by  $\delta_S$ . Another  $\delta$ -like term is obtained from the regular part  $\text{reg}(\mathcal{G})$  which is discontinuous across  $S$ . Taking the derivative  $\nabla_3 \text{reg}(\mathcal{G})^3_b$  of the regular part we obtain, therefore, its jump of it across  $S$  multiplied by the Dirac delta:  $[\text{reg}(\mathcal{G})^3_b] \delta_S$ . Finally, the singular part of the Bianchi identities is the sum of the above two expressions:

$$\nabla_\mu \mathbf{G}^\mu_b = ([\text{reg}(\mathcal{G})^\perp_b] + \bar{\nabla}_a \mathbf{G}^a_b) \delta(x^3) \equiv 0, \quad (3.10)$$

where the last identity is just the Gauss-Codazzi Eq. (3.9). Hence, we have shown that identities  $\nabla_\mu \mathbf{G}^\mu_b \equiv 0$  are also fulfilled for space-times with a singular curvature.

#### IV. DYNAMICS OF THE SHELL + GRAVITY SYSTEM

Dynamical equations of the physical system composed of a matter shell and the surrounding gravitational field will be derived from the action principle  $\delta \mathcal{A} = 0$ , where

$$\mathcal{A} = \mathcal{A}_{\text{grav}}^{\text{reg}} + \mathcal{A}_{\text{grav}}^{\text{sing}} + \mathcal{A}_{\text{matter}} \quad (4.1)$$

is the sum of the gravitational action and the matter action. Gravitational action, defined as the integral of the Hilbert Lagrangian  $L = \frac{1}{16\pi} \sqrt{|g|} R$ , splits into the regular  $\mathcal{A}_{\text{grav}}^{\text{reg}}$  and the singular part  $\mathcal{A}_{\text{grav}}^{\text{sing}}$ , according to the decomposition of the curvature  $R = \text{reg}(R) + \text{sing}(R)$  (a similar ‘‘mixture’’ of a ‘‘bulk action’’ and a singular ‘‘body action’’ concentrated on a submanifold was recently used by C. Barrabès and W. Israel—see [9]—to derive brane dynamics in general relativity).

Using formulas (2.5)–(2.7), we express the singular part of  $R$  in terms of the singular part of the Einstein tensor:

$$\begin{aligned} 16\pi L_{\text{grav}}^{\text{sing}} &= \sqrt{|g|} \text{sing}(R) = -\text{sing}(\mathcal{G}) = -\mathbf{G}^{\mu\nu} g_{\mu\nu} \delta(x^3) \\ &= -\mathbf{G}^{ab} g_{ab} \delta(x^3). \end{aligned} \quad (4.2)$$

Hence the total action is the sum of three integrals:

$$\mathcal{A} = \int_D L_{\text{grav}}^{\text{reg}} + \int_D L_{\text{grav}}^{\text{sing}} + \int_{D \cap S} L_{\text{matter}}, \quad (4.3)$$

where  $D$  is a spatially compact four-dimensional region with boundary in  $M$ , which is possibly cut by a three-dimensional surface  $S$  (actually, because of the Dirac-delta factor, the second term reduces to integration over  $D \cap S$ ).

Variation is taken with respect to the space-time metric tensor  $g_{\mu\nu}$  and to the matter fields  $z^K$  living on  $S$ . At the moment we do not specify the nature of the matter fields. It is enough to assume that they are geometric objects living on this surface and that matter Lagrangian  $L_{\text{mat}}$  is a scalar density on  $S$ , which depends locally on the values of those fields, their derivatives along  $S$  and the metric of the surface  $S$ .

From the point of view of the two regular half-space-times (which are tailored across  $S$ ) the singular part of the action arises as the sum of the boundary contributions from both the sides of the shell. On the other hand, the action (4.3) does not contain any surface term at infinity. This is due to the techniques used here (cf. [3]), where we first derive the field dynamics within a spatially compact region  $D$  and then shift its boundary  $\partial D$  to the space-infinity. Of course, the boundary manipulations at infinity are still necessary but in our approach they arise as a Legendre transformation between different control modes at the boundary  $\partial D$  (see Sec. ).

There are many ways to calculate variation of the Hilbert Lagrangian. Here, we use a method proposed by one of us (see [10]). It is based on the following, simple observation:

$$\begin{aligned} \delta \left( \frac{1}{16\pi} \sqrt{|g|} g^{\mu\nu} R_{\mu\nu} \right) &= -\frac{1}{16\pi} \mathbf{G}^{\mu\nu} \delta g_{\mu\nu} + \frac{1}{16\pi} \\ &\quad \times \sqrt{|g|} g^{\mu\nu} \delta R_{\mu\nu} \\ &= -\frac{1}{16\pi} \mathbf{G}^{\mu\nu} \delta g_{\mu\nu} + \pi^{\mu\nu} \delta R_{\mu\nu}. \end{aligned} \quad (4.4)$$

Expressing  $R_{\mu\nu}$  in (4.4) by the connection coefficients  $\Gamma_{\mu\nu}^\lambda$  and their derivatives, it is easy to show that the last term on the right-hand side is a complete divergence due to the following identity:

$$\begin{aligned} \pi^{\mu\nu} \delta R_{\mu\nu} &\equiv \partial_\kappa (\pi_\lambda^{\mu\nu\kappa} \delta \Gamma_{\mu\nu}^\lambda) \\ &= (\partial_\kappa \pi_\lambda^{\mu\nu\kappa}) \delta \Gamma_{\mu\nu}^\lambda + \pi_\lambda^{\mu\nu\kappa} \delta \Gamma_{\mu\nu,\kappa}^\lambda, \end{aligned} \quad (4.5)$$

where, besides of the quantity (2.9), we have introduced the following notation:

$$\pi_\lambda^{\mu\nu\kappa} := \pi^{\mu\nu} \delta_\lambda^\kappa - \pi^{\kappa(\nu} \delta^{\mu)}_\lambda, \quad (4.6)$$

and

$$\Gamma_{\mu\nu,\kappa}^\lambda := \partial_\kappa \Gamma_{\mu\nu}^\lambda. \quad (4.7)$$

Proof of the identity (4.5) is straightforward if one uses the fact that  $\Gamma_{\mu\nu}^\lambda$  are not independent quantities but the connection coefficients, i.e., combinations of the metric components  $g_{\mu\nu}$  and their derivatives. This means that the covariant derivative  $\nabla \pi$  with respect to connection  $\Gamma$  vanishes identically. Taking into account the fact that  $\pi$  is a tensor density we have that

$$\partial_\kappa \pi_\lambda^{\mu\nu\kappa} \equiv \pi_\alpha^{\mu\nu\kappa} \Gamma_{\lambda\kappa}^\alpha - \pi_\lambda^{\alpha\nu\kappa} \Gamma_{\alpha\kappa}^\mu - \pi_\lambda^{\mu\alpha\kappa} \Gamma_{\alpha\kappa}^\nu, \quad (4.8)$$

which implies identity (4.5) in a simple way see [10]).

Finally, variation of the regular part of the Hilbert Lagrangian is following:

$$\delta\left(\frac{1}{16\pi}\sqrt{|g|}\text{reg}(R)\right) = -\frac{1}{16\pi}\text{reg}(\mathcal{G})^{\mu\nu}\delta g_{\mu\nu} + \text{reg}(\partial_\kappa(\pi_\lambda^{\mu\nu\kappa}\delta\Gamma_{\mu\nu}^\lambda)). \quad (4.9)$$

We are going to show now that the same equation holds for the singular part of  $R$ , i.e.,

$$\begin{aligned} \delta L_{\text{grav}}^{\text{sing}} &= \delta\left(\frac{1}{16\pi}\sqrt{|g|}\text{sing}(R)\right) \\ &= -\frac{1}{16\pi}\text{sing}(\mathcal{G})^{\mu\nu}\delta g_{\mu\nu} + \text{sing}(\partial_\kappa(\pi_\lambda^{\mu\nu\kappa}\delta\Gamma_{\mu\nu}^\lambda)). \end{aligned} \quad (4.10)$$

*Proof of (4.10).*—Calculate the singular part of  $\partial_\kappa(\pi_\lambda^{\mu\nu\kappa}\delta\Gamma_{\mu\nu}^\lambda)$ . Because all these quantities are invariant, geometric objects ( $\delta\Gamma$  is a tensor), we may calculate them in an arbitrary coordinate system. Hence, we may use our adapted coordinate system, where coordinate  $x^3$  is constant on  $S$ . Taking into account the continuity of  $\pi_\lambda^{\mu\nu\kappa}$  across  $S$  we obtain

$$\begin{aligned} \text{sing}(\partial_\kappa(\pi_\lambda^{\mu\nu\kappa}\delta\Gamma_{\mu\nu}^\lambda)) &= \delta(x^3)\pi_\lambda^{\mu\nu\perp}\delta[\Gamma_{\mu\nu}^\lambda] \\ &= \delta(x^3)\pi_\lambda^{\mu\nu 3}\delta[\Gamma_{\mu\nu}^\lambda] \\ &= \delta(x^3)\pi^{\mu\nu}\delta[A_{\mu\nu}^3]. \end{aligned} \quad (4.11)$$

From the definition (2.10) of the object  $\tilde{Q}^{\mu\nu}$  it follows immediately that

$$\pi^{\mu\nu}\delta A_{\mu\nu}^3 = -\frac{1}{16\pi}g_{\mu\nu}\delta\tilde{Q}^{\mu\nu}. \quad (4.12)$$

Using (2.11) we have that

$$[\tilde{Q}^{\mu\nu}]\delta(x^3) = \text{sing}(\mathcal{G})^{\mu\nu}. \quad (4.13)$$

Putting these formulas together we obtain

$$\begin{aligned} \delta(x^3)\pi_\lambda^{\mu\nu\perp}\delta[\Gamma_{\mu\nu}^\lambda] &= -\frac{1}{16\pi}g_{\mu\nu}\delta\text{sing}(\mathcal{G})^{\mu\nu} \\ &= \delta L_{\text{grav}}^{\text{sing}} + \frac{1}{16\pi}\text{sing}(\mathcal{G})^{\mu\nu}\delta g_{\mu\nu}, \end{aligned} \quad (4.14)$$

which ends the proof of (4.10).  $\square$

Summing up the regular part (4.9) and the singular part (4.10) we obtain variation of the whole gravitational Lagrangian:

$$\delta L_{\text{grav}} = -\frac{1}{16\pi}\mathcal{G}^{\mu\nu}\delta g_{\mu\nu} + \partial_\kappa(\pi_\lambda^{\mu\nu\kappa}\delta\Gamma_{\mu\nu}^\lambda), \quad (4.15)$$

which generalizes the corresponding formula from Ref. [3] to the case of space-times with a singular curvature.

## A. Matter Lagrangian and the singular energy momentum

Now, consider the matter part of the action. As was mentioned already, we assume that all the dynamical properties of matter fields are described by a Lagrangian density  $L_{\text{mat}}$ , which is an invariant, three-dimensional scalar density on  $S$ . It depends upon some matter fields  $z^K$  living on  $S$ , their first derivatives along  $S$ :  $z^K_a := \partial_a z^K$ , and the metric tensor  $g_{ab}$  on  $S$ :

$$L_{\text{mat}} = L_{\text{mat}}(z^K; z^K_a; g_{ab}). \quad (4.16)$$

We calculate the variation of the matter Lagrangian by

$$\begin{aligned} \delta L_{\text{mat}} &= \frac{\partial L_{\text{mat}}}{\partial g_{ab}}\delta g_{ab} + \frac{\partial L_{\text{mat}}}{\partial z^K}\delta z^K + \frac{\partial L_{\text{mat}}}{\partial z^K_a}\partial_a\delta z^K \\ &= \frac{1}{2}\tau^{ab}\delta g_{ab} + \left(\frac{\partial L_{\text{mat}}}{\partial z^K} - \partial_a\frac{\partial L_{\text{mat}}}{\partial z^K_a}\right)\delta z^K \\ &\quad + \partial_a(p_K^a\delta z^K), \end{aligned} \quad (4.17)$$

where we have introduced the following three-dimensional symmetric energy-momentum tensor density  $\tau^{ab}$  on  $S$ :

$$\tau^{ab} := 2\frac{\partial L}{\partial g_{ab}}, \quad (4.18)$$

and the momenta  $p_K^a$  canonically conjugate to material variables  $z^K$ :

$$p_K^a := \frac{\partial L_{\text{mat}}}{\partial z^K_a}. \quad (4.19)$$

Finally, we obtain the following formula for the variation of the total (“matter + gravity”) Lagrangian:

$$\begin{aligned} \delta L &= -\frac{1}{16\pi}\text{reg}(\mathcal{G})^{\mu\nu}\delta g_{\mu\nu} + \delta(x^3)\left(\frac{\partial L_{\text{mat}}}{\partial z^K} - \partial_a\frac{\partial L_{\text{mat}}}{\partial z^K_a}\right) \\ &\quad \times \delta z^K - \delta(x^3)\frac{1}{16\pi}(\mathbf{G}^{ab} - 8\pi\tau^{ab})\delta g_{ab} \\ &\quad + \partial_\kappa(\pi_\lambda^{\mu\nu\kappa}\delta\Gamma_{\mu\nu}^\lambda) + \delta(x^3)\partial_a(p_K^a\delta z^K). \end{aligned} \quad (4.20)$$

In this section we assume that both  $\delta g_{\mu\nu}$  and  $\delta z^K$  vanish in a neighborhood of the boundary  $\partial D$  of the space-time region  $D$  (this assumption will be later relaxed, when deriving Hamiltonian structure of the theory). Hence, the last two boundary terms of the above formula vanish when integrated over  $D$ . Vanishing of the variation  $\delta\mathcal{A} = 0$  with fixed boundary data implies, therefore, the Euler-Lagrange equations for the matter field  $z^K$ , together with Einstein equations for gravitational field. The regular part of Einstein equations

$$\text{reg}(\mathcal{G})^{\mu\nu} = 0 \quad (4.21)$$

is satisfied in the whole space-time, whereas the singular part

$$\mathbf{G}^{ab} = 8\pi\tau^{ab}, \quad (4.22)$$

must be fulfilled on  $S$ . This way the Israel equations for the shell dynamics have been derived from the variational principle.

Information encoded in the three-dimensional object  $\tau^{ab}$  may be expressed also in terms of a four-dimensional energy-momentum tensor  $\mathcal{T}^{\mu\nu} := \delta(x^3)\tau^{\mu\nu}$ , where, in agreement with (2.7), the transversal components of  $\tau$  vanish by definition:  $\tau^{\perp\nu} \equiv 0$  (in our adopted coordinate system it simply means that  $\tau^{3\nu} \equiv 0$ , but this condition in its previous form does not depend on the coordinate system). Summing up the singular and the regular parts of the gravitational field, we write the variation of the total Lagrangian as

$$\begin{aligned} \delta L = & \frac{1}{16\pi} (\mathcal{G}^{\mu\nu} - 8\pi\mathcal{T}^{\mu\nu}) \delta g_{\mu\nu} \\ & + \delta(x^3) \left( \frac{\partial L_{\text{mat}}}{\partial z^K} - \partial_a \frac{\partial L_{\text{mat}}}{\partial z_a^K} \right) \delta z^K + \partial_\kappa (\pi_\lambda^{\mu\nu\kappa} \delta \Gamma_{\mu\nu}^\lambda) \\ & + \delta(x^3) \partial_a (p_K^a \delta z^K), \end{aligned} \quad (4.23)$$

which is the starting point of our derivation of the dynamics of the system. We stress that this equation is an *identity*, implied by the structure of the action (4.1).

Field equations of the theory, i.e., Einstein equations for gravitational field and the Euler-Lagrange equations for matter fields, are equivalent to vanishing of the volume part of the above variation (i.e., of the first two terms):

$$\mathcal{G}^{\mu\nu} = 8\pi\mathcal{T}^{\mu\nu}, \quad (4.24)$$

$$\frac{\partial L_{\text{mat}}}{\partial z^K} = \partial_a \frac{\partial L_{\text{mat}}}{\partial z_a^K}. \quad (4.25)$$

This, in turn, is equivalent to the fact that for arbitrary variations (nonnecessarily vanishing on the boundary) of the independent fields  $\delta g_{\mu\nu}$  and  $\delta z^K$ , variation of the Lagrangian reduces to the boundary part:

$$\delta L = \partial_\kappa (\pi_\lambda^{\mu\nu\kappa} \delta \Gamma_{\mu\nu}^\lambda) + \delta(x^3) \partial_a (p_K^a \delta z^K). \quad (4.26)$$

The whole dynamics of the system matter + gravity is, therefore, equivalent to the above equation. Similarly as in Eq. (4.11), we can use the definition of  $\pi_\lambda^{\mu\nu\kappa}$  and express it in terms of contravariant tensor density  $\pi^{\mu\nu}$  obtaining

$$\pi_\lambda^{\mu\nu\kappa} \delta \Gamma_{\mu\nu}^\lambda = \pi^{\mu\nu} \delta A_{\mu\nu}^\kappa. \quad (4.27)$$

Hence field equations can be written in the following way:

$$\delta L = \partial_\kappa (\pi^{\mu\nu} \delta A_{\mu\nu}^\kappa) + \delta(x^3) \partial_a (p_K^a \delta z^K). \quad (4.28)$$

We complete this section with the Noether theorem for the energy-momentum tensor (4.18) which, due to Bianchi identities (3.9), provides the necessary consistency condition for the Einstein Eqs. (4.22) or, equivalently, (4.24). In fact, due to the regular part (4.21) of Einstein equations, Bianchi identity (3.9) reduces to:  $\bar{\nabla}_a \mathbf{G}^a_b \equiv 0$ . Hence, the

following identity is necessary and sufficient for the consistency of the singular part (4.22) of Einstein equations:

*Theorem 1. (Noether)*— For any field configuration  $(z^K(x))$  satisfying the matter dynamics (4.25), the energy-momentum tensor (4.18) carried by this configuration satisfies the following identity:

$$\bar{\nabla}_a \tau^a_b \equiv 0. \quad (4.29)$$

The proof of the Noether identity is given in the next section, just after the Belinfante-Rosenfeld theorem.

## V. HAMILTONIAN STRUCTURE OF THE THEORY

The above form of field equations is analogous to the Lagrangian form of the dynamics in theoretical mechanics, which may be written as follows:

$$\delta L(q, \dot{q}) = \frac{d}{dt} (p \delta q) \equiv \dot{p} \delta q + p \delta \dot{q}, \quad (5.1)$$

and contains relation between momenta and velocities:

$$p = \frac{\partial L}{\partial \dot{q}},$$

as well as Newton equations:

$$\dot{p} = \frac{\partial L}{\partial q}.$$

This formula is a starting point of derivation of the Hamiltonian form of the dynamics. It is sufficient to perform Lagrange transformation between  $p$  and  $\dot{q}$ , putting:

$$p \delta \dot{q} = \delta(p \dot{q}) - \dot{q} \delta p,$$

and move the total derivative  $\delta(p \dot{q})$  to the left-hand side of the Eq. (5.1). This way we obtain the Hamiltonian formula:

$$-\delta H(p, q) = \dot{p} \delta q - \dot{q} \delta p, \quad (5.2)$$

where we have put  $H(p, q) := p \dot{q} - L$ . This formula is equivalent to the Hamiltonian form of the equations of motion:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

In order to derive the Hamiltonian formulation of the field theory, we perform a similar Legendre transformation between time derivatives of the fields and corresponding momenta. For this purpose we have to fix a  $(3+1)$ -decomposition of the space-time  $M$ . This way the theory becomes a Hamiltonian system, with the space of Cauchy data on each of the three-dimensional surfaces  $\Sigma_t := \{t = \text{const}\}$  playing the role of an infinite-dimensional phase space. Unlike in the case of the classical mechanics, the dynamics of such a system is not uniquely defined, unless we control also boundary data for the field in an appropriate way.

In the present paper we consider the case of an asymptotically flat space-time and assume that also the leaves  $\Sigma_t$  of our (3 + 1)-decomposition are asymptotically flat at infinity. To keep control over two-dimensional surface integrals at spatial infinity, we first consider dynamics of our matter + gravity system in a finite world tube  $\mathcal{T}$ , whose boundary carries a nondegenerate metric of signature  $(-, +, +)$ . At the end of the day we shall shift the boundary  $\partial\mathcal{T}$  of the tube to space-infinity. We assume that the tube contains the surface  $S$  together with our matter travelling over it.

To simplify calculations we choose the coordinate system adapted to this (3 + 1)-decomposition. This means that the time variable  $t = x^0$  is constant on three-dimensional surfaces of this foliation. We assume that these surfaces are spacelike. To obtain Hamiltonian formulation of our theory we shall simply integrate Eq. (4.26) [or, equivalently, (4.28)] over a finite piece  $V$  of the Cauchy surface  $C \subset M$  and then perform Legendre transformation between time derivatives and the corresponding momenta.

Denoting by  $V := \mathcal{T} \cap C$  the portion of the Cauchy hypersurface  $C$  which is contained in the tube  $\mathcal{T}$ , we thus integrate (4.28) over the finite volume  $V \subset C$  and keep surface integrals on the boundary  $\partial V$  of  $V$ . They will produce the ADM mass as the Hamiltonian of the total matter + gravity system when we pass to infinity with  $\partial V = C \cap \partial\mathcal{T}$ . Because our approach is geometric and does not depend upon the choice of coordinate system, we may further simplify our calculations using coordinate  $x^3$  adapted to both  $S$  and to the boundary  $\partial\mathcal{T}$  of the tube. We thus assume that  $x^3$  is constant on both these surfaces.

Integrating (4.28) over the volume  $V$  we thus obtain:

$$\delta \int_V L = \int_V \partial_\kappa (\pi^{\mu\nu} \delta A_{\mu\nu}^\kappa) + \int_V \delta(x^3) \partial_a (p_K^a \delta z^K) \quad (5.3)$$

$$= \int_V (\pi^{\mu\nu} \delta A_{\mu\nu}^0) \cdot + \int_{\partial V} \pi^{\mu\nu} \delta A_{\mu\nu}^\perp + \int_{V \cap S} (p_K^0 \delta z^K) \cdot, \quad (5.4)$$

where by “dot” we denote the time derivative. In this formula we have skipped the two-dimensional divergencies which vanish when integrated over surfaces  $\partial V$  and  $V \cap S$  (see Fig. 1 below).

To further simplify our formalism, we denote by  $p_K := p_K^0$  the timelike component of the momentum canonically conjugate to the field variable  $z^K$ . Now we perform the Legendre transformation in material variables:

$$(p_K \delta z^K) \cdot = \dot{p}_K \delta z^K - \dot{z}^K \delta p_K + \delta(p_K \dot{z}^K). \quad (5.5)$$

The last term, put on the left-hand side of (5.3), meets the matter Lagrangian and produces the matter Hamiltonian (with minus sign), according to the formula

$$L_{\text{mat}} - p_K \dot{z}^K = L_{\text{mat}} - p_K^0 z_0^K = -T^0_0, \quad (5.6)$$

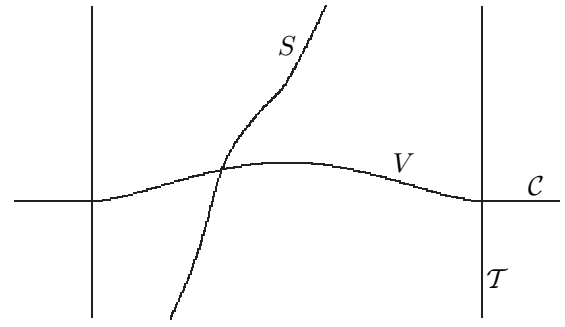


FIG. 1. Integrated portion of space-time.

where the canonical energy-momentum tensor is defined as follows:

$$T^\mu_\nu := p_K^\mu z_\nu^K - \delta_\nu^\mu L_{\text{mat}}.$$

Even if singular, the above quantity satisfies the standard Rosenfeld-Belinfante identity (cf. [11]), which states that the *canonical* energy-momentum tensor is equal (modulo a minus sign, due to the convention used here) to the *symmetric* energy-momentum tensor (4.18).

*Theorem 2. (Rosenfeld-Belinfante)*—: Symmetric and canonical energy-momentum tensors are, essentially, the same. More precisely, the following identity holds:

$$T^\mu_\nu = -\tau^{\mu\lambda} g_{\lambda\nu}, \quad (5.7)$$

or, equivalently,

$$T^a_b = -\tau^{ac} g_{cb}, \quad (5.8)$$

because both the transversal parts  $T^\perp_\nu$  and  $\tau^{\perp\nu}$  vanish identically from the definition.

*Proof.*—We remember that  $L_{\text{mat}}$  is a scalar density on  $S$  and, therefore, may be written as  $L_{\text{mat}} = \sqrt{\det g_{ab}} \Lambda$ , where  $\Lambda$  is a scalar function, depending exclusively upon quantities  $(z^K; z^K_a; g_{ab})$ . But the only way to produce a scalar from the partial derivatives  $z^K_a$  is to take the following combination:  $F^{KN} := z^K_a z^N_b \hat{g}^{ab}$ . We conclude that

$$L_{\text{mat}} = \sqrt{\det g_{ab}} \Lambda(z^K, F^{KN}).$$

This implies identity [11] in a straightforward way (cf. also [2]).

Now, we are ready for the proof of the Noether theorem (4.29):

*Proof.*—The invariant character of the matter Lagrangian  $L_{\text{mat}} = L_{\text{mat}}(z^K; z^K_a; g_{ab})$  means that, for any vector field  $X$  on  $S$ , dragging the arguments  $(z^K; z^K_a; g_{ab})$  along  $X$  produces the same effect on the Lagrangian that dragging it directly as a scalar density does. Choosing any coordinates  $(x^a)$  on  $S$  and choosing  $X = \partial_a$  we obtain, therefore, the following identity:

$$\begin{aligned}
 \partial_a L_{\text{mat}} &= \frac{\partial L_{\text{mat}}}{\partial z^K} z^K{}_a + \frac{\partial L_{\text{mat}}}{\partial z^K_b} \partial_a z^K_b + \frac{\partial L_{\text{mat}}}{\partial g_{cd}} \partial_a g_{cd} \\
 &= \left( \frac{\partial L_{\text{mat}}}{\partial z^K} - \partial_b \frac{\partial L_{\text{mat}}}{\partial z^K_b} \right) z^K{}_a + \partial_b \left( \frac{\partial L_{\text{mat}}}{\partial z^K_b} z^K{}_a \right) \\
 &\quad + \frac{1}{2} \tau^{cd} \partial_a g_{cd}, \tag{5.9}
 \end{aligned}$$

where we have used the symmetry of second derivatives:  $\partial_a z^K_b = \partial_b z^K_a$  and the definition (4.18) of the symmetric energy-momentum tensor  $\tau$ . Putting now the last two terms on the left-hand side and using the Rosenfeld-Belinfante theorem we obtain:

$$\partial_b \tau^b{}_a - \frac{1}{2} \tau^{cd} \partial_a g_{cd} = \left( \frac{\partial L_{\text{mat}}}{\partial z^K} - \partial_b \frac{\partial L_{\text{mat}}}{\partial z^K_b} \right) z^K{}_a. \tag{5.10}$$

It may be easily checked that the left-hand side is precisely the covariant divergence  $\bar{\nabla}_b \tau^b{}_a$  on  $S$  (remember that  $\tau$  is *not* a tensor but the tensor density). Hence, we obtain a kinematic identity, fulfilled by arbitrary field configurations, not only those fulfilling field equations:

$$\bar{\nabla}_b \tau^b{}_a = \left( \frac{\partial L_{\text{mat}}}{\partial z^K} - \partial_b \frac{\partial L_{\text{mat}}}{\partial z^K_b} \right) z^K{}_a. \tag{5.11}$$

This completes the proof.  $\square$

Now, integrating Eq. (5.6) over the two-dimensional region  $V \cap S$ , we obtain the ‘‘material’’ part of the total (matter + gravity) Hamiltonian:

$$\int_{V \cap S} L_{\text{mat}} - p_K \dot{z}^K = - \int_{V \cap S} T^0_0 = \int_{V \cap S} \tau^0_0 = \int_V \mathcal{T}^0_0, \tag{5.12}$$

where  $\mathcal{T}^{\mu\nu} := \delta(x^3) \tau^{\mu\nu} \cosh$ . Hence, the Legendre transformation in material degrees of freedom gives us the following formula:

$$\begin{aligned}
 \delta \int_V (\mathcal{T}^0_0 + L_{\text{grav}}) &= \int_V (\pi^{\mu\nu} \delta A^0_{\mu\nu}) + \int_{\partial V} \pi^{\mu\nu} \delta A^\perp_{\mu\nu} \\
 &\quad + \int_V (\tilde{\pi}_K \delta z^K - \dot{z}^K \delta \pi_K), \tag{5.13}
 \end{aligned}$$

where we denote

$$\pi_K := p_K \delta(x^3). \tag{5.14}$$

Here, the matter degrees of freedom are already described in the Hamiltonian picture (with the matter Hamiltonian ‘‘ $-\mathcal{T}^0_0$ ’’ on the left-hand side) and the gravitational degrees of freedom still remain on the Lagrangian level.

### A. Legendre transformation in the gravitational degrees of freedom

To perform also Legendre transformation in gravitational degrees of freedom—analogue to transformation (5.5) in material degrees of freedom—we follow here a method proposed by one of us (see [10]). We show in the Appendix that, after the transformation, formula (5.13)

assumes the following form:

$$\delta \int_V \left( \mathcal{T}^0_0 - \frac{1}{8\pi} \mathcal{G}^0_0 \right) + \frac{1}{16\pi} \delta \int_{\partial V} (\mathcal{Q}^{00} g_{00} - \mathcal{Q}^{AB} g_{AB}) \tag{5.15}$$

$$\begin{aligned}
 &= \frac{1}{16\pi} \int_V (\dot{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl}) + \frac{1}{16\pi} \int_{\partial V} (\dot{\lambda} \delta \alpha - \dot{\alpha} \delta \lambda) \\
 &\quad + \int_V (\tilde{\pi}_K \delta z^K - \dot{z}^K \delta \pi_K) - \frac{1}{16\pi} \int_{\partial V} g_{ab} \delta \mathcal{Q}^{ab}. \tag{5.16}
 \end{aligned}$$

Here, we have introduced the following notation:  $P^{kl}$  denotes the external curvature of  $\Sigma$  written in the ADM form, i.e., given by the equations:

$$\begin{aligned}
 P^{kl} &:= \sqrt{\det g_{mn}} (K \tilde{g}^{kl} - K^{kl}), \\
 K_{kl} &:= - \frac{1}{\sqrt{|g^{00}|}} \Gamma^0_{kl} = - \frac{1}{\sqrt{|g^{00}|}} A^0_{kl}, \tag{5.17}
 \end{aligned}$$

and  $\tilde{g}^{kl}$  stands for three-dimensional contravariant metric, invariant to the metric  $g_{kl}$  induced on the Cauchy surface  $V$ . Similarly,  $\mathcal{Q}^{ab}$  denotes the external curvature<sup>2</sup> of the tube  $\partial \mathcal{T}$  written in the ADM form, i.e., three-dimensional tensor density given by equations similar to (3.1) and (3.2):

$$\mathcal{Q}^{ab} := \sqrt{|\det g_{cd}|} (L \hat{g}^{ab} - L^{ab}), \tag{5.18}$$

$$L_{ab} := - \frac{1}{\sqrt{g^{33}}} \Gamma^3_{ab} = - \frac{1}{\sqrt{g^{33}}} A^3_{ab}, \tag{5.19}$$

and  $\hat{g}^{ab}$  is a three-dimensional contraindicant metric on the tube  $\partial \mathcal{T}$ , invariant to the induced metric  $g_{ab}$ . Moreover,  $\lambda = \sqrt{\det g_{AB}}$  denotes the two-dimensional volume form on  $\partial V$ , whereas

$$\alpha := \operatorname{arcsinh} \left( \frac{g^{30}}{\sqrt{|g^{00} g^{33}|}} \right), \tag{5.20}$$

is the hyperbolic angle between the Cauchy surface  $V$  and the tube  $\partial \mathcal{T}$ .

The formula (5.16) has been derived in Ref. [3] for a wide class of matter Lagrangian (including also gauge fields), but only for models with the continuous matter distribution. Now we have proved its validity also in case of a singular matter, concentrated on a two-dimensional shell  $S$ , whose internal metric is nondegenerate and carries signature  $(-, +, +)$ .

Observe that the first term on the left-hand side of (5.15) vanishes identically due to Einstein equations<sup>3</sup> which means that the volume part of the total gravity + matter

<sup>2</sup>We use the symbol  $\mathcal{Q}$  for denoting external curvature of the world tube  $\partial \mathcal{T}$  to distinguish it from external curvature of the shell  $S$ , which is denoted by  $Q$ .

<sup>3</sup>The quantity  $(\mathcal{G}^0_0 - 8\pi \mathcal{T}^0_0)$  in (5.15) is often denoted by  $N\mathcal{H} + N^k \mathcal{H}_k$ , where  $\mathcal{H}$  and  $\mathcal{H}_k$  are the scalar and the vector constraints, respectively.



energy vanishes identically. This does not mean that the energy vanishes, because there is also another, surface contribution to the Hamiltonian. Indeed, a detailed analysis (see [10]) shows that we are not allowed to control freely the tube external curvature  $Q^{ab}$  in the last integral of (5.16) because of constraints which occur here. The simplest way to overcome this difficulty is to perform another Legendre transformation in the expression  $g_{ab}\delta Q^{ab} = g_{00}\delta Q^{00} + 2g_{0A}\delta Q^{0A} + g_{AB}\delta Q^{AB}$ . Namely, we write

$$g_{AB}\delta Q^{AB} = \delta(g_{AB}Q^{AB}) - Q^{AB}\delta g_{AB}, \quad (5.21)$$

and put the complete derivative  $\delta(g_{AB}Q^{AB})$  on the left-hand side of (5.15). This way we obtain the ‘‘quasilocal’’ (i.e., assigned to the two-surface  $\partial V$ ) Hamiltonian of the system. Finally, we have

$$-\delta\mathcal{M}_{\partial V} = \frac{1}{16\pi} \int_V (\dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl}) + \frac{1}{16\pi} \int_{\partial V} (\dot{\lambda}\delta\alpha - \dot{\alpha}\delta\lambda) \quad (5.22)$$

$$+ \int_V (\dot{\pi}_K\delta z^K - \dot{z}^K\delta\pi_K) - \frac{1}{16\pi} \int_{\partial V} g_{00}\delta Q^{00} + 2g_{0A}\delta Q^{0A} - Q^{AB}\delta g_{AB}, \quad (5.23)$$

where the quasilocal Hamiltonian (mass) assigned to the two-dimensional surface  $\partial V$  is defined as follows:

$$\mathcal{M}_{\partial V} = E_0(\partial V) - \frac{1}{16\pi} \int_{\partial V} (Q^{00}g_{00}), \quad (5.24)$$

and the additive constant  $E_0(\partial V)$  is arbitrary. It turns out (cf. [10]) that it may be chosen in such a way that the quasilocal mass vanishes for the flat Minkowski space initial data.<sup>4</sup> The final step of our derivation consists in shifting the tube  $\mathcal{T}$  to infinity. For this purpose we limit ourselves to the asymptotically flat case and assume that the limiting case of  $V$  is equal to an asymptotically flat Cauchy three-surface  $C$ , i.e.,  $V \rightarrow C$ . It may be easily checked that the limit of our quasilocal mass is equal to the ADM global mass, i.e., that  $\mathcal{M}_{\partial V} \rightarrow \mathcal{M}_{\text{ADM}}$ . Moreover, all the surface terms on the right-hand side of our Hamiltonian formula (5.22)–(5.23) vanish. This way we obtain the following global Hamiltonian formula, fully analogous to the mechanical formula (5.2), with the ADM mass  $\mathcal{M}_{\text{ADM}}$  playing the role of the total Hamiltonian of the matter + gravity system:

$$-\delta\mathcal{M}_{\text{ADM}} = \frac{1}{16\pi} \int_C (\dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl}) + \int_C (\dot{\pi}_K\delta z^K - \dot{z}^K\delta\pi_K). \quad (5.25)$$

<sup>4</sup>Actually, the quasilocal Hamiltonian formula (5.22) needs some minor improvements, see [10], but it is sufficient for global purposes of the present paper and this is why we do not discuss them here.

The phase space of our system is thus described by the gravitational variables  $(P^{kl}, g_{kl})$  on the Cauchy three-surface  $C$  and by the material variables  $(\pi_K, z^K)$  concentrated on the two-surface  $S \cap C$ ; cf. (5.14).

## VI. CONSTRAINTS

As usual, Gauss-Codazzi equations imply constraints, which must be fulfilled by the Cauchy data  $(P^{kl}, g_{kl}, \pi_K, z^K)$  on a three-surface  $C$ . Outside of the shell these are standard, vacuum constraints. In this section we are going to derive the complete description of constraints, valid not only for the regular but also for the singular part of the data. We denote by  $\tilde{g}^{kl}$  the three-dimensional metric inverse to the metric  $g_{kl}$  and put  $\gamma := \sqrt{\det g_{kl}}$ . By  ${}^{(3)}R$  we denote the three-dimensional scalar curvature of  $g_{kl}$ ,  $P := P^{kl}g_{kl}$  and ‘‘ $\int$ ’’ is the three-dimensional covariant derivative with respect to  $g_{kl}$ .

Outside of  $S$  Gauss-Codazzi equations relate the components  $\mathcal{G}^0{}_\mu$  of the Einstein tensor density with the Cauchy data in the standard way. The spatial part of these constraints, tangent to  $C_t$ , reads as

$$\mathcal{G}^0{}_l = -P^k{}_{lk}, \quad (6.1)$$

and the timelike part—normal to  $C_t$ , as

$$2\mathcal{G}^0{}_\mu n^\mu = -\gamma^{(3)}R + \left(P^{kl}P_{kl} - \frac{1}{2}P^2\right)\frac{1}{\gamma}, \quad (6.2)$$

where  $n$  denotes the future oriented, orthonormal vector to the Cauchy surface  $C_t$ :

$$n^\mu = -\frac{g^{0\mu}}{\sqrt{-g^{00}}}. \quad (6.3)$$

Vacuum Einstein equations outside and inside of  $S$  imply vanishing of the regular part of  $\mathcal{G}^0{}_\mu$ . Hence, the regular part of the vector constraint on  $C$  reads:

$$\text{reg}(P^k{}_{lk}) = 0, \quad (6.4)$$

whereas the regular part of the scalar constraint reduces to

$$\text{reg}\left(\gamma^{(3)}R - \left(P^{kl}P_{kl} - \frac{1}{2}P^2\right)\frac{1}{\gamma}\right) = 0. \quad (6.5)$$

In the case of discontinuous matter, the above standard constraint shall be completed by their singular part, with support on the intersection  $S_t = C_t \cap S$  in the following way. The singular part of three-dimensional derivatives of the ADM momentum  $P_{kl}$  consists of derivatives in the direction of  $x^3$ :

$$\text{sing}(P_{lk}{}^k) = \text{sing}(\partial_3 P_l{}^3) = \delta(x^3)[P_l{}^3], \quad (6.6)$$

so the full Gauss-Codazzi Eqs. (6.1) take the form

$$\mathcal{G}^0{}_l = -\text{sing}(P_{lk}{}^k) = -[P_l{}^3]\delta(x^3). \quad (6.7)$$

Components of the ADM momentum  $P^{kl}$  are regular,

hence the singular part of the term  $(P^{kl}P_{kl} - \frac{1}{2}P^2)$  vanishes. The singular part of the three-dimensional scalar curvature consists of derivatives in the direction of  $x^3$  of the (three-dimensional) connection coefficients:

$$\begin{aligned} \text{sing}({}^{(3)}R) &= \text{sing}(\partial_3(\Gamma_{kl}^3 \tilde{g}^{kl} - \Gamma_{ml}^m \tilde{g}^{3l})) \\ &= \delta(x^3)[\Gamma_{kl}^3 \tilde{g}^{kl} - \Gamma_{ml}^m \tilde{g}^{3l}], \end{aligned}$$

and expression in the square brackets may be reduced to the following term

$$\begin{aligned} \gamma[\Gamma_{kl}^3 \tilde{g}^{kl} - \Gamma_{ml}^m \tilde{g}^{3l}] &= -2\sqrt{\tilde{g}^{33}} \left[ \partial_3(\gamma\sqrt{\tilde{g}^{33}}) \right] \\ &= -2\sqrt{\tilde{g}^{33}} \left[ \partial_k \left( \frac{\gamma \tilde{g}^{3k}}{\sqrt{\tilde{g}^{33}}} \right) \right], \end{aligned} \quad (6.8)$$

because derivatives tangent to  $S$  are continuous. But the expression in square brackets is equal to the external curvature scalar  $k$  for the two-dimensional surface  $S_t \subset C_t$ :

$$\gamma k = -\partial_k \left( \frac{\gamma \tilde{g}^{3k}}{\sqrt{\tilde{g}^{33}}} \right). \quad (6.9)$$

This implies that

$$\text{sing}(\gamma^{(3)}R) = 2\gamma\sqrt{\tilde{g}^{33}}[k]\delta(x^3) = 2[\lambda k]\delta(x^3).$$

Finally the total spacelike Gauss-Codazzi Eq. (6.2) takes the following form:

$$\begin{aligned} 2\mathcal{G}^0_{\mu n^\mu} &= -\text{sing} \left( \gamma^{(3)}R - \left( P^{kl}P_{kl} - \frac{1}{2}P^2 \right) \frac{1}{\gamma} \right) \\ &= -2[\lambda k]\delta(x^3). \end{aligned} \quad (6.10)$$

Equations (6.7) and (6.10) give a generalization (in the sense of distributions) of the usual vacuum constraints (vector and scalar, respectively).

Now, we will show how the distributional matter located on  $S_t$  determines the four surface quantities  $[P^3_k]$  and  $[\lambda k]$ , entering into the singular part of the constraints. The tangent (to  $S$ ) part of  $\mathcal{G}^0_{\mu}$  splits into the two-dimensional part tangent to  $S_t$  and the transversal part.

The tangent to  $S_t$  part of Einstein equations gives the following:

$$\mathcal{G}^0_k = 8\pi\delta(x^3)\tau^0_k, \quad (6.11)$$

which, due to (6.1) and (6.7), implies the following constraints:

$$[P^3_k] = -8\pi\tau^0_k. \quad (6.12)$$

The remaining null tangent part of Einstein equations reads:

$$\mathcal{G}^0_{\mu n^\mu} = 8\pi\mathcal{T}^0_{\mu n^\mu},$$

where  $E := \mathcal{T}^0_{\mu n^\mu} = \tau^0_{\mu n^\mu} \delta(x^3) = \delta(x^3)\varepsilon$  describes

matter density on  $C$ , and  $\varepsilon = \tau^0_{\mu n^\mu}$  is a surface energy density on  $S$ . Hence  $\text{sing}(\mathcal{G}^0_{\mu n^\mu}) = 8\pi\varepsilon$ , and scalar constraint takes the form

$$8\pi\varepsilon + [\lambda k] = 0. \quad (6.13)$$

Finally, summing up the regular part of the constraints [i.e., Eqs. (6.4) or (6.5), respectively] together with their singular parts [i.e., Eqs. (6.6) or (6.10), respectively] we may finally write down both constraints in their distributional forms:

$$P^k_{|l} = -8\pi\tau^0_k \delta(x^3), \quad (6.14)$$

$$\gamma^{(3)}R - \left( P^{kl}P_{kl} - \frac{1}{2}P^2 \right) = -16\pi\varepsilon\delta(x^3), \quad (6.15)$$

where the matter momentum  $\tau^0_k$  and energy  $\varepsilon$  must be expressed in terms of material variables  $(p_K, z^K)$  via the matter constitutive equations (4.18).

## VII. CONCLUSIONS

We have proved that the general scheme, used in [3,10] to describe any continuous, self-gravitating matter may be extended also to the singular matter, concentrated on a two-dimensional shell. The main result of this paper: The Hamiltonian of the complete matter + gravity system is always equal numerically to the ADM mass at infinity, similarly as in continuous models. The above structure was used in [12] to derive the canonical formulation of a spherically symmetric dust shell. Recently, it was proved that this result may be easily extended far beyond the dust case (see [13]). We stress that the ADM mass generates the Hamiltonian evolution of the system with respect to the asymptotic time variable at space-infinity, whereas the local redefinition of the Cauchy surface  $C_t := \{t = \text{const}\}$ , which does not change it at infinity, is merely a gauge transformation. If we want to use another time (i. e. the Minkowski time inside the shell or the shell's proper time) there will be another Hamiltonian generating the evolution in the new parametrization. These issues were thoroughly discussed in [14].

## APPENDIX

To prove formula (5.15) via Legendre transformation in gravitational degrees of freedom take metricity conditions (3.4)–(3.5) [equivalent to Eqs. (2.12) and (2.13)] for the connection  $\Gamma$  on the surface  $\partial\mathcal{T} = \{x^3 = \text{const}\}$  and plug them into expression  $\pi^{\mu\nu} \delta A^3_{\mu\nu} = \pi^{33} \delta A^3_{33} + 2\pi^{3a} \delta A^3_{3a} + \pi^{ab} \delta A^3_{ab}$ . A straightforward calculation leads to the following result:

$$\begin{aligned} \pi^{\mu\nu} \delta A^1_{\mu\nu} &= \pi^{\mu\nu} \delta A^3_{\mu\nu} \\ &= -\frac{1}{16\pi} g_{ab} \delta Q^{ab} + \partial_a \left( \pi^{33} \delta \left( \frac{\pi^{3a}}{\pi^{33}} \right) \right), \end{aligned} \quad (A1)$$

where  $Q^{ab}$  denotes the external curvature of the tube  $\partial\mathcal{T}$

written in the ADM form, i.e., a three-dimensional tensor density given by equations similar to (3.1) and (3.2):

$$\mathcal{Q}^{ab} := \sqrt{|\det g_{cd}|} (L\hat{g}^{ab} - L^{ab}), \quad (\text{A2})$$

$$L_{ab} := -\frac{1}{\sqrt{g^{33}}}\Gamma_{ab}^3 = -\frac{1}{\sqrt{g^{33}}}A_{ab}^3, \quad (\text{A3})$$

and  $\hat{g}^{ab}$  is a three-dimensional contraindicant metric on the tube  $\partial\mathcal{T}$ , inverse to the induced metric  $g_{ab}$ .

Replacing now  $x^3$  by  $x^0$ , we obtain analogous metricity conditions on the surface  $V = \{x^0 = \text{const}\}$ :

$$A_{00}^0 = \frac{1}{\pi^{00}}(\partial_k \pi^{0k} + A_{kl}^3 \pi^{kl}), \quad (\text{A5})$$

$$A_{0k}^0 = -\frac{1}{2\pi^{00}}(\partial_k \pi^{00} + 2A_{kl}^0 \pi^{0l}). \quad (\text{A5})$$

Plugging them into the expression  $\pi^{\mu\nu} \delta A_{\mu\nu}^0 = \pi^{00} \delta A_{00}^0 + 2\pi^{0k} \delta A_{0k}^0 + \pi^{kl} \delta A_{kl}^0$  we obtain immediately the following identity:

$$\pi^{\mu\nu} \delta A_{\mu\nu}^0 = -\frac{1}{16\pi} g_{kl} \delta P^{kl} + \partial_k \left( \pi^{00} \delta \left( \frac{\pi^{0k}}{\pi^{00}} \right) \right), \quad (\text{A6})$$

where  $P^{kl}$  denotes the external curvature of  $\Sigma$  written in the ADM form, i.e., given by the equations:

$$\begin{aligned} P^{kl} &:= \sqrt{\det g_{mn}} (K\tilde{g}^{kl} - K^{kl}), \\ K_{kl} &:= -\frac{1}{\sqrt{|g^{00}|}}\Gamma_{kl}^0 = -\frac{1}{\sqrt{|g^{00}|}}A_{kl}^0, \end{aligned} \quad (\text{A7})$$

and  $\tilde{g}^{kl}$  stands for three-dimensional contravariant metric, inverse to the metric  $g_{kl}$  induced on the Cauchy surface  $V$ .

Using these results and skipping the two-dimensional divergencies which vanish after integration over  $\partial V$ , we may rewrite the gravitational part of (5.4) in the following way:

$$\begin{aligned} &\int_V (\pi^{\mu\nu} \delta A_{\mu\nu}^0) \cdot + \int_{\partial V} \pi^{\mu\nu} \delta A_{\mu\nu}^\perp \\ &= -\frac{1}{16\pi} \int_V (g_{kl} \delta P^{kl}) \cdot - \frac{1}{16\pi} \int_{\partial V} g_{ab} \delta \mathcal{Q}^{ab} \\ &\quad + \int_{\partial V} \left( \pi^{00} \delta \left( \frac{\pi^{03}}{\pi^{00}} \right) + \pi^{33} \delta \left( \frac{\pi^{30}}{\pi^{33}} \right) \right) \cdot. \end{aligned} \quad (\text{A8})$$

Next, we use the following, obvious identity

$$\pi^{00} \delta \left( \frac{\pi^{03}}{\pi^{00}} \right) + \pi^{33} \delta \left( \frac{\pi^{30}}{\pi^{33}} \right) = 2\sqrt{|\pi^{00}\pi^{33}|} \delta \frac{\pi^{30}}{\sqrt{|\pi^{00}\pi^{33}|}}. \quad (\text{A9})$$

Then we denote

$$q := \frac{\pi^{30}}{\sqrt{|\pi^{00}\pi^{33}|}} = \frac{g^{30}}{\sqrt{|g^{00}g^{33}|}}. \quad (\text{A10})$$

We have

$$2\sqrt{|\pi^{00}\pi^{33}|} = \frac{2}{16\pi} \sqrt{|g|} \sqrt{|g^{00}g^{33}|} = \frac{1}{8\pi} \frac{\lambda}{\sqrt{1+q^2}}, \quad (\text{A11})$$

where  $\lambda := \sqrt{\det g_{AB}}$ . Hence, we obtain

$$\pi^{00} \delta \left( \frac{\pi^{03}}{\pi^{00}} \right) + \pi^{33} \delta \left( \frac{\pi^{30}}{\pi^{33}} \right) = \frac{1}{8\pi} \lambda \delta \alpha, \quad (\text{A12})$$

where  $\alpha := \text{arc sinh}(q)$  and, consequently,

$$\begin{aligned} &\int_V (\pi^{\mu\nu} \delta A_{\mu\nu}^0) \cdot + \int_{\partial V} \pi^{\mu\nu} \delta A_{\mu\nu}^\perp \\ &= -\frac{1}{16\pi} \int_V (g_{kl} \delta P^{kl}) \cdot - \frac{1}{16\pi} \int_{\partial V} g_{ab} \delta \mathcal{Q}^{ab} + \frac{1}{8\pi} \\ &\quad \times \int_{\partial V} (\lambda \delta \alpha) \cdot. \end{aligned}$$

Now we perform Legendre transformation between time derivatives and the corresponding canonical momenta. This transformation is performed both in volume

$$(g_{kl} \delta P^{kl}) \cdot = (\dot{g}_{kl} \delta P^{kl} - \dot{P}^{kl} \delta g_{kl}) + \delta(g_{kl} \dot{P}^{kl}),$$

and on the boundary  $(\lambda \delta \alpha) \cdot = (\dot{\lambda} \delta \alpha - \dot{\alpha} \delta \lambda) + \delta(\lambda \dot{\alpha})$ . The sum of the two total derivatives which arise here may be calculated easily using the same arguments as in Ref. [10]:

$$\begin{aligned} &-\frac{1}{16\pi} \delta \int_V (g_{kl} \dot{P}^{kl}) + \frac{1}{8\pi} \delta \int_{\partial V} \lambda \dot{\alpha} \\ &= \frac{1}{8\pi} \delta \int_V \sqrt{|g|} R^0_0 + \frac{1}{16\pi} \delta \int_{\partial V} (\mathcal{Q}^{AB} g_{AB} - \mathcal{Q}^{00} g_{00}). \end{aligned} \quad (\text{A13})$$

Moving the first (volume) quantity to the left-hand side of formula (5.13) and collecting it with the gravitational part of the Lagrangian, we obtain

$$\begin{aligned} &\frac{1}{16\pi} \int_V L_{\text{grav}} - \frac{1}{8\pi} \int_V \sqrt{|g|} R^0_0 \\ &= \frac{1}{16\pi} \int_V \sqrt{|g|} (R - 2R^0_0) = -\frac{1}{8\pi} \int_V \mathcal{G}_0^0, \end{aligned} \quad (\text{A14})$$

which may be treated as the ‘‘volume part of the gravitational Hamiltonian.’’ It meets the ‘‘matter Hamiltonian’’ (5.13). Their sum (the ‘‘volume part of the total Hamiltonian’’) vanishes identically as a consequence of Einstein equations. This completes the proof of formula (5.15).

- [1] W. Israel, *Nuovo Cimento B* **44**, 1 (1966).
- [2] P. Hájíček and J. Kijowski, *Phys. Rev. D* **57**, 914 (1998).
- [3] J. Kijowski, *Gen. Relativ. Gravit.* **29**, 307 (1997).
- [4] J. Jezierski, J. Kijowski, and E. Czuchry, *Phys. Rev. D* **65**, 064036 (2002).
- [5] E. Czuchry, Ph.D. thesis (in Polish), University of Warsaw, 2002.
- [6] J. Kijowski, *Gen. Relativ. Gravit.* **9**, 857 (1978).
- [7] R. Arnowitt, S. Deser, and C. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), p. 227.
- [8] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, Calif., 1973).
- [9] C. Barrabès and W. Israel, *Phys. Rev. D* **71**, 064008 (2005).
- [10] J. Kijowski, in *Proceedings of Journées Relativistes 1983, Pitagora Editrice, Bologna* (1985), p. 205; *Lect. Notes Phys.*, **212**, 40 (1984).
- [11] F.J. Belinfante, *Physica (Amsterdam)* **6**, 887 (1939); *Physica (Amsterdam)* **7**, 449 (1940); L. Rosenfeld, *Acad. Roy. Belg. Mem. Class. Sci.* **18**, 1 (1940).
- [12] J. Kijowski, *Acta Phys. Pol. B* **29**, 1001 (1998).
- [13] J. Kijowski, G. Magli, and D. Malafarina, *Int. J. Geom. Meth. Mod. Phys.* **2**, 1 (2005).
- [14] P. Hájíček and J. Kijowski, *Phys. Rev. D* **62**, 044025 (2000).