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Unification via intermediate symmetry breaking scales with the quartification gauge group

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The idea of quark-lepton universality at high energies has been introduced as a natural extension to the standard model. This is achieved by endowing leptons with new degrees of freedom—leptonic color, an analogue of the familiar quark color. Grand and partially unified models which utilize this new gauge symmetry $SU(3)_\ell$ have been proposed in the context of the quartification gauge group $SU(3)^4$. Phenomenologically successful gauge coupling constant unification without supersymmetry has been demonstrated for cases where the symmetry breaking leaves a residual $SU(2)_\ell$ unbroken. Though attractive, these schemes either incorporate *ad hoc* discrete symmetries and nonrenormalizable mass terms, or achieve only partial unification. We show that grand unified models can be constructed where the quartification group can be broken fully [i.e. no residual $SU(2)_\ell$] to the standard model gauge group without requiring additional discrete symmetries or higher dimension operators. These models also automatically have suppressed nonzero neutrino masses. We perform a systematic analysis of the renormalization-group equations for all possible symmetry breaking routes from $SU(3)^4 \to SU(3)_q \otimes SU(2)_L \otimes U(1)_Y$. This analysis indicates that gauge coupling unification can be achieved for several different symmetry breaking patterns and we outline the requirements that each gives on the unification scale. We also show that the unification scenarios of those models which leave a residual $SU(2)_\ell$ symmetry are not unique. In both symmetry breaking cases, some of the scenarios require new physics at the TeV scale, while others do not allow for new TeV phenomenology in the fermionic sector.

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I. INTRODUCTION

Grand unified theories (GUTs) are an important class of extensions to standard model (SM) physics, with most theories attempting to unify the strong and electroweak interactions within the framework of a single, larger gauge symmetry G. The simple groups $SU(5)$ [1] and $SO(10)$ [2], which may be derived from a possible underlying E_6 [3], have been the common groups of interest. $SU(5)$ is the smallest group with complex representations that can accommodate the SM gauge structure, with the fermions placed in the $\mathbf{1} \oplus \bar{\mathbf{5}} \oplus \mathbf{10}$ representation. As both quarks and leptons are contained in both the $\bar{\mathbf{5}}$ and the $\mathbf{10}$, gauge-mediated quark-lepton transformations exist, giving rise to baryon number violation. Similarly, $SO(10)$ can house an entire generation of fermions, including the right-handed neutrino, in a single **16**. This enumeration of fermions provides great simplicity but also places unrealistic bounds on proton stability once a unification scale is identified.

Additionally, these groups are plagued by a lack of phenomenologically successful coupling constant unification. As these unified theories are based upon a simple group G, a single gauge coupling constant describes the strength of all gauge interactions. The three coupling parameters of the low-energy SM field theory need to separately evolve as a function of energy until concordance at some possible unification energy scale results and we have only one effective coupling constant [4]. Whether unification of the gauge coupling constants can be achieved represents a crucial test for the feasibility of a GUT. The running of the coupling constants in $SU(5)$ and $SO(10)$ theories fails to satisfy this criterion unless appropriate new physics at an intermediate scale, such as supersymmetry, is invoked. This motivates the application of product groups $G \otimes G \otimes \ldots$, augmented by a discrete symmetry permuting the G factors, as an alternative class of unified theories. These models need not have gauge boson mediated proton decay as the quarks and leptons are often in separate representations, however even if so, proton instability can still originate through Higgs-fermion Yukawa interactions.

The smallest such group consistent with SM phenomenology at low energy is trinification, based on $SU(3)_q \otimes SU(3)_L \otimes SU(3)_R$, which has shown promise both within and independent of a supersymmetric context. It can also be obtained from an E_6 theory and as a result has been studied extensively [5,6].

A natural extension to trinification is quartification.[1] Quartification was first proposed by Volkas and Joshi [8] and then independently revisited by Babu, Ma and Willenbrock [9]. These theories represent an implementation of the idea of quark-lepton universality at high energies postulated by Foot and Lew [10]. In the low-energy world described by the SM, there are significant disparities between the quarks and leptons. They have different electric charges, and the quarks are confined by color interactions whereas the leptons are not. Extended models

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[1] See also Ref. [7] for an extension of trinification to $3N$ $SU(3)$ factors, where N is a positive integer.

employing discrete quark-lepton symmetry [10] allow quarks and leptons to become indistinguishable above energy scales as low as a few TeV. To achieve this, one must introduce new degrees of freedom for the leptons which are embodied in a separate gauge group $SU(3)_\ell$. With this gauge group supplementing the familiar quark color $SU(3)_q$ group, a discrete exchange symmetry between the quarks and leptons can be imposed. The quark-lepton indistinguishability afforded by this scheme does not require gauge coupling constant unification. Quartification is the simplest known way to extend such models to also provide for full coupling constant unification.

The gauge group of these theories is $SU(3)^4$ with an anomaly-free fermion assignment. The symmetry breaking is accomplished with Higgs multiplets in a certain 36-dimensional representation of $SU(3)^4$, and in Refs. [8,9] the symmetry is broken down to $SU(3)_q \otimes SU(2)_L \otimes U(1)_Y \otimes SU(2)_\ell$. In the low-energy limit, there is an $SU(2)_\ell$ remnant of leptonic color left unbroken. Each standard lepton has heavy exotic partners, which following tradition we call "liptons," in an $SU(2)_\ell$ doublet. If the liptons are heavier than a few 100 GeV, then the existence of the new unbroken gauge symmetry $SU(2)_\ell$ is hidden, though potentially to be found at Large Hadron Collider energies.

The original proposal of Ref. [8] succeeds only in partial unification [11], with two independent gauge coupling constants at the unification scale. The model of Babu *et al.* [9] is a variant that achieves full unification, with the running coupling constants meeting at around 4×10^{11} GeV without invoking supersymmetry. Vital to this result is the presence of the liptons at the TeV scale, to help ensure appropriate running for the coupling constants, assisted by a second light Higgs doublet. In nonquartified Foot-Lew–type models [10], TeV-scale liptons are a natural possibility. In quartification, however, the default situation sees the liptons acquiring unification-scale masses. This is precisely why the original paper [8] proposed partial unification only. By contrast, Babu *et al.* [9] impose an additional discrete symmetry for the sole purpose of avoiding ultralarge lipton masses. While it is of course technically natural, the additional discrete symmetry is in other respects very much an afterthought. It is compounded by the resolutions employed (nonrenormalizable operators) to obtain realistic TeV-scale lipton masses. There are also issues with neutrino mass generation. This scheme is reviewed more fully in Sec. 2.

The purpose of this paper is to show how to achieve full unification through quartification without the imposition of the additional discrete symmetry. In fact, we demonstrate that the unification scheme of Ref. [9] is not unique, and can be obtained without restricting Yukawa couplings simply by introducing intermediate stages in the symmetry breaking. Importantly, we show that heavy exotic liptons need not spoil unification. In Sec. III we describe the matter content of our models and state any assumptions associated with the Higgs sector. We outline in Sec. 3 all possible choices of symmetry breaking which leave a residual $SU(2)_\ell$, while in the subsequent subsection we break this symmetry entirely [12] and resolve the neutrino mass issues. We systematically solve the renormalization-group equations for each, showing the choices that give rise to successful unification. In Sec. V we comment on the low-energy phenomenology in those scenarios in which unification is possible. We conclude in Sec. VI.

II. THE QUARTIFICATION MODEL

The quartification gauge group is

$$G_4 = SU(3)_q \otimes SU(3)_L \otimes SU(3)_\ell \otimes SU(3)_R. \tag{2.1}$$

A Z_4 symmetry which cyclically permutes the gauge groups as per $q \to L \to \ell \to R \to q$ is imposed, ensuring a single gauge coupling constant g_4. The fermions are contained within a left-handed **36** of Eq. (2.1),

$$\mathbf{36} = (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}) + (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{3}), \tag{2.2}$$

$$\equiv \quad q \quad \oplus \quad \ell \quad \oplus \quad \ell^c \quad \oplus \quad q^c, \tag{2.3}$$

where $q(\ell)$ denotes the left-handed quarks (leptons) and $q^c(\ell^c)$ the left-handed antiquarks (antileptons). Under G_4, these have the transformations

$$\begin{aligned} q &\to U_q q U_L^\dagger, & q &\to U_R q^c U_q^\dagger, \\ \ell &\to U_L \ell U_\ell^\dagger, & \ell^c &\to U_\ell \ell^c U_R^\dagger, \end{aligned} \tag{2.4}$$

where $U_{q,L,\ell,R} \in SU(3)_{q,L,\ell,R}$, and the multiplets are represented by 3×3 matrices:

$$\begin{aligned} q \sim (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}) &= \begin{pmatrix} d & u & h \\ d & u & h \\ d & u & h \end{pmatrix}, \\ q^c \sim (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{3}) &= \begin{pmatrix} d^c & d^c & d^c \\ u^c & u^c & u^c \\ h^c & h^c & h^c \end{pmatrix}, \\ \ell \sim (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) &= \begin{pmatrix} x_1 & x_2 & \nu \\ y_1 & y_2 & e \\ z_1 & z_2 & N \end{pmatrix}, \\ \ell^c \sim (\mathbf{1}, \mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}) &= \begin{pmatrix} x_1^c & y_1^c & z_1^c \\ x_2^c & y_2^c & z_2^c \\ \nu^c & e^c & N^c \end{pmatrix}. \end{aligned} \tag{2.5}$$

Note the existence of exotic particles as a necessary ingredient in the representations. Defining the generator of electric charge Q as

$$Q = I_{3L} - \frac{Y_L}{2} - \frac{Y_\ell}{2} + I_{3R} - \frac{Y_R}{2}, \tag{2.6}$$

we identify N, N^c as neutral particles, $h(h^c)$ as charge $Q = -\frac{2}{3}(\frac{2}{3})$ exotic quarks and the liptons (x, y, z) to have charges $(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$.

In Babu *et al.*'s model [9], the Higgs fields are contained in two different **36**'s, and shall be denoted

$$\begin{aligned} &\Phi_a \sim (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{1}, \mathbf{3}), \qquad \Phi_b \sim (\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}}, \mathbf{1}),\\ &\Phi_c \sim (\mathbf{1}, \mathbf{3}, \mathbf{1}, \bar{\mathbf{3}}), \qquad \Phi_d \sim (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{3}, \mathbf{1}),\\ &\Phi_\ell \sim (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}), \qquad \Phi_{\ell^c} \sim (\mathbf{1}, \mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}),\\ &\Phi_{q^c} \sim (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{3}), \qquad \Phi_q \sim (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}) \end{aligned} \tag{2.7}$$

in this paper (Ref. [9] used different subscripts). Note that $\Phi_a \sim \Phi_c^\dagger$; this effective replication of a Higgs multiplet is achieved as a natural consequence of the Z_4 symmetry. These fields, which comprise two sets of multiplets that are closed under the Z_4, are sufficient to break the quartification symmetry and generate realistic fermion masses and mixings. The vacuum expectation value (VEV) pattern is given by

$$\langle\Phi_a\rangle \sim \langle\Phi_c^\dagger\rangle \sim \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ v & 0 & v \end{pmatrix}, \qquad \langle\Phi_\ell\rangle \sim \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix},$$

$$\langle\Phi_{\ell^c}\rangle \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ v & 0 & v \end{pmatrix}, \tag{2.8}$$

$$\langle\Phi_b\rangle = \langle\Phi_d\rangle = \langle\Phi_{q^c}\rangle = \langle\Phi_q\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.9}$$

where the u's and v's are at the electroweak and unification scales, respectively. This VEV structure induces the strong symmetry breaking $SU(3)_q \otimes SU(3)_L \otimes SU(3)_\ell \otimes SU(3)_R \to SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes U(1)_Y$ in a single step, and then instigates electroweak symmetry breaking. To do this, a delicate hierarchy within some of these multiplets must exist which necessitates unnatural fine-tuning. (This unwelcome feature also occurs in trinification models and in the new quartification schemes we propose below.)

The coupling to the fermions is described by the Z_4-invariant Lagrangian[2]

$$\begin{aligned} \mathcal{L} = {} & Y_1 \,\mathrm{Tr}(\ell\ell^c\Phi_a + \ell^c q^c\Phi_b + qq^c\Phi_c + q\ell\Phi_d) \\ & + Y_2 \,\mathrm{Tr}(\ell\ell^c\Phi_c^\dagger + \ell^c q^c\Phi_d^\dagger + qq^c\Phi_a^\dagger + q\ell\Phi_b^\dagger) \\ & + \mathrm{H.c.} \end{aligned} \tag{2.10}$$

[2]The notation $\ell\ell^c\Phi_a$ means $\bar{\ell}_R\ell_L\Phi_a$, etc.

The theory also admits couplings of the type $\lambda\ell\ell\Phi_\ell$ and cyclic permutations. These terms, however, give GUT-scale masses to the liptons x_1, x_2, y_1, y_2, x_1^c, x_2^c, y_1^c and y_2^c, and, according to [9], it is essential that these particles remain light for gauge coupling constant unification. An additional Z_4' symmetry defined by

$$\begin{aligned} &(q, \ell, q^c, \ell^c) \to i(q, \ell, q^c, \ell^c), \\ &\Phi_{a-d} \to -\Phi_{a-d} \quad \text{and} \quad \Phi_{\ell,\ell^c,q,q^c} \to \Phi_{\ell,\ell^c,q,q^c} \end{aligned} \tag{2.11}$$

is then imposed to forbid these terms, reducing the Yukawa Lagrangian exclusively to Eq. (2.10). As a consequence of this Z_4', after symmetry breaking, the massless particle spectrum contains the x, x^c and y, y^c liptons in addition to the minimal SM particles. This massless spectrum, which contains two SM Higgs doublets, is sufficient to unify the coupling constants within experimental error at approximately 4×10^{11} GeV.

If the discrete symmetry of Eq. (2.11) was not imposed, then the intersection of the three SM coupling constants would not occur. A less nice aspect of this scheme is that there is no natural origin for this symmetry—why, on theoretical grounds, should certain Yukawa terms be forbidden and not others? This unattractive feature is exacerbated by the need to introduce nonrenormalizable terms of the form $\epsilon_{jkl}\epsilon_{mnp}\ell^{jm}\ell^{kn}(\Phi_a^\dagger\Phi_{\ell^c}^\dagger)^{lp}$ and $\epsilon_{jkl}\epsilon_{mnp}(\ell^c)^{jm} \times (\ell^c)^{kn}(\Phi_\ell^\dagger\Phi_c)^{lp}$. These terms must exist to give TeV-scale masses to the liptons, otherwise their mass terms would be indistinguishable from the ordinary leptons at the electroweak level. Although the proton decay mediated by these terms is predicted at a realistic rate, from a model-building point of view they are a little *ad hoc*. Babu *et al.* then introduce another ingredient to resolve the issue of neutrino mass. In the bare model, neutrinos naturally acquire Dirac masses of the same order as the charged fermions. This is circumvented by the addition of a Higgs singlet whose coupling with the right-handed neutrino induces the seesaw mechanism. We comment later on how neutrinos naturally acquire light masses without a Higgs singlet when $SU(2)_\ell$ is broken. All in all, the very pleasing gauge unification property of this scheme is partially spoilt by the additional discrete symmetry required, the nonrenormalizable operators, and the extra Higgs singlet.

III. MASS SPECTRUM

A. $SU(2)_\ell$ unbroken

The matter content and mass thresholds of a unified theory govern the running of the coupling constants. It is thus important to elucidate at what energy scales the various particles gain mass and how they contribute to the renormalization-group equations. We employ the same fermion and Higgs multiplet assignments as Ref. [9], summarized by Eqs. (2.4), (2.5), and (2.7).

Given that we are aiming for as natural a quartification model as possible, one needs to be aware of the most obvious approach in determining the Higgs VEV structure short of performing a minimization analysis of a Higgs potential. First, those Higgs fields of Eq. (2.7) which are not singlets under quark color necessarily cannot acquire VEVs, and we also naturally assume them to have mass of unification scale always. Thus the fields Φ_b, Φ_d, Φ_q and Φ_{q^c} have no influence on the renormalization-group equations and can be ignored for now. For the remaining fields our policy is the following: We first choose a symmetry breaking cascade. At a given stage in the symmetry breaking chain, those components that can acquire a VEV consistent with the symmetry breakdown pattern do so at that scale, and that the corresponding Higgs masses are also at that same scale.

With this in mind, consider the breaking

$$SU(3)_q \otimes SU(3)_L \otimes SU(3)_\ell \otimes SU(3)_R \xrightarrow{v} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes U(1)_Y \xrightarrow{u} SU(3)_q \otimes SU(2)_\ell \otimes U(1)_Q. \tag{3.1}$$

The VEV pattern which achieves this is

$$\langle\Phi_\ell\rangle = \begin{pmatrix} 0 & 0 & u_\ell \\ 0 & 0 & 0 \\ 0 & 0 & v_\ell \end{pmatrix} \qquad \langle\Phi_{\ell^c}\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ v_{\ell_1^c} & 0 & v_{\ell_2^c} \end{pmatrix}$$

$$\langle\Phi_a\rangle = \langle\Phi_c^\dagger\rangle = \begin{pmatrix} u_{a1} & 0 & u_{a2} \\ 0 & u_{a3} & 0 \\ v_{a1} & 0 & v_{a2} \end{pmatrix}. \tag{3.2}$$

(If we were to introduce intermediate steps in the breaking, then the v's would be of different orders.) It is unfortunate that this VEV structure has an intramultiplet hierarchy, with entries of both order u and v contained in a single multiplet. We shall accept this as we think it would be more unnatural for the Higgs fields to get smaller VEVs than the symmetry breaking scheme requires. Solving this hierarchy problem would require additional Higgs fields (not necessarily in the same representations as those above), which would necessarily imply a larger Higgs potential with a greater number of arbitrary parameters, result in only a partially unified theory as in Ref. [8], or require a completely different symmetry breaking mechanism, such as the employment of inhomogeneous scalar field configurations [13] or orbifold symmetry breaking in a brane-world setting [14] (see Refs. [15,16] for applications of the former and latter, respectively, to trinification models).

To give all Yukawa couplings even grounding, we remove the discrete symmetry employed in Ref. [9], leaving four independent Yukawa interactions which can endow the fermions with mass. These are

$$\lambda_q \,\mathrm{Tr}[q^c q \Phi_a] \qquad \lambda_\ell \,\mathrm{Tr}[\ell\ell^c \Phi_c], \tag{3.3}$$

$$\lambda_L \epsilon^{jkl}\epsilon^{mnp}\ell^{jm}\ell^{kn}(\Phi_\ell^\dagger)^{lp},\ \lambda_R \epsilon^{jkl}\epsilon^{mnp}(\ell^c)^{jm}(\ell^c)^{kn}(\Phi_{\ell^c})^{lp}, \tag{3.4}$$

giving the quark mass term

$$\mathcal{L}_{\text{quark mass}} = \lambda_q (d \quad h)\begin{pmatrix} u_{a1} & u_{a2} \\ v_{a1} & v_{a2} \end{pmatrix}\begin{pmatrix} d^c \\ h^c \end{pmatrix} + (\lambda_q u_{a3}) u u^c + \text{H.c.} \tag{3.5}$$

The up quarks acquire electroweak-scale Dirac masses, while the d and h quarks are mixed. Upon diagonalization of this mass matrix, we have only one $Q = -1/3$ quark per family with electroweak-scale mass to be identified as the down quark, and the exotic quark gains a GUT-scale mass. Note that mixing between the h and d quarks is suppressed by u/v.

The mass terms of the leptons are solely of Dirac nature. The $Q = -1$ charged leptons gain masses of electroweak order and do not mix with any other states. The liptons x_1, x_2, y_1^c, y_2^c, z_1 and z_2 have charge $Q = +1/2$ and pair up with the charge $-1/2$ liptons x_1^c, x_2^c, y_1, y_2, z_1^c and z_2^c, to acquire GUT-scale masses. The electrically neutral leptons ν, N, N^c and ν^c also only have Dirac mass terms. One sector is of GUT scale, identified as heavy neutral leptons, and the other, the ordinary neutrinos, is of electroweak scale. We see that we encounter the same problem as did Babu *et al.* with respect to obtaining a light neutrino mass.

In summary, all SM particles including the Dirac neutrinos have electroweak-scale masses, and all exotics have GUT-scale masses. If the symmetry breaking occurs via intermediate scales, then the masses of the exotic particles will be at the unification or one of these intermediate scales.

In determining the running of the gauge coupling constants, we must also know the full structure of the light Higgs spectrum at each stage of symmetry breaking. The VEV structure above neither provides enough information to define all the masses of the Higgs' components nor how many SM doublets there are. One is forced to make an assumption to deal with this, and, again, we adopt as natural a one as possible. The assumption chosen involves looking at the branchings, and particularly at what scale components branch away from those components that acquire VEVs. If a component gains a VEV, then the SM multiplet in which it is contained is taken to get a mass at the same scale. In the case where $SU(2)_\ell$ remains unbroken, there are SM multiplets which have no VEVs but are embedded within a quartification multiplet that does. We assume that these gain mass at the scale of the largest VEV in the quartification multiplet. For example, the VEV

$$\langle\Phi_\ell\rangle = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix} \tag{3.6}$$

implies that the components $(\Phi_\ell)_3^1$, $(\Phi_\ell)_3^2$ have masses of order u, while the remaining components all have mass of order v. This gives us seven candidate light Higgs doublets at the SM level: one from Φ_ℓ and three each from Φ_a and Φ_c. The Higgs doublet multiplicity has a beneficial effect on the achievement of gauge coupling constant unification [6]. Although the Higgs sector of our models has been burdened with these assumptions, we have avoided the introduction of an additional discrete symmetry.

B. $SU(2)_\ell$ broken

The Higgs fields of Eq. (2.7) also have the capacity to break the leptonic color symmetry completely, leaving no residual $SU(2)_\ell$ gauge group unbroken [12]. Consider the breaking cascade

$$SU(3)_q \otimes SU(3)_L \otimes SU(3)_\ell \otimes SU(3)_R \xrightarrow{v} SU(3)_q \otimes SU(2)_L \otimes U(1)_Y \xrightarrow{u} SU(3)_q \otimes U(1)_Q. \tag{3.7}$$

With this symmetry breaking pattern, the electric charge generator is given by

$$Q = I_{3L} - \frac{Y_L}{2} + I_{3\ell} - \frac{Y_\ell}{2} + I_{3R} - \frac{Y_R}{2}. \tag{3.8}$$

Notice that the three spontaneously broken $SU(3)$ factors contribute in a symmetric manner to the electric charge generator. This breakdown pattern is accomplished by Higgs fields obtaining VEVs of the form

$$\langle\Phi_\ell\rangle = \begin{pmatrix} u_{\ell 1} & 0 & u_{\ell 2} \\ 0 & u_{\ell 3} & 0 \\ v_{\ell 1} & 0 & v_{\ell 2} \end{pmatrix},$$
$$\langle\Phi_{\ell^c}\rangle = \begin{pmatrix} v_{\ell^c 1} & 0 & v_{\ell^c 2} \\ 0 & v_{\ell^c 3} & 0 \\ v_{\ell^c 4} & 0 & v_{\ell^c 5} \end{pmatrix}, \tag{3.9}$$
$$\langle\Phi_a\rangle = \langle\Phi_c^\dagger\rangle = \begin{pmatrix} u_{a1} & 0 & u_{a2} \\ 0 & u_{a3} & 0 \\ v_{a1} & 0 & v_{a2} \end{pmatrix},$$

where all Higgs components that can acquire a VEV at a given scale do so. The Higgs mass spectrum here is "derived" in a more obvious fashion than before. All members of SM multiplets which get a VEV acquire masses at that scale. This leaves nine light, left-handed Higgs doublets at the standard model level, three each from Φ_ℓ, Φ_a and Φ_c.

As before, Eq. (3.3) describes the Yukawa couplings. Breaking leptonic color completely has no impact on the quarks as they are singlets under this gauge group: the quark masses are identical irrespective of whether or not $SU(2)_\ell$ is broken. The leptons, however, possess leptonic color and their electric charges are altered due to the different electric charge generator of Eq. (3.8), and their mass terms are greatly influenced by the different VEV pattern of Eq. (3.9). The components that were previously half-integrally charged liptons now acquire integral charges $Q = \pm 1, 0$, so they are no longer liptons but are instead charged and neutral heavy leptons.

The leptons with a charge of $+1$ are the e^c, y_1^c, z_2 and x_2 components of Eq. (2.5). They mix and form Dirac mass terms with the charge -1 lepton components e, y_1, z_2^c and x_2^c, in the manner

$$(e \quad y_1 \quad z_2^c \quad x_2^c) \begin{pmatrix} u_{a3} & 0 & -u_{\ell 1} & v_{\ell 1} \\ 0 & u_{a3} & u_{\ell 2} & -v_{\ell 2} \\ -v_{\ell^c 1} & v_{\ell^c 4} & v_{a2} & u_{a2} \\ v_{\ell^c 2} & -v_{\ell^c 5} & v_{a1} & u_{a1} \end{pmatrix} \begin{pmatrix} e^c \\ y_1^c \\ z_2 \\ x_2 \end{pmatrix} + \text{H.c.} \tag{3.10}$$

There are three Dirac mass eigenvalues (per family) of GUT scale, and one eigenvalue (per family) of electroweak scale corresponding to the e, μ and τ masses.

The leptonic components N, N^c, ν, ν^c, x_1, x_1^c, y_2, y_2^c, z_1 and z_1^c are all neutral. Unlike the former case these ten leptons gain Majorana masses, as per

$$(N \quad N^c \quad \nu \quad \nu^c \quad x_1 \quad x_1^c \quad y_2 \quad y_2^c \quad z_1 \quad z_1^c) \times \begin{pmatrix} 0 & v_{a2} & 0 & v_{a1} & u_{L3} & 0 & u_{L1} & 0 & 0 & 0 \\ v_{a2} & 0 & u_{a2} & 0 & 0 & v_{\ell^c 3} & 0 & v_{\ell^c 1} & 0 & 0 \\ 0 & u_{a2} & 0 & u_{a1} & 0 & 0 & -u_{\ell 1} & 0 & -u_{L3} & 0 \\ v_{a1} & 0 & u_{a1} & 0 & 0 & 0 & 0 & -v_{\ell^c 2} & 0 & -v_{\ell^c 3} \\ u_{\ell 3} & 0 & 0 & 0 & 0 & u_{a1} & v_{\ell 2} & 0 & 0 & 0 \\ 0 & v_{\ell^c 3} & 0 & 0 & u_{a1} & 0 & 0 & v_{\ell^c 5} & 0 & 0 \\ u_{\ell 1} & 0 & -v_{\ell 1} & 0 & v_{\ell 2} & 0 & 0 & u_{a3} & -u_{\ell 2} & 0 \\ 0 & v_{\ell^c 1} & 0 & -v_{\ell^c 2} & 0 & v_{\ell^c 5} & u_{a3} & 0 & 0 & 0 \\ 0 & 0 & -u_{\ell 3} & 0 & 0 & 0 & -u_{\ell 2} & 0 & 0 & v_{a2} \\ 0 & 0 & 0 & -v_{\ell^c 3} & 0 & 0 & 0 & 0 & v_{a2} & 0 \end{pmatrix} \begin{pmatrix} N \\ N^c \\ \nu \\ \nu^c \\ x_1 \\ x_1^c \\ y_2 \\ y_2^c \\ z_1 \\ z_1^c \end{pmatrix}. \tag{3.11}$$

Nine of the resulting mass eigenvalues are of the order of the GUT scale, and the tenth has is a small mass of the order of $\frac{u^2}{v}$, which is precisely the mass scale that would result from a regular seesaw mechanism [17]. This particle displays the correct weak coupling with the electron to be identified as the neutrino, and all interactions involving the light leptons with the heavy leptons are very suppressed. When intermediate scales are introduced, some of the order v entries decrease in size and thus some of the large eigenvalues also decrease. One anticipates that this raises the value of the smallest eigenvalue.

In summary, the VEV patterns of Eq. (3.9) through the Yukawa coupling terms provide large masses to exotic fermions, electroweak-scale masses for standard charged fermions, and a seesaw suppressed masses for the neutrinos.

IV. RENORMALIZATION-GROUP EQUATIONS

A. $SU(2)_\ell$ unbroken

We begin by analyzing the renormalization-group equations for the schemes featuring a remnant of the leptonic color symmetry at low energy. There is no physical reason why the symmetry breaking has to directly proceed via Eq. (3.1). In fact in this case, without the restrictions proposed in Ref. [9] imposed on the Yukawa sector, the gauge coupling constants only come within the vicinity of intersecting if the Higgs sector is enlarged significantly.

An alternative to the one-step scenario is the introduction of intermediate symmetry breaking scales. There are four independent symmetry breaking routes from $G_4 \rightarrow SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes U(1)_Y$ which can be achieved with our Higgs sector. They are labeled as

1. $$G_4 \xrightarrow{v} SU(3)_q \otimes SU(3)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_1} \xrightarrow{w} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_2} \xrightarrow{x} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes U(1)_Y \quad (4.1)$$

2. $$G_4 \xrightarrow{v} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(3)_R \otimes U(1)_{X_1} \xrightarrow{w} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_2} \xrightarrow{x} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes U(1)_Y \quad (4.2)$$

3. $$G_4 \xrightarrow{v} SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(2)_R \otimes U(1)_{X_1} \xrightarrow{w} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_2} \xrightarrow{x} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes U(1)_Y \quad (4.3)$$

4. $$G_4 \xrightarrow{v} SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(2)_R \otimes U(1)_{X_1} \xrightarrow{w} SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes U(1)_{X_2} \xrightarrow{x} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes U(1)_Y. \quad (4.4)$$

For each cascade, the symmetry breaking is achieved by the VEVs of Eq. (3.2) where the energies of the v_i VEV entries are not uniform. Each of these cascades is of course followed by electroweak symmetry breakdown. As we now have several scales, the masses of the exotic fermions and Higgs bosons will stagger with energy and they will have varying contributions to the renormalization-group equations. The exact nature of the VEV entries and at which scale the fermions gain masses are detailed in Table I. Notice that the x_1, x_2, y_1, y_2 particles, which had to be at the TeV scale in Ref. [9], can only potentially be light in the case of cascade four.

TABLE I. The energy scale of the VEV entries of Eq. (3.2) with $v \geq w \geq x$, and the enumeration of the fermion masses for the four cascades of Eqs. (4.2), (4.3), (4.4), and (4.5) where the quartification gauge symmetry is broken down to $G_{SM} \otimes SU(2)_\ell$ in stages. Notice that cascade four is the only symmetry breaking route that will potentially permit light liptons.

Cascade	Energy scale of VEVs	Masses at v	Masses at w	Masses at x
1	$v_{R_2} \sim v$, $v_{a2} \sim w$, $v_L \sim w$, $v_{R_1} \sim x$, $v_{a1} \sim x$	$x_1^c, x_2^c, y_1^c, y_2^c$	$h, h^c, z_1, z_1^c z_2, z_2^c$, $N, N^c, x_1, x_2, y_1, y_2$	None
2	$v_L \sim v$, $v_{R_2} \sim w$, $v_{a2} \sim w$, $v_{R_1} \sim x$, $v_{a1} \sim x$	x_1, x_2, y_1, y_2	$h, h^c, z_1, z_1^c z_2, z_2^c$, $N, N^c, x_1^c, x_2^c, y_1^c, y_2^c$	None
3	$v_{a2} \sim v$, $v_L \sim w$, $v_{R2} \sim w$, $v_{a1} \sim x$, $v_{R1} \sim x$	h, h^c, z_1, z_2, N, z_1^c, z_2^c, N^c	x_1, x_2, y_1, y_2, $x_1^c, x_2^c, y_1^c, y_2^c$	None
4	$v_{a2} \sim v$, $v_{a1} \sim w$ $v_L \sim x$, $v_{R2} \sim x$	h, h^c, z_1, z_2, N, z_1^c, z_2^c, N^c	None	x_1, x_2, y_1, y_2, $x_1^c, x_2^c, y_1^c, y_2^c$

Defining the fine-structure constants as α_q, α_L, α_ℓ and α_Y, respectively, for quark color, weak $SU(2)_L$, leptonic color and hypercharge, the one-loop renormalization-group equations which describe their evolution have the form

$$\frac{1}{\alpha_i(M_1)} = \frac{1}{\alpha_i(M_2)} + \frac{b_i}{2\pi}\ln\left(\frac{M_1}{M_2}\right). \tag{4.5}$$

$i = q, L, \ell, Y$, $M_{1,2}$ denote two mass scales of our theory, and the b factor is given by

$$b = -\frac{11}{3}T(\text{gauge bosons}) + \frac{2}{3}T(\text{Weyl fermions}) + \frac{1}{3}T(\text{complex scalars}). \tag{4.6}$$

The T's are group theoretical properties which depend on the gauge group representations and are defined by the generators λ^a in the representation R as

$$\mathrm{Tr}\,(\lambda^a\lambda^b) = T_R\delta^{ab}. \tag{4.7}$$

At each stage of the symmetry breaking, b harbors all knowledge of particles with masses lighter than that particular scale. Our labeling scheme for these factors is best illustrated by an example: b_{q_1} refers to the cumulative effect of fields which possess quark color between v and w; while b_{q_2} is concerned with the energy range $x \leftrightarrow w$; and b_{q_3} the range $M_{EW} \leftrightarrow x$. The b-factors for the ℓ, L and R sectors are denoted similarly, with q replaced by the appropriate subscript. The quantity b_{u_j}, $j = 1, 2, 3$ shall denote the running of the $U(1)$ coupling and takes into account the normalization of the generators defining X_1, X_2 and Y. The generator of hypercharge is taken to be the conventional embedding and is given by

$$Y = I_{3R} + \frac{2}{\sqrt{3}}(\lambda_{L8} + \lambda_{\ell 8} + \lambda_{R8}), \tag{4.8}$$

where the λ's are the usual Gell-Mann generators.

For all cascades, the running of the α_q, α_L and α_ℓ constants have the same generic form given by

$$\frac{1}{\alpha_i(v)} = \frac{1}{\alpha_i(M_{EW})} - \frac{b_{i_1}}{2\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{b_{i_1} - b_{i_2}}{2\pi}\ln\left(\frac{w}{M_{EW}}\right) + \frac{b_{i_2} - b_{i_3}}{2\pi}\ln\left(\frac{x}{M_{EW}}\right). \tag{4.9}$$

This arises as the $SU(2)_{L,\ell}$ groups have the same coupling constants as $SU(3)_{L,\ell}$ because of the way in which these subgroups are embedded within their parent $SU(3)$. At each scale, the coupling constants are analyzed and we can determine a relationship between the different energy scales v, w, x and the values of the fine-structure constants at the electroweak scale M_{EW}. The evolutions of the $U(1)$ coupling constants are different for each cascade as they depend on the specific linear combinations of generators defining X_1 and X_2, and so the forms of the renormalization-group equations depend on the symmetry breaking pattern. The relationship between the $U(1)$ electroweak-scale fine-structure constant and the electroweak-scale coupling constants for $SU(2)_\ell$ and $SU(2)_L$ is

$$\frac{1}{\alpha_Y} = \frac{1}{3\alpha_\ell} + \frac{5}{3\alpha_L} + \frac{3(b_{u_1} + b_{R_1}) - b_{\ell_1} - 4b_{L_1}}{6\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{3(b_{u_2} + b_{R_2} - b_{u_1} - b_{R_1}) + b_{\ell_1} - b_{\ell_2} + 4b_{L_1} - 5b_{L_2}}{6\pi} \times \ln\left(\frac{w}{M_{EW}}\right) + \frac{3(b_{u_3} - b_{u_2} - b_{R_2}) + b_{\ell_2} - b_{\ell_3} + 5(b_{L_2} - b_{L_3})}{6\pi}\ln\left(\frac{x}{M_{EW}}\right) \tag{4.10}$$

for cascade one, but takes a different form for the other cascades. The experimental values for the fine-structure constants at M_{EW} are [18]

$$\alpha_q = 0.1172, \qquad \alpha_L = 0.0338, \qquad \alpha_Y = 0.0102, \tag{4.11}$$

where remember we have absorbed the normalization into the b_{u_j}'s.

In this section we shall focus on the renormalization-group equation analysis for cascade one and present only a summary of results for the other symmetry breaking routes. The full details of their equations are relegated to Appendix A. After the first stage of symmetry breaking, the particles x_1^c, x_2^c, y_1^c and y_2^c gain mass and the light Higgs spectrum is

$$\Phi_\ell \sim (\mathbf{1},\mathbf{3},\mathbf{2},\mathbf{1})\left(\frac{1}{3}\right) \oplus (\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})\left(-\frac{2}{3}\right), \quad \Phi_{\ell^c} \sim (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1), \tag{4.12}$$

$$\Phi_a \sim (\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{2})\left(\frac{1}{3}\right) \oplus (\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})\left(-\frac{2}{3}\right), \qquad \Phi_c \sim \Phi_a^\dagger. \tag{4.13}$$

The remaining exotic fermions gain masses at the w scale and the light Higgs spectrum is

$$\Phi_\ell \sim (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1), \qquad \Phi_{\ell^c} \sim (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1), \tag{4.14}$$

$$\Phi_a \sim (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1), \quad \Phi_c \sim \Phi_a^\dagger. \tag{4.15}$$

After the final breaking particles of the minimal SM and ν^c are massless and

$$\Phi_\ell \sim (\mathbf{1}, \mathbf{2}, \mathbf{1})(-1), \tag{4.16}$$

$$\Phi_a \sim (\mathbf{1}, \mathbf{2}, \mathbf{1})(-1) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1})(1) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1})(-1), \qquad \Phi_c \sim \Phi_a^\dagger, \tag{4.17}$$

giving us seven SM Higgs doublets.

Defining N_H to be the multiplicity of the Higgs fields in Eq. (2.7)[3] and summing over three generations of fermions, this spectrum of particles defines the values of the b quantities as

$$\begin{array}{lllll} b_{q_1} = -5, & b_{L_1} = -5 + \frac{3N_H}{2}, & b_{\ell_1} = -\frac{10}{3} + \frac{N_H}{2}, & b_{R_1} = -\frac{10}{3} + \frac{7N_H}{6}, & b_{u_1} = 4 + \frac{2N_H}{3}, \\ b_{q_2} = -7, & b_{L_2} = -\frac{10}{3} + \frac{7N_H}{6}, & b_{\ell_2} = -\frac{22}{3}, & b_{R_2} = -\frac{10}{3} + \frac{7N_H}{6}, & b_{u_2} = \frac{8}{3} + N_H, \\ b_{q_3} = b_{q_2}, & b_{L_3} = b_{L_2}, & b_{\ell_3} = b_{\ell_2}, & & b_{u_3} = \frac{20}{3} + \frac{7N_H}{6}. \end{array} \tag{4.18}$$

Substituting these numbers into the renormalization-group equations, we have

$$\frac{1}{\alpha_q(v)} = \frac{1}{\alpha_q} + \frac{5}{2\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{1}{\pi}\ln\left(\frac{w}{M_{EW}}\right), \tag{4.19}$$

$$\frac{1}{\alpha_L(v)} = \frac{1}{\alpha_L} + \frac{10 - 3N_H}{4\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{N_H - 5}{6\pi} \times \ln\left(\frac{w}{M_{EW}}\right), \tag{4.20}$$

$$\frac{1}{\alpha_\ell(v)} = \frac{1}{\alpha_\ell} + \frac{20 - 3N_H}{12\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{8 + N_H}{4\pi} \times \ln\left(\frac{w}{M_{EW}}\right), \tag{4.21}$$

$$\frac{1}{\alpha_Y} = \frac{1}{3\alpha_\ell} + \frac{5}{3\alpha_L} + \frac{76 - 3N_H}{18\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{5N_H - 10}{18\pi}\ln\left(\frac{w}{M_{EW}}\right) + \frac{22 - 3N_H}{6\pi}\ln\left(\frac{x}{M_{EW}}\right). \tag{4.22}$$

With these inputs, we have a large degree of freedom in the unification of the couplings, with the simplest scheme being that in which $N_H = 1$.

Given that we have no particles gaining mass at the final stage of breaking, the scale x affects only the evolution of the hypercharge fine-structure constant. As a result, this scale can be as low as a few TeV and as large as 6×10^7 GeV without spoiling the unification. Taking $x \equiv x_{\min} \sim 1$ TeV, unification occurs for

$$w \sim 2.7 \times 10^{12}\ \text{GeV}, \qquad v \sim 1.2 \times 10^{17}\ \text{GeV}. \tag{4.23}$$

At this unification scale, the value of the fine-structure constant for our unified theory is $\alpha_{G_4}^{-1} \sim 43.85$, giving $\alpha_\ell \sim 0.0912$ at the electroweak scale, which has a value between the weak $SU(2)_L$ and the strong couplings. If we allow x to increase in energy, then the GUT scale decreases and the w scale increases until $v = w \sim 7.5 \times 10^{13}$ GeV at $x_{\max} \sim 6.4 \times 10^7$ GeV. This unification scheme gives $\alpha_{G_4}^{-1} \sim 39.02$ and $\alpha_\ell \sim 0.1408$. It is interesting to note here that the scale w cannot be low for unification purposes, and subsequently all our exotic fermions will be heavy, leaving only the $SU(2)_\ell$ gauge bosons and additional Higgs fields as light particles foreign to the standard model.

Unification of the gauge coupling constants can also be obtained for the other three breaking patterns with the range of possible, consistent scales summarized in Table II. Unlike the first case, all the intermediate symmetry breaking scales for the other patterns must be high. The lowest the final intermediate breaking scale can be is about 6×10^5 GeV in option two, which is significantly higher than the electroweak scale. Furthermore, this choice requires the unification scale to be larger than the Planck scale, which is unacceptable. So, realistically, in this scheme we would have to consider higher values of x so as to lower the unification scale. As a result, the theories prescribed by cascades two, three and four will contain very heavy exotic particles that do not lie within reach of future colliders. Note also that the value of the fine-structure constant for leptonic color $SU(2)_\ell$ at the electroweak scale is generally always larger than that describing quark color.

Recall that the fourth symmetry breaking option is the only route that returns exotic particles with masses of order x. However, the value of x can only be pushed down in energy to 6×10^8 GeV if unification is to be preserved. Consequently, the possible existence of new low-energy phenomenology suggested by the presence of exotic fermions at this last stage of breaking is denied by the

[3]We take the same multiplicity N_H for *both* of the **36**'s representing our Higgs fields. This can obviously be generalized, but we shall have no need to do this because we shall focus only on the simplest possible schemes in which the multiplicity of each representation is precisely $N_H = 1$. Happily, it turns out $N_H > 1$ is not required.

TABLE II. Range of energy scales of symmetry breaking that yield unification of the gauge coupling constants. There is only one scenario which allows for a TeV-level breaking scale, while the scales offered by the other choices are quite similar to each other.

Cascade	x	w	v	$\alpha_{G_4}^{-1}$	α_ℓ
1	$x_{\rm min} \sim 1$ TeV	2.7×10^{12} GeV	1.2×10^{17} GeV	43.85	0.0912
	$x_{\rm max} \sim 6.4 \times 10^{7}$ GeV	7.5×10^{13} GeV	7.5×10^{13} GeV	39.02	0.1408
2	$x_{\rm min} \sim 6.5 \times 10^{5}$ GeV	6.5×10^{5} GeV	3.9×10^{19} GeV	43.55	0.0526
	$x_{\rm max} \sim 6.5 \times 10^{7}$ GeV	7.4×10^{13} GeV	7.4×10^{13} GeV	39.01	0.1407
3	$x_{\rm min} \sim 6.3 \times 10^{7}$ GeV	7.7×10^{13} GeV	7.7×10^{13} GeV	39.02	0.1412
	$x_{\rm max} \sim 4.9 \times 10^{10}$ GeV	4.9×10^{10} GeV	7×10^{12} GeV	36.35	0.1210
4	$x_{\rm min} \sim 6.2 \times 10^{8}$ GeV	1.7×10^{12} GeV	1.7×10^{12} GeV	34.77	0.111
	$x_{\rm max} \sim 4.8 \times 10^{10}$ GeV	4.8×10^{10} GeV	7×10^{12} GeV	36.35	0.1208

demands placed on the energy scale by the unification of the gauge coupling constants.

We shall comment further on the phenomenology of our models in Sec. V

B. $SU(2)_\ell$ broken

By demanding that the leptonic color symmetry is broken entirely, the number of symmetry breaking routes from the quartification gauge group to the SM broadens to eight independent cascades. These are labeled as

1. $$\begin{aligned} G_4 &\xrightarrow{v} SU(3)_q \otimes SU(3)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_1} \\ &\xrightarrow{w} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_2} \\ &\xrightarrow{x} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes U(1)_{X_3} \\ &\xrightarrow{y} SU(3)_q \otimes SU(2)_L \otimes U(1)_Y \end{aligned} \quad (4.24)$$

2. $$\begin{aligned} G_4 &\xrightarrow{v} SU(3)_q \otimes SU(3)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_1} \\ &\xrightarrow{w} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_2} \\ &\xrightarrow{x} SU(3)_q \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{X_3} \\ &\xrightarrow{y} SU(3)_q \otimes SU(2)_L \otimes U(1)_Y \end{aligned} \quad (4.25)$$

3. $$\begin{aligned} G_4 &\xrightarrow{v} SU(3)_q \otimes SU(3)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_1} \\ &\xrightarrow{w} SU(3)_q \otimes SU(3)_L \otimes U(1)_{X_2} \\ &\xrightarrow{y} SU(3)_q \otimes SU(2)_L \otimes U(1)_Y \end{aligned} \quad (4.26)$$

4. $$\begin{aligned} G_4 &\xrightarrow{v} SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(2)_R \otimes U(1)_{X_1} \\ &\xrightarrow{w} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_2} \\ &\xrightarrow{x} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes U(1)_{X_3} \\ &\xrightarrow{y} SU(3)_q \otimes SU(2)_L \otimes U(1)_Y \end{aligned} \quad (4.27)$$

5. $$\begin{aligned} G_4 &\xrightarrow{v} SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(2)_R \otimes U(1)_{X_1} \\ &\xrightarrow{w} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_2} \\ &\xrightarrow{x} SU(3)_q \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{X_3} \\ &\xrightarrow{y} SU(3)_q \otimes SU(2)_L \otimes U(1)_Y \end{aligned} \quad (4.28)$$

6. $$\begin{aligned} G_4 &\xrightarrow{v} SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(2)_R \otimes U(1)_{X_1} \\ &\xrightarrow{w} SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes U(1)_{X_2} \\ &\xrightarrow{x} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes U(1)_{X_3} \\ &\xrightarrow{y} SU(3)_q \otimes SU(2)_L \otimes U(1)_Y \end{aligned} \quad (4.29)$$

7. $$\begin{aligned} G_4 &\xrightarrow{v} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(3)_R \otimes U(1)_{X_1} \\ &\xrightarrow{w} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_2} \\ &\xrightarrow{x} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes U(1)_{X_3} \\ &\xrightarrow{y} SU(3)_q \otimes SU(2)_L \otimes U(1)_Y \end{aligned} \quad (4.30)$$

8. $$\begin{aligned} G_4 &\xrightarrow{v} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(3)_R \otimes U(1)_{X_1} \\ &\xrightarrow{w} SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_2} \\ &\xrightarrow{x} SU(3)_q \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{X_3} \\ &\xrightarrow{y} SU(3)_q \otimes SU(2)_L \otimes U(1)_Y, \end{aligned} \quad (4.31)$$

where the generator of hypercharge now has the form

$$Y = I_{3R} + I_{3\ell} + \frac{2}{\sqrt{3}}(\lambda_{L8} + \lambda_{\ell 8} + \lambda_{R8}). \quad (4.32)$$

Of these eight choices, there are seven which can deliver unification of the gauge coupling constants. Cascade three does not offer a viable model assuming a minimal Higgs sector is used, so we eliminate it from further consideration. In all models, the light particle spectrum consists of the standard model particles and nine candidate SM Higgs doublets. The exotic fermions gain masses either of order v

or w in all cases except for cascade six. The VEV structure of cascade six endows the particles x_1, x_2, y_1, y_2, x_1^c, x_2^c, y_1^c and y_2^c with masses at the x scale, which at first sight could potentially result in lighter masses than the other scenarios.

For all cascades, the full details of the particle spectra, including the Higgs VEV patterns instigating the breaking, and the analysis of the renormalization-group equations, are contained in Appendix B. Table III provides a summary of results for the possible ranges of energy scales which give unification of the gauge coupling constants for $N_H = 1$.

Only a subset of these seven symmetry breaking schemes allows for a flexible range of unification and intermediate breaking scales. The final breaking to the SM gauge group can be as low as a TeV for six of these seven options, with cascades one, two, seven and eight demanding that this scale be precisely of TeV order. In fact, $y \equiv y_{\max} \sim 7.1 \times 10^2$ GeV is the highest scale at which this breaking occurs for these options. This choice of y-scale offers only two intermediate scales with unification requiring $x = y$ and $w = v \sim 1.3 \times 10^{13}$ GeV. The symmetry breaking patterns for these four cascades thus become equivalent, reducing to $G_4 \rightarrow SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1) \rightarrow SU(3)_q \otimes SU(2)_L \otimes U(1)_Y$.

Unification can still be achieved in options four and five if the y scale is as high as $\sim 10^6$ GeV. Furthermore, once at the $SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)$ level, the choice of which $SU(2)_{\ell,R}$ factor to break first has no significant influence on the outcome of the unification and intermediate scales. It turns out that for all viable y values, x must be very close to y. When the final breaking occurs at $y_{\min} \sim 1$ TeV, then the unification scale is of order 10^{13} GeV, whereas at $y_{\max} \sim 10^6$ GeV, the unification scale is at a lower energy, of order 10^{11} GeV, which could potentially be more dangerous with respect to proton decay. Nevertheless, for all these cascades, the highest the unification scale can be is $\sim 10^{13}$ GeV which is much lower than the GUT energies possible when $SU(2)_\ell$ remains unbroken at low energy.

Cascade six affords the most tantalizing spectrum of masses for the exotic fermions, with masses of order x resulting from the breaking $G_{SM} \otimes SU(3)_\ell \rightarrow G_{SM} \otimes SU(2)_\ell$. This scale can be as low as $\sim 10^4$ GeV with unification occurring for

$$y_{\min} \sim 1 \text{ TeV}, \qquad x \sim 8.8 \times 10^3 \text{ GeV},$$
$$w = v \equiv v_{\min} \sim 3.6 \times 10^{10} \text{ GeV}. \tag{4.33}$$

A 10 TeV scale for some of the exotic fermion masses provides hope for possible discovery at the LHC. However, this choice also requires the unification scale $v \sim 10^{10}$ GeV, which may be low enough to be troubling with respect to proton decay. If we allow x to increase, then we obtain

$$y_{\min} \sim 1 \text{ TeV}, \qquad x = w \sim 4.2 \times 10^7 \text{ GeV},$$
$$v \equiv v_{\max} \sim 3.8 \times 10^{11} \text{ GeV}. \tag{4.34}$$

There is flexibility in y; it can be pushed up to

$$y_{\max} = x = w \sim 1.2 \times 10^6 \text{ GeV},$$
$$v \sim 1.4 \times 10^{11} \text{ GeV}, \qquad \alpha_{G_4}^{-1} = 32.02. \tag{4.35}$$

With this choice, it becomes equivalent to the upper bound of unification for cascades four and five, with the symmetry breaking now described by $G_4 \rightarrow SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(2)_R \otimes U(1)_{X_1} \rightarrow G_{SM}$. Since the x-scale is now at about 10^6 GeV, we see that, while this cascade is consistent with exotic fermion masses of about 10 TeV, they can also be significantly higher without spoiling unification.

V. PHENOMENOLOGY

We now round out our discussion of the phenomenological consequences of the various schemes above. It is beyond the scope of this paper to provide a rigorous quantitative analysis of phenomenological bounds for all of these models, so our remarks shall be qualitative and our analysis necessarily incomplete.

TABLE III. The range of energies for the symmetry breaking scales that will consistently give unification of the gauge coupling constants, when $N_H = 1$. Four of the schemes become equivalent if unification is to be demanded, and the last stage of symmetry breaking has to occur below a TeV. The other three choices allow for a range in the intermediate scales while still preserving unification. When $y_{\max}$ is chosen for cascades four, five and six, they also become equivalent.

	y	x	w	v	$\alpha_{G_4}^{-1}$
1 and 2	$y_{\max} \sim 7.1 \times 10^2$ GeV	7.1×10^2 GeV	1.3×10^{13} GeV	1.3×10^{13} GeV	37.05
4 and 5	$y_{\min} \sim 1$ TeV	1 TeV	6.2×10^{12} GeV	$v_{\max} \sim 1.1 \times 10^{13}$ GeV	36.82
		4.2×10^7 GeV	4.2×10^7 GeV	$v_{\max} \sim 3.8 \times 10^{11}$ GeV	33.11
	$y_{\max} \sim 1.2 \times 10^6$ GeV	1.2×10^6 GeV	1.2×10^6 GeV	1.4×10^{11} GeV	32.02
6	$y_{\min} \sim 1$ TeV	8.8×10^3 GeV	3.6×10^{10} GeV	$v_{\min} \sim 3.6 \times 10^{10}$ GeV	30.48
		4.2×10^7 GeV	4.2×10^7 GeV	$v_{\max} \sim 3.8 \times 10^{11}$ GeV	33.11
	$y_{\max} \sim 1.2 \times 10^6$ GeV	1.2×10^6 GeV	1.2×10^6 GeV	1.4×10^{11} GeV	32.02
7 and 8	$y_{\max} \sim 7.1 \times 10^2$ GeV	7.1×10^2 GeV	1.3×10^{13} GeV	1.3×10^{13} GeV	37.05

We first deal with the four cascades featuring a remnant $SU(2)_\ell$. All of them feature seven electroweak Higgs doublets. We emphasize that this multiplicity is not due to duplication of the fundamental Higgs multiplets. Rather, they are an integral part of the minimal Higgs sector required for quartification. Obviously, questions about Higgs-induced flavor-changing neutral processes arise. Without a detailed analysis we can only make the simple remark that some of the doublets will have to acquire TeV-scale masses or have somewhat small Yukawa coupling constants. It is certainly interesting, though, that multiple Higgs doublets are a generic prediction of quartification models.

Cascade one is the only one that allows an intermediate scale as low as a TeV. This scale is a right-handed weak-isospin breaking scale, so the immediate phenomenological consequences are right-handed W bosons and a corresponding Z' at the TeV level. Since this scale can be raised above 10^4 TeV without spoiling unification, it is clear that it can be made phenomenologically acceptable. However, it has no necessary new physics at LHC energies, apart from the multi-Higgs doublet feature it shares with the other quartification schemes. Recall from the earlier discussion that it also has no TeV-scale exotic fermions. The unification scale is in the range 10^{14-17} GeV, so we would guess that it is safe from too-rapid Higgs-induced proton decay.

Cascades two, three and four all have high x-scales, so their only characteristic TeV-level feature is the seven Higgs doublets. The unification scales lie in the range $7 \times 10^{12-13}$ GeV, which should be safe from a proton-decay point of view.

We now turn to the schemes having no leptonic color remnant symmetry. As noted earlier, cascade three is unsuccessful and hence discarded. All the cascades feature nine electroweak Higgs doublets.

The requirement of unification makes cascades one, two, seven and eight identical, with the lowest breaking scale being at about 700 GeV. This scheme is possibly ruled out, because it results in quite light right-handed W-bosons and other light gauge particles including Z' states. The 700 GeV scale follows from the central values for the electroweak-scale gauge coupling constants; it can be pushed up to the TeV range by varying these values within the experimental error range. All the exotic fermions, however, are quite heavy, gaining masses at the w-scale which is about 10^{13} GeV. If detailed study were to show it is not yet falsified, then it would be an interesting situation in regards to possible discovery of new gauge bosons below 1 TeV. The unification scale of 10^{13} GeV may be sufficient to suppress proton decay.

Cascades four and five can feature, respectively, $SU(2)_\ell$ or $SU(2)_R$ gauge bosons at the TeV level, although they need not. It would be interesting to study their GUT-scale proton-decay phenomenology, as a lower y-value implies a higher GUT-scale, as summarized in Table III. Conceivably, the suppression of Higgs-induced proton decay might favor new TeV-scale physics for these cascades.

As already noted, cascade six is unique in that it can have new fermions at the relatively low scale of about 10 TeV. If so, this would be correlated with a low breaking scale for $SU(2)_\ell$ and hence the presence of exotic gauge bosons coupling leptons to exotic leptons. There is also extended neutral current phenomenology. The danger for cascade six is the low range for the unification scale, which is not allowed to be much higher than 10^{11} GeV.

In summary, there is obviously a wealth of phenomenology to be explored within these schemes, both at the TeV scale and at the GUT scale. The overall impression is that the schemes that totally break leptonic color are more constrained, either from TeV-scale considerations or from the GUT regime or both. This makes them more exciting, more easily tested; some are possibly already ruled out. Note that we have not yet attempted a systematic study of the Higgs-induced proton-decay question, so our concerns about some of the lower unification scales are generic rather than specific.

We have also not yet attempted a study of the Higgs potential and the minimization conditions. It almost goes without saying that all proposed quartification schemes suffer from the gauge hierarchy problem. In our opinion, however, the overall framework has considerable appeal, despite this standard defect common to all nonsupersymmetric GUTs.

Another interesting topic for future work is neutrino mass generation for the totally broken leptonic color scenarios, to understand the effect of the intermediate scales on the seesaw suppression given by Eq. (3.11).

VI. CONCLUSION

Quartification schemes offer an alternative route to grand unification. They are conceptually rather appealing, with the fundamental fermion and Higgs multiplets taking relatively simple and elegant forms. As we have shown in this paper, there are a variety of symmetry breaking cascades consistent with successful gauge coupling constant unification. None of them require supersymmetry, though all of them of course require intermediate scales. The nontrivial result is that appropriate intermediate scales are a natural possibility. Our results add to the important observation of Babu, Ma and Willenbrock [9] that complete unification is possible in quartification models, rather than having to settle for the partial unification originally proposed by Joshi and Volkas [8].

The various schemes have different phenomenological consequences, though all have the existence of several electroweak Higgs doublets as a feature. This multiplicity is not due to a replication of fundamental Higgs multiplets, but is rather an inherent feature of the minimal Higgs sector required for quartification. Depending on the scheme, rich phenomenology at LHC energies such as additional gauge bosons and fermions is possible and in some cases required. In addition, the models may have Higgs-induced proton decay, though detailed analyses of this and the new physics at the TeV scale have yet to be carried out.

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APPENDIX A: RGEs FOR $SU(2)_\ell$ UNBROKEN

Here we summarize the analysis of the renormalization-group equations (RGEs) analysis for cascades two, three and four within the class that leaves the leptonic color remnant symmetry $SU(2)_\ell$ unbroken.

1. Cascade 2

The renormalization-group equations are

$$\frac{1}{\alpha_i(v)} = \frac{1}{\alpha_i(M_{EW})} - \frac{b_{i_1}}{2\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{b_{i_1}-b_{i_2}}{2\pi}\ln\left(\frac{w}{M_{EW}}\right) + \frac{b_{i_2}-b_{i_3}}{2\pi}\ln\left(\frac{x}{M_{EW}}\right), \qquad i = q, L, \ell \tag{A1}$$

$$\frac{1}{\alpha_Y} = \frac{1}{3\alpha_\ell} + \frac{5}{3\alpha_L} + \frac{3b_{u_1}+4b_{R_1}-b_{\ell_1}-5b_{L_1}}{6\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{3b_{u_2}+3b_{R_2}-3b_{u_1}-4b_{R_1}+b_{\ell_1}-b_{\ell_2}+5b_{L_1}-5b_{L_2}}{6\pi} \times \ln\left(\frac{w}{M_{EW}}\right) + \frac{3b_{u_3}-3b_{u_2}-3b_{R_2}+b_{\ell_2}-b_{\ell_3}+5b_{L_2}-5b_{L_3}}{6\pi}\ln\left(\frac{x}{M_{EW}}\right). \tag{A2}$$

As illustrated in Table I, the leptons x_1, x_2, y_1 and y_2 have masses of order v while the remaining exotic fermions gain mass at w. The *light* Higgs sector has the structure

$$\begin{aligned}
&\Phi_\ell \xrightarrow{v} (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1) \xrightarrow{x} (\mathbf{1},\mathbf{2},\mathbf{1})(-1),\\
&\Phi_{\ell^c} \xrightarrow{v} (\mathbf{1},\mathbf{1},\mathbf{2},\bar{\mathbf{3}})\left(-\frac{1}{3}\right) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\bar{\mathbf{3}})\left(\frac{2}{3}\right) \xrightarrow{w} (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1), \xrightarrow{x} \text{nothing},\\
&\Phi_a \xrightarrow{v} (\mathbf{1},\mathbf{2},\mathbf{1},\bar{\mathbf{3}})\left(-\frac{1}{3}\right) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\bar{\mathbf{3}})\left(\frac{2}{3}\right) \xrightarrow{w} (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1),\\
&\quad \xrightarrow{x} (\mathbf{1},\mathbf{2},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(-1),\\
&\Phi_c \sim \Phi_a^\dagger.
\end{aligned} \tag{A3}$$

The resulting spectrum of particle masses implies that the b quantities are

$$\begin{aligned}
&b_{q_1} = -5, && b_{L_1} = -\tfrac{10}{3} + \tfrac{7N_H}{6}, && b_{\ell_1} = -\tfrac{10}{3} + \tfrac{N_H}{2}, && b_{R_1} = -5 + \tfrac{3N_H}{2}, && b_{u_1} = 4 + \tfrac{2N_H}{3},\\
&b_{q_2} = -7, && b_{L_2} = -\tfrac{10}{3} + \tfrac{7N_H}{6}, && b_{\ell_2} = -\tfrac{22}{3}, && b_{R_2} = -\tfrac{10}{3} + \tfrac{7N_H}{6}, && b_{u_2} = \tfrac{8}{3} + N_H,\\
&b_{q_3} = b_{q_2}, && b_{L_3} = b_{L_2}, && b_{\ell_3} = b_{\ell_2}, && && b_{u_3} = \tfrac{20}{3} + \tfrac{7N_H}{6}.
\end{aligned} \tag{A4}$$

Substituting these in, the equations describing the evolution of the gauge coupling constants are

$$\frac{1}{\alpha_q(v)} = \frac{1}{\alpha_q} + \frac{5}{2\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{1}{\pi}\ln\left(\frac{w}{M_{EW}}\right), \tag{A5}$$

$$\frac{1}{\alpha_L(v)} = \frac{1}{\alpha_L} + \frac{20-7N_H}{12\pi}\ln\left(\frac{v}{M_{EW}}\right), \tag{A6}$$

$$\frac{1}{\alpha_\ell(v)} = \frac{1}{\alpha_\ell} + \frac{20-3N_H}{12\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{8+N_H}{4\pi}\ln\left(\frac{w}{M_{EW}}\right), \tag{A7}$$

$$\frac{1}{\alpha_Y} = \frac{1}{3\alpha_\ell} + \frac{5}{3\alpha_L} + \frac{36+5N_H}{18\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{10-N_H}{6\pi}\ln\left(\frac{w}{M_{EW}}\right) + \frac{22-3N_H}{6\pi}\ln\left(\frac{x}{M_{EW}}\right). \tag{A8}$$

As a consequence, we have

$$x_{\min} = w \sim 6.5 \times 10^5 \text{ GeV}, \qquad v \sim 3.9 \times 10^{19} \text{ GeV}, \qquad \alpha_{G_4}^{-1} = 43.55, \qquad \alpha_\ell = 0.0526 \tag{A9}$$

as the minimum possible value at which the last breaking can occur. The maximum value of this energy scale is

$$x_{\max} \sim 6.5 \times 10^7 \text{ GeV}, \qquad w = v \sim 7.4 \times 10^{13} \text{ GeV}, \qquad \alpha_{G_4}^{-1} = 39.01, \qquad \alpha_\ell = 0.1407. \tag{A10}$$

The GUT unification scale is quite high in this scenario.

2. Cascade 3

The general form of the equations for this symmetry breaking pattern is given by Eq. (A1) together with

$$\frac{1}{\alpha_Y} = \frac{1}{3\alpha_\ell} + \frac{5}{3\alpha_L} + \frac{3(b_{u_1} + b_{R_1}) - 5b_{L_1}}{6\pi} \ln\left(\frac{v}{M_{EW}}\right) + \frac{3(b_{u_2} + b_{R_2} - b_{u_1} - b_{R_1}) - b_{\ell_2} + 5(b_{L_1} - b_{L_2})}{6\pi} \ln\left(\frac{w}{M_{EW}}\right) + \frac{3(b_{u_3} - b_{u_2} - b_{R_2}) + b_{\ell_2} - b_{\ell_3} + 5(b_{L_2} - b_{L_3})}{6\pi} \ln\left(\frac{x}{M_{EW}}\right). \tag{A11}$$

The light Higgs spectrum has the form

$$\begin{aligned}
&\Phi_\ell \xrightarrow{v} (\mathbf{1}, \mathbf{2}, \bar{\mathbf{3}}, \mathbf{1})\left(-\frac{1}{3}\right) \oplus (\mathbf{1}, \mathbf{1}, \bar{\mathbf{3}}, \mathbf{1})\left(\frac{2}{3}\right) \xrightarrow{w} (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})(-1) \xrightarrow{x} (\mathbf{1}, \mathbf{2}, \mathbf{1})(-1),\\
&\Phi_{\ell^c} \xrightarrow{v} (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{2})\left(\frac{1}{3}\right) \oplus (\mathbf{1}, \mathbf{1}, \bar{\mathbf{3}}, \mathbf{1})\left(-\frac{2}{3}\right) \xrightarrow{w} (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})(1), \xrightarrow{x} \text{nothing},\\
&\Phi_a \xrightarrow{v,w} (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})(0) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})(1) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})(-1),\\
&\quad \xrightarrow{x} (\mathbf{1}, \mathbf{2}, \mathbf{1})(-1) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1})(1) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1})(-1),\\
&\Phi_c \sim \Phi_a^\dagger,
\end{aligned} \tag{A12}$$

specifying the b's as

$$\begin{array}{lllll}
b_{q_1} = -7, & b_{L_1} = -\frac{4}{3} + \frac{3N_H}{2}, & b_{\ell_1} = -7 + N_H, & b_{R_1} = -\frac{4}{3} + \frac{3N_H}{2}, & b_{u_1} = \frac{4}{3} + N_H,\\
b_{q_2} = b_{q_1}, & b_{L_2} = -\frac{10}{3} + \frac{7N_H}{6}, & b_{\ell_2} = -\frac{22}{3}, & b_{R_2} = -\frac{10}{3} + \frac{7N_H}{6}, & b_{u_2} = \frac{8}{3} + N_H,\\
b_{q_3} = b_{q_1}, & b_{L_3} = b_{L_2}, & b_{\ell_3} = b_{\ell_2}, & & b_{u_3} = \frac{20}{3} + \frac{7N_H}{6},
\end{array} \tag{A13}$$

The renormalization-group equations reduce to

$$\frac{1}{\alpha_q(v)} = \frac{1}{\alpha_q} + \frac{7}{2\pi} \ln\left(\frac{v}{M_{EW}}\right), \tag{A14}$$

$$\frac{1}{\alpha_L(v)} = \frac{1}{\alpha_L} + \frac{8 - 9N_H}{12\pi} \ln\left(\frac{v}{M_{EW}}\right) + \frac{6 + N_H}{6\pi} \ln\left(\frac{w}{M_{EW}}\right), \tag{A15}$$

$$\frac{1}{\alpha_\ell(v)} = \frac{1}{\alpha_\ell} + \frac{7 - N_H}{2\pi} \ln\left(\frac{v}{M_{EW}}\right) + \frac{1 + 3N_H}{6\pi} \ln\left(\frac{w}{M_{EW}}\right), \tag{A16}$$

$$\frac{1}{\alpha_Y} = \frac{1}{3\alpha_\ell} + \frac{5}{3\alpha_L} + \frac{20}{18\pi} \ln\left(\frac{v}{M_{EW}}\right) + \frac{23 + N_H}{9\pi} \ln\left(\frac{w}{M_{EW}}\right) + \frac{22 - 3N_H}{6\pi} \ln\left(\frac{x}{M_{EW}}\right). \tag{A17}$$

These unify for the range

$$x_{\min} \sim 6.3 \times 10^7 \text{ GeV}, \qquad w = v \sim 7.7 \times 10^{13} \text{ GeV}, \tag{A18}$$

$$x_{\max} = w \sim 4.9 \times 10^{10} \text{ GeV}, \qquad v \sim 7 \times 10^{12} \text{ GeV}. \tag{A19}$$

3. Cascade 4

Again the evolution of the $SU(N)$ fine-structure constants is given by Eq. (A1) and the $U(1)$ charge equation has the form

$$\frac{1}{\alpha_Y}=\frac{1}{3\alpha_\ell}+\frac{5}{3\alpha_L}+\frac{3b_{u_1}+3b_{R_1}-5b_{L_1}}{6\pi}\ln\left(\frac{v}{M_{EW}}\right)+\frac{3b_{u_2}-3b_{u_1}-3b_{R_1}+5b_{L_1}-5b_{L_2}}{6\pi}\ln\left(\frac{w}{M_{EW}}\right)+\frac{3b_{u_3}-3b_{u_2}-b_{\ell_3}+5b_{L_2}-5b_{L_3}}{6\pi}\ln\left(\frac{x}{M_{EW}}\right). \tag{A20}$$

The light Higgs spectrum goes as

$$\begin{aligned}
&\Phi_\ell\xrightarrow{v,w}(\mathbf{1},\mathbf{2},\bar{\mathbf{3}},\mathbf{1})\left(-\frac{1}{3}\right)\oplus(\mathbf{1},\mathbf{1},\bar{\mathbf{3}},\mathbf{1})\left(\frac{2}{3}\right)\xrightarrow{x}(\mathbf{1},\mathbf{2},\mathbf{1})(-1),\\
&\Phi_{\ell^c}\xrightarrow{v}(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{2})\left(\frac{1}{3}\right)\oplus(\mathbf{1},\mathbf{1},\bar{\mathbf{3}},\mathbf{1})\left(-\frac{2}{3}\right)\xrightarrow{w}(\mathbf{1},\mathbf{1},\mathbf{3})\left(-\frac{2}{3}\right)\oplus(\mathbf{1},\mathbf{1},\mathbf{3})\left(\frac{4}{3}\right)\oplus(\mathbf{1},\mathbf{1},\mathbf{3})\left(-\frac{2}{3}\right),\\
&\Phi_a\xrightarrow{v}(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})(0)\oplus(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1)\oplus(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1),\\
&\quad\xrightarrow{w,x}(\mathbf{1},\mathbf{2},\mathbf{1})(-1)\oplus(\mathbf{1},\mathbf{2},\mathbf{1})(1)\oplus(\mathbf{1},\mathbf{2},\mathbf{1})(-1),\\
&\Phi_c\sim\Phi_a^\dagger,
\end{aligned} \tag{A21}$$

resulting in the quantities

$$\begin{aligned}
&b_{q_1}=-7, && b_{L_1}=-\tfrac{4}{3}+\tfrac{3N_H}{2}, && b_{\ell_1}=-7+N_H, && b_{R_1}=-\tfrac{4}{3}+\tfrac{3N_H}{2}, && b_{u_1}=\tfrac{4}{3}+N_H,\\
&b_{q_2}=b_{q_1}, && b_{L_2}=b_{L_1}, && b_{\ell_2}=b_{\ell_1}, && && b_{u_2}=\tfrac{22}{3}+\tfrac{11N_H}{6},\\
&b_{q_3}=b_{q_1}, && b_{L_3}=-\tfrac{10}{3}+\tfrac{7N_H}{6}, && b_{\ell_3}=-\tfrac{22}{3}, && && b_{u_3}=\tfrac{20}{3}+\tfrac{7N_H}{6},
\end{aligned} \tag{A22}$$

and the equations

$$\frac{1}{\alpha_q(v)}=\frac{1}{\alpha_q}+\frac{7}{2\pi}\ln\left(\frac{v}{M_{EW}}\right), \tag{A23}$$

$$\frac{1}{\alpha_L(v)}=\frac{1}{\alpha_L}+\frac{8-9N_H}{12\pi}\ln\left(\frac{v}{M_{EW}}\right)+\frac{6+N_H}{6\pi}\ln\left(\frac{x}{M_{EW}}\right), \tag{A24}$$

$$\frac{1}{\alpha_\ell(v)}=\frac{1}{\alpha_\ell}+\frac{7-N_H}{2\pi}\ln\left(\frac{v}{M_{EW}}\right)+\frac{1+3N_H}{6\pi}\ln\left(\frac{x}{M_{EW}}\right), \tag{A25}$$

$$\frac{1}{\alpha_Y}=\frac{1}{3\alpha_\ell}+\frac{5}{3\alpha_L}+\frac{20}{18\pi}\ln\left(\frac{v}{M_{EW}}\right)+\frac{11-N_H}{3\pi}\ln\left(\frac{w}{M_{EW}}\right)+\frac{46-N_H}{18\pi}\ln\left(\frac{x}{M_{EW}}\right). \tag{A26}$$

These unify in a similar range of energy scales to cascade three.

APPENDIX B: RGEs FOR $SU(2)_\ell$ BROKEN

We now provide the technical details for the eight cascades featuring completely broken leptonic color.

1. Cascade 1

The VEV pattern that induces the breaking of cascade one is

$$\langle\Phi_\ell\rangle=\begin{pmatrix}u&0&u\\0&u&0\\y&0&w\end{pmatrix},\qquad \langle\Phi_{\ell^c}\rangle=\begin{pmatrix}y&0&y\\0&y&0\\x&0&v\end{pmatrix},\qquad \langle\Phi_a\rangle=\langle\Phi_c^\dagger\rangle=\begin{pmatrix}u&0&u\\0&u&0\\x&0&w\end{pmatrix}, \tag{B1}$$

where $v\geq w\geq x\geq y\geq u$, and u instigates the electroweak symmetry breaking. After the first stage of symmetry breaking, the particles x_1^c, x_2^c, y_1^c and y_2^c gain Dirac masses and our light Higgs spectrum is

$$\Phi_\ell \sim (\mathbf{1},\mathbf{3},\mathbf{2},\mathbf{1})\left(\frac{1}{3}\right) \oplus (\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})\left(-\frac{2}{3}\right), \qquad \Phi_{\ell^c} \sim (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1),$$
$$\Phi_a \sim (\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{2})\left(\frac{1}{3}\right) \oplus (\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})\left(-\frac{2}{3}\right), \qquad \Phi_c \sim (\mathbf{1},\bar{\mathbf{3}},\mathbf{1},\mathbf{2})\left(-\frac{1}{3}\right) \oplus (\mathbf{1},\bar{\mathbf{3}},\mathbf{1},\mathbf{1})\left(\frac{2}{3}\right). \tag{B2}$$

The second stage of breaking sees the remaining charged exotic fermions gaining Dirac masses of order w, and the neutral exotic particle N, N^c gains a w scale Majorana mass. The components of the Higgs multiplets which remain light are

$$\Phi_\ell \sim (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})(0) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1), \qquad \Phi_{\ell^c} \sim (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1),$$
$$\Phi_a \sim (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1), \qquad \Phi_c \sim (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(-1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(1). \tag{B3}$$

There are no fermion mass terms of order x, but the light Higgs sector reduces to

$$\Phi_\ell \sim (\mathbf{1},\mathbf{2},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(-1), \qquad \Phi_{\ell^c} \sim (\mathbf{1},\mathbf{1},\mathbf{2})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})(-1),$$
$$\Phi_a \sim (\mathbf{1},\mathbf{2},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(-1), \qquad \Phi_c \sim (\mathbf{1},\mathbf{2},\mathbf{1})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(1). \tag{B4}$$

After the final stage of breaking down to the standard model gauge group, the left-handed antineutrino ν^c gains a y scale mass and we have nine light Higgs doublets with $Y = \pm 1$. This spectrum of particles defines the b quantities as

$$\begin{array}{lllll} b_{q_1} = -5, & b_{L_1} = -5 + \frac{3N_H}{2}, & b_{\ell_1} = -\frac{10}{3} + N_H, & b_{R_1} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{u_1} = 4 + \frac{5N_H}{6}, \\ b_{q_2} = -7, & b_{L_2} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{\ell_2} = -\frac{22}{3} + N_H, & b_{R_2} = b_{R_1}, & b_{u_2} = \frac{8}{3} + \frac{4N_H}{3}, \\ b_{q_3} = b_{q_2}, & b_{L_3} = b_{L_2}, & b_{\ell_3} = b_{\ell_2}, & & b_{u_3} = \frac{20}{3} + \frac{11N_H}{6}, \\ b_{q_4} = b_{q_2}, & b_{L_4} = b_{L_2}, & & & b_{u_4} = \frac{20}{3} + \frac{3N_H}{2}. \end{array} \tag{B5}$$

The relationship between the fine-structure constants and the symmetry breaking scales has the general form

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} + \frac{3(b_{\ell_1} + b_{R_1} + b_{u_1}) - 8b_{L_1}}{6\pi} \ln\left(\frac{v}{M_{EW}}\right) + \frac{3(b_{\ell_2} + b_{R_2} + b_{u_2} - b_{\ell_1} - b_{R_1} - b_{u_1}) + 8b_{L_1} - 9b_{L_2}}{6\pi} \times \ln\left(\frac{w}{M_{EW}}\right) + \frac{b_{\ell_3} + 3b_{L_2} + b_{u_3} - b_{\ell_2} - b_{R_2} - b_{u_2} - 3b_{L_3}}{2\pi} \ln\left(\frac{x}{M_{EW}}\right) + \frac{b_{u_4} - b_{\ell_3} - b_{u_3} + 3b_{L_3} - 3b_{L_4}}{2\pi} \times \ln\left(\frac{y}{M_{EW}}\right). \tag{B6}$$

Inputting these values, the renormalization-group equations reduce to

$$\frac{1}{\alpha_q(v)} = \frac{1}{\alpha_q} + \frac{5}{2\pi} \ln\left(\frac{v}{M_{EW}}\right) + \frac{1}{\pi} \ln\left(\frac{w}{M_{EW}}\right) \tag{B7}$$

$$\frac{1}{\alpha_L(v)} = \frac{1}{\alpha_L} + \frac{10 - 3N_H}{4\pi} \ln\left(\frac{v}{M_{EW}}\right) - \frac{5}{6\pi} \ln\left(\frac{w}{M_{EW}}\right) \tag{B8}$$

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} + \frac{16 - N_H}{3\pi} \ln\left(\frac{v}{M_{EW}}\right) - \frac{13}{3\pi} \ln\left(\frac{w}{M_{EW}}\right) + \frac{22 - 3N_H}{6\pi} \ln\left(\frac{x}{M_{EW}}\right) + \frac{11 - 2N_H}{3\pi} \ln\left(\frac{y}{M_{EW}}\right). \tag{B9}$$

Unification of the coupling constants at v can only be achieved if $y_{\max} \sim 7.1 \times 10^2$ GeV, with the configuration of our energy scales being

$$y_{\max} = x \sim 7.1 \times 10^2 \text{ GeV}, \qquad w = v \sim 1.3 \times 10^{13} \text{ GeV}. \tag{B10}$$

In this symmetry breaking scheme, the unification scale is of order 10^{13} GeV and does not have much scope to change if we want the coupling constants to intersect. There now are only two symmetry breaking stages, with the breaking proceeding via

$$G_4 \to SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)_{X_1} \to SU(3)_q \otimes SU(2)_L \otimes U(1)_Y. \tag{B11}$$

2. Cascade 2

The symmetry breaking of cascade two is generated by Higgs VEVs of the form

$$\langle\Phi_\ell\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ x & 0 & w \end{pmatrix}, \qquad \langle\Phi_{\ell^c}\rangle = \begin{pmatrix} y & 0 & x \\ 0 & y & 0 \\ y & 0 & v \end{pmatrix}, \qquad \langle\Phi_a\rangle = \langle\Phi_c^\dagger\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ y & 0 & w \end{pmatrix}. \tag{B12}$$

The spectrum of fermion masses is identical to cascade one, and the light Higgs fields have a similar form but the b's will differ as we have three multiplets which transform nontrivially under $SU(3)_R$. This change in the b's is as

$$\begin{array}{lllll} b_{q_1} = -5, & b_{L_1} = -5 + \frac{3N_H}{2}, & b_{\ell_1} = -\frac{10}{3} + N_H, & b_{R_1} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{u_1} = 4 + \frac{5N_H}{6}, \\ b_{q_2} = -7, & b_{L_2} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{\ell_2} = -\frac{22}{3} + N_H, & b_{R_2} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{u_2} = \frac{8}{3} + \frac{4N_H}{3}, \\ b_{q_3} = b_{q_2}, & b_{L_3} = b_{L_2}, & & b_{R_3} = b_{R_2}, & b_{u_3} = \frac{8}{3} + \frac{5N_H}{3}, \\ b_{q_4} = b_{q_2}, & b_{L_4} = b_{L_2}, & & & b_{u_4} = \frac{20}{3} + \frac{3N_H}{2}. \end{array} \tag{B13}$$

The evolution of the strong and weak couplings is identical to that of cascade one, however, the Abelian-charge fine-structure constant has a different running with energy as evident in its equation

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} + \frac{3b_{\ell_1} + 3b_{R_1} + 3b_{u_1} - 8b_{L_1}}{6\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{3(b_{\ell_2} + b_{R_2} + b_{u_2} - b_{\ell_1} - b_{R_1} - b_{u_1}) + 8b_{L_1} - 9b_{L_2}}{6\pi}\ln\left(\frac{w}{M_{EW}}\right) + \frac{b_{R_3} + 3b_{L_2} + b_{u_3} - b_{\ell_2} - b_{R_2} - b_{u_2} - 3b_{L_3}}{2\pi}\ln\left(\frac{x}{M_{EW}}\right) + \frac{b_{u_4} - b_{R_3} - b_{u_3} + 3b_{L_3} - 3b_{L_4}}{2\pi}\ln\left(\frac{y}{M_{EW}}\right) \tag{B14}$$

$$= \frac{3}{\alpha_L} + \frac{16 - N_H}{3\pi}\ln\left(\frac{v}{M_{EW}}\right) - \frac{13}{3\pi}\ln\left(\frac{w}{M_{EW}}\right) + \frac{11 - N_H}{3\pi}\ln\left(\frac{x}{M_{EW}}\right) + \frac{22 - 5N_H}{6\pi}\ln\left(\frac{y}{M_{EW}}\right). \tag{B15}$$

This cascade has an identical range of scales for unification as the previous scheme, the two cascades becoming equivalent once the unification scales have been identified.

3. Cascade 3

The Higgs VEV pattern which induces the breaking of cascade three is

$$\langle\Phi_\ell\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ y & 0 & y \end{pmatrix}, \qquad \langle\Phi_{\ell^c}\rangle = \begin{pmatrix} w & 0 & w \\ 0 & w & 0 \\ w & 0 & v \end{pmatrix},$$
$$\langle\Phi_a\rangle = \langle\Phi_c^\dagger\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ y & 0 & y \end{pmatrix}. \tag{B16}$$

This symmetry breaking scheme is the one option that does not allow the unification of the gauge coupling constants, with the renormalization-group equations

$$\frac{1}{\alpha_q(v)} = \frac{1}{\alpha_q} + \frac{5}{2\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{1}{\pi}\ln\left(\frac{y}{M_{EW}}\right) \tag{B17}$$

$$\frac{1}{\alpha_L(v)} = \frac{1}{\alpha_L} + \frac{10 - 3N_H}{4\pi}\ln\left(\frac{v}{M_{EW}}\right) - \frac{5}{6\pi}\ln\left(\frac{y}{M_{EW}}\right) \tag{B18}$$

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} + \frac{16 - N_H}{3\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{16 - 2N_H}{3\pi}\ln\left(\frac{w}{M_{EW}}\right) - \frac{14 + 3N_H}{6\pi}\ln\left(\frac{y}{M_{EW}}\right), \tag{B19}$$

failing to intersect unless $N_H > 1$.

4. Cascade 4

The Higgs VEV pattern

$$\langle\Phi_\ell\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ y & 0 & w \end{pmatrix}, \qquad \langle\Phi_{\ell^c}\rangle = \begin{pmatrix} y & 0 & y \\ 0 & y & 0 \\ x & 0 & w \end{pmatrix},$$
$$\langle\Phi_a\rangle = \langle\Phi_c^\dagger\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ x & 0 & v \end{pmatrix}, \tag{B20}$$

instigates the breaking of cascade four. After the first stage of breaking the h, h^c, z_1, z_2, N, z_1^c, z_2^c and N^c particles gain mass, and the light Higgs spectrum is

$$\begin{aligned} \Phi_\ell &\sim (\mathbf{1}, \mathbf{2}, \bar{\mathbf{3}}, \mathbf{1})\left(-\frac{1}{3}\right) \oplus (\mathbf{1}, \mathbf{1}, \bar{\mathbf{3}}, \mathbf{1})\left(\frac{2}{3}\right), \\ \Phi_{\ell^c} &\sim (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{2})\left(\frac{1}{3}\right) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1})\left(-\frac{2}{3}\right), \end{aligned} \tag{B21}$$

$$\Phi_a \sim (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1), \quad \Phi_c \sim \Phi_a^\dagger. \tag{B22}$$

At w, the remaining charged fermions acquire mass, and ν^c gets an order y mass. The components of the Higgs multiplets that remain light are

$$\Phi_\ell \sim (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})(0) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1), \tag{B23}$$

$$\Phi_{\ell^c} \sim (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1), \tag{B24}$$

$$\Phi_a \sim (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1), \quad \Phi_c \sim \Phi_a^\dagger \tag{B25}$$

at w, and

$$\Phi_\ell \sim (\mathbf{1},\mathbf{2},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(-1), \tag{B26}$$

$$\Phi_{\ell^c} \sim (\mathbf{1},\mathbf{1},\mathbf{2})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})(1), \tag{B27}$$

$$\Phi_a \sim (\mathbf{1},\mathbf{2},\mathbf{1})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(-1), \quad \Phi_c \sim \Phi_a^\dagger \tag{B28}$$

at x, and at y

$$\Phi_\ell \sim (\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{2})(-1), \tag{B29}$$

$$\Phi_a \sim (\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{2})(-1), \qquad \Phi_c \sim \Phi_a^\dagger. \tag{B30}$$

This spectrum of particles defines the b's as follows:

$$\begin{array}{lllll} b_{q_1} = -7, & b_{L_1} = -\frac{4}{3} + \frac{3N_H}{2}, & b_{\ell_1} = -7 + N_H, & b_{R_1} = -\frac{4}{3} + \frac{3N_H}{2}, & b_{u_1} = \frac{4}{3} + N_H, \\ b_{q_2} = b_{q_1}, & b_{L_2} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{\ell_2} = -\frac{22}{3} + N_H, & b_{R_2} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{u_2} = \frac{8}{3} + \frac{4N_H}{3}, \\ b_{q_3} = b_{q_1}, & b_{L_3} = b_{L_2}, & b_{\ell_3} = b_{\ell_2}, & & b_{u_3} = \frac{20}{3} + \frac{11N_H}{6}, \\ b_{q_4} = b_{q_1}, & b_{L_4} = b_{L_2}, & & & b_{u_4} = \frac{20}{3} + \frac{3N_H}{2}. \end{array} \tag{B31}$$

The evolution of the $U(1)$ factor yields the relation

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} + \frac{3b_{u_1} + 3b_{R_1} + 4b_{\ell_1} - 9b_{L_1}}{6\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{3(b_{u_2} + b_{R_2} + b_{\ell_2} - b_{u_1} - b_{R_1}) - 4b_{\ell_1} + 9b_{L_1} - 9b_{L_2}}{6\pi}\ln\left(\frac{w}{M_{EW}}\right) + \frac{b_{u_3} + b_{\ell_3} - b_{u_2} - b_{R_2} - b_{\ell_2} + 3(b_{L_2} - b_{L_3})}{2\pi}\ln\left(\frac{x}{M_{EW}}\right) + \frac{b_{u_4} - b_{u_3} - b_{\ell_3} + 3(b_{L_3} - b_{L_4})}{2\pi}\ln\left(\frac{y}{M_{EW}}\right), \tag{B32}$$

which gives the renormalization-group equations to be

$$\frac{1}{\alpha_q(v)} = \frac{1}{\alpha_q} + \frac{7}{2\pi}\ln\left(\frac{v}{M_{EW}}\right), \tag{B33}$$

$$\frac{1}{\alpha_L(v)} = \frac{1}{\alpha_L} + \frac{8 - 9N_H}{12\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{1}{\pi}\ln\left(\frac{w}{M_{EW}}\right), \tag{B34}$$

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} - \frac{8 + N_H}{3\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{11}{3\pi}\ln\left(\frac{w}{M_{EW}}\right) + \frac{22 - 3N_H}{6\pi}\ln\left(\frac{x}{M_{EW}}\right) + \frac{11 - 2N_H}{3\pi}\ln\left(\frac{y}{M_{EW}}\right). \tag{B35}$$

This cascade is less restrictive than the previous three. The final breaking stage can occur at the TeV scale, but unlike the first two cascades, this low a value is not necessary for unification. Again, if we choose $y_{\rm min} \sim 1$ TeV, then choosing $x_{\rm min} \sim y$, yields the maximum scale of unification given by $w \sim 6.2 \times 10^{12}$ GeV and $v \sim 1.1 \times 10^{13}$ GeV, giving the coupling constant at v as $\alpha_{G_4}^{-1} = 36.82$. If x increases, then both w and v decrease as does w/v until we reach $x_{\rm max} = w \sim 4.2 \times 10^7$ GeV, and $v \sim 3.8 \times 10^{11}$ GeV. This gives a large range of unification possibilities, with the quartification gauge coupling constant equal to $\alpha_{G_4}^{-1} = 33.12$ at this upper bound.

The final stage of symmetry breaking can occur up to an energy of $y_{\rm max} \sim 1.2 \times 10^6$ GeV while still preserving the unification. As y increases, $x_{\rm max}$ decreases as do both w and v. At the value $y_{\rm max}$, we must have $x = w \sim 1.2 \times 10^6$ GeV and $v \sim 1.4 \times 10^{11}$ GeV for unification, giving the effective coupling $\alpha_{G_4}^{-1} = 32.02$.

5. Cascade 5

The Higgs VEV pattern which induces the breaking of cascade five is

$$\langle\Phi_\ell\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ x & 0 & w \end{pmatrix}, \qquad \langle\Phi_{\ell^c}\rangle = \begin{pmatrix} y & 0 & x \\ 0 & y & 0 \\ y & 0 & w \end{pmatrix},$$

$$\langle\Phi_a\rangle = \langle\Phi_c^\dagger\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ y & 0 & v \end{pmatrix}. \tag{B36}$$

The fermion mass spectrum is the same as the previous cascade. The light Higgs spectrum has the branching

$$\Phi_\ell \xrightarrow{v} (\mathbf{1},\mathbf{2},\bar{\mathbf{3}},\mathbf{1})\left(-\frac{1}{3}\right) \oplus (\mathbf{1},\mathbf{1},\bar{\mathbf{3}},\mathbf{1})\left(\frac{2}{3}\right)$$
$$\xrightarrow{w} (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})(0) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})(1)$$
$$\xrightarrow{x} (\mathbf{1},\mathbf{2},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(-1)$$
$$\xrightarrow{y} (\mathbf{1},\mathbf{2})(-1) \oplus (\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{2})(-1) \qquad \text{(B37)}$$

$$\Phi_{\ell^c} \xrightarrow{v} (\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{2})\left(\frac{1}{3}\right) \oplus (\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1})\left(-\frac{2}{3}\right)$$
$$\xrightarrow{w} (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1)$$
$$\xrightarrow{x} (\mathbf{1},\mathbf{1},\mathbf{2})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})(-1)$$
$$\xrightarrow{y} \text{nothing}$$
$$\Phi_a \xrightarrow{v,w} (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1) \qquad \text{(B38)}$$

$$\Phi_a \xrightarrow{v,w} (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1)$$
$$\xrightarrow{x} (\mathbf{1},\mathbf{2},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})(1)$$
$$\xrightarrow{y} (\mathbf{1},\mathbf{2})(-1) \oplus (\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{2})(-1) \qquad \text{(B39)}$$

$$\Phi_c \sim \Phi_a^\dagger. \qquad \text{(B40)}$$

The general form for the relationship between the structure constants and the breaking scales is

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} + \frac{3b_{u_1} + 3b_{R_1} + 4b_{\ell_1} - 9b_{L_1}}{6\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{3(b_{u_2} + b_{R_2} + b_{\ell_2} - b_{u_1} - b_{R_1}) - 4b_{\ell_1} + 9b_{L_1} - 9b_{L_2}}{6\pi}\ln\left(\frac{w}{M_{EW}}\right) + \frac{b_{u_3} + b_{R_3} - b_{u_2} - b_{R_2} - b_{\ell_2} + 3(b_{L_2} - b_{L_3})}{2\pi}\ln\left(\frac{x}{M_{EW}}\right) + \frac{b_{u_4} - b_{u_3} - b_{R_3} + 3(b_{L_3} - b_{L_4})}{2\pi}\ln\left(\frac{y}{M_{EW}}\right), \qquad \text{(B41)}$$

where the b's are defined as

$$\begin{array}{lllll} b_{q_1} = -7, & b_{L_1} = -\frac{4}{3} + \frac{3N_H}{2}, & b_{\ell_1} = -7 + N_H, & b_{R_1} = -\frac{4}{3} + \frac{3N_H}{2}, & b_{u_1} = \frac{4}{3} + N_H, \\ b_{q_2} = b_{q_1}, & b_{L_2} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{\ell_2} = -\frac{22}{3} + N_H, & b_{R_2} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{u_2} = \frac{8}{3} + \frac{4N_H}{3}, \\ b_{q_3} = b_{q_1}, & b_{L_3} = b_{L_1}, & & b_{R_3} = b_{R_2}, & b_{u_3} = \frac{8}{3} + \frac{5N_H}{3}, \\ b_{q_4} = b_{q_1}, & b_{L_4} = b_{L_1}, & & & b_{u_4} = \frac{20}{3} + \frac{3N_H}{2}. \end{array} \qquad \text{(B42)}$$

The renormalization-group equations are

$$\frac{1}{\alpha_q(v)} = \frac{1}{\alpha_q} + \frac{7}{2\pi}\ln\left(\frac{v}{M_{EW}}\right), \qquad \text{(B43)}$$

$$\frac{1}{\alpha_L(v)} = \frac{1}{\alpha_L} + \frac{8 - 9N_H}{12\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{1}{\pi}\ln\left(\frac{w}{M_{EW}}\right), \qquad \text{(B44)}$$

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} - \frac{8 + N_H}{3\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{11}{3\pi}\ln\left(\frac{w}{M_{EW}}\right) + \frac{11 - N_H}{3\pi}\ln\left(\frac{x}{M_{EW}}\right) + \frac{22 - 5N_H}{6\pi}\ln\left(\frac{y}{M_{EW}}\right), \qquad \text{(B45)}$$

where only the last equation is different from those of the previous cascade. This difference is compensated in the b's and we obtain a very similar spectrum of energy scales to cascade four which yield unification.

6. Cascade 6

The VEV pattern that induces the breaking in cascade six is

$$\langle\Phi_\ell\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ y & 0 & x \end{pmatrix}, \qquad \langle\Phi_{\ell^c}\rangle = \begin{pmatrix} y & 0 & y \\ 0 & y & 0 \\ x & 0 & x \end{pmatrix},$$
$$\langle\Phi_a\rangle = \langle\Phi_c^\dagger\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ w & 0 & v \end{pmatrix}. \qquad \text{(B46)}$$

The particles h, h^c, z_1, z_2, N, z_1^c, z_2^c and N^c gain order v masses, while there are no new masses at w. At x the remaining exotic fermions acquire mass, and as usual, ν^c has mass of order y. The Higgs spectrum which is light has the branching

$$\Phi_\ell \xrightarrow{v,w} (\mathbf{1},\mathbf{2},\bar{\mathbf{3}},\mathbf{1})\left(-\frac{1}{3}\right) \oplus (\mathbf{1},\mathbf{1},\bar{\mathbf{3}},\mathbf{1})\left(\frac{2}{3}\right) \xrightarrow{x} (\mathbf{1},\mathbf{2},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})(1) \xrightarrow{y} (\mathbf{1},\mathbf{2})(-1) \oplus (\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{2})(-1) \tag{B47}$$

$$\Phi_{\ell^c} \xrightarrow{v} (\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{2})\left(\frac{1}{3}\right) \oplus (\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1})\left(-\frac{2}{3}\right) \xrightarrow{w} (\mathbf{1},\mathbf{1},\mathbf{3})\left(-\frac{2}{3}\right) \oplus (\mathbf{1},\mathbf{1},\mathbf{3})\left(\frac{4}{3}\right) \oplus (\mathbf{1},\mathbf{1},\mathbf{3})\left(-\frac{2}{3}\right) \xrightarrow{x} (\mathbf{1},\mathbf{1},\mathbf{2})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})(-1) \xrightarrow{y} \text{nothing} \tag{B48}$$

$$\Phi_a \xrightarrow{v} (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1) \xrightarrow{w,x,y} (\mathbf{1},\mathbf{2},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(-1) \tag{B49}$$

$$\Phi_c \sim \Phi_a^\dagger. \tag{B50}$$

The relationship between the fine-structure constants and the breaking scales is

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} + \frac{3b_{u_1} + 3b_{R_1} + 4b_{\ell_1} - 9b_{L_1}}{6\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{3(b_{u_2} - b_{u_1} - b_{R_1}) + 4(b_{\ell_2} - b_{\ell_1}) + 9b_{L_1} - 9b_{L_2}}{6\pi}\ln\left(\frac{w}{M_{EW}}\right) + \frac{3(b_{u_3} + b_{\ell_3} - b_{u_2}) - 4b_{\ell_2} + 9(b_{L_2} - b_{L_3})}{6\pi}\ln\left(\frac{x}{M_{EW}}\right) + \frac{b_{u_4} - b_{u_3} - b_{\ell_3} + 3(b_{L_3} - b_{L_4})}{2\pi}\ln\left(\frac{y}{M_{EW}}\right), \tag{B51}$$

with the b's defined by

$$\begin{array}{lllll} b_{q_1} = -7, & b_{L_1} = -\frac{4}{3} + \frac{3N_H}{2}, & b_{\ell_1} = -7 + N_H, & b_{R_1} = -\frac{4}{3} + \frac{3N_H}{2}, & b_{u_1} = \frac{4}{3} + N_H \\ b_{q_2} = b_{q_1}, & b_{L_2} = -\frac{4}{3} + \frac{3N_H}{2}, & b_{\ell_2} = b_{\ell_1}, & & b_{u_2} = \frac{22}{3} + \frac{11N_H}{6} \\ b_{q_3} = b_{q_1}, & b_{L_3} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{\ell_3} = -\frac{22}{3} + N_H, & & b_{u_3} = \frac{20}{3} + \frac{11N_H}{6}, \\ b_{q_4} = b_{q_1}, & b_{L_4} = b_{L_3}, & & & b_{u_4} = \frac{20}{3} + \frac{3N_H}{2}. \end{array} \tag{B52}$$

The equations reduce to

$$\frac{1}{\alpha_q(v)} = \frac{1}{\alpha_q} + \frac{7}{2\pi}\ln\left(\frac{v}{M_{EW}}\right), \tag{B53}$$

$$\frac{1}{\alpha_L(v)} = \frac{1}{\alpha_L} + \frac{8 - 9N_H}{12\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{1}{\pi}\ln\left(\frac{x}{M_{EW}}\right), \tag{B54}$$

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} - \frac{8 + N_H}{3\pi}\ln\left(\frac{v}{M_{EW}}\right) + \frac{11 - N_H}{3\pi}\ln\left(\frac{w}{M_{EW}}\right) + \frac{22 - N_H}{6\pi}\ln\left(\frac{x}{M_{EW}}\right) + \frac{11 - 2N_H}{3\pi}\ln\left(\frac{y}{M_{EW}}\right). \tag{B55}$$

Again unification can be achieved for a range of values for the lower breaking scales.

7. Cascade 7

The VEV pattern that induces the breaking pattern of cascade seven is

$$\langle\Phi_\ell\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ y & 0 & v \end{pmatrix}, \qquad \langle\Phi_{\ell^c}\rangle = \begin{pmatrix} y & 0 & x \\ 0 & y & 0 \\ y & 0 & w \end{pmatrix}, \qquad \langle\Phi_a\rangle = \langle\Phi_c^\dagger\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ x & 0 & w \end{pmatrix}. \tag{B56}$$

After the first stage of symmetry breaking the exotic fermions x_1, x_2, y_1 and y_2 gain GUT-scale Dirac masses. The remaining exotic fermions gain w scale masses and ν^c an order y mass. The light Higgs spectrum has the branching

$$\Phi_\ell \xrightarrow{v,w,x} (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})(0) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})(1) \xrightarrow{y} (\mathbf{1},\mathbf{2})(-1) \oplus (\mathbf{1},\mathbf{2})(1) \oplus (\mathbf{1},\mathbf{2})(-1) \tag{B57}$$

$$\Phi_{\ell^c} \xrightarrow{v} (\mathbf{1},\mathbf{1},\mathbf{2},\bar{\mathbf{3}})\left(-\frac{1}{3}\right) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\bar{\mathbf{3}})\left(\frac{2}{3}\right) \xrightarrow{w} (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1) \tag{B58}$$

$$\Phi_a \xrightarrow{v} (\mathbf{1},\mathbf{2},\bar{\mathbf{3}},\mathbf{1})\left(-\frac{1}{3}\right) \oplus (\mathbf{1},\mathbf{1},\bar{\mathbf{3}},\mathbf{1})\left(\frac{2}{3}\right) \xrightarrow{w} (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})(0) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})(1)$$
$$\xrightarrow{x,y} (\mathbf{1},\mathbf{2},\mathbf{1})(-1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(1) \oplus (\mathbf{1},\mathbf{2},\mathbf{1})(-1)$$

$$\Phi_c \sim \Phi_a^\dagger. \tag{B59}$$

The b's are then

$$\begin{array}{lllll} b_{q_1} = -5, & b_{L_1} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{\ell_1} = -\frac{10}{3} + N_H, & b_{R_1} = -5 + \frac{3N_H}{2}, & b_{u_1} = 4 + \frac{5N_H}{6}, \\ b_{q_2} = -7, & b_{L_2} = b_{L_1}, & b_{\ell_2} = -\frac{22}{3} + N_H, & b_{R_2} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{u_2} = \frac{8}{3} + \frac{4N_H}{3}, \\ b_{q_3} = b_{q_2}, & b_{L_3} = b_{L_1}, & b_{\ell_3} = b_{\ell_2}, & & b_{u_3} = \frac{20}{3} + \frac{11N_H}{6}, \\ b_{q_4} = b_{q_2}, & b_{L_4} = b_{L_1}, & & & b_{u_4} = \frac{20}{3} + \frac{3N_H}{2}. \end{array} \tag{B60}$$

The fine-structure constants at the electroweak level are related to the breaking scales via

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} + \frac{3b_{u_1} + 4b_{R_1} + 3b_{\ell_1} - 9b_{L_1}}{6\pi} \ln\left(\frac{v}{M_{EW}}\right) + \frac{3(b_{u_2} + b_{R_2} + b_{\ell_2} - b_{u_1} - b_{\ell_1}) - 4b_{R_1} + 9b_{L_1} - 9b_{L_2}}{6\pi} \ln\left(\frac{w}{M_{EW}}\right)$$
$$+ \frac{b_{u_3} + b_{\ell_3} - b_{u_2} - b_{R_2} - b_{\ell_2} + 3(b_{L_2} - b_{L_3})}{2\pi} \ln\left(\frac{x}{M_{EW}}\right) + \frac{b_{u_4} - b_{u_3} - b_{\ell_3} + 3(b_{L_3} - b_{L_4})}{2\pi} \ln\left(\frac{y}{M_{EW}}\right), \tag{B61}$$

giving renormalization-group equations of the form

$$\frac{1}{\alpha_q(v)} = \frac{1}{\alpha_q} + \frac{5}{2\pi} \ln\left(\frac{v}{M_{EW}}\right) + \frac{1}{\pi} \ln\left(\frac{w}{M_{EW}}\right), \tag{B62}$$

$$\frac{1}{\alpha_L(v)} = \frac{1}{\alpha_L} + \frac{20 - 9N_H}{12\pi} \ln\left(\frac{v}{M_{EW}}\right), \tag{B63}$$

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} + \frac{6 - N_H}{3\pi} \ln\left(\frac{v}{M_{EW}}\right) - \frac{1}{\pi} \ln\left(\frac{w}{M_{EW}}\right) + \frac{22 - 3N_H}{6\pi} \ln\left(\frac{x}{M_{EW}}\right) + \frac{11 - 2N_H}{3\pi} \ln\left(\frac{y}{M_{EW}}\right). \tag{B64}$$

The GUT scale must be of order 10^{13} GeV for unification. The lowest two breaking scales are forced to be around the TeV scale, with the only freedom coming into the choice of the w scale. The range of scales for which unification can be achieved are the same as cascades one and two, with the symmetry breaking patterns becoming identical.

8. Cascade 8

As previously noted, choosing whether or not to break $SU(2)_\ell$ or $SU(2)_R$ first from the $SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)$ level has no significant difference on the unification scales and our fermion mass spectrum is identical to that above. Subsequently, we just list the equations below.

The VEV pattern that induces the breaking pattern of cascade eight is

$$\langle\Phi_\ell\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ x & 0 & v \end{pmatrix}, \qquad \langle\Phi_{\ell^c}\rangle = \begin{pmatrix} y & 0 & x \\ 0 & y & 0 \\ y & 0 & w \end{pmatrix},$$
$$\langle\Phi_a\rangle = \langle\Phi_c^\dagger\rangle = \begin{pmatrix} u & 0 & u \\ 0 & u & 0 \\ y & 0 & w \end{pmatrix}. \tag{B65}$$

The general form of the relationship between the fine-structure constants at low energy and the breaking scales is

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} + \frac{3b_{u_1} + 4b_{R_1} + 3b_{\ell_1} - 9b_{L_1}}{6\pi} \ln\left(\frac{v}{M_{EW}}\right) + \frac{3(b_{u_2} + b_{R_2} + b_{\ell_2} - b_{u_1} - b_{\ell_1}) - 4b_{R_1} + 9b_{L_1} - 9b_{L_2}}{6\pi} \ln\left(\frac{w}{M_{EW}}\right)$$
$$+ \frac{b_{u_3} + b_{R_3} - b_{u_2} - b_{R_2} - b_{\ell_2} + 3(b_{L_2} - b_{L_3})}{2\pi} \ln\left(\frac{x}{M_{EW}}\right) + \frac{b_{u_4} - b_{u_3} - b_{R_3} + 3(b_{L_3} - b_{L_4})}{2\pi} \ln\left(\frac{y}{M_{EW}}\right). \tag{B66}$$

The b's are defined by

$$\begin{array}{lllll} b_{q_1} = -5, & b_{L_1} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{\ell_1} = -\frac{10}{3} + N_H, & b_{R_1} = -5 + \frac{3N_H}{2}, & b_{u_1} = 4 + \frac{5N_H}{6}, \\ b_{q_2} = -7, & b_{L_2} = b_{L_1}, & b_{\ell_2} = -\frac{22}{3} + N_H, & b_{R_2} = -\frac{10}{3} + \frac{3N_H}{2}, & b_{u_2} = \frac{8}{3} + \frac{4N_H}{3}, \\ b_{q_3} = b_{q_2}, & b_{L_3} = b_{L_1}, & & b_{R_3} = b_{R_2}, & b_{u_3} = \frac{8}{3} + \frac{5N_H}{3}, \\ b_{q_4} = b_{q_2}, & b_{L_4} = b_{L_1}, & & & b_{u_4} = \frac{20}{3} + \frac{3N_H}{2}, \end{array} \tag{B67}$$

and the RGEs

$$\frac{1}{\alpha_q(v)} = \frac{1}{\alpha_q} + \frac{5}{2\pi} \ln\left(\frac{v}{M_{EW}}\right) + \frac{1}{\pi} \ln\left(\frac{w}{M_{EW}}\right), \tag{B68}$$

$$\frac{1}{\alpha_L(v)} = \frac{1}{\alpha_L} + \frac{20 - 9N_H}{12\pi} \ln\left(\frac{v}{M_{EW}}\right), \tag{B69}$$

$$\frac{1}{\alpha_Y} = \frac{3}{\alpha_L} + \frac{6 - N_H}{3\pi} \ln\left(\frac{v}{M_{EW}}\right) - \frac{1}{\pi} \ln\left(\frac{w}{M_{EW}}\right) + \frac{11 - N_H}{3\pi} \ln\left(\frac{x}{M_{EW}}\right) + \frac{22 - 5N_H}{6\pi} \ln\left(\frac{y}{M_{EW}}\right). \tag{B70}$$

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