SU(3) relations and the CP asymmetry in $B \to K_S K_S K_S$

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The CP asymmetry in the $B \to K_S K_S K_S$ decay is being measured by the two B factories. A large deviation of the CP asymmetry $S_{K_S K_S K_S}$ from $-S_{\psi K_S}$ and/or of $C_{K_S K_S K_S}$ from zero would imply new physics in $b \to s$ transitions. We try to put upper bounds on the Standard Model size of these deviations, using SU(3) flavor relations and experimental data on the branching ratios of various decay modes that proceed via $b \to d$ transitions. We point out several subtleties that distinguish the case of three-body final states from two body ones. We present several simple relations that can become useful once all relevant modes are measured accurately enough.

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I. INTRODUCTION

The Belle [1] and Babar [2] experiments have recently presented their first results on the CP asymmetries in $B \to K_S K_S K_S$ decays. The average of the two measurements is given by [3]

$$S_{K_S K_S K_S} = -0.26 \pm 0.34, \qquad C_{K_S K_S K_S} = -0.41 \pm 0.22. \tag{1}$$

The $B \to K_S K_S K_S$ decay is a flavor changing neutral current process and, consequently, does not proceed via tree level diagrams. Within the Standard Model, the $b \to s$ penguin contributions are dominated by a single weak phase, that is the phase of $V_{cb}^* V_{cs}$. The effects of a second phase, that is the phase of $V_{ub}^* V_{us}$, are CKM suppressed by $\mathcal{O}(\lambda^2)$. Neglecting the latter contributions, and taking into account that the $K_S K_S K_S$ state is purely CP even [4], the SM predictions are then as follows:

$$S_{K_S K_S K_S} \approx -S_{\psi K_S} \qquad C_{K_S K_S K_S} \approx 0, \tag{2}$$

where, experimentally, $S_{\psi K_S} = 0.726 \pm 0.037$ [3]. These predictions are valid also in extensions of the SM where the $B^0 - \bar{B}^0$ mixing amplitude is possibly affected by new phases, but the $b \to s$ decay amplitudes are not.

In order to understand whether violations of Eq. (2) signal new physics, it is necessary to estimate or, at least, put an upper bound on the CKM-suppressed SM contributions. Such a calculation involves, however, in addition to the CKM factors, hadronic physics. Currently, no first principle method for calculating hadronic matrix elements has been proven to work to a high level of precision. Furthermore, existing methods (for example, [5–8]) have only been applied to two body final states, while our interest here lies in the three-body mode $B \to K_S K_S K_S$.

In this work we use the approximate $SU(3)$ of the strong interactions to constrain the relevant hadronic matrix elements [9]. While this method has the advantage of being hadronic-model independent, it has the following two weaknesses. First, $SU(3)$ breaking effects can be of order 30%, so that our results cannot be trusted to better accuracy than that. Second, since we have no information about the strong phases that are involved, we make the most conservative assumption, whereby all amplitudes interfere constructively. This leads to upper bounds on the deviations from Eq. (2) that are often much weaker than the actual deviations expected in the Standard Model. Furthermore, the quality of our upper bounds depends on the precision of current measurements. Thus, one should not think of our bounds as estimates of the deviation expected within the Standard Model. They are only approximate (to $\mathcal{O}(0.3)$) and, in most cases, very conservative upper bounds (with the advantage of being model independent).

This type of analysis has been previously applied to CP asymmetries in decays into two body final states using the full $SU(3)$ symmetry or into three-body final states using an $SU(2)$ subgroup [9–14]. For the mode of interest to us, an $SU(2)$ analysis is not enough. We thus study three-body decays in the framework of the full $SU(3)$ group.

The analysis of three-body final states involves several subtleties and technical complications. We have developed methods to overcome these difficulties that are of more general applicability than just the $K_S K_S K_S$ mode. As concerns our final results, we find that current experimental data give no constraint on the CP asymmetry in $B \to K_S K_S K_S$ using only $SU(3)$. It is possible, however, that future measurements of branching ratios of a few additional three-body modes, together with an improvement in the constraints on a few other, will lead to useful constraints. (When we add a rather mild dynamical assumption to our $SU(3)$ analysis, we do obtain a bound with present data. The experimental range of the CP asymmetries is consistent with this bound.)

The plan of this paper is as follows. In Sec. II we introduce formalism and notations that are specifically suitable for three-body decays. In Sec. III we explain the principles of how to obtain $SU(3)$ relations that constrain the CP asymmetries in three-body decays. In Sec. IV we focus on the mode of interest, $B \to K_S K_S K_S$, and give a

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few concrete examples of amplitude relations, as well as a Table that allows one to derive all relevant relations. We conclude in Sec. V. Technical details are further discussed in two appendices. In App. A we derive the relations between the CP asymmetries and the parameters that we define in Sec. II, and we justify the approximations that we use. In App. B we describe the techniques that we developed to deal with the complicated $SU(3)$ decomposition of the decay amplitudes. Appendix C contains a list of relevant branching ratios.

II. NOTATIONS AND FORMALISM

In this section we show how to modify and generalize the analysis of Ref. [9] so that it can be applied to three-body decays.

Unlike two body decays, the final state in three-body decays is not uniquely determined by the identity of the final mesons. Additional quantum numbers (for example, the momenta) are needed to specify the state. We use abstract vector notation, e.g. $\vec{A}_{K_SK_SK_S}$, where the vector index runs over all possible values for the quantum numbers, to describe the various states. The total decay rate is given by

$$\Gamma(B^0 \to K_SK_SK_S) = \| \vec{A}_{K_SK_SK_S} \|^2 . \tag{3}$$

This equation defines the normalization of the decay amplitudes $\vec{A}_f$. The norm in the right hand side of Eq. (3) represents a sum over all possible final states. If we choose to describe the different final states using definite linear momenta, the norm is actually calculated by an integral over all momentum configurations. We stress that the norm is the same, no matter which basis we choose to span the final states with.

In order to derive $SU(3)$ relations, we choose to span the final states in a basis with definite linear momenta. Our convention is that the order in which we write the three final mesons corresponds to their momentum configuration:

$$|M_iM_jM_k\rangle \equiv |M_i(p_1)M_j(p_2)M_k(p_3)\rangle. \tag{4}$$

We further define symmetrized states, $|S(f)\rangle$, as follows:

$$\begin{aligned} |S(M_1M_1M_1)\rangle &\equiv |M_1M_1M_1\rangle, \\ |S(M_1M_1M_2)\rangle &\equiv \frac{1}{\sqrt{3}}(|M_1M_1M_2\rangle + |M_1M_2M_1\rangle \\ &\quad + |M_2M_1M_1\rangle), \\ |S(M_1M_2M_3)\rangle &\equiv \frac{1}{\sqrt{6}}(|M_1M_2M_3\rangle + |M_2M_3M_1\rangle \\ &\quad + |M_3M_1M_2\rangle + |M_3M_2M_1\rangle \\ &\quad + |M_2M_1M_3\rangle + |M_1M_3M_2\rangle). \end{aligned} \tag{5}$$

In Eq. (5), M_1, M_2 and M_3 stand for different mesons.

Focussing on the mode of interest to us, namely B^0 decay into a final $|K_SK_SK_S\rangle$ state, we note it can proceed via any of the three $\Delta s = -1$ transitions whereby a B^0 meson decays into $|K^0K^0\bar{K}^0\rangle$, $|K^0\bar{K}^0K^0\rangle$ or $|\bar{K}^0K^0K^0\rangle$. Owing to the symmetry of the $|K_SK_SK_S\rangle$ state under exchange of any two of the final mesons, it can only come from the totally symmetric combination of the three states, $|S(K^0K^0\bar{K}^0)\rangle$. Neglecting CP violation in the neutral kaon mixing (the experimental measurement of ε_K guarantees that this approximation is good to $\mathcal{O}(10^{-3})$), we have

$$\langle K_SK_SK_S|S(K^0K^0\bar{K}^0)\rangle = \sqrt{\frac{3}{8}}\frac{V_{cs}^*V_{cd}}{|V_{cs}^*V_{cd}|}, \tag{6}$$

for every set of values for the momenta p_1, p_2, p_3. There are two additional combinations of $|K^0K^0\bar{K}^0\rangle$, $|K^0\bar{K}^0K^0\rangle$ and $|\bar{K}^0K^0K^0\rangle$ which are orthogonal to $|S(K^0K^0\bar{K}^0)\rangle$. However, since the projection of these combinations on $|K_SK_SK_S\rangle$ is zero, we can write

$$\vec{A}_{K_SK_SK_S} = \sqrt{3/8}[(V_{cs}^*V_{cd})/|V_{cs}^*V_{cd}|]\vec{A}_{S(K^0K^0\bar{K}^0)}. \tag{7}$$

Within the Standard Model, the violation of CP is encoded in the complex phases of the CKM elements. It is therefore convenient, for the purpose of discussing CP asymmetries, to have the CKM dependence explicit. Following the discussion above, we thus write the $B^0 \to K_SK_SK_S$ decay amplitudes as follows:

$$\begin{aligned} \vec{A}_{K_SK_SK_S} &= (V_{cb}^*V_{cs}\vec{a}^c_{S(K^0K^0\bar{K}^0)} + V_{ub}^*V_{us}\vec{a}^u_{S(K^0K^0\bar{K}^0)}) \\ &\quad \times \sqrt{3/8}[(V_{cs}^*V_{cd})/|V_{cs}^*V_{cd}|]. \end{aligned} \tag{8}$$

Here, and for all other processes discussed below, the amplitudes for the CP-conjugate processes, $\overline{B}^0 \to \overline{f}$, have the CKM factors complex-conjugated, while the $\vec{a}_f^{u,c}$ factors remain the same.

Generalizing [9], we introduce a parameter ξ:

$$\xi \equiv \frac{|V_{ub}^*V_{us}|}{|V_{cb}^*V_{cs}|}\frac{\vec{a}^c_{S(K^0K^0\bar{K}^0)} \cdot \vec{a}^u_{S(K^0K^0\bar{K}^0)}}{\| \vec{a}^c_{S(K^0K^0\bar{K}^0)} \|^2}, \tag{9}$$

where the dot product of complex vectors is defined by $\vec{X} \cdot \vec{Y} \equiv \sum_\nu X_\nu^* Y_\nu$. Another useful parameter, $|\overline{\xi}|$, is defined as follows:

$$|\overline{\xi}| \equiv \frac{|V_{ub}^*V_{us}|}{|V_{cb}^*V_{cs}|}\frac{\| \vec{a}^u_{S(K^0K^0\bar{K}^0)} \|}{\| \vec{a}^c_{S(K^0K^0\bar{K}^0)} \|}. \tag{10}$$

We have

$$\frac{|\xi|}{|\overline{\xi}|} = \frac{|\vec{a}^c_{S(K^0K^0\bar{K}^0)} \cdot \vec{a}^u_{S(K^0K^0\bar{K}^0)}|}{\| \vec{a}^c_{S(K^0K^0\bar{K}^0)} \| \cdot \| \vec{a}^u_{S(K^0K^0\bar{K}^0)} \|} \le 1. \tag{11}$$

The parameter $|\overline{\xi}|$ is the one which can be constrained by $SU(3)$ relations, and that would lead, through Eq. (11), to a constraint on $|\xi|$.

The case of two body decays [9] constitutes a specific example of our more general notation (9), where the vectors are simply one dimensional and, as can be seen from Eq. (11), $|\xi| = |\bar{\xi}|$. The way in which ξ of Ref. [9] is defined differs, however, by a weak phase factor: ξ(Ref. [9]) $= e^{i\gamma}\xi$ (Eq. (9)) .

Before concluding this section, we introduce one more definition. Experiments often measure charge-averaged rates,

$$\Gamma(B \to f) = \frac{1}{2}[\Gamma(B^0 \to f) + \Gamma(\bar{B}^0 \to \bar{f})], \quad (12)$$

where $\bar{f}$ is the CP-conjugate state of f. For CP eigenstates, $\bar{f} = \pm f$. When a single weak phase dominates, the CP-conjugate rates are equal, $\Gamma(B^0 \to f) = \Gamma(\bar{B}^0 \to \bar{f})$, and there is no reason to make a distinction between $\Gamma(B \to f)$ and $\Gamma(B^0 \to f)$.

III. CONSTRAINING THE CP ASYMMETRIES

As we show in App. A, we can write, to first order in $\mathcal{R}e(\xi)$ and $\mathcal{I}m(\xi)$,

$$-S_{K_S K_S K_S} - S_{\psi K_S} = 2\cos 2\beta \sin\gamma \mathcal{R}e(\xi), \quad (13)$$

$$C_{K_S K_S K_S} = -2\sin\gamma \mathcal{I}m(\xi). \quad (14)$$

The significance of the parameter ξ is that it encodes all hadronic physics that affects the deviation of $-S_{K_S K_S K_S}$ from $\sin 2\beta$ and of $C_{K_S K_S K_S}$ from zero. The other parameters, β and γ, are weak phases that can be determined rather accurately from other measurements. We learn that if we are able to put an upper bound, $|\xi| \leq |\xi|^{\max}$, we will obtain an unambiguous test of the Standard Model CP violation by asking whether the relation [9]

$$[(S_{K_S K_S K_S} + S_{\psi K_S})/\cos 2\beta]^2 + C^2_{K_S K_S K_S} \leq 4\sin^2\gamma(|\xi|^{\max})^2 \quad (15)$$

is fulfilled. As mentioned above, the parameter that appears in the $SU(3)$ relations is actually $|\bar{\xi}|$. In this and the next section, we assume that $SU(3)$ is exact, and use it to constrain $|\xi|$.

In order to constrain $|\bar{\xi}|$ we consider $\Delta s = 0$ decay amplitudes and write, using our vector notation,

$$\vec{A}_f = V^*_{cb} V_{cd} \vec{b}^c_f + V^*_{ub} V_{ud} \vec{b}^u_f. \quad (16)$$

$SU(3)$ relations lead to amplitude relations of the form

$$\vec{a}^{\,q}_{S(K^0 K^0 \bar{K}^0)} = \sum_f X'_f \vec{b}^q_f \ (q = u \text{ or } c). \quad (17)$$

Most generally, in the sum over f, states which are permutations of each other are treated as different states (for example, $K^+ K^- \pi^0$ and $K^+ \pi^0 K^-$ are different states). However, since the state $|S(K^0 K^0 \bar{K}^0)\rangle$ is completely symmetric, the strongest constraint is obtained from summing only over completely symmetric terms. Making this choice for Eq. (17) allows us to rewrite it as follows:

$$\vec{a}^{\,q}_{S(K^0 K^0 \bar{K}^0)} = \sum_f X_f \vec{b}^q_{S(f)}. \quad (18)$$

The X_f's of Eq. (18) are related to the X'_f's of Eq. (17) by symmetry factors. Taking the norm of Eq. (17) needs to be done with care: the sum can involve states with different symmetry properties, and the corresponding norms have different meanings. On the other hand, there is no ambiguity in taking the norm of Eq. (18). Consequently, we can write

$$\| \vec{a}^q_{S(K^0 K^0 \bar{K}^0)} \| \leq \sum_f |X_f| \, \| \vec{b}^q_{S(f)} \| . \quad (19)$$

We provisionally assume, for simplicity, that $\Delta s = \pm 1$ decays are dominated by the $\vec{a}^c$ terms (this assumption is justified if $|\bar{\xi}|$ is small, see Eq. (10)), while $\Delta s = 0$ decays are dominated by the $\vec{b}^u$ terms. (Below we obtain our constraints without making these assumptions, in a fashion similar to [9].) Then the amplitudes are related to the decay rates by

$$|V^*_{cb} V_{cs}| \, \| \vec{a}^c_{S(K^0 K^0 \overline{K}^0)} \| \approx \sqrt{(8/3)\Gamma(B \to K_S K_S K_S)}, \quad (20)$$

$$|V^*_{ub} V_{ud}| \, \| \vec{b}^u_{S(f)} \| \lesssim \sqrt{\Gamma(B \to f)}. \quad (21)$$

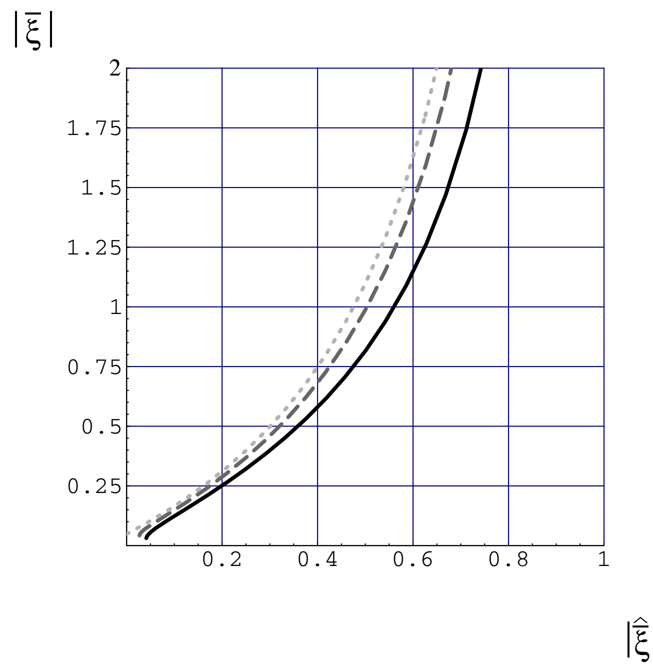

FIG. 1 (color online). The upper bound that can be placed on $|\bar{\xi}|$ as a function of the upper bound on $|\hat{\bar{\xi}}|$, according to Eq. (26). The three curves correspond to different ways of treating the weak phase γ: $\gamma = 59.0°$, the experimental central value (solid black); $\gamma \in [35.9°, 80.1°]$, the 3σ range [20] (dark-gray dashed); γ unconstrained (light-gray dotted).

The inequality in Eq. (21) comes from the fact that we consider the symmetrized state, rather then a generic state, for which $|V_{ub}^* V_{ud}| \parallel \vec{b}_f^u \parallel \approx \sqrt{\Gamma(B \to f)}$. Combining Eqs. (19)–(21), we get

$$|\bar{\xi}| \lesssim \left|\frac{V_{us}}{V_{ud}}\right| \sum_f |X_f| \sqrt{\frac{\Gamma(B \to f)}{(8/3)\Gamma(B \to K_S K_S K_S)}}. \tag{22}$$

We now proceed without making the assumptions of $\vec{a}_f^c$- and $\vec{b}_f^u$-dominance which led to Eqs. (20) and (21) [9]. Instead of $|\bar{\xi}|$, we constrain a new parameter, $|\hat{\bar{\xi}}|$, defined by

$$|\hat{\bar{\xi}}|^2 \equiv \left|\frac{V_{us}}{V_{ud}}\right|^2 \frac{\|V_{cb}^* V_{cd} \vec{a}^c_{S(K^0K^0\bar{K}^0)} + V_{ub}^* V_{ud} \vec{a}^u_{S(K^0K^0\bar{K}^0)}\|^2 + \|V_{cb} V_{cd}^* \vec{a}^c_{S(K^0K^0\bar{K}^0)} + V_{ub} V_{ud}^* \vec{a}^u_{S(K^0K^0\bar{K}^0)}\|^2}{\|V_{cb}^* V_{cs} \vec{a}^c_{S(K^0K^0\bar{K}^0)} + V_{ub}^* V_{us} \vec{a}^u_{S(K^0K^0\bar{K}^0)}\|^2 + \|V_{cb} V_{cs}^* \vec{a}^c_{S(K^0K^0\bar{K}^0)} + V_{ub} V_{us}^* \vec{a}^u_{S(K^0K^0\bar{K}^0)}\|^2}. \tag{23}$$

The numerator and denominator of $|\hat{\bar{\xi}}|^2$ are related to charge-averaged rates:

$$\| V_{cb}^* V_{cd} \vec{a}^c_{S(K^0K^0\bar{K}^0)} + V_{ub}^* V_{ud} \vec{a}^u_{S(K^0K^0\bar{K}^0)} \|^2 + \| V_{cb} V_{cd}^* \vec{a}^c_{S(K^0K^0\bar{K}^0)} + V_{ub} V_{ud}^* \vec{a}^u_{S(K^0K^0\bar{K}^0)} \|^2 \leq 2\left(\sum_f |X_f| \sqrt{\Gamma(B \to f)}\right)^2, \tag{24}$$

$$\| V_{cb}^* V_{cs} \vec{a}^c_{S(K^0K^0\bar{K}^0)} + V_{ub}^* V_{us} \vec{a}^u_{S(K^0K^0\bar{K}^0)} \|^2 + \| V_{cb} V_{cs}^* \vec{a}^c_{S(K^0K^0\bar{K}^0)} + V_{ub} V_{us}^* \vec{a}^u_{S(K^0K^0\bar{K}^0)} \|^2 = (16/3)\Gamma(B \to K_S K_S K_S). \tag{25}$$

Using the measured charge-averaged rates, a constraint on $|\hat{\bar{\xi}}|^2$ is obtained without any further assumptions.

The $|\hat{\bar{\xi}}|$ and $|\bar{\xi}|$ parameters are related as follows:

$$|\hat{\bar{\xi}}|^2 = \frac{|\frac{V_{us}V_{cd}}{V_{cs}V_{ud}}|^2 + |\bar{\xi}|^2 + 2\cos\gamma \mathcal{R}e(\frac{V_{us}V_{cd}}{V_{cs}V_{ud}}\xi)}{1 + |\bar{\xi}|^2 + 2\cos\gamma \mathcal{R}e(\xi)}. \tag{26}$$

The relation (26) is a generalization of the relation in [9], Eq. (14).[1] It has the important property that for $\lambda^2 \lesssim |\hat{\bar{\xi}}| \leq 1$ we get a constraint on $|\bar{\xi}|$, for any ξ (of course, within the allowed range, $|\xi| \leq |\bar{\xi}|$, see Eq. (11)). Since we do not know the value of ξ, we should consider the weakest constraint, which corresponds to $\mathcal{R}e(\xi) = |\bar{\xi}|$ (the $(V_{us}V_{cd})/(V_{cs}V_{ud})$ term is experimentally known to be real to a good approximation). We show in Fig. 1 the relation between the upper bound on $|\hat{\bar{\xi}}|$ and the resulting upper bound on $|\bar{\xi}|$, for three ranges of the weak phase γ. The weakest bound, which corresponds to $\mathcal{R}e(\xi) = |\bar{\xi}|$ and $\gamma = 0$, is the curve $|\hat{\bar{\xi}}| = (|\bar{\xi}| - \lambda^2)/(1 + |\bar{\xi}|)$. Note that the translation from $|\hat{\bar{\xi}}|$ to $|\bar{\xi}|$ is nonlinear, a point which was not stressed in [9], although it is true there as well.

IV. $SU(3)$ RELATIONS FOR $K_S K_S K_S$

The simplest way to find $SU(3)$ relations is to express the decay amplitudes using invariant $SU(3)$ reduced matrix elements. While the number of $SU(3)$ independent reduced matrix elements in three-body decays is quite large, a significant simplification is obtained by the fact that we only consider completely symmetric final states. In particular, for generic f's we have 40 independent reduced matrix elements, while for $S(f)$'s there are only 7.

By scanning over all possible contractions of the relevant $SU(3)$ tensors, we are able to obtain all matrix element relations in a systematic way, avoiding the need to discuss $SU(3)$ properties of tensor products. We give more details on the calculation in App. B.

The main results of our work are summarized in Table I where we list the dependence of the symmetrized three-body decay amplitudes on $SU(3)$ reduced matrix elements. We give here only B^0 and B^+ decays, but it is straightforward to add B_s decays in a similar way. We stress that, since this table includes only totally symmetric states, it is only applicable to constrain the totally symmetric final states such as $K_S K_S K_S$.

A simple examination of Table I reveals that all the relations of the form of Eq. (18) involve at least one $\Delta s = 0$ decay into a final state with an η_8 meson. In the exact $SU(3)$ limit, this corresponds to a state with a final η meson. We would like to emphasize two points in this regard:

(1) The use of $SU(3)$ relations involving amplitudes with final η_8 and/or η_1 mesons was recently criticized in Ref. [15], on the basis that $SU(3)$ breaking effects in this system are large. The phenomenological value of the octet-singlet mixing angle is $\sin\theta \approx 0.27$ [16]. $SU(3)$ breaking in the decay constants is parametrized by $2(f_s - f_q)/(f_s + f_q) \approx 0.22$ [16]. The breaking effects are thus consistent with our estimated accuracy of $\mathcal{O}(0.3)$, and are not $\mathcal{O}(1)$, as suggested in Ref. [15].

[1] Reference [9] uses the rates $\Gamma(B^0 \to f)$ to define $|\hat{\xi}|$, while we use the charge-averaged rates $\Gamma(B \to f)$. If we leave γ unconstrained, the resulting upper bound on $|\bar{\xi}|$ is the same, but for $\gamma \neq 0$, our expression gives a stronger bound, as can be seen in Fig. 1.

TABLE I. $SU(3)$ decomposition of $A_{S(f)}$. The different blocks refer to different degrees of symmetrization needed for each state.

$S(f)$	A_1	A_2	A_3	A_4	A_5	A_6	A_7
$S(K^0\bar{K}^0K^0)$	1	0	0	0	0	0	0
$S(K^+K^-\pi^0)$	0	1	0	0	0	0	0
$S(K^0\bar{K}^0\pi^0)$	0	0	1	0	0	0	0
$S(\pi^+\pi^-\pi^0)$	0	0	0	1	0	0	0
$S(K^+\pi^-\bar{K}^0)$	0	0	0	0	1	0	0
$S(K^-\pi^+K^0)$	0	0	0	0	0	1	0
$S(K^+K^-\pi^+)$	0	0	0	0	0	0	1
$S(K^+\bar{K}^0\pi^0)$	0	-1	0	1	$\frac{1}{\sqrt{2}}$	0	0
$S(K^0\bar{K}^0\pi^+)$	0	0	$-\sqrt{2}$	$\sqrt{2}$	0	-1	1
$S(K^0\bar{K}^0\eta_8)$	$\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	0	0	0	0
$S(K^+K^-\eta_8)$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}}$	0	0	0
$S(\pi^+\pi^-\eta_8)$	$\frac{1}{\sqrt{3}}$	0	$\frac{2}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	0
$S(K^+\bar{K}^0\eta_8)$	0	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{6}}$	0	0
$S(\pi^+\pi^0\eta_8)$	0	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	0	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	0
$S(\pi^+\pi^+\pi^-)$	0	0	0	0	0	0	$\sqrt{2}$
$S(\pi^+\pi^0\pi^0)$	0	0	0	0	0	0	$\frac{1}{\sqrt{2}}$
$S(\pi^0\pi^0\eta_8)$	$\frac{1}{\sqrt{6}}$	$\sqrt{\frac{2}{3}}$	0	$-\frac{1}{\sqrt{6}}$	0	0	0
$S(\pi^0\eta_8\eta_8)$	0	$\frac{2\sqrt{2}}{3}$	$\frac{2\sqrt{2}}{3}$	$-\frac{5}{3\sqrt{2}}$	0	0	0
$S(\pi^+\eta_8\eta_8)$	0	$\frac{-2}{3}$	$\frac{-2}{3}$	$\frac{4}{3}$	$\frac{\sqrt{2}}{3}$	$-\frac{\sqrt{2}}{3}$	$\frac{1}{\sqrt{2}}$
$S(\pi^0\pi^0\pi^0)$	0	0	0	$\sqrt{\frac{3}{2}}$	0	0	0
$S(\eta_8\eta_8\eta_8)$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{2}}{3}$	$\frac{2\sqrt{2}}{3}$	$-\frac{1}{\sqrt{2}}$	0	0	0

(2) If the relevant decays into final states involving η' are found to be enhanced compared to the corresponding states involving η, then the effects of the octet-singlet mixing on our results may be significant. These effects can, however, be taken into account by using both η and η' data, in a way similar to Ref. [9].

We now present several interesting specific relations. Note that, since we do not know the values of the strong phases, in deriving our bounds we must add the various $\Delta s = 0$ amplitudes constructively, see Eq. (19). This conservative procedure may weaken the bound considerably. Therefore, relations involving a smaller number of $\Delta s = 0$ amplitudes are more likely to give strong bounds.

A. A Single $\Delta s = 0$ Amplitude

There is no amplitude relation of the form of Eq. (18) that involves only a single $\Delta s = 0$ decay amplitude of B^0 or B^+. Such relations would have had the potential to lead to a tight constraint. Note that, in general, we would still get an upper bound on, rather than an estimate of, $|\hat{\xi}|$. The reason is that we consider the symmetrized final state while experiments measure nonsymmetrized final states. However, there exists such a relation involving B_s decays:

$$\vec{a}^{\,q}_{B_d\to S(K^0K^0\bar{K}^0)} = \vec{b}^{q}_{B_s\to S(\bar{K}^0\bar{K}^0K^0)}. \tag{27}$$

This relation is, in fact, due to the U-spin subgroup of $SU(3)$ and it holds for the nonsymmetrized states as well. Since in B_s decays the $|K_SK_SK_S\rangle$ state can only come from $|S(\bar{K}^0\bar{K}^0K^0)\rangle$, in a way similar to Eq. (7), Eq. (27) implies relations between the $B_d \to K_SK_SK_S$ and $B_s \to K_SK_SK_S$ decay amplitudes, $V_{cs}V_{cd}^*\vec{a}^{q}_{B_d\to K_SK_SK_S} = V_{cs}^*V_{cd}\vec{b}^{q}_{B_s\to K_SK_SK_S}$, leading to

$$|\hat{\xi}| = \left|\frac{V_{us}}{V_{ud}}\right|\sqrt{\frac{\Gamma(B_s\to K_SK_SK_S)}{\Gamma(B_d\to K_SK_SK_S)}}. \tag{28}$$

B. Two $\Delta s = 0$ Amplitudes

We find a single amplitude relation involving only two $\Delta s = 0$ amplitudes:

$$\vec{a}^{\,q}_{S(K^0K^0\bar{K}^0)} = \sqrt{3}\vec{b}^{q}_{S(K^0\bar{K}^0\eta_8)} - \vec{b}^{q}_{S(K^0\bar{K}^0\pi^0)}. \tag{29}$$

The fact that we are interested only in symmetrized states is helpful here in yet another way. Let us write

$$\langle S(K^0\bar{K}^0X)|S(K_SK_SX)\rangle = -\langle S(K^0\bar{K}^0X)|S(K_LK_LX)\rangle = \frac{1}{\sqrt{2}}. \tag{30}$$

(X here can be any meson except K^0, $\bar{K}^0$, K_S or K_L.) Since in B decays the $|S(K_SK_SX)\rangle$ and $|S(K_LK_LX)\rangle$ states can only come from an $|S(K^0\bar{K}^0X)\rangle$ state, we can write, similarly to Eq. (7),

$$\vec{A}_{S(K_SK_SX)} = -\vec{A}_{S(K_LK_LX)} = \frac{1}{\sqrt{2}}\vec{A}_{S(K^0\bar{K}^0X)}. \tag{31}$$

Consequently, the relation (29) leads to the following relation, which is more practical from the experimental point of view:

$$\vec{a}^{\,q}_{S(K^0K^0\overline{K}^0)} = \sqrt{6}\vec{b}^{q}_{S(K_SK_S\eta_8)} - \sqrt{2}\vec{b}^{q}_{S(K_SK_S\pi^0)}. \tag{32}$$

(K_SK_S can be replaced by K_LK_L.) There is yet no measurement of the modes in Eq. (32).

C. Three $\Delta s = 0$ Amplitudes

We find several amplitude relations which involve three $\Delta s = 0$ amplitudes, for example,

$$\vec{a}^{\,q}_{S(K^0K^0\bar{K}^0)} = \sqrt{6}\vec{b}^{q}_{S(\pi^0\pi^0\eta_8)} - 2\vec{b}^{q}_{S(K^+K^-\pi^0)} + \vec{b}^{q}_{S(\pi^+\pi^-\pi^0)}. \tag{33}$$

At present, the branching ratio $\mathcal{B}(\pi^0\pi^0\eta)$ is not yet constrained, while $\mathcal{B}(K^+K^-\pi^0)$ and $\mathcal{B}(\pi^+\pi^-\pi^0)$ have rather weak upper bounds. For the relation (33) to become useful, the branching ratio of the first mode must be constrained, and the bounds on the latter two must be improved.

D. Measured $\Delta s = 0$ Amplitudes

There are relations which involve only modes which have been measured. Given the experimental data in App. C, the strongest bound is obtained by using the following relation:

$$\vec{a}^{\,q}_{S(K^0K^0\overline{K}^0)} = -\sqrt{2}\vec{b}^q_{S(K^+\pi^-\overline{K}^0)} + \sqrt{3}\vec{b}^q_{S(\pi^+\pi^-\eta_8)} - \vec{b}^q_{S(\pi^+\pi^-\pi^0)} - \sqrt{2}\vec{b}^q_{S(K^+K^-\pi^+)} + \sqrt{2}\vec{b}^q_{S(K^0\overline{K}^0\pi^+)}. \tag{34}$$

Using the definition (23), we get

$$|\hat{\bar{\xi}}| \leq 0.22\sqrt{\frac{3}{8}}\left(\sqrt{\frac{2\mathcal{B}(K^+\overline{K}^0\pi^-)}{\mathcal{B}(K_SK_SK_S)}} + \sqrt{\frac{3\mathcal{B}(\pi^+\pi^-\eta)}{\mathcal{B}(K_SK_SK_S)}} + \sqrt{\frac{\mathcal{B}(\pi^+\pi^-\pi^0)}{\mathcal{B}(K_SK_SK_S)}} + \sqrt{\frac{2\mathcal{B}(K^+K^-\pi^+)}{\mathcal{B}(K_SK_SK_S)}} + \sqrt{\frac{4\mathcal{B}(K_SK_S\pi^+)}{\mathcal{B}(K_SK_SK_S)}}\right) \leq 1.28. \tag{35}$$

As explained above, we take here the $SU(3)$ limit in replacing η_8 with η. We also use $\mathcal{B}(S(K^0\bar{K}^0\pi^+)) = 2\mathcal{B}(S(K_SK_S\pi^+))$.

We see that the strongest bound we can currently put on $|\hat{\bar{\xi}}|$ is too weak to bound $|\bar{\xi}|$ and the CP asymmetries. It is possible, however, that an improvement in experimental data, as well as measurements of additional modes, will eventually lead to a significant bound.

E. Dynamical Assumptions

One can use simplifying dynamical assumptions and neglect the effect of small contributions from exchange, annihilation, and penguin annihilation diagrams [17]. Practically, this means that all reduced matrix elements in which the spectator (the B triplet) is contracted with a Hamiltonian operator are put to zero. More details are given in App. B.

Such a simplification does lead to new relations. Most notably, there is now an amplitude relation involving a single $\Delta s = 0$ mode:

$$\vec{a}^{\,q}_{S(K^0K^0\bar{K}^0)} = \sqrt{2}\vec{b}^q_{S(K^0\bar{K}^0\pi^+)}. \tag{36}$$

This relation leads to the following upper bound:

$$|\hat{\bar{\xi}}| \leq 0.22\sqrt{\frac{3}{2}}\sqrt{\frac{\mathcal{B}(K_SK_S\pi^+)}{\mathcal{B}(K_SK_SK_S)}} \leq 0.20 \quad \longrightarrow \quad |\bar{\xi}| \leq 0.31. \tag{37}$$

The observed CP asymmetries are well within this bound. We conclude that to uncover a signal of new physics with our methods will require improved experimental data.

V. CONCLUSIONS

We use the approximate $SU(3)$ flavour symmetry to constrain the SM pollution and the CP asymmetries in three-body decays. This is an extension of previous works that considered two body final states. One important difference is that two body final states are entirely defined by the identity of the final mesons, while in three-body decays additional quantum numbers (such as momenta or angular momenta) are required to characterize the final state. In the absence of an experimental spatial analysis, the measured quantities are always a sum over all possible final states. On the other hand, the $SU(3)$ relations hold for each final state separately. The application of $SU(3)$ relations to three-body final states should therefore be done with care.

The case of $B \to K_SK_SK_S$ decay is special since the final state is symmetric under the exchange of any two mesons. We showed how this leads to a significant simplification in the $SU(3)$ analysis, allowing us to consider only final states with the same symmetry and considerably reducing the number of independent $SU(3)$ reduced matrix elements.

Still, decomposing the decay amplitudes for three-body final states into reduced matrix elements with well defined $SU(3)$ transformation properties is a difficult task in terms of group theoretical calculation. Since, however, we are eventually interested only in relations between physical decay amplitudes, we were able to scan systematically over all possible reduced matrix elements and find an independent subset of them. Using this method, our reduced matrix elements bear no clean $SU(3)$ interpretation, but their guaranteed independence is all that matters for the task of finding amplitude relations. The same method can be used to simplify other $SU(3)$ calculations where the goal is obtaining physical amplitude relations.

Whether a numerical upper bound is achieved (and whether this bound is strong enough to be in conflict with a measured CP asymmetry) depends on the available experimental data. Currently, no such bound can be obtained with no additional assumptions, and the bound which is obtained when additional dynamical assumptions are used is not strong enough to be in conflict with the measured CP asymmetry. However, our work shows which new measurements have the potential to lead to a constraint.

The hope is that, given more and better experimental data, three-body decays and $SU(3)$ relations will provide us with an additional unambiguous test of the SM mechanism of CP violation.

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APPENDIX A: THE ξ DEPENDENCE OF $S_{K_SK_SK_S}$ AND $C_{K_SK_SK_S}$

In order to derive the dependence of CP asymmetries in $B \to K_S K_S K_S$ on ξ, we work in the basis of states with definite angular momentum between two of the three final K_S mesons. While it is difficult to write $SU(3)$ relations in this basis, the advantage of using it is that any basis-state is manifestly CP even [4]. (In other bases, final states which are related by CP would have related decay amplitudes such that only the combinations which correspond to CP even final states have a nonzero amplitude. The discussion, however, is much simpler in the basis we choose.)

We denote the various components of the vector $\vec{A}_{K_SK_SK_S}$ in this basis by A_ν where ν is some collective index which runs over all possible final states. We define for every final state the parameter

$$\lambda_\nu \equiv e^{-i\phi_B} \frac{\bar{A}_\nu}{A_\nu}, \tag{A1}$$

where ϕ_B is the phase of the $B^0 - \bar{B}^0$ mixing amplitude (see the review on CP violation in [18] and, in particular, Eq. (58)).

Currently, the experimental time dependent CP asymmetry is measured with no distinction between various final $K_SK_SK_S$ states. The expression for the measured CP asymmetry involves therefore a sum over all final $K_SK_SK_S$ states:

$$\mathcal{A}_{K_SK_SK_S}(t) = \frac{\sum_\nu \Gamma(\bar{B}^0(t) \to \nu) - \sum_\nu \Gamma(B^0(t) \to \nu)}{\sum_\nu \Gamma(\bar{B}^0(t) \to \nu) + \sum_\nu \Gamma(B^0(t) \to \nu)}. \tag{A2}$$

Writing

$$\mathcal{A}_{K_SK_SK_S}(t) = S_{K_SK_SK_S} \sin(\Delta m t) - C_{K_SK_SK_S} \cos(\Delta m t), \tag{A3}$$

we use the definition (A1) to get

$$S_{K_SK_SK_S} = \frac{\sum_\nu |A_\nu|^2 2\mathrm{Im}\lambda_\nu}{\sum_\nu |A_\nu|^2(1 + |\lambda_\nu|^2)}, \tag{A4}$$

$$C_{K_SK_SK_S} = \frac{\sum_\nu |A_\nu|^2(1 - |\lambda_\nu|^2)}{\sum_\nu |A_\nu|^2(1 + |\lambda_\nu|^2)}. \tag{A5}$$

We can make the CKM dependence of each amplitude explicit:

$$\begin{aligned} A_\nu &= (V_{cb}^* V_{cs} a_\nu^c + V_{ub}^* V_{us} a_\nu^u)[(V_{cs}^* V_{cd})/|V_{cs}^* V_{cd}|] \\ &= A_\nu^c (1 + \xi_\nu e^{i\gamma}), \end{aligned} \tag{A6}$$

where γ is the weak phase between $V_{cb}^* V_{cs}$ and $V_{ub}^* V_{us}$, and ξ_ν therefore contains only strong phases and is defined by

$$\xi_\nu \equiv \left| \frac{V_{ub}^* V_{us}}{V_{cb}^* V_{cs}} \right| \frac{a_\nu^u}{a_\nu^c}. \tag{A7}$$

Assuming that $|\xi_\nu|$ is small for every ν (we justify this assumption below), we expand (A4) and (A5) to first order in $|\xi_\nu|$:

$$\begin{aligned} S_{K_SK_SK_S} &= \sin 2\beta \frac{\sum_\nu |A_\nu^c|^2 \eta_\nu}{\sum_\nu |A_\nu^c|^2} \\ &\quad + 2\cos 2\beta \sin\gamma \frac{\sum_\nu |A_\nu^c|^2 \eta_\nu \mathcal{R}e(\xi_\nu)}{\sum_\nu |A_\nu^c|^2}, \end{aligned} \tag{A8}$$

$$C_{K_SK_SK_S} = -2\sin\gamma \frac{\sum_\nu |A_\nu^c|^2 \mathcal{I}m(\xi_\nu)}{\sum_\nu |A_\nu^c|^2}. \tag{A9}$$

At this point, the fact that all final states are CP even plays an important role as it dictates that $\eta_\nu = 1$ for all ν. Switching now to vector notation we have

$$\sum_\nu |A_\nu^c|^2 = |V_{cb}^* V_{cs}|^2 \parallel \vec{a}^c \parallel^2, \tag{A10}$$

$$\sum_\nu |A_\nu^c|^2 \mathcal{R}e(\xi_\nu) = |V_{cb}^* V_{cs} V_{ub}^* V_{us}| \mathcal{R}e(\vec{a}^c \cdot \vec{a}^u), \tag{A11}$$

$$\sum_\nu |A_\nu^c|^2 \mathcal{I}m(\xi_\nu) = |V_{cb}^* V_{cs} V_{ub}^* V_{us}| \mathcal{I}m(\vec{a}^c \cdot \vec{a}^u). \tag{A12}$$

Using the definition (9) we therefore get Eqs. (13) and (14) to first order in ξ.

We still need to justify the expansions (A8) and (A9). In making it, we assumed that for every ν we have $|\xi_\nu| < 1$. Unlike in two body decay, the mere smallness of $|\bar{\xi}|$ is not enough to validate this assumption for every ν. Nevertheless, we show next that the smallness of $|\bar{\xi}|$ does guarantee that the branching ratio of final states in which $|\xi_\nu| \geq 1$ is constrained by $\mathcal{O}(|\bar{\xi}|^2)$. The terms that are omitted in writing Eqs. (A8) and (A9) are therefore of $\mathcal{O}(|\bar{\xi}|^2)$.

Our starting point is the definition of $|\bar{\xi}|$, Eq. (10), leading to

$$\begin{aligned} |\bar{\xi}|^2 &= \frac{|V_{ub}^* V_{us}|^2}{|V_{cb}^* V_{cs}|^2} \frac{\parallel \vec{a}^u \parallel^2}{\parallel \vec{a}^c \parallel^2} \\ &= \frac{|V_{ub}^* V_{us}|^2 \parallel \vec{a}^u \parallel^2}{\parallel V_{cb}^* V_{cs} \vec{a}^c + V_{ub}^* V_{us} \vec{a}^u \parallel^2} + \mathcal{O}(|\bar{\xi}|^3). \end{aligned} \tag{A13}$$

We divide the index ν into two groups: The group S in which $|\xi_\nu|$ are small and the expansions (A8) and (A9) are justified, and the group $\bar{S}$ in which $|\xi_\nu| \geq 1$ and the expansions are not justified. We are interested in the sum over $\bar{S}$ only. We have

$$\frac{\sum_{\nu\in\bar{S}}|V^*_{ub}V_{us}|^2|a^u_\nu|^2}{\| V^*_{cb}V_{cs}\vec{a}^c + V^*_{ub}V_{us}\vec{a}^u \|^2} \leq \frac{\sum_{\nu\in\bar{S}}|V^*_{ub}V_{us}|^2|a^u_\nu|^2 + \sum_{\nu\in S}|V^*_{ub}V_{us}|^2|a^u_\nu|^2}{\| V^*_{cb}V_{cs}\vec{a}^c + V^*_{ub}V_{us}\vec{a}^u \|^2} = |\bar{\xi}|^2 + \mathcal{O}(|\bar{\xi}|^3). \tag{A14}$$

However, in the group $\bar{S}$, where $|V^*_{cb}V_{cs}a^c_\nu| \leq |V^*_{ub}V_{us}a^u_\nu|$, the full amplitude $|A_\nu|^2$ can be at most $4|V^*_{ub}V_{us}a^u_\nu|^2$. We therefore find that

$$\frac{\sum_{\nu\in\bar{S}}|A_\nu|^2}{\sum_\nu|A_\nu|^2} \leq \frac{4\sum_{\nu\in\bar{S}}|V^*_{ub}V_{us}|^2|a^u_\nu|^2}{\| V^*_{cb}V_{cs}\vec{a}^c + V^*_{ub}V_{us}\vec{a}^u \|^2} \leq 4|\bar{\xi}|^2 + \mathcal{O}(|\bar{\xi}|^3). \tag{A15}$$

One can easily see that omitting these small $|A_{\nu\in\bar{S}}|^2$ terms from the sums in the expressions (A4) and (A5) corresponds to omitting terms of $\mathcal{O}(|\bar{\xi}|^2)$ in the expansions (A8) and (A9).

APPENDIX B: THE $SU(3)$ ANALYSIS

Finding $SU(3)$ amplitude relations can be done systematically using tensor methods. We write down the (π, K, η) meson octet as

$$(P_8)^i_j = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta_8 \end{pmatrix}, \tag{B1}$$

and the B meson triplet as

$$(B_3)_i = \begin{pmatrix} B^+ & B_d & B_s \end{pmatrix}. \tag{B2}$$

We combine the $\Delta s = 0$ and $\Delta s = -1$ Hamiltonian operators into three rank 3 tensors [17,19]:

$$((H_3)^i)^j_k = (\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} \lambda^d_q&0&0\\0&\lambda^d_q&0\\0&0&\lambda^d_q \end{pmatrix}, \begin{pmatrix} \lambda^s_q&0&0\\0&\lambda^s_q&0\\0&0&\lambda^s_q \end{pmatrix}), \tag{B3}$$

$$((H_{\bar{6}})^i)^j_k = (\begin{pmatrix} 0&0&0\\ \lambda^d_q&0&0\\ \lambda^s_q&0&0 \end{pmatrix}, \begin{pmatrix} -\lambda^d_q&0&0\\0&0&0\\0&-\lambda^s_q&\lambda^d_q \end{pmatrix}, \begin{pmatrix} -\lambda^s_q&0&0\\0&\lambda^s_q&-\lambda^d_q\\0&0&0 \end{pmatrix}), \tag{B4}$$

$$((H_{15})^i)^j_k = (\begin{pmatrix} 0&0&0\\ 3\lambda^d_q&0&0\\ 3\lambda^s_q&0&0 \end{pmatrix}, \begin{pmatrix} 3\lambda^d_q&0&0\\0&-2\lambda^d_q&0\\0&-\lambda^s_q&-\lambda^d_q \end{pmatrix}, \begin{pmatrix} 3\lambda^s_q&0&0\\0&-\lambda^s_q&-\lambda^d_q\\0&0&-2\lambda^s_q \end{pmatrix}), \tag{B5}$$

where $\lambda^{q'}_q = V^*_{qb}V_{qq'}$.

Usually, one proceeds by combining all mesons into irreducible representations of $SU(3)$ and contracting the Hamiltonian operators in all possible ways. This would require a large amount of multiplications of irreducible representations. Instead, we obtain a set of independent reduced matrix elements by summing systematically over all possible permutations:

$$\sum_{\text{Permutations } p}\left(\sum_{i_1,i_2,i_3,i_4,i_5=1}^{3}[A^{\mathbf{p}}_{\mathbf{3}}(P_8)^{i_1}_{i_{p1}}(P_8)^{i_2}_{i_{p2}}(P_8)^{i_3}_{i_{p3}}(B_3)_{i_{p4}}(H_3)^{i_4i_5}_{i_{p5}} + A^{\mathbf{p}}_{\bar{\mathbf{6}}}(P_8)^{i_1}_{i_{p1}}(P_8)^{i_2}_{i_{p2}}(P_8)^{i_3}_{i_{p3}}(B_3)_{i_{p4}}(H_{\bar{6}})^{i_4i_5}_{i_{p5}} + A^{\mathbf{p}}_{\mathbf{15}}(P_8)^{i_1}_{i_{p1}}(P_8)^{i_2}_{i_{p2}}(P_8)^{i_3}_{i_{p3}}(B_3)_{i_{p4}}(H_{15})^{i_4i_5}_{i_{p5}}]\right). \tag{B6}$$

We remind the reader that the order of the final states mesons is important as it corresponds to different momentum configurations.

All together there are 120 permutations and therefore 360 parameters $(A^{\mathbf{p}}_{\mathbf{3}}, A^{\mathbf{p}}_{\bar{\mathbf{6}}}, A^{\mathbf{p}}_{\mathbf{15}})$. However, these 360 free parameters appear in only a small number of combinations which correspond to independent reduced matrix elements. Automating the calculation, it is straightforward to obtain the set of such independent combinations. Using Young diagrams, we verify that we obtain the correct number of independent reduced matrix elements. We get 40 independent reduced matrix elements: 10 for H_3 in the

Hamiltonian (in other words, there are 10 reduced matrix elements $A_{\mathbf{3}}^{\mathbf{r_i}}$ where $\mathbf{r_i}$ stands for the 10 different representations in $\mathbf{8} \times \mathbf{8} \times \mathbf{8}$ which have nonzero reduced matrix elements involving H_3), 12 for $H_{\bar{6}}$, and 18 for H_{15}. The predictive power of the $SU(3)$ symmetry is manifest when one realizes that there are 95 $\Delta s = 0$ decays of B^+ and B^0 decays to which we can relate the 3 $\Delta s = -1$ modes of interest ($B^0 \to K^0 K^0 \bar{K}^0$, $K^0 \bar{K}^0 K^0$ and $\bar{K}^0 K^0 K^0$).

In this work, only totally symmetric final states play a role. This situation can be used to simplify the analysis. We replace the combination $(P_8)^{i_1}_{i_{p1}} (P_8)^{i_2}_{i_{p2}} (P_8)^{i_3}_{i_{p3}}$ in Eq. (B6) with the symmetrized combination:

$$\begin{aligned} &(P_8)^{i_1}_{i_{p1}} (P_8)^{i_2}_{i_{p2}} (P_8)^{i_3}_{i_{p3}} + (P_8)^{i_2}_{i_{p2}} (P_8)^{i_3}_{i_{p3}} (P_8)^{i_1}_{i_{p1}} \\ &\quad + (P_8)^{i_3}_{i_{p3}} (P_8)^{i_1}_{i_{p1}} (P_8)^{i_2}_{i_{p2}} + (P_8)^{i_2}_{i_{p2}} (P_8)^{i_1}_{i_{p1}} (P_8)^{i_3}_{i_{p3}} \\ &\quad + (P_8)^{i_1}_{i_{p1}} (P_8)^{i_3}_{i_{p3}} (P_8)^{i_2}_{i_{p2}} + (P_8)^{i_3}_{i_{p3}} (P_8)^{i_2}_{i_{p2}} (P_8)^{i_1}_{i_{p1}}. \end{aligned} \tag{B7}$$

Then the number of independent matrix elements is reduced to 7. The predictive power is maintained, since there are 20 symmetrized $\Delta s = 0$ modes to which we relate the single $B^0 \to S(K^0 K^0 \bar{K}^0)$ decay amplitude. Note, however, that the symmetric states are not properly normalized and so their normalization needs to be introduced by hand (see Eq. (5)).

Table I lists the independent reduced matrix elements. The names of the matrix elements (A_1, A_2 etc.) bear no significance. The different blocks are divided according to the form of the symmetrized final state. The symmetrized states in the table are all normalized.

In this work we also consider a simplifying dynamical assumption by which we neglect the effect of exchange, annihilation and penguin annihilation diagram [17]. The implementation of this assumption is straightforward in our calculation. One should just drop all permutations in which the B-meson triplet is contracted with the Hamiltonian operator. In other words, one takes Eq. (B6) and drops from the sum all permutations in which $p4 = 4$ or $p4 = 5$. When this procedure is applied to the symmetrized combination (B7), there are only 5 independent reduced matrix elements.

APPENDIX C: EXPERIMENTAL DATA

We quote experimental data relevant to three pseudoscalar final states. Measurements where resonant contributions are removed from the sample are denoted by (NR). The currently measured $\Delta s = \pm 1$ modes are [3]:

$$\begin{aligned} \mathcal{B}(K_S K_S K_S) &= (5.8 \pm 1.0) \times 10^{-6}, \\ \mathcal{B}(K^+ \pi^+ \pi^-) &= (53.5 \pm 3.5) \times 10^{-6}, \\ \mathcal{B}^{(NR)}(K^+ \pi^+ \pi^-) &= (4.9 \pm 1.5) \times 10^{-6}, \\ \mathcal{B}(K^+ K^- K^+) &= (30.1 \pm 1.9) \times 10^{-6}, \\ \mathcal{B}(K^+ K_S K_S) &= (11.5 \pm 1.3) \times 10^{-6}, \\ \mathcal{B}(\eta K^+ \pi^-) &= (33.4^{+4.1}_{-3.8}) \times 10^{-6}, \\ \mathcal{B}(K^0 \pi^+ \pi^-) &= (44.9 \pm 4.0) \times 10^{-6}, \\ \mathcal{B}(K^+ \pi^- \pi^0) &= (35.6^{+3.4}_{-3.3}) \times 10^{-6}, \\ \mathcal{B}(K^+ K^- K^0) &= (24.7 \pm 2.3) \times 10^{-6}. \end{aligned} \tag{C1}$$

The currently measured or constrained $\Delta s = 0$ modes are [2,3,18,21]:

$$\begin{aligned} \mathcal{B}(\pi^+ \pi^- \pi^+) &= (16.2 \pm 2.5) \times 10^{-6}, \\ \mathcal{B}(\pi^+ \pi^- \eta) &= (16.6^{+3.8}_{-3.4}) \times 10^{-6}, \\ \mathcal{B}(K^+ K^- \pi^+) &< 6.3 \times 10^{-6}, \\ \mathcal{B}(K_S K_S \pi^+) &< 3.2 \times 10^{-6}, \\ \mathcal{B}(K^+ \bar{K}^0 \pi^0) &< 24 \times 10^{-6}, \\ \mathcal{B}(K^0 K^- \pi^+) &< 21.0 \times 10^{-6}, \\ \mathcal{B}(K^+ K^- \pi^0) &< 19 \times 10^{-6}, \\ \mathcal{B}(K^+ \bar{K}^0 \pi^-) &< 18 \times 10^{-6}, \\ \mathcal{B}^{(NR)}(\pi^+ \pi^- \pi^0) &< 7.3 \times 10^{-6}. \end{aligned} \tag{C2}$$

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