PHYSICAL REVIEW D **72,** 083504 (2005)

Inflation with violation of the null energy condition

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(Received 3 June 2005; published 6 October 2005)

Inflation may have been driven by a component which violated the Null-Energy Condition, thereby leading to *super inflation*. We provide the formalism to study cosmological perturbations when such a component is described by a scalar field with arbitrary Lagrangian. Since the background curvature grows with time, gravitational waves always have a blue spectrum. We apply our formalism to the case of phantom inflation with an exponential potential (whose polelike inflationary stage is an attractor for inhomogeneous cosmological models for any value of the potential slope) as an example. We finally compare the predictions of super inflation with those of standard inflation stressing the role of gravitational waves.

DOI: 10.1103/PhysRevD.72.083504 PACS numbers: 98.80.Cq

I. INTRODUCTION

Inflation is the most promising theory of structure formation. Its predictions for the simplest case of a single scalar-field model with a nearly scale-invariant spectrum of Gaussian curvature perturbations are in good agreement with observational data. The "smoking gun" for inflation is the detection of the stochastic background of relic gravitational waves, which carries information on the energy scale of inflation. For a scalar field described by a canonical Lagrangian, the amplitude P_T and spectral index n_T (at a certain scale k_0) of gravitational waves are locked to the amplitude of scalar perturbations P_S by the *consistency relation* $r \equiv P_T/P_S = -8n_T$. While this relation is violated in multifield inflationary models, Garriga and Mukhanov [1] showed how it can be violated also in single scalar-field models with a noncanonical Lagrangian.

What remains a generic prediction of inflation—single or multifield models—in Einstein theories is a *red spectrum* for gravitational waves. Such a prediction is related to the decrease of the Hubble parameter H during inflation [2]. In this paper we focus on the possibility that H may grow during inflation, as it occurs when the Null-Energy Condition (NEC), i.e. $p + \rho \geq 0$ (with p the pressure and ρ the energy density), is violated. Although known forms of matter seem to respect NEC, there is no evidence that the universe as a whole does at the present time [3]: it is then conceivable and interesting to explore the consequences of such a violation in the early past, i.e. during inflation. For this purpose, we provide the treatment of scalar perturbations for general scalar-field theory, extending previous works [1]. We show that, in contrast to common belief, NEC violating theories can be completely stable at the classical level when gravitational perturbations are self-consistently taken into account.

The distinctive signature of *super inflation* is a *blue-tilted* spectrum for relic gravitational waves, which offers better chances to be detected indirectly in cosmic microwave background (CMB) anisotropy measurements. Specific realizations of this model may even lead to the possibility of direct detection by space-borne interferometers, such as LISA [4] or BBO [5].

II. BASIC EQUATIONS

Let us consider the action for gravity plus a scalar field ϕ with the generic Lagrangian $p(\phi, \chi)$,

$$S \equiv \int d^4x \mathcal{L} = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} + p(\phi, \chi)\right], \quad (1)$$

where $\kappa^2 = 8\pi G$, $\chi \equiv -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. The background homogeneous equation of motion is

$$\Delta(\ddot{\phi} + 3Hc_s^2\dot{\phi}) + \frac{\partial^2 p}{\partial\phi\partial\chi}\dot{\phi}^2 - \frac{\partial p}{\partial\phi} = 0, \quad (2)$$

where

$$\Delta \equiv \frac{\partial p}{\partial\chi} + \dot{\phi}^2\frac{\partial^2 p}{\partial\chi^2}, \qquad c_s^2 \equiv \frac{\partial p}{\partial\rho}\bigg|_\phi = \frac{\frac{\partial p}{\partial\chi}}{\Delta}, \quad (3)$$

supplemented by the Hubble law

$$3H^2 = \kappa^2\rho = \kappa^2\left[2\chi\frac{\partial p}{\partial\chi} - p\right]. \quad (4)$$

We note that NEC is violated when $\partial p/\partial\chi < 0$. For simplicity we study scalar and tensor perturbations in the uniform curvature gauge around a flat Robertson-Walker (RW) line-element

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$$ds^2 = -(1+2\alpha)dt^2 - a\beta_{,i}dtdx^i + a^2(\delta_{ij} + h_{ij}^{TT})dx^i dx^j. \tag{5}$$

The equation of motion for the scalar-field fluctuation is

$$\ddot{\delta\phi} + \left[\frac{\dot{\Delta}}{\Delta} + 3H\right]\dot{\delta\phi} + \left[-\frac{c_s^2}{a^2}\nabla^2 - \frac{1}{\Delta}\frac{\partial^2 p}{\partial\phi^2} + \frac{1}{\Delta a^3} \times \left(a^3\frac{\partial^2 p}{\partial\phi\partial\chi}\dot{\phi}\right)^{\cdot}\right]\delta\phi = \left[\frac{\dot{\Delta}}{\Delta}\dot{\phi} + 2\ddot{\phi} + 3H\dot{\phi} + 3Hc_s^2\dot{\phi}\right]\alpha + \dot{\phi}\,\dot{\alpha} - \frac{c_s^2}{2a}\dot{\phi}\nabla^2\beta. \tag{6}$$

By using the energy and momentum constraints from Einstein's equations [6], and going to Fourier space, we obtain

$$\ddot{\delta\phi}_{\mathbf{k}} + \left[3H + \frac{\dot{\Delta}}{\Delta}\right]\dot{\delta\phi}_{\mathbf{k}} + \left[c_s^2\frac{k^2}{a^2} - \frac{1}{a^3\Delta x}(a^3\Delta\dot{x})^{\cdot}\right]\delta\phi_{\mathbf{k}} = 0, \tag{7}$$

where $x = \dot{\phi}/H$. This equation is one of the main results of our paper. When $c_s^2 < 0$ fluctuations are unstable on small scales. However, $c_s^2 > 0$ when $\Delta < 0$ and NEC is violated at the same time.

By defining

$$v = a\delta\phi\sqrt{|\Delta|} \tag{8}$$

$$z = a\frac{\dot{\phi}}{H}\sqrt{|\Delta|} = a\frac{|\rho + p|^{1/2}}{c_s H} \tag{9}$$

we find that Eq. (7) takes the form

$$v_{\mathbf{k}}'' + \left(c_s^2 k^2 - \frac{z''}{z}\right)v_{\mathbf{k}} = 0. \tag{10}$$

The latter equation and the definitions in Eqs. (8) and (9) agree with those in Ref. [1], but are extended to the separate region where $\Delta < 0$ and $\rho + p < 0$.

For the configurations whose dynamics evolves across the boundary $\rho + p = 0$ [7–9], Eq. (7) multiplied by Δ can be used as a regular equation. The long-wavelength solution for $\delta\phi_{\mathbf{k}}$ is

$$\delta\phi_{\mathbf{k}} = C(k)\frac{\dot{\phi}}{H} + D(k)\frac{\dot{\phi}}{H}\int dt\frac{H^2}{a^3\dot{\phi}^2\Delta} = C(k)\frac{\dot{\phi}}{H} + D(k)\frac{\dot{\phi}}{2HM_{\rm pl}^2}\int dt\frac{c_s^2}{a^3\epsilon_1}, \tag{11}$$

where the second term is the decaying mode, and in the second line we have introduced $\epsilon_1 \equiv -\dot{H}/H^2$ and the reduced Planck mass $M_{\rm pl} = \kappa^{-1}$.

The phantom crossing (if it indeed occurs [7]) affects ϕ fluctuations through Δ and not through $\partial p/\partial\chi$ (which is instead related to the parameter of state $w = p/\rho$). As it can be seen from Eq. (11), if $\Delta \propto (t - t_*)$ in the vicinity of the crossing $\dot{H}(t_*) = 0$, the decaying mode diverges logarithmically at $t = t_*$. This fact has already been observed in a different gauge for theories with $c_s = 1$ in [9]. Nevertheless, this divergence in field fluctuations may be softened (if not removed) in two cases: if c_s^2 also vanishes at the crossing [as it can be seen in the second line of Eq. (11)] or if $\Delta \propto (t - t_*)^{1/m}$, with $m > 1$ being an odd integer. We also note that scalar metric perturbations α and β,

$$\frac{H}{a}\nabla^2\beta = \kappa^2\Delta\frac{\dot{\phi}^2}{H}\left(\frac{H}{\dot{\phi}}\delta\phi\right)^{\cdot}, \qquad 2H\partial_i\alpha = \kappa^2\frac{\partial p}{\partial\chi}\dot{\phi}\partial_i\delta\phi, \tag{12}$$

remain finite across the transition $\Delta = 0$ irrespective of possible singularities in $\delta\phi$.

In terms of the *horizon flow functions* ϵ_i [defined as $\epsilon_{n+1} = \dot{\epsilon}_n/(H\epsilon_n)$, for $i \geq 2$] the potential can be written as

$$\frac{z''}{z} = F + G \tag{13}$$

where F is the usual (canonical) expansion term

$$F = a^2H^2\left(2 - \epsilon_1 + \frac{3}{2}\epsilon_2 - \frac{1}{2}\epsilon_1\epsilon_2 + \frac{1}{4}\epsilon_2^2 + \frac{1}{2}\epsilon_2\epsilon_3\right) \tag{14}$$

and G depends on the sound speed,

$$G = 2\left(\frac{c_s'}{c_s}\right)^2 - \frac{c_s''}{c_s} - 2aH\frac{c_s'}{c_s} - aH\frac{c_s'}{c_s}\epsilon_2. \tag{15}$$

Equations (13)–(15) agree with Ref. [10] for $\epsilon_1 > 0$, but their validity is extended to the region $\epsilon_1 < 0$, corresponding to NEC violating models.

The amplitude h^{TT} of gravitational waves satisfies

$$\ddot{h}_{\mathbf{k}}^{TT} + 3H\dot{h}_{\mathbf{k}}^{TT} + \frac{k^2}{a^2}h_{\mathbf{k}}^{TT} = 0, \tag{16}$$

with the usual prediction $n_T = -2\epsilon_1$. For super inflation gravitational waves have therefore a blue spectrum [2,11] with the slow-roll tensor-to-scalar ratio $r = 8c_s n_T$.

III. A TOY MODEL

As the simplest example of super inflation we consider the case with an exponential potential

$$\rho = \sigma_K\frac{\dot{\phi}^2}{2} + \sigma_V V_0 e^{-\lambda(\phi/M_{\rm pl})}, \tag{17}$$

where $\sigma_K, \sigma_V = \pm 1$. For $\sigma_K = \sigma_V = 1$ the stable phase-space trajectories for homogeneous cosmologies occur for $\lambda < \sqrt{6}$ [12]. Power-law inflation [13] ($\lambda < \sqrt{2}$) is a local asymptotic attractor among inhomogeneous cosmologies [14].

The case with $\sigma_K = \sigma_V = -1$ leads to solutions in Euclidean time and we do not consider them here.

The case with negative potential ($\sigma_K = -\sigma_V = 1$) leads to the simplest single field realization of the Ekpyrotic scenario [15]. It represents a contracting solution with ultrastiff matter ($w > 1$) which is stable for $\lambda > \sqrt{6}$ both at homogeneous [16,17] and inhomogeneous levels [17] and solves the horizon problem. Unfortunately, this single field Ekpyrotic model cannot lead to the observed nearly scale-invariant spectrum for cosmological fluctuations [18].

The case $\sigma_K = -\sigma_V = -1$ leads to a stage of polelike inflation:

$$a(t) \sim (-t)^p, \qquad t < 0, \qquad p < 0,$$
$$\phi(t) = \frac{2}{\lambda} M_{\rm pl} \log(-M_{\rm pl} t), \qquad V_0 = M_{\rm pl}^4 p(3p - 1), \tag{18}$$

where $p = -2/\lambda^2$ (i.e. $p < 0$). Such a solution is characterized by a constant state parameter $w = -1 + 2/(3p) < -1$ and $\dot{H} > 0$ (we remind that $H = p/t$, where $t = 0$ corresponds to the big rip singularity).

IV. CLASSICAL STABILITY

The stage of polelike inflation in Eq. (18) is stable for *any negative p*. It is stable within the RW backgrounds: it is easy to show that the phase points

$$\left(\frac{\kappa \dot{\phi}}{\sqrt{6} H}, \frac{\kappa \sqrt{V}}{\sqrt{3} H}\right) = \left(-\frac{\lambda}{\sqrt{6}}, \sqrt{1 + \frac{\lambda^2}{6}}\right) \tag{19}$$

are stable. This stability analysis can be extended to the case of inhomogeneous space-times following closely the analysis in Ref. [14], by introducing

$$ds^2 = -dt^2 + (-t)^{2p} h_{ij} dx^i dx^j,$$
$$h_{ij}(t, \mathbf{x}) = a_{ij} + \sum_n b_{ij}^{(n)} (-t)^n, \tag{20}$$
$$\phi(t, \mathbf{x}) = \frac{2}{\lambda} M_{\rm pl} \log(-M_{\rm pl} t) - \sum_n \Phi^{(n)} (-t)^n,$$

where $a_{ij}, b_{ij}^{(n)}, \Phi^{(n)}$ are arbitrary space-dependent functions, $n \in \{k n_1 + l n_2 + m n_3\}$, n_1, n_2, n_3 are non-negative integers, with at least a positive one, and k, l, m are positive real numbers [14]. Inserting the expansion of Eq. (20) in the Einstein equations, along the lines of Ref. [14], we obtain four solutions, two of which are the residual gauge modes of the synchronous gauge.

On the basis defined by k, l, m, we first consider $n = k$, obtaining $k = 1 - 3p$ and

$$\Phi^{(k)} = -\frac{b^{(k)}}{2\lambda}, \qquad \tilde{b}_{ij}^{(k)} \text{ arbitrary}, \tag{21}$$

where $\tilde{b}_{ij}^{(k)}$ is the trace free part of the tensor $b_{ij}^{(k)}$. For $n = l$ we obtain $l = 2(1 - p)$ and (for $p \neq -1$)

$$\Phi^{(l)} = \frac{P}{2\lambda p(p + 1)(2p - 3)},$$
$$b^{(l)} = \frac{p - 3}{2(p^2 - 1)(2p - 3)} P, \qquad \tilde{b}_{ij}^{(l)} = \frac{\tilde{P}_{ij}}{p^2 - 1}, \tag{22}$$

where P and $\tilde{P}_{ij}$ are the trace and traceless parts of the three-dimensional Ricci tensor associated to a_{ij}. The case $n = m$ with the double solution $m = 0, -1$ corresponds to a gauge mode which is not fixed by the synchronous gauge in Eq. (20) (we note that for power-law inflation the solution is also twofold with $m = 0, 1$).

Therefore we have demonstrated that phantom inflation with an exponential potential described by Eq. (18) is a local attractor (towards the big rip singularity) among inhomogeneous space-times for *any* slope of the potential, at the classical level. This independence on the slope of the potential may be interesting for a theoretical motivation of exponential potentials [19]. As for ordinary power-law inflation [14], only the field becomes smooth while the metric retains the initial $a_{ij}(x_0)$, stretched to ultralarge cosmological scales. It would be of great interest if quantum effects [20] lead to the avoidance of the singularity [21] and drive the Universe into a radiation dominated era. More in general, some physical mechanism—a second field, for instance—driving the Universe out of the inflationary epoch has to be invoked [7,22], as for power-law inflation.

V. PREDICTIONS OF THE TOY MODEL

It is interesting to study cosmological perturbations on such a stable background (see also [22] for phantom inflation with a generic potential). For an exponential potential, both scalar and tensor fluctuations satisfy the equation for massless fields, Eq. (16).

In the background given by Eq. (18) the solution for X (X can be either the amplitude h of gravitational waves or the Mukhanov variable Q) is

$$X_k = A(-\eta)^\nu H_\nu^{(1)}(-k\eta) \tag{23}$$

where A is the normalization factor and the index ν of the Hankel function is given by

$$\nu = \frac{1}{2} \frac{3p - 1}{p - 1} = \frac{3}{2} + \frac{1}{p - 1}. \tag{24}$$

The spectrum of fluctuations is blue tilted with respect to scale invariance, but the tilt is suppressed for large $|p|$.

As it happens also in the Ekpyrotic model, gauge invariant scalar fluctuations satisfy an equation in which the long-wavelength instabilities of field fluctuations in rigid space-time are removed due to the opposite sign of the kinetic and potential terms: this means that a consistent gravitational embedding leads to a self-regulation of these flat space-time instabilities.

Both the Ekpyrotic scenario and the super inflation sketched here violate the condition $|p| \leq \rho$ and lead to a singularity (in contrast with power-law inflation). It is therefore necessary to solve the graceful exit problem in both cases. However there is an important difference: the exit from super inflation needs to reverse sign to $\dot{H}$ to match with a radiation era, while in the Ekpyrotic case the graceful exit is characterized by a switch in the sign of H (as for the pre-big bang scenario [23] in the conformal frame). This difference reflects in the perturbation sector: in super inflation for $p < -1$ (i.e. for $n_T < 1$) growing and decaying modes remain as such before and after the graceful exit, while in bouncing models growing and decaying modes may invert their role before and after the bounce.

VI. GRAVITATIONAL WAVES WITH A BLUE SPECTRUM

The detection of the tensor contribution to CMB anisotropies is a major observational challenge. In Fig. 1 we show that a blue spectral index $n_T > 0$ *increases* the possibility of detection of gravitational waves, since the tensor power-spectrum (in temperature and polarization) is increased at intermediate multipoles ℓ (before the tensor contribution is cut off by the decay of gravitational waves inside the Hubble radius), for a fixed tensor-to-scalar ratio on the largest scales. The tensor contribution to CMB anisotropies depends on n_T, as it can be seen from Fig. 1: for $r = 0.38$ (a value compatible with the WMAP 2σ bound $r = 0.43$, with no running spectral indices [24]) the tensor polarization signal in super inflation with $n_T = 0.1$ is more than twice the standard inflation one with $n_T = -0.1$. For comparison we also show the difference in the tensor contribution between $n_T = 0.9$ and $n_T = -0.1$ in Fig. 2: increasing n_T, the increase in polarization amplitude is larger than in the temperature. Moreover, we also note that the peak of the B-mode spectrum shifts to larger ℓ increasing n_T. Other inflationary models beyond Einstein theories (scalar-tensor theories or higher order gravity theories) may also display a blue spectrum for gravitational waves.

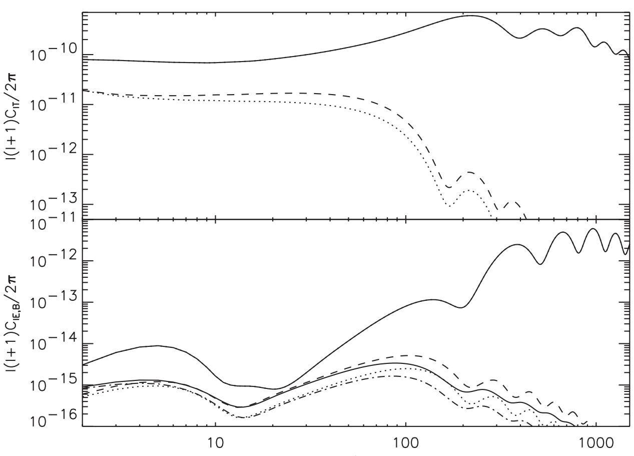

FIG. 1. Comparison of the CMB anisotropy power-spectrum for conventional inflation and super inflation. For scalar perturbations a blue spectrum $n_S = 1.1$ is assumed (as predicted by hybrid models in conventional inflation) for both standard and super inflation. For tensors $n_T = -0.1$ is assumed for inflation and $n_T = 0.1$ for super inflation. The tensor-to-scalar ratio $r \sim 0.38$ has been assumed. For reionization, the subroutine *recfast* has been used, with optical depth $\tau = 0.1$. The other parameters used are $\Omega_{CDM} = 0.26$, $\Omega_b = 0.04$, $\Omega_\Lambda = 0.7$ and $H_0 = 72\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$. In the top panel, besides the temperature scalar spectrum, the dashed (dotted) line corresponds to tensor modes in super (standard) inflation. In the bottom panel, besides the scalar E-mode (solid), the tensor E-mode is the dashed (dotted) line and the B-mode is the triple dot-dashed (dot-dashed) line for super (standard) inflation.

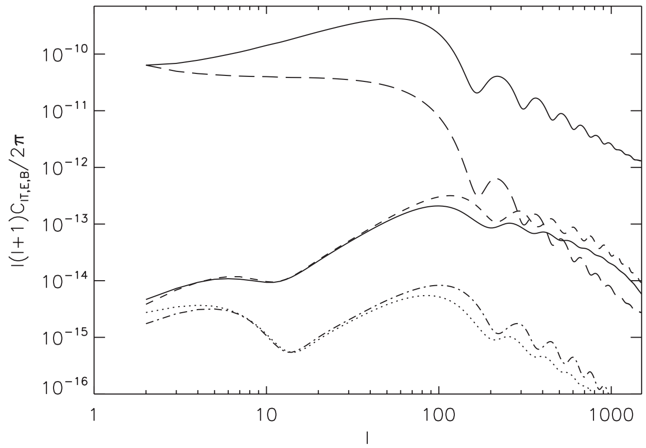

FIG. 2. Comparison of tensor contribution to CMB anisotropy for super inflation with $n_T = 0.9$ and standard inflation $n_T = -0.1$ (with the same amplitude in temperature anisotropies on large scales). Besides the temperature spectrum as a triple dot-dashed (long dashed) line, the E-mode spectrum is dashed (dot-dashed) and the B-mode one is solid (dotted) for super (standard) inflation. Note the different increase in the spectra and the generic shift at larger ℓ of the first peak when increasing n_T.

VII. CONCLUSIONS

In this paper we have presented a consistent framework to study super inflation and given exact solution for a toy model with an exponential potential. While these models deserve further investigation both in connection with quantum instabilities [20], nonperturbative effects and the exit from the accelerated stage, they provide the distinctive prediction that the tensor perturbation spectrum is blue tilted (independently on the number of scalar fields involved). This might also open new windows for the direct detection of the stochastic gravitational-wave background by interferometric antennas. Indeed, the ratio with respect to a scale-invariant tensor spectrum of the contribution to closure energy density in gravitational waves (per unit log frequency) scales like $(k/k_*)^{n_T} \sim (5 \times 10^{16})^{n_T}(k/\mathrm{Hz})^{n_T}$, if the tensor spectrum is normalized to CMB observations at $k_* = 0.002\ \mathrm{Mpc}^{-1}$ [24]. This may lead to a relevant increase of the signal for positive tilt, even without requiring large deviations from scale invariance, unlike the Ekpyrotic scenario with $n_T = 3$ or pre-big bang with $n_T = 4$. Given the completely general scalar-field theory we have considered, such a blue spectrum for gravitational waves does not constrain the slope of the scalar perturbation spectrum, therefore allowing agreement with observational data.

ACKNOWLEDGMENTS

We wish to thank A. A. Starobinsky for discussions. F. F. wishes to thank INFN for support.

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