Characterization of Schwarzschildean initial data

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A theorem providing a characterization of Schwarzschildean initial data sets on slices with an asymptotically Euclidean end is proved. This characterization is based on the proportionality of the Weyl tensor and its D'Alambertian that holds for some vacuum Petrov type D spacetimes (e.g. the Schwarzschild spacetime, the C-metric, but not the Kerr solution). The $3 + 1$ decomposition of this proportionality condition renders necessary conditions for an initial data set to be a Schwarzschildean initial set. These conditions can be written as quadratic expressions of the electric and magnetic parts of the Weyl tensor, and thus involve only the freely specifiable data. In order to complete our characterization, a study of which vacuum static Petrov type D spacetimes admit asymptotically Euclidean slices is undertaken. Furthermore, a discussion of the Arnowitt-Deser-Misner (ADM) 4-momentum for boost-rotation symmetric spacetimes is given. As a by-product of our analysis a certain characterization of the Schwarzschild spacetime is obtained. Finally, a generalization of our characterization, valid for Schwarzschildean hyperboloidal initial data sets is put forward.

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I. INTRODUCTION

This article is concerned with answering the following question: given a 3-dimensional manifold, S, and a pair (h_{ij}, K_{ij}) of symmetric tensors on S satisfying the Einstein vacuum constraint equations

$$r + K^2 - K_{ij}K^{ij} = 0, \tag{1}$$

$$D^j K_{ij} - D_i K = 0, \tag{2}$$

how do we know that the triplet (S, h_{ij}, K_{ij}) corresponds to a slice of the Schwarzschild space-time? Above, as well as in the sequel, D and r denote, respectively, the connection and the Ricci scalar of the 3-metric h_{ij}, and we have written $K = K^i{}_i$ for the trace of extrinsic curvature K_{ij}.

The problem stated above is of interest because although the Schwarzschild spacetime is, arguably, fairly well understood, several aspects of its $3 + 1$ decomposition (relevant for numerical investigations) are still open. Among what is known, one should mention the examples of time asymmetric slices given by Reinhardt and Estabrook *et al.*, [1,2], and the constant mean curvature (CMC) slicing found by Beig and O'Murchadha [3]. Examples of foliations with a harmonic time function have been given in [4], and conditions for the embedding of spherically symmetric slices in a Schwarzschild spacetime have been considered in [5]. On the other hand, however, boosted slices in the Schwarzschild spacetime constitute, essentially, an uncharted territory. It is not known, for example, if there are boosted slices which are maximal—the available examples, e.g. that given by York in [6], are not. That these slices cannot be boosted can be proved by the methods used in [7].

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We note that in the case of the Minkowski spacetime, the Codazzi equations readily provide a pointwise,i.e. local, answer to the analogue question. Namely, a pair (h_{ij}, K_{ij}) of symmetric tensors correspond (locally) to the first and second fundamental form of a slice S in Minkowski spacetime if and only if

$$D_{[i}K_{j]l} = 0, \tag{3a}$$

$$r_{ijkl} = -2K_{k[i}K_{j]l}, \tag{3b}$$

where D_i and r_{ijkl} denote, respectively, the connection and the Riemann tensor associated to the 3-metric h_{ij}.

If the spacetime has a nonvanishing curvature, the situation is fundamentally more complicated, and in order to obtain a local answer in a systematic way, one would have to resort to some (yet unavailable) $3 + 1$ formulation of the equivalence problem.

Almost any invariant characterization of the Schwarzschild spacetime has to make use, *a fortiori*, of the fact that it is of Petrov type D, see e.g. [8,9][1]. However, the Petrov type, a neat 4-dimensional property of spacetime, tends to project into complicated expressions when attempting a $3 + 1$ decomposition of its defining relations. The point is, then, to find a description (if any) of the fact that a spacetime is of Petrov type D with a neat $3 + 1$ decomposition. A description of the desired sort is given by a proportionality relation between the D'Alambertian of

[1]The Petrov classification is an algebraic characterization of the Weyl tensor based on the solutions of a certain eigenvalue problem. In particular, a spacetime is said to be of Petrov type D if there are two vectors k^μ and l^μ, *the principal null directions*, such that

$$C_{\mu\nu\lambda[\rho}k_{\sigma]}k^\nu k^\lambda = 0, \qquad C_{\mu\nu\lambda[\rho}l_{\sigma]}l^\nu l^\lambda = 0.$$

For further details on the theory of the Petrov classification see e.g. [10].

the Weyl tensor and the Weyl tensor itself satisfied by some vacuum Petrov type D spacetimes, Schwarzschild included, found by Zakharov [11,12], see Eq. (10).

In what follows, by *the Schwarzschild spacetime* it will be understood the Schwarzschild-Kruskal maximal extension, $(\mathcal{M}, g_{\mu\nu})$, of the Schwarzschild spacetime [13]. Accordingly, by a *slice* of the Schwarzschild spacetime it will be understood that there exists an embedding $\phi: S \longrightarrow \mathcal{M}$ such that $h_{ij} = (\phi^* g)_{ij}$, and $K_{ij} = \frac{1}{2}(\phi^* \mathcal{L}_n h)_{ij}$, where n^μ is the (timelike) g-unit normal of $\phi(S)$, $\mathcal{L}$ is the Lie derivative, and ϕ^* denotes the pullback of tensor fields from $\mathcal{M}$ to S. Furthermore, let $C_{\mu\nu\lambda\rho}$ denote the Weyl tensor of the metric $g_{\mu\nu}$, and denote by $E_{\mu\nu}$ and $B_{\mu\nu}$, respectively, the n-electric and n-magnetic parts of $C_{\mu\nu\lambda\rho}$. As $E_{\mu\nu}$ and $B_{\mu\nu}$ are spatial tensors, we shall be writing $E_{ij} = (\phi^* E)_{ij}$ and $B_{ij} = (\phi^* B)_{ij}$; the tensors E_{ij} and B_{ij} can be expressed purely in terms of h_{ij} and K_{ij}.

In terms of the above language, the answer we want to provide to the question raised in the opening paragraph is given by the following:

Theorem 1—Let S be a 3-manifold with at least one asymptotically Euclidean flat end, and let (h_{ij}, K_{ij}) be a solution to the Einstein vacuum constraint equations decaying on the asymptotically Euclidean end as

$$h_{ij} - \delta_{ij} = \mathcal{O}_k(r^{-\beta}), \qquad K_{ij} = \mathcal{O}_k(r^{-1-\beta}), \tag{4}$$

for some $k \geq 2$ and $\beta > 1/2$. Let the ADM 4-momentum associated to the asymptotic end be nonvanishing. If there is a function α such that

$$6\Big(E_{ik}E^k{}_j - \frac{1}{3}h_{ij}E^{kl}E_{kl}\Big) - 6\Big(B_{ik}B^k{}_j - \frac{1}{3}h_{ij}B^{kl}B_{kl}\Big) = \alpha E_{ij}, \tag{5a}$$

$$12\Big(E^k{}_{(i}B_{j)k} - \frac{1}{3}h_{ij}E_{kl}B^{kl}\Big) = \alpha B_{ij}, \tag{5b}$$

then the triplet (S, h_{ij}, K_{ij}) corresponds to a (spacelike) slice of the Schwarzschild spacetime. Conversely, for any slice of the Schwarzschild spacetime the conditions (5a) and (5b) hold with

$$\alpha = -\frac{6m}{r^3}, \tag{6}$$

where r is the radial coordinate in the standard Schwarzschild coordinates.

In the previous theorem by an asymptotically Euclidean end it is understood a portion of S which is diffeomorphic to

$$\Big\{x^i \in \mathbb{R}^3 | r = |x| = \Big(\sum_{i=1}^3 (x^i)^2\Big)^{1/2} > r_0\Big\}, \tag{7}$$

where r_0 is some positive real number. Note that the (spacelike) slices covered by the latter theorem are not necessarily Cauchy hypersurfaces. However, hyperboloidal hypersurfaces not intersecting one of the two spatial infinities of the Kruskal extension are excluded, see Fig. 1.

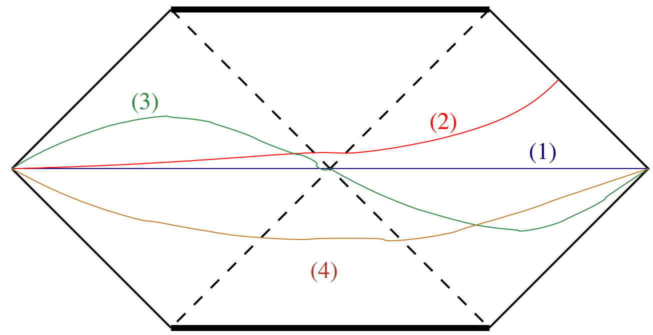

FIG. 1 (color online). Schematic description of types of hypersurfaces covered by our main theorem: (1) a time symmetric Cauchy hypersurface, (2) a hyperboloid which also reaches one of the two spatial infinities, (3) a boosted slice, (4) a generic nontime symmetric Cauchy hypersurface.

The decay conditions (4) with the prescribed values of the constants k and β are of technical nature. Some of the arguments leading to the main theorem are performed in what is called *boost-type domains*—see Sec. IV, Eq. (27) for a definition. Decay conditions in this type of domains are naturally given in terms of the notation $\mathcal{O}_k$ which is explained in Appendix A. In particular, the decay conditions (4) ensure that (see e.g. [14,15]) the Arnowitt-Deser-Misner (ADM) 4-momentum [16] given via the integrals

$$p_0 = \frac{1}{16\pi}\int_{S_\infty}(\partial_j h_{ij} - \partial_i h)dS^i, \tag{8a}$$

$$p_i = \frac{1}{8\pi}\int_{S_\infty}(K_{ij} - K\delta_{ij})dS^j, \tag{8b}$$

where $h = h_{ij}\delta^{ij}$ is well defined. The latter integrals are tailored to yield

$$(p_1)^2 + (p_2)^2 + (p_3)^2 - (p_0)^2 = -m^2, \tag{9}$$

for any Cauchy slice S in the Schwarzschild spacetime which satisfies the decay conditions (4). This can seen by calculating the integrals (8a) and (8b) on the time symmetric slice which can be obtained directly from the Schwarzschild metric in isotropic coordinates. The transformation properties of the ADM 4-momentum then yields (9) for any other slice, see e.g. [6,15].

Given a hypersurface S satisfying the conditions (5a) and (5b), the assumption of the existence of an asymptotically flat end with a nonvanishing ADM mass is sharp in order to be able to single out Schwarzschildean data. If, for example, no statement is made about the ADM momentum, then the initial data set can be either a Schwarzschildean one, or one corresponding to the C-metric. In this sense, our characterization contains a global element. In order to obtain a purely local characterization of Schwarzschild data, one would have to undertake, for example, a 3 + 1 decomposition of the characterization of the Schwarzschild spacetime in terms of concomitants of the Weyl tensor obtained by Ferrando and Sáez [8]; this will be presented elsewhere.

The article is structured as follows: in Sec. II, we discuss the property of the D'Alambertian of the Weyl tensor of some vacuum Petrov type D spacetimes which is the keystone of our characterization, the *Zakharov property.* A relation of the Petrov type D spacetimes satisfying this property is given. In Sec. III we consider the 3 + 1 decomposition of the Zakharov property. In Sec. IV a discussion of which vacuum static Petrov type D spacetimes admit asymptotically Euclidean slices is given. Section V is concerned with the ADM 4-momentum of boost-rotation symmetric spacetimes. Section VI contains a brief comments on the propagation of the Zakharov property from an initial hypersurface to the spacetime. Because of its inherent interest, a full discussion of these matters will be given elsewhere. Finally, in Sec. VII the main results of the previous sections—propositions 1, 2, 3, 4, and 5—are recalled and put in context to render our main result, theorem 1. The shortcomings of our characterization are discussed briefly, and a generalization of the characterization, valid for hyperboloidal data is given, see the conjecture in Sec. VII. There are, also, two appendices. In Appendix A some notational issues are addressed. Appendix B contains a proof that the Ehlers-Kundt solutions A2, A3, B1, B2, B3 are not asymptotically flat. This fact is required in the proof of the main theorem.

II. A RESULT ON TYPE D SPACETIMES

Let $R_{\mu\nu\lambda\rho}$ denote the Riemann tensor of the metric $g_{\mu\nu}$. Our point of departure is the following curious result to be found in the *Exact Solutions* book [10]:

Theorem 2 (Zakharov 1965, 1970, 1972)—Vacuum fields satisfying the equation

$$R_{\mu\nu\lambda\rho;\sigma}{}^{\sigma} = \alpha R_{\mu\nu\lambda\rho}, \tag{10}$$

for a certain function α are either type N ($\alpha = 0$) or type D ($\alpha \neq 0$).

The proof of this theorem follows immediately from $R_{\mu\nu} = 0$ and the identity [12]

$$R_{\mu\nu\lambda\rho;\sigma}{}^{\sigma} = R^{\sigma}{}_{\tau\mu\nu}R^{\tau}{}_{\sigma\lambda\rho} + 2(R^{\sigma}{}_{\mu\rho\tau}R^{\tau}{}_{\lambda\nu\sigma} - R^{\sigma}{}_{\nu\rho\tau}R^{\tau}_{\lambda\mu\sigma}), \tag{11}$$

written down with respect to a principal tetrad, see [11]. The theorem 2 stems from attempts due to A. L. Zel'manov (in the case $\alpha = 0$) of obtaining a characterization of spacetimes containing gravitational radiation.

A direct evaluation using computer algebra shows that the property (10) (which we shall call the *Zakharov property*) is satisfied by the Schwarzschild spacetime, but, for example, not by the Kerr solution. As the vacuum Petrov type D spacetimes are all known thanks to the work of Kinnersley [17], it is not too taxing to perform a casuistic analysis to see which are the ones satisfying the property (10). Kinnersley's analysis made use of the Newman-Penrose (NP) formalism [18] and divides naturally into two cases: those solutions for which the NP spin coefficient ρ (the expansion) vanishes and those for which it does not. The case with $\rho \neq 0$ divides, in turn, into 9 subcases. The solutions in case I have, in general, a nonvanishing Newman-Unti-Tamburino (NUT) parameter, l. If $l = 0$, then one obtains the Ehlers-Kundt solutions A1, A2, A3, see [19]. These solutions are static, and save the solution A1 (Schwarzschild) they are not asymptotically flat in the sense that there are no constants $k \geq 2$, $\beta > 1/2$ for which

$$g_{\mu\nu} - \eta_{\mu\nu} = \mathcal{O}_k(r^{-\beta}). \tag{12}$$

The latter definition of asymptotic flatness has been borrowed from [20,21] and will turn out to be most convenient for our endeavors. The $\rho \neq 0$ case II.A to II.F contain the Kerr-NUT solution and also other (nonasymptotically flat) solutions describing spinning bodies. The cases $\rho \neq 0$ III.A and III.B correspond, respectively, to the C-metric and its generalization, the spinning C-metric. These solutions are known to be compatible (for particular ranges of the parameters) with the notion of asymptotic flatness, see [22–24]. Finally, the solutions with $\rho = 0$ divide, in turn, in the two classes A and B. The class A corresponds to the Ehlers-Kundt solutions (B1)–(B3) and are not asymptotically flat in the sense given by Eq. (12). The solutions of class A are spinning generalisations of class B. A summary of which of the vacuum Petrov type D spacetimes satisfy the Zakharov property, Eq. (10), is given in Table I. From there, we derive the following:

Proposition 1—The only type D solutions satisfying the Zakharov property, Eq. (10), are the Ehlers-Kundt solutions A1, B1, B2, B3, and C.

TABLE I. Relation of the vacuum, type D spacetimes satisfying the Zakharov property. The description of the different cases follows the discussion given in Kinnersley's analysis—see [17,25]. The case I with $l = 0$ corresponds to Ehlers-Kundt solutions (A1) (Schwarzschild), A2, A3. The case A corresponds to the Ehlers-Kundt solutions (B1)–(B3) [19].

$\rho \neq 0$	Case I (NUT metrics)	only if $l = 0$
	Case II.A (Kerr-NUT)	no
	Case II.B	no
	Case II.C	no
	Case II.D	no
	Case II.E	no
	Case II.F	no
	Case III.A (C-metric)	yes
	Case III.B (twisting C-metric)	no
$\rho = 0$	Case A	yes
	Case B	no

Arguably, of the spacetimes in Table I satisfying the property (10) those of most interest are the Schwarzschild spacetime and the C-metric. For the Schwarzschild spacetime in the standard coordinates (t, r, θ, φ) the line element assumes the form

$$g_S = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{13}$$

and the proportionality function is given by

$$\alpha_S = -\frac{6m}{r^3}. \tag{14}$$

On the other hand, for the C-metric in the coordinates (t, x, y, p) (see e.g. [26]) such that

$$g_C = \frac{1}{A^2(x+y)^2}\left(F(y)dt^2 - \frac{dx^2}{G(x)} - \frac{dy^2}{F(y)} - G(x)dp^2\right), \tag{15}$$

where

$$G(x) = 1 - x^2 - 2mAx^3, \quad F(y) = -1 + y^2 - 2mAy^3, \tag{16}$$

one has that

$$\alpha_C = -6A^3 m(x+y)^3. \tag{17}$$

III. A 3 + 1 DECOMPOSITION

The property (10) in theorem 2 provides the cornerstone for a characterization of the Schwarzschild spacetime that projects neatly under a 3 + 1 decomposition. The crucial observation is that in vacuum, the tensor

$$Z_{\mu\nu\lambda\rho} = R_{\mu\nu\lambda\rho;\sigma}{}^{\sigma} = C_{\mu\nu\lambda\rho;\sigma}{}^{\sigma}, \tag{18}$$

where $C_{\mu\nu\lambda\rho}$ is the Weyl tensor of $g_{\mu\nu}$, is Weyl-like, that is, it is tracefree; $Z_{\mu\nu\lambda\rho} = Z_{\lambda\rho\mu\nu} = -Z_{\nu\mu\lambda\rho} = -Z_{\mu\nu\rho\lambda}$; and satisfies the first Bianchi identity $Z_{\mu\nu\lambda\rho} + Z_{\lambda\mu\nu\rho} + Z_{\nu\lambda\mu\rho} = 0$.

Let n^μ be an unit timelike vector, and let us denote by $h^\mu{}_\nu = g^\mu{}_\nu - n^\mu n_\nu = \delta^\mu{}_\nu - n^\mu n_\nu$ the associated projector. Following the notation and conventions of [27], we decompose the Weyl tensor as

$$C_{\mu\nu\lambda\rho} = 2(l_{\mu[\lambda}E_{\rho]\nu} - l_{\nu[\lambda}E_{\rho]\mu} - n_{[\lambda}B_{\rho]\tau}\epsilon^{\tau}{}_{\mu\nu} - n_{[\mu}B_{\nu]\tau}\epsilon^{\tau}{}_{\lambda\rho}), \tag{19}$$

where

$$E_{\tau\sigma} = C_{\mu\nu\lambda\rho}h^\mu{}_\tau n^\nu h^\lambda{}_\sigma n^\rho, \tag{20a}$$

$$B_{\tau\sigma} = C^*_{\mu\nu\lambda\rho}h^\mu{}_\tau n^\nu h^\lambda{}_\sigma n^\rho, \tag{20b}$$

denote, respectively, the *n-electric* and *n-magnetic* parts of $C_{\mu\nu\lambda\rho}$, $\epsilon_{\tau\lambda\rho} = \epsilon_{\sigma\tau'\lambda'\rho'}n^\sigma h^{\tau'}{}_\tau h^{\lambda'}{}_\lambda h^{\rho'}{}_\rho$ is the spatial Levi-Civita tensor, $l_{\mu\nu} = h_{\mu\nu} + n_\mu n_\nu$, and $C^*_{\mu\nu\lambda\rho} = \frac{1}{2}C_{\mu\nu\tau\sigma}\epsilon^{\tau\sigma}{}_{\lambda\rho}$ denotes the dual of $C_{\mu\nu\lambda\rho}$. The electric and magnetic parts of $C_{\mu\nu\lambda\rho}$ are symmetric, $E_{\mu\nu} = E_{\nu\mu}$, $B_{\mu\nu} = B_{\nu\mu}$, and traceless $E^\mu{}_\mu = B^\mu{}_\mu = 0$. Moreover, they are spatial tensors in the sense that $E_{\mu\nu'}h^{\nu'}{}_\nu = B_{\mu\nu'}h^{\nu'}{}_\nu = 0$; and $C_{\mu\nu\lambda\rho} = 0$ if and only if $E_{\mu\nu} = B_{\mu\nu} = 0$.

Using the embedding ϕ, we can calculate the pullbacks of the electric and magnetic parts of $C_{\mu\nu\lambda\rho}$ to the hypersurface S. Consequently, let us write $E_{ij} = (\phi^*E)_{ij}$ and $B_{ij} = (\phi^*B)_{ij}$. It is a direct consequence of the Codazzi equations that one can write

$$E_{ij} = r_{ij} + KK_{ij} - K_{ik}K^k{}_j, \tag{21a}$$

$$B_{ij} = -2\epsilon_i{}^{kl}D_k K_{lj}, \tag{21b}$$

where r_{ij} denotes the Ricci tensor of the 3-metric $h_{ij} = (\phi^*h)_{ij}$. Thus, on S, the electric and magnetic parts of the Weyl tensor can be entirely written in terms of the initial data (h_{ij}, K_{ij}). Note, that, in particular, for time symmetric spacetimes one has $B_{ij} = 0$ as $K_{ij} = 0$.

The tensor $Z_{\mu\nu\lambda\rho}$, being Weyl-like, admits a similar decomposition in terms of n-electric and n-magnetic parts, which we shall denote by $D_{\mu\nu}$ and $H_{\mu\nu}$, respectively. Hence, we write

$$Z_{\mu\nu\lambda\rho} = 2(l_{\mu[\lambda}D_{\rho]\nu} - l_{\nu[\lambda}D_{\rho]\mu} - n_{[\lambda}H_{\rho]\tau}\epsilon^{\tau}{}_{\mu\nu} - n_{[\mu}H_{\nu]\tau}\epsilon^{\tau}{}_{\lambda\rho}), \tag{22}$$

where

$$D_{\tau\sigma} = Z_{\mu\nu\lambda\rho}h^\mu{}_\tau n^\nu h^\lambda{}_\sigma n^\rho, \tag{23a}$$

$$H_{\tau\sigma} = Z^*_{\mu\nu\lambda\rho}h^\mu{}_\tau n^\nu h^\lambda{}_\sigma n^\rho, \tag{23b}$$

and $Z^*_{\mu\nu\lambda\rho} = \frac{1}{2}Z_{\mu\nu\tau\sigma}\epsilon^{\tau\sigma}{}_{\lambda\rho}$. As in the case of $E_{\mu\nu}$ and $B_{\mu\nu}$, one has that $D_{\mu\nu} = D_{\nu\mu}$, $H_{\mu\nu} = H_{\nu\mu}$, $D^\mu{}_\mu = H^\mu{}_\mu = 0$, $D_{\mu\nu'}h^{\nu'}{}_\nu = H_{\mu\nu'}h^{\nu'}{}_\nu = 0$; and $Z_{\mu\nu\lambda\rho} = 0$ if and only if $D_{\mu\nu} = H_{\mu\nu} = 0$.

For vacuum spacetimes, the identity (11) allows to write the tensors $D_{\mu\nu}$ and $H_{\mu\nu}$ as quadratic expressions of $E_{\mu\nu}$ and $B_{\mu\nu}$. A lengthy, but straightforward calculation renders the remarkably simple expressions:

$$D_{\mu\nu} = 6\left(E_{\mu\sigma}E^\sigma{}_\nu - \frac{1}{3}h_{\mu\nu}E^{\sigma\tau}E_{\sigma\tau}\right) - 6\left(B_{\mu\sigma}B^\sigma{}_\nu - \frac{1}{3}h_{\mu\nu}B^{\sigma\tau}B_{\sigma\tau}\right), \tag{24a}$$

$$H_{\mu\nu} = 12\left(E^\sigma{}_{(\mu}B_{\nu)\sigma} - \frac{1}{3}h_{\mu\nu}E_{\sigma\tau}B^{\sigma\tau}\right). \tag{24b}$$

These expressions can be pulled back to the hypersurface S by means of the embedding ϕ to obtain the following:

Proposition 2—Necessary conditions for an initial data set (S, h_{ij}, K_{ij}) to be a Schwarzschildean initial data set are

$$D_{ij} = \alpha E_{ij}, \tag{25a}$$

$$H_{ij} = \alpha B_{ij}, \tag{25b}$$

where $\alpha = -6m/r^3$, where r is the standard Schwarzschild radial coordinate.

It is noted that the C-metric satisfies an analogous theorem with $\alpha = -6A^3m(x+y)^3$.

Finally, we bring to attention that the scalar α can be expressed entirely in terms of the n-electric and n-magnetic parts of the Weyl tensor. Indeed, contracting Eq. (25b) with E^{ij} one obtains

$$\alpha = 12\frac{E_i{}^k B_{jk} E^{ij}}{E_{ij}B^{ij}} = 12J/I, \tag{26}$$

where I and J are the usual scalars of the Weyl tensor, see e.g. [10].

IV. ASYMPTOTIC FLATNESS AND STATIC TYPE D SPACETIMES

In order to be able to discern Schwarzschildean data from among all those vacuum type D initial data sets satisfying the conditions $D_{ij} = \alpha E_{ij}$ and $H_{ij} = \alpha B_{ij}$, we require a couple of further results. Our first task is to get rid of those spacetimes which admit no slices with asymptotically Euclidean ends. Intuitively, it seems clear that a static spacetime which is not asymptotically flat should not admit slices with asymptotically flat ends. More precisely, one has the following:

Proposition 3—If a vacuum static spacetime is not asymptotically flat in the sense given by Eq. (12), i.e. it belongs to the Ehlers-Kundt classes A2, A3, B1, B2,B3, then it admits no slices with asymptotically Euclidean ends for which the decay conditions (4) hold.

That the Ehlers-Kundt solutions A2, A3, B1, B2, B3 are not asymptotically flat can be proof by means of a careful revisiting of Kinnersley's construction of all vacuum, type D spacetimes. In order not to clog the discussion, a proof of this is presented in Appendix B.

The proof of the proposition 3 is by contradiction. Assume that our nonasymptotically flat, static spacetime, $\mathcal{M}$, admits a slice, S, with an asymptotically Euclidean end for which the asymptotic decay conditions (4) hold. By construction, in this slice one has that $h_{ij} - \delta_{ij} = \mathcal{O}_k(r^{-\beta})$ and $K_{ij} = \mathcal{O}_{k-1}(r^{-1-\beta})$ with $\beta > 1/2$ and $k \geq 2$. For this type of initial data the solution to the *boost problem* (see [28]) ensures the existence of a *boost-type domain* $\Omega_{r_0,\theta}$ of the form

$$\Omega_{r_0,\theta} = \{(t, x^i) \in \mathbb{R} \times \mathbb{R}^3 \,|\, |x| \geq r_0, |t| \leq \theta|x|\}, \tag{27}$$

for some constants r_0 and θ, such that $r_0 > 0$ and $0 < \theta < 1$. From the fact that $\mathcal{M}$ is static, it follows that the slice S possesses a static *Killing initial data set* (KID). That is, there exists a pair (N, X^μ), where N is a scalar field and X^μ is a spatial vector field ($X^\mu h_{\mu\nu} = 0$) such that $\xi^\mu|_S = Nn^\mu + X^\mu$, where ξ^μ denotes the static Killing vector of the spacetime $\mathcal{M}$, and n^μ is the normal to S. In what follows, let X^i denote the pullback of X^μ, i.e. $X^i = (\phi^* X)^i$. Now, it is natural to consider the evolution of the initial data set (h_{ij}, K_{ij}) along the flow given by the static Killing vector ξ^μ. Thus, in $\Omega_{r_0,\theta}$ one has that the spacetime metric is given by

$$g = -N^2dt^2 + h_{ij}(dx^i + X^i)(dx^j + X^j). \tag{28}$$

Recall that along this flow one has that $\partial_t h_{ij} = 0$. Furthermore, see theorem 2.1 in [21] and also theorem 2.1 in [20]; the lapse N and shift X^i behave asymptotically as

$$N = 1 + \mathcal{O}_k(r^{-\beta}), \tag{29a}$$

$$X^i = \mathcal{O}_k(r^{-\beta}), \tag{29b}$$

with $\beta > 1/2$ and $k \geq 2$. Thus, it follows that in $\Omega_{r_0,\theta}$

$$g_{\mu\nu} - \eta_{\mu\nu} = \mathcal{O}_k(r^{-\beta}). \tag{30}$$

This is a contradiction to the assumption that spacetime is not asymptotically flat.

V. THE ADM MASS OF THE C-METRIC

The proposition 3 reduces our task of characterising Schwarzschildean initial data to finding a way of distinguishing between initial data corresponding to the C-metric and those corresponding to the Schwarzschild spacetime.

The C-metric belongs to the so-called class of boost-rotation symmetric spacetimes (see [24,29,30]), that is, it possesses two commuting, hypersurface orthogonal Killing vectors. One of them is axial, and the other is of boost type. An argument outlined by Dray in [31] leads to

Proposition 4 (Dray, 1982)—The ADM 4-momentum of a boost-rotation symmetric spacetimes which is asymptotically flat (in the sense of Eq. (12)) vanishes.

This (at first sight) surprising result deserves a comment. From the mass positivity theorem of Schoen and Yau [32–34] we know that if the ADM 4-momentum of *regular* Cauchy hypersurface S vanishes then the hypersurface corresponds to a slice in Minkowski spacetime. In view of this, Dray's result on the vanishing of the ADM 4-momentum of the C-metric can be seen as a manifestation of the existence of strut (naked) singularities in the interior of the spacetime which produce *nonregular* Cauchy hypersurfaces. This is a feature that haunts all boost-rotation symmetric solutions [29].

Our strategy will be to make use of the latter result to discern between initial data sets corresponding to the C-metric, and those of the Schwarzschild spacetime.

Dray's original argument lacks of some technical details, which we now proceed to fill. Let $(\mathcal{M}, g)$ denote a boost-rotation symmetric spacetime, and let us denote by χ^μ, ξ^μ, respectively, the axial and boost Killing vectors of the spacetime. The vectors χ^μ and ξ^μ commute. From the general theory of boost-rotation symmetric spacetimes given in [29] we know that there is a region of the spacetime (the one below the so-called *roof*) where the spacetime is static. The portion of the spacetime below the roof admits a boost-type domain, $\Omega_{r_0,\theta}$, like the one in (27). From the fact that static spacetimes (and, in general, flat stationary spacetimes) admit a smooth null infinity, see e.g. [35], and from the analysis of [21] it follows that on $\Omega_{r_0,\theta}$ there exist matrices $\sigma_{\mu\nu} = \sigma_{[\mu\nu]}$, $\rho_{\mu\nu} = \rho_{[\mu\nu]}$ such that

$$\chi^\mu - \sigma^\mu{}_\nu x^\nu = \mathcal{O}_k(r^{-\beta}), \tag{31a}$$

$$\xi^\mu - \rho^\mu{}_\nu x^\nu = \mathcal{O}_k(r^{-\beta}), \tag{31b}$$

for $k \geq 2$ and $\alpha > 1/2$, with $\sigma^\mu{}_\nu \equiv \eta^{\mu\lambda}\sigma_{\lambda\nu}$, $\rho^\mu{}_\nu \equiv \eta^{\mu\lambda}\rho_{\lambda\nu}$, and $\eta_{\mu\nu}$ denoting the Minkowski metric. Without loss of generality assume that the axis of symmetry of the axial Killing vector lies along the x^3 axis. Accordingly,

$$\sigma_{\mu\nu}x^\nu = (0, -x^2, x^1, 0), \tag{32a}$$

$$\rho_{\mu\nu}x^\nu = (-x^3, 0, 0, -t). \tag{32b}$$

Thus, from the commuting nature of the two Killing vectors χ^μ and ξ^μ it follows that

$$\sigma^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{33}$$

One can associate to the boost-type domain $\Omega_{r_0,\theta}$, provided that $k \geq 2$ and $\beta > 1/2$, in an unique way an ADM 4-momentum vector p^μ, see e.g. [14,15]. It follows from the theory developed in [21] that

$$\sigma^\mu{}_\nu p^\nu = \rho^\mu{}_\nu p^\nu = 0, \tag{34}$$

whence necessarily

$$p^\mu = 0, \tag{35}$$

which is the observation made by Dray in [31]. As a side remark, note that the above result needs not to hold if the Killing vectors are noncommuting.

VI. PROPAGATION OF THE ZAKHAROV'S CONDITIONS

The last step in our argumentation is a result guaranteeing the propagation of the Zakharov's conditions (25a) and (25b). This is necessary for one could have, for example, a spacetimetime of a more general algebraic type which degenerate precisely at the initial hypersurface S. More precisely, we require a result stating that if the conditions (25a) and (25b) hold initially on an initial hypersurface S, then they will also hold at latter times. More precisely, we have the following

Proposition 5—Let (S, h_{ij}, K_{ij}) be an initial data set satisfying both the decay conditions (4) and the Zakharov's conditions (25a) and (25b). Then, in the boost-type domain $\Omega_{r_0,\theta}$ the Zakharov property, Eq. (10).

In order to prove the last statement one requires to construct a subsidiary system for the propagation of the Zakharov's conditions (25a) and (25b) with an α given by Eq. (26). Because of the complicated algebraic structure of α it seems more convenient to work either with a frame formalism or with a spinorial one. This is complicated and technical argumentation. Because of the inherent interest of this construction, the details of this construction and those of the proof of proposition 5 will be given elsewhere.

VII. CONCLUDING REMARKS

Our main theorem (see the introductory section) follows directly from the propositions 1, 2, 3, 4, and 5. As a by-product, our results imply the following (peculiar) characterization of the Schwarzschild spacetime:

Theorem 3—Let $\Omega_{r_0,\theta}$ be an asymptotically flat boost-type domain, in the sense that there are $k \geq 2$ and $\beta > 1/2$ for which Eq. (12) holds, with nonvanishing ADM 4-momentum. Then $\Omega_{r_0,\theta}$ is isometric to a portion of the Schwarzschild spacetime.

It is clear from the argumentation that the conditions to single out the Schwarzschild solution are sharp. In particular, as seen from proposition 4 if no hypothesis on the ADM 4-momentum is made, initial data for the C-metric is included. Precisely because of this condition, it is that our argumentation can not be extended to include hyperboloidal initial data sets not intersecting spatial infinity like the ones discussed in [36]. Intuitively, in the case of hyperboloidal data one would try to replace the condition on the ADM 4-momentum by some condition regarding the Bondi 4-momentum. However, it is well known that the Bondi mass of boost-rotation symmetric spacetimes is nonvanishing, see e.g. [24]. An alternative is to replace the condition on the ADM 4-momentum by a condition on the so-called Newman-Penrose (NP) constants [37,38]. The NP constants vanish for the Schwarzschild spacetime, see e.g. [39], but are nonvanishing for the C-metric, confront e.g. [40]. Friedrich and Kánnár [41] have shown how these quantities defined at null infinity can be expressed in terms of Cauchy initial data. In principle, the NP constants are also expressible in terms of hyperboloidal data—the details of this have not yet been worked out, and will be pursued elsewhere. Accordingly, we state (without going fully into the details) the following:

Conjecture—Let S be a 3-manifold with a hyperboloidal end, and let (h_{ij}, K_{ij}) be a pair of symmetric tensors on S satisfying the Einstein vacuum constraints. Assume that the Newman-Penrose constants associated to the hyperboloid S vanish. If there is a function α such that the conditions (5a) and (5b) hold, i.e.

$$D_{ij} = \alpha E_{ij}, \qquad H_{ij} = \alpha B_{ij}, \tag{36}$$

then the triplet (S, h_{ij}, K_{ij}) corresponds to an hyperboloid of the Schwarzschild spacetime.

For a discussion on the appropriate boundary conditions giving rise to a hyperboloidal end, the reader is remitted to [42]—see also [43] and references therein.

The question whether the theorems 1 or 2 can be used to construct Schwarzschildean initial data sets with special properties (for example, boosted slices with vanishing mean curvature, if these exist) remains open. In any case, the conditions (5a) and (5b) are necessary conditions for an initial data set to be Schwarzschildean. Also, it would be of interest to see if it is possible to obtain a reformulation of (5a) and (5b) which does not contain the function α. These ideas will be pursued elsewhere.

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APPENDIX A: THE NOTATIONS $\mathcal{O}_k$ AND $\mathcal{O}^\infty$

The arguments leading to the propositions 3 and 4 require assumptions on the decay of the gravitational field on so-called boost-type domains $\Omega_{r_0,\theta}$, see Eq. (27). As the name points out, this kind of asymptotic domains contains the asymptotic ends of boosted and nonboosted Cauchy hypersurfaces. However, no hypersurfaces opening towards null infinity are contained. Accordingly, the usual notions of decay of fields measure, say, in terms of affine parameters along the generators of outgoing light cones cannot be used. As mentioned in the introduction, a natural way of discussing the decay of fields on this kind of domains is by means of the $\mathcal{O}_k$ notation introduced in [21]. Given a function ϕ on the boost-type domain $\Omega_{r_0,\theta}$, we say that $\phi = \mathcal{O}_k(r^\varepsilon)$, for $\varepsilon \in \mathbb{R}$, if $\phi \in C^k(\Omega_{r_0,\theta})$ and there is a function $C(t)$ such that

$$|\partial_{\alpha_1} \cdots \partial_{\alpha_i} \phi| \leq C(t)(1 + |x|)^{\varepsilon - i}, \qquad 0 \leq i \leq k. \tag{A1}$$

By means of the boost theorem [28], and assuming the decay conditions (4) it is possible to guarantee the existence of solutions to the Einstein field equations on the domain $\Omega_{r_0,\theta}$. In order to be able to express the decay of the fields in the standard way, i.e. by means of affine parameters along generators of light cones, one needs to guarantee the existence of portion of spacetime which is bigger than a boost-type domain and which contains a portion of null infinity. This task is, in general, much more complicated and may require extra conditions on the initial data. However, in the case of static/stationary initial data that it can be done without further complications. This is discussed in the Appendix B.

In Appendix B, we shall also make use of the $\mathcal{O}^\infty$ notation. A function Φ is said to be of $\mathcal{O}^\infty(f(r))$ if there is a C^∞ function $f(r)$ such that $|\Phi| \leq |f(r)|$, $|\partial_\mu \Phi| \leq |\partial f/\partial r|$, $\partial_\mu \partial_\nu \Phi| \leq |\partial^2 f/\partial r^2|$, etc.

APPENDIX B: ON THE (NON-) ASYMPTOTICAL FLATNESS OF THE EHLERS-KUNDT SOLUTIONS A2, A3, B1, B2, B3

In this appendix we show that the static Ehlers-Kundt solutions A2, A3, B1, B2, B3 are not asymptotically flat. More precisely, we have the following:

Theorem 4—There is no coordinate system (t, x^i) in the boost-type domain $\Omega_{r_0,\theta}$ for which the metrics corresponding to the solutions A2, A3, B1, B2, B3 can be written as

$$g_{\mu\nu} - \eta_{\mu\nu} = \mathcal{O}_k(r^{-\beta}),$$

for $k \geq 2$ and $\beta > 1/2$, where $r = |x| = \delta_{ij} x^i x^j$.

Newman and collaborators and Kinnersley rederived the solutions in question by making use of the Newman-Penrose formalism. The solutions in question correspond to the Kinnersley classes $\rho \neq 0$, I (A2, A3) and $\rho = 0$, case IV.A (B1, B2, B3). In order to show the nonasymptotic flatness we will need to revisit Kinnersley's derivation of the solutions. In the sequel, acquaintance with the Newman-Penrose spin formalism will be assumed, see [44] and references therein.

1. Staticity and asymptotic flatness

As a first step we will show that if a static metric decays as $g_{\mu\nu} - \eta_{\mu\nu} = \mathcal{O}_k(r^{-\beta})$ with $k \geq 2$ and $\beta > 1/2$—in, say, the boost domain $\Omega_{r_0,\theta}$ (see Eq. (27))—then the metric actually admits an analytic null infinity. Let ξ^μ denote the static Killing vector. It follows from [21] that there is a constant vector $A^\mu = (1, 0, 0, 0)$ such that on

$$\xi^\mu - A^\mu = \mathcal{O}_k(r^{-\beta}). \tag{B1}$$

From here it follows that the *quotient metric*, $\gamma_{\mu\nu}$, satisfies

$$\gamma_{\mu\nu} = g_{\mu\nu} - \frac{1}{\lambda} \xi_\mu \xi_\nu, \tag{B2a}$$

$$= \eta_{\mu\nu} + \delta^0_\mu \delta^0_\nu + \mathcal{O}_k(r^{-\beta}), \tag{B2b}$$

where $\lambda = \xi^\mu \xi_\mu$. The latter implies for the pullback $\gamma_{ij} = (\phi^* \gamma)_{ij}$ of the quotient metric to the hypersurface S that

$$\gamma_{ij} = \delta_{ij} + \mathcal{O}_k(r^{-\beta}), \qquad \lambda = -1 + \mathcal{O}_k(r^{-\beta}). \tag{B3}$$

with $k \geq 2$ and $\beta > 1/2$. For such a quotient metric and norm of the static Killing vector, the analysis of Beig and Simon [45] (see also [46,47]) shows that there is a coordinate system $(\tilde{x}^i)$ and a gauge in the asymptotic end such that

$$\gamma_{ij} = \delta_{ij} + \mathcal{O}^\infty(\tilde{r}^{-1}), \qquad \lambda = -1 + \mathcal{O}^\infty(\tilde{r}^{-1}), \quad \text{(B4)}$$

where $\tilde{r} = \delta_{ij}\tilde{x}^i\tilde{x}^j$. That is, the quotient admits, in the asymptotic end, a convergent expansion in powers of $1/\tilde{r}$. From here, well known results (see e.g. [48] and also [35]) imply that the static spacetime admits an analytic null infinity.

Asymptotic expansions for spacetimes with a smooth null infinity are well known. These are generally given in terms of the Newman-Penrose (NP) formalism, see e.g. [49,50]. In the case of static spacetimes, the leading term in the expansion of the NP spin coefficient $\hat{\sigma}$ can be set to zero [38][2]. In the sequel we shall make use of these expansions in the form given in Stewart's book [50]. Coordinates, $(\hat{u}, \hat{r}, \hat{\theta}, \hat{\varphi})$ are constructed as follows: a retarded time, $\hat{u}$, is used to label null hypersurfaces intersecting null infinity on a cut $C_{\hat{u}}$. On each generator of these hypersurfaces we choose an affine parameter, $\hat{r}$. Finally, one each cut $C_{\hat{u}}$, we choose spherical coordinates $(\hat{\theta}, \hat{\varphi})$ and propagate them into the spacetime by requiring that the remain constant along the generators of the hypersurfaces $\hat{u} =$ const. Associated to these coordinates, we construct a NP null tetrad $\{\hat{l}^\mu, \hat{n}^\mu, \hat{m}^\mu, \hat{\overline{m}}^\mu\}$. The vector $\hat{l}^\mu$ is tangent to the generators of the $\hat{u} =$ const. hypersurfaces. It is future pointing and orthogonal to the 2-surfaces $\hat{u} =$ const., $\hat{r} =$ const., $Z_{\hat{u},\hat{r}}$. There is precisely only one other null direction with this property. The vector $\hat{n}^\mu$ is parallel to it. Finally, $\hat{m}^\mu$, $\hat{\overline{m}}^\mu$ span the tangent space to $Z_{\hat{u},\hat{r}}$. From the construction on has that

$$\hat{l}^\mu = (0, 1, 0, 0), \quad \text{(B5a)}$$
$$\hat{n}^\mu = (1, Q, C^2, C^3), \quad \text{(B5b)}$$
$$\hat{m}^\mu = (0, 0, P^2, P^3). \quad \text{(B5c)}$$

Now, choosing a conformal factor Ω such that

$$\Omega = 1/\hat{r}, \quad \text{(B6)}$$

we have that the components of the Weyl tensor behave, asymptotically as

$$\hat{\Psi}_0 = \Psi_0^0\Omega^5 + \mathcal{O}(\Omega^6), \quad \text{(B7a)}$$
$$\hat{\Psi}_1 = \Psi_1^0\Omega^4 - \bar{\eth}\Psi_0^0\Omega^5 + \mathcal{O}(\Omega^6), \quad \text{(B7b)}$$
$$\hat{\Psi}_2 = \Psi_2^0\Omega^3 - \bar{\eth}\Psi_1^0\Omega^4 + \mathcal{O}(\Omega^5), \quad \text{(B7c)}$$
$$\hat{\Psi}_3 = -\bar{\eth}\Psi_2^0\Omega^3 + \mathcal{O}(\Omega^4), \quad \text{(B7d)}$$
$$\hat{\Psi}_4 = \mathcal{O}(\Omega^3). \quad \text{(B7e)}$$

For the NP spin coefficients one has

$$\hat{\rho} = -\Omega + \mathcal{O}(\Omega^5), \quad \text{(B8a)}$$
$$\hat{\sigma} = \mathcal{O}(\Omega^4), \quad \text{(B8b)}$$
$$\hat{\alpha} = q\Omega + \mathcal{O}(\Omega^3), \quad \text{(B8c)}$$
$$\hat{\beta} = -q\Omega + \mathcal{O}(\Omega^3), \quad \text{(B8d)}$$
$$\hat{\pi} = \mathcal{O}(\Omega^3), \quad \text{(B8e)}$$
$$\hat{\tau} = \mathcal{O}(\Omega^3), \quad \text{(B8f)}$$
$$\hat{\gamma} = -\frac{1}{2}\Psi_2^0\Omega^2 + \mathcal{O}(\Omega^3), \quad \text{(B8g)}$$
$$\hat{\lambda} = \mathcal{O}(\Omega^3), \quad \text{(B8h)}$$
$$\hat{\mu} = -\frac{1}{2}\Omega + \Psi_2^0\Omega^2 + \mathcal{O}(\Omega^3), \quad \text{(B8i)}$$
$$\hat{\nu} = \bar{\eth}\Psi_2^0\Omega^2 + \mathcal{O}(\Omega^3). \quad \text{(B8j)}$$

Finally, for the frame components the expansions yield

$$Q = -\frac{1}{2} - \Psi_2^0\Omega + \mathcal{O}(\Omega^2), \quad \text{(B9a)}$$
$$C^A = \mathcal{O}(\Omega^3), \quad \text{(B9b)}$$
$$P^A = p^A\Omega + \mathcal{O}(\Omega^3), \quad \text{(B9c)}$$

where $A = 2, 3$. In the above expansions $q = -2^{-3/2}\cot\hat{\theta}$ and $p^2 = 2^{-1/2}$, $p^3 = -2^{-1/2}i\csc\hat{\theta}$. Because of the static character of the spacetime, the coefficients in the expansions are $\hat{u}$-independent and $\Psi_2^0 = \overline{\Psi}_2^0$.

2. Asymptotic flatness and the Kinnersley's construction

In this subsection we revisit part of Kinnersley's construction of all vacuum, type D spacetimes under the assumption of asymptotic flatness. From this analysis it will follow that the Ehlers-Kundt static solutions A2, A2, B1, B2, and B3 cannot be asymptotic flat.

Kinnersley's construction makes use of the NP formalism with a null tetrad $\{l^\nu, n^\mu, m^\mu, \overline{m}^\mu\}$ adapted to the the algebraic character of the spacetimes[3]. It is well known that for a type D spacetime, the vectors l^μ and n^μ can be chosen to point, respectively, along each of the two repeated null principal directions. It follows that with respect to such a

[2]Here, and in the sequel, a hat indicates that the quantity in question is given a certain gauge, the NP gauge, which is adapted to null infinity.

[3]In order to differentiate the NP objects belonging to Kinnersley's tetrad from those in the gauge discussed in the previous section and which is adapted to null infinity, the former will have no hat.

frame, there is only one nonvanishing component of the Weyl tensor, namely, Ψ_2. Further, because of the Goldberg-Sachs theorem, see e.g. [10], one has that

$$\kappa = \sigma = \nu = \lambda = 0. \tag{B10}$$

Let $\check{r}$ be an affine parameter along l^μ. The components of the null tetrad read

$$l^\mu = (0, 1, 0, 0), \tag{B11a}$$

$$n^\mu = (X^0, U, X^2, X^3), \tag{B11b}$$

$$m^\mu = (\varsigma^0, \omega, \varsigma^2, \varsigma^3). \tag{B11c}$$

The remaining freedom in this tetrad consists of a spin and a boost. One can use them to set the spin coefficient ϵ to zero.

The tetrads $\{\hat{l}^\mu, \hat{n}^\mu, \hat{m}^\mu, \hat{\overline{m}}^\mu\}$ and $\{l^\mu, n^\mu, m^\mu, \overline{m}^\mu\}$ are related to each other by means of a Lorentz transformation which can be written as the composition of: a null rotation about l^μ with parameter $a \in \mathbb{C}$, $\mathfrak{l}_a$; a null rotation about $\mathfrak{l}_a(n^\mu)$ with parameter $b \in \mathbb{C}$, $\mathfrak{n}_b$; a boost with parameter $c \in \mathbb{R}$, $\mathfrak{b}_c$; and a spin with parameter $\vartheta \in \mathbb{R}$, $\mathfrak{s}_\vartheta$. A complete list of the action of the transformations $\mathfrak{l}_a$, $\mathfrak{n}_b$, $\mathfrak{b}_c$, and $\mathfrak{s}_\vartheta$ on the NP scalars can be found in [51].

The parameters a and b of the null rotations can be determined from (B10). That is, by requiring

$$\mathfrak{n}_b\mathfrak{l}_a(\hat{\kappa}) = \mathfrak{n}_b\mathfrak{l}_a(\hat{\sigma}) = \mathfrak{n}_b\mathfrak{l}_a(\hat{\nu}) = \mathfrak{n}_b\mathfrak{l}_a(\hat{\lambda}) = 0. \tag{B12}$$

Note that the boost $\mathfrak{b}_c$ and the spin $\mathfrak{s}_\vartheta$ have not been used here for the scalars κ, σ, ν, and λ transform homogeneously under them, and thus they cannot be used to set these particular spin coefficients to zero. Some experimentation reveals that the parameters a and b have to be of the form:

$$a = a_{-1}\Omega^{-1} + a_0 + \mathcal{O}^\infty(\Omega), \tag{B13a}$$

$$b = b_1\Omega + b_2\Omega^2 + \mathcal{O}^\infty(\Omega^3), \tag{B13b}$$

where the coefficients a_{-1}, a_0, b_1, and b_2 are Ω-independent. It is noted, by passing, that a and b have, respectively, spin-weights -1 and 1. The boost and spin $\mathfrak{b}_c$ and $\mathfrak{s}_\vartheta$ are determined by the condition

$$\epsilon = \mathfrak{b}_c\mathfrak{s}_\vartheta\mathfrak{n}_b\mathfrak{l}_a(\hat{\epsilon}) = 0. \tag{B14}$$

We shall consider the following Ansatz for the expansions of c and ϑ:

$$c = c_0 + c_1\Omega + \mathcal{O}^\infty(\Omega^2), \tag{B15a}$$

$$\vartheta = \vartheta_0 + \vartheta_1\Omega + \mathcal{O}^\infty(\Omega^2). \tag{B15b}$$

Hence,

$$\mathfrak{b}_c\mathfrak{s}_\vartheta\mathfrak{n}_b\mathfrak{l}_a(\hat{\epsilon}) = \left(b_1c_0q - \overline{b}_1c_0q - c_0\overline{a}_0b_1 - \frac{1}{2}c_1 - \frac{1}{2}ic_0\vartheta_1\right)\Omega^2 + \mathcal{O}^\infty(\Omega^3). \tag{B16}$$

Now, extracting real and imaginary parts one finds that:

$$c_1 = -c_0(\overline{a}_0b_1 + a_0\overline{b}_1), \tag{B17a}$$

$$\vartheta_1 = i(a_0\overline{b}_1 - \overline{a}_0b_1). \tag{B17b}$$

Note that the coefficients c_0 and θ_0 remain, up to this point, undetermined.

a. The Ehlers-Kundt solutions B_1, B_2, and B_3

These solutions are characterized by the fact that in addition to the Goldberg-Sachs conditions (B10) one also has

$$\rho = \mu = 0, \tag{B18}$$

which, in turn, imply that $\mathfrak{n}_b\mathfrak{l}_a(\hat{\mu}) = \mathfrak{n}_b\mathfrak{l}_a(\hat{\rho}) = 0$. However, the expansions (B13a) and (B13b) render

$$\mathfrak{n}_b\mathfrak{l}_a(\hat{\mu}) = |a_{-1}|^2\Omega^{-1} + \mathcal{O}^\infty(\Omega^0), \tag{B19a}$$

$$\mathfrak{n}_b\mathfrak{l}_a(\hat{\rho}) = (-1 - |b_1|^2|a_{-1}|^2 - b_1\overline{a}_{-1} - \overline{b}_1a_1)\Omega + \mathcal{O}^\infty(\Omega^2). \tag{B19b}$$

Hence, it follows that $\mathfrak{n}_b\mathfrak{l}_a(\hat{\mu})$ and $\mathfrak{n}_b\mathfrak{l}_a(\hat{\rho})$ cannot vanish simultaneously if the spacetime is assumed to be asymptotically flat. Thus, the Ehlers-Kundt solutions B_1, B_2 and B_3 are not asymptotically flat in the sense discussed in Appendix B 1.

b. The Ehlers-Kundt solutions $A1$, $A2$, and $A3$

The Ehlers-Kundt solutions $A1$, $A2$, and $A3$ corresponds to the case I in Kinnersley's analysis. The solutions are such that $\rho \neq 0$, that is, they are expanding. Furthermore, they satisfy

$$\pi = \tau = 0. \tag{B20}$$

The action of $\mathfrak{n}_b\mathfrak{l}_a$ on $\hat{\tau}$ and $\hat{\pi}$ (π and τ transform homogenously under a spin and a boost, so there is, again, no need to consider these type of transformations) yield

$$\mathfrak{n}_b\mathfrak{l}_a(\hat{\tau}) = (-a_{-1} - a_{-1}\overline{a}_{-1}b_1) + \mathcal{O}^\infty(\Omega^0), \tag{B21a}$$

$$\mathfrak{n}_b\mathfrak{l}_a(\hat{\pi}) = (\overline{a}_{-1} + a_{-1}\overline{a}_{-1}\overline{b}_1) + \mathcal{O}^\infty(\Omega^0), \tag{B21b}$$

so that either $a_{-1} = 0$ or $b_1 = -1/\overline{a}_{-1}$. The solution $b_1 = -1/\overline{a}_{-1}$ renders, in virtue of Eq. (B19b), $\rho = \mathcal{O}^\infty(\Omega)$, which as we shall see below (confront Equation (B27)) is inconsistent with Kinnersley's analysis. So, in what follows, we shall stick to the $a_{-1} = 0$ solution. Under this assumption, one has that

$$\mathfrak{n}_b\mathfrak{l}_a(\hat{\nu}) = \dot{\overline{a}}_0 + \mathcal{O}^\infty(\Omega), \tag{B22a}$$

$$\mathfrak{n}_b\mathfrak{l}_a(\hat{\lambda}) = (\eth\overline{a}_0 - \overline{a}_0^2 + \overline{b}_1\dot{\overline{a}}_0)\Omega + \mathcal{O}^\infty(\Omega^2), \tag{B22b}$$

where $\dot{}$ denotes derivation with respect to the retarded time u. Thus, the Goldberg-Sachs conditions (B10) imply that

$$\dot{a}_0 = 0, \tag{B23a}$$

$$\bar{\eth} a_0 = a_0^2. \tag{B23b}$$

More crucially for our purposes, one finds that

$$\rho = \mathfrak{b}_c \mathfrak{s}_\vartheta \mathfrak{n}_b \mathfrak{l}_a(\hat{\rho}) = -c_0\Omega + \mathcal{O}^\infty(\Omega^2), \tag{B24a}$$

$$\mu = \mathfrak{b}_c \mathfrak{s}_\vartheta \mathfrak{n}_b \mathfrak{l}_a(\hat{\mu}) = \frac{1}{c_0}\left(\bar{\eth}\bar{a}_0 - \frac{1}{2}\right)\Omega + \mathcal{O}^\infty(\Omega^2), \tag{B34b}$$

$$\gamma = \mathfrak{b}_c \mathfrak{s}_\vartheta \mathfrak{n}_b \mathfrak{l}_a(\hat{\gamma}) = \frac{1}{2c_0^2}(\dot{c}_0 + \dot{\vartheta}_0 c_0 i) + \mathcal{O}^\infty(\Omega), \tag{B24c}$$

where the relation (B23a) has already been used.

In Kinnersley's analysis the NP scalar ρ can be readily calculated from the NP equation

$$D\rho = \rho^2. \tag{B25}$$

The latter can be written in terms of hatted derivative operators as

$$\hat{D}\rho + b\hat{\bar{\delta}}\rho + \bar{b}\,\hat{\delta}\,\rho + b\bar{b}\,\hat{\Delta}\,\rho = \rho^2. \tag{B26}$$

Now, $D\Omega = -\Omega^2$, and furthermore, $b\hat{\bar{\delta}}$, $\bar{b}\,\hat{\delta}$, $b\bar{b}\,\hat{\Delta}$ are of order $\mathcal{O}^\infty(\Omega^2)$, confront expansion (B13b). Whence,

$$\rho = -\Omega + \mathcal{O}^\infty(\Omega). \tag{B27}$$

Comparing the latter equation with Eq. (B24a) we see that $c_0 = 1$. Once the scalar ρ is known, other radial NP field equations can be integrated. In particular one has that,

$$\Psi_2 = \Psi^0\rho^3, \tag{B28a}$$

$$\gamma = \gamma^0 + \frac{1}{2}\Psi^0\rho^2, \tag{B28b}$$

$$\mu = \mu^0\bar{\rho} + \frac{1}{2}\Psi^0(\rho^2 + \rho\bar{\rho}), \tag{B28c}$$

where Ψ^0, γ^0, and μ^0 are Ω-independent functions. In Kinnersley's original analysis (see also [25]) the remaining freedom in the construction of the tetrad—a constant spin—can be used to set $\gamma^0 = 0$. This is equivalent (see Eq. (B24c)) to requiring $\vartheta_0 =$ const.. It can be shown that the latter actually implies $\gamma = \mathfrak{b}_c \mathfrak{s}_\vartheta \mathfrak{n}_b \mathfrak{l}_a(\hat{\gamma}) = \mathcal{O}^\infty(\Omega^2)$, consistent with Eq. (B28b). Further, the coefficient μ^0 is found to be a numerical constant taking the values $1/2$, 0, or $-1/2$. The case $\mu^0 = 1/2$ corresponds to the Schwarzschild solution.

Comparing Eq. (B34b) and (B28c), we find that

$$\bar{\eth}\bar{a}_0 - \frac{1}{2} = -\mu^0, \tag{B29}$$

from where it follows that a_0 must be a numerical constant. However, the only constant solution of Eq. (B23b) is the zero solution. Thus $a_0 = 0$ and

$$\mu^0 = 1/2. \tag{B30}$$

Hence, the Schwarzschild solution is the only among the Ehlers-Kundt solutions A1, A2, A3 which is asymptotically flat.

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