

QCD analytic perturbation theory: From integer powers to any power of the running coupling

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We propose a new generalized version of the QCD analytic perturbation theory of Shirkov and Solovtsov for the computation of higher-order corrections in inclusive and exclusive processes. We construct nonpower series expansions for the analytic images of the running coupling and its powers for any fractional (real) power and complete the linear space of these solutions by constructing the index derivative. Using the Laplace transformation in conjunction with dispersion relations, we are able to derive at the one-loop order closed-form expressions for the analytic images in terms of the Lerch function. At the two-loop order we provide approximate analytic images of products of powers of the running coupling and logarithms—typical in higher-order perturbative calculations and when including evolution effects. Moreover, we supply explicit expressions for the two-loop analytic coupling and the analytic images of its powers in terms of one-loop quantities that can strongly simplify two-loop calculations. We also show how to resum powers of the running coupling while maintaining analyticity, a procedure that captures the generic features of Sudakov resummation. The algorithmic rules to obtain analytic-coupling expressions within the proposed fractional analytic perturbation theory from the standard QCD power-series expansion are supplied ready for phenomenological applications and numerical comparisons are given for illustration.

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I. INTRODUCTION

A fundamental goal of perturbative QCD is to provide a microscopic description of hadronic short-distance phenomena that yields reliable predictions to be compared with experimental data of increasing precision. While singularities on the timelike axis in the complex Q^2 plane of hadronic observables are related to physical particles (or resonances), the appearance of singularities on the spacelike axis are unphysical and may violate causality. On the other hand, the expansion of hadronic quantities at large momentum transfer Q^2 can be safely calculated in terms of a power-series expansion in the running strong coupling $\alpha_s(Q^2)$ by virtue of asymptotic freedom. But the one-loop running coupling contains at $Q^2 = \Lambda_{\text{QCD}}^2$ ($\Lambda_{\text{QCD}} \equiv \Lambda$ in the following) a ghost singularity—the Landau pole—that spoils its analyticity structure. To restore analyticity and ensure causality in the whole Q^2 plane, this pole has to be removed. With most available experimental data on several exclusive processes being at rather low Q^2 values, the Landau-singularity problem is not only of academic interest, but affects significantly perturbative predictions in the low-to-medium Q^2 domain. The reason is that—lacking all-order perturbative expressions—one has to resort to a renormalization-scheme choice that makes the uncalculated higher-order corrections negligible and adopt a renormalization scale that reflects the typical parton virtual-

ities in the considered process. The latter procedure, however, may result into a scale in the region of only a few Λ , where the application of perturbation theory for the conventional running coupling without infrared (IR) protection against the Landau pole becomes inapplicable—a prominent example being the Brodsky–Lepage–Mackenzie scale-setting procedure [1]. Different strategies have been suggested over the years as how to minimize the dependence on the renormalization scheme and scale setting—unavoidable in any perturbative calculation beyond the leading order—and obtain reliable and stable results in the low-momentum regime (see, for example, Ref. [2] for a recent extensive discussion of these issues in terms of the electromagnetic pion form factor and references cited therein).

In a series of papers during the last few years Shirkov and Solovtsov (SS) [3–7] have developed an approach which enables the removal of the Landau singularity without introducing extraneous IR regulators, like an effective gluon mass [8–12]. The analyticity of the coupling in the spacelike region is achieved by a nonperturbative, power-behaved term that contains no other scale than Λ and leaves the ultraviolet (UV) behavior of the running coupling unchanged. At zero-momentum transfer the Shirkov–Solovtsov APT coupling assumes a universal value that depends solely on renormalization-group constants. Using dispersion relations, this scheme was both generalized (in approximate form) to higher-loop orders and also extended to the timelike regime [6,7,13–23], encompassing previous incomplete attempts [24,25] in this direction, and amounting to the theoretical framework of analytic pertur-

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bation theory (APT). There have been a number of parallel developments by various authors during the past several years to avoid the Landau pole using different “analytization” techniques, prime examples being Refs. [17–19,24–35].

Two other major challenges, connected with—first—the implementation of the Shirkov-Solovtsov APT analytization to three-point functions beyond the leading order of perturbation theory and—second—the extension to noninteger powers of the coupling, remained open, or at least partially open. Indeed, in the first case, extensive analyses [2,36,37] have shown that the analytization principle has to be generalized to accommodate a second scale, serving as a factorization scale, or in order to include evolution effects comprising typical logarithms to some fractional power. Technically speaking, this means to extend the assertion of analyticity from the level of the coupling (and its powers) to the level of the whole reaction amplitude. This requirement was formalized by Karanikas and Stefanis (KS) in [38,39] in an attempt to calculate power corrections to the pion form factor and the Drell–Yan process. The systematic development of a perturbative expansion in terms of fractional powers of the coupling—the second major challenge—is the goal of the present investigation, the main focus being placed on the methodology towards improving perturbative higher-order calculations in QCD. This goal has been accomplished and will be described in this paper. A specific application of the KS analytization principle to the pion’s electromagnetic form factor at NLO accuracy is given in fully worked out detail in [40]. Other applications will follow in future publications in conjunction with the inclusion of heavy-flavor thresholds and the extension to the timelike regime.

The paper is organized as follows. In Sec. II we first review the key features of the original analytic perturbation theory of Shirkov and Solovtsov, highlighting those properties pertaining to the generalization of the approach to fractional powers of the coupling. The actual extension of the approach to fractional—in fact, real—powers of the coupling is performed in Sec. III. This section describes in three subsections the new analytization technique, based on the Laplace transform, the verification of the analytic properties of the obtained results, and the way to include products of powers of the coupling with powers of logarithms. Moreover, we provide here approximate expressions for two-loop quantities in terms of one-loop analytic-coupling images and their index derivatives that can be extremely useful in practical calculations. Section IV is devoted to the validation of the developed theoretical framework of the fractional analytic perturbation theory (FAPT) and includes a table where we collect the algorithmic rules to connect the new analytic framework to the standard QCD perturbative power-series expansion. Our conclusions are drawn in Sec. V, while important technical details are presented in three appendices.

II. ORIGINAL ANALYTIC PERTURBATION THEORY

In the analytic perturbation-theory approach of Shirkov and Solovtsov, the power-series expansion in the running coupling is given up in favor of a nonpower series (functional) expansion. This can be written generically in terms of numerical coefficients d_m in the following way [4,41]:

$$\sum_m d_m a_{(l)}^m(Q^2) \Rightarrow \sum_m d_m \mathcal{A}_m^{(l)}(Q^2), \quad (2.1)$$

where the “normalized” coupling $a = b_0 \alpha_s / (4\pi)$ (b_0 is the first coefficient of the QCD β -function—see Appendix B) has been introduced instead of α_s in order to simplify intermediate calculations and because then the analytic-coupling \mathcal{A}_1 is bounded from above by unity [4]. In the above expression, the superscript m on $a_{(l)}^m$ appears on the left-hand side as a power, whereas on the right-hand side (rhs) the subscript m on $\mathcal{A}_m^{(l)}$ denotes the index of the functional expansion¹; (l) denotes the loop order. For the sake of simplicity, we will avoid to indicate the loop-order index explicitly because we mostly work in the one-loop approximation; deviations, if needed, will be labeled by appropriate superscripts or subscripts in parentheses, as in Eq. (2.1). The conversion to analytic images of the coupling is achieved in terms of the functions

$$\mathcal{A}_m^{(l)}(Q^2) \equiv [a_{(l)}^m(Q^2)]_{\text{an}} \quad (2.2)$$

according to the general prescription

$$[f(Q^2)]_{\text{an}} = \frac{1}{\pi} \int_0^\infty \frac{\text{Im}[f(-\sigma)]}{\sigma + Q^2 - i\epsilon} d\sigma. \quad (2.3)$$

For the one-loop running coupling

$$a = \frac{1}{\ln(Q^2/\Lambda^2)}, \quad (2.4)$$

we have

$$\mathcal{A}_1(Q^2) \equiv [a(Q^2)]_{\text{an}} = \frac{1}{\ln(Q^2/\Lambda^2)} - \frac{1}{Q^2/\Lambda^2 - 1} \quad (2.5)$$

and $\mathcal{A}_1^{(1)}(0) = 1$. Employing the variable $L = \ln(Q^2/\Lambda^2)$, which naturally appears in perturbative QCD (pQCD) calculations, we can recast a and \mathcal{A} in terms of L to obtain

$$a^1(L) = \frac{1}{L}, \quad \mathcal{A}_1(L) = \frac{1}{L} - \frac{1}{e^L - 1}. \quad (2.6)$$

In this context, amplitudes (depending on a single scale Q^2) perturbatively expanded in terms of the powers of the running coupling map on a nonpower series expansion [4,41]:

¹In the following, a calligraphic notation is used to denote analytic images.

$$F(a) = \sum_n f_n a^n(L) \Rightarrow \mathcal{F}(L) = \sum_n f_n \mathcal{A}_n(L), \quad (2.7)$$

where f_n are numbers in minimal subtraction renormalization schemes. By construction, the set $\{\mathcal{A}_n\}$ constitutes a linear space, which, however, is not equipped with the multiplication operation of its elements. Therefore, the product $\mathcal{A}_n \cdot \mathcal{A}_m$ has no rigorous meaning here. The standard algebra is recovered only for the main asymptotic contribution [cf. Eq. (2.6)] at $L \rightarrow \infty$, when $\{\mathcal{A}_n\} \rightarrow \{a^n\}$ [3–5].

Let us now turn to the properties of this map and of the space $\{\mathcal{A}_n\}$. There are several points to note about them.

- (1) The map should have the property of isomorphism, i.e., it should conserve the linear structure of the original space:

$$a^0 \Rightarrow \mathcal{A}_0 \equiv 1. \quad (2.8)$$

- (2) Renormalization-group summation leads to contributions like

$$f(a) = a^\nu, \quad \text{where } \nu \text{ is real,} \quad (2.9)$$

necessitating the introduction of the analytic images of $f(a)$: $[f(a)]_{\text{an}} = [a^\nu]_{\text{an}}$. These are exactly those terms needed to supply the original linear space $\{\mathcal{A}_n\}$ with the completeness property as regards the differential operator with respect to the real index ν .

- (3) Motivated by the typical logarithmic contributions, appearing in loop calculations in standard pQCD, we consider the analytization of terms of the sort

$$f(a) = \begin{cases} a^\nu \ln(a); \\ a^\nu L^m, m = 1, 2, \end{cases} \quad (2.10)$$

giving rise to the corresponding analytic images

$$\mathcal{L}_\nu(a) \equiv [a^\nu \ln(a)]_{\text{an}}, \quad \mathcal{L}_{\nu,m}(L) \equiv [a^\nu L^m]_{\text{an}}. \quad (2.11)$$

Expression (2.9) is universal and allows one to apply any one-loop renormalization-group results to APT. In fact, the corresponding renormalization factor Z , associated with the renormalizable quantity B , $B(Q^2) = B(\mu^2)Z(Q^2)/Z(\mu^2)$, reduces in the one-loop approximation to

$$Z \sim a^\nu(L)|_{\nu=\nu_0=\gamma_0/(2b_0)},$$

where γ_0 is the coefficient of the one-loop anomalous dimension. Therefore, we have

$$[B(Q^2)]_{\text{an}} \sim [a^\nu(L)]_{\text{an}} = \mathcal{A}_\nu(L). \quad (2.12)$$

The next two functions $\mathcal{L}_\nu(a)$ and $\mathcal{L}_{\nu,m}(L)$ appear in NLO of pQCD and also in light-cone sum rules [42,43] and reflect the specific features of these calculations. An ex-

ample of the first kind in connection with a NLO calculation of the electromagnetic pion form factor is treated in [40], while the investigation of such terms in the context of light-cone sum rules will be considered in a future publication. To be more specific, we will consider below terms of the form $(a_{(2)})^\nu L$ and $(a_{(2)})^\nu L^2$ —rather than deal with the series containing the constant coefficients f_n , given in Eq. (2.7).

One possible way in generalizing the presented original APT formalism to noninteger (fractional) values of the index ν is to construct the spectral density

$$\rho_\nu(\sigma) = \frac{1}{\pi} \mathbf{Im}[a^\nu(-\sigma)] \quad (2.13)$$

for $\nu \in \mathbb{R}$. Indeed, substituting

$$a(-\sigma) = \frac{1}{L(\sigma) - i\pi}, \quad L(\sigma) = \ln(\sigma/\Lambda^2), \quad (2.14)$$

for the one-loop running coupling into Eq. (2.13), we can obtain by a straightforward calculation a closed-form expression for the spectral density in the form

$$\rho_\nu(\sigma) = \frac{1}{\pi} \frac{\sin(\nu\varphi)}{[\pi^2 + L^2(\sigma)]^{\nu/2}}, \quad (2.15)$$

$$\varphi = \arccos\left(\frac{L(\sigma)}{\sqrt{L^2(\sigma) + \pi^2}}\right).$$

A result similar to that has been derived in the context of Electrodynamics in the early article of Ref. [44]. It was later reinvented in QCD by Oehme [45] and used by Magradze in [20]. To get now the desired analytic coupling for some fractional index, one has to insert this expression back into Eq. (2.3) and perform the integral numerically, loosing, alas, this way the possibility to reveal the mathematical properties of this function. Let us emphasize at this point that the extension of this procedure to the two-loop order for the first integer values of ν has been done in Refs. [4,5,20], while the inclusion of still higher-loops [46] seems feasible.² However, this approach, based on the spectral density (2.13), is restricted to the specific structure of the Shirkov-Solovtsov APT.

III. FRACTIONAL ANALYTIC PERTURBATION THEORY

A. A new generalization technique to include fractional indices

In this subsection, we formulate and outline another procedure to continue the integer index of the analytic coupling to fractional values. First, to generate higher indices at the one-loop level of the analytic images within APT, or, equivalently, higher powers of the standard run-

²Indeed a partial result for a few fractional ν values has already been obtained [47].

ning coupling within conventional pQCD, we follow [4] and write

$$\left(\frac{\mathcal{A}_n(L)}{a^n(L)}\right) = \frac{1}{(n-1)!} \left(-\frac{d}{dL}\right)^{n-1} \left(\frac{\mathcal{A}_1(L)}{a^1(L)}\right). \quad (3.1)$$

Second, to facilitate the transition to fractional index values, it is instrumental to employ the Laplace representation of both types of couplings—the analytic, $\mathcal{A}_1(L)$, and the conventional one, $a(L)$ (both at the one-loop order)—and define at $L > 0$

$$\left(\frac{\mathcal{A}_1(L)}{a^1(L)}\right) = \int_0^\infty e^{-Lt} \left(\frac{\tilde{\mathcal{A}}_1(t)}{\tilde{a}_1(t)}\right) dt. \quad (3.2)$$

The advantage of this representation is that it transforms the result of a differential operator into an algebraic expression containing monomials. Then, applying Eq. (3.1) to (3.2), we get

$$\begin{aligned} \mathcal{A}_n(L) &= \frac{1}{(n-1)!} \left(-\frac{d}{dL}\right)^{n-1} \mathcal{A}_1(L) \\ &= \int_0^\infty e^{-Lt} \left[\frac{t^{n-1}}{(n-1)!} \cdot \tilde{\mathcal{A}}_1(t)\right] dt, \end{aligned} \quad (3.3)$$

so that we establish the correspondence

$$\mathcal{A}_n(L) \longleftrightarrow \tilde{\mathcal{A}}_n(t) = \frac{t^{n-1}}{(n-1)!} \cdot \tilde{\mathcal{A}}_1(t), \quad (3.4)$$

whereas for the case of the conventional pQCD coupling, one has the evident Laplace conjugates \tilde{a}_n

$$a^1(L) \equiv \frac{1}{L} \longleftrightarrow \tilde{a}_1 = 1, \quad (3.5)$$

$$a^n(L) \longleftrightarrow \tilde{a}_n = \frac{t^{n-1}}{(n-1)!} \cdot \tilde{a}_1. \quad (3.6)$$

Equation (3.4) enables us to generalize $\mathcal{A}_n(L)$ to any real index ν . To do so, let us introduce the following definition for the Laplace conjugate $\tilde{\mathcal{A}}_\nu(t)$:

$$\tilde{\mathcal{A}}_\nu(t) \stackrel{\text{def}}{=} \frac{t^{\nu-1}}{\Gamma(\nu)} \cdot \tilde{\mathcal{A}}_1(t). \quad (3.7)$$

At this stage of the continuation in the index ν , we have based our considerations solely on the first relation in Eq. (3.3). Therefore, the Laplace conjugate (3.7) remains valid for any nonpower perturbative expansion satisfying this relation, reiterating that this holds true at the one-loop level. To complete the generalization process, we should obtain an expression for $\tilde{\mathcal{A}}_1(t)$, based on Eq. (2.6). This gives the result

$$\mathcal{A}_1(L) = \frac{1}{L} - \frac{1}{e^L - 1} \longleftrightarrow \tilde{\mathcal{A}}_1(t) = 1 - \sum_{m=1}^\infty \delta(t - m), \quad (3.8)$$

which can be verified by a straightforward calculation.

Let us pause for a moment to make some useful remarks concerning the behavior of the two parts of Eq. (3.8). One should note the strong difference in the behavior of these functions with respect to the logarithmic term of standard perturbation theory, on the one hand,

$$\frac{1}{L} \longleftrightarrow 1,$$

and the pole remover appearing in APT, on the other,

$$\frac{1}{e^L - 1} \longleftrightarrow \sum_{m=1}^\infty \delta(t - m).$$

Thus, one can define $\mathcal{A}_\nu(L)$ according to Eq. (3.3), and, then, using Eqs. (3.7) and (3.8), arrive at

$$\begin{aligned} \mathcal{A}_\nu(L) &= \int_0^\infty e^{-Lt} \frac{t^{\nu-1}}{\Gamma(\nu)} \cdot \tilde{\mathcal{A}}_1(t) dt \\ &= \frac{1}{L^\nu} - \frac{1}{\Gamma(\nu)} \cdot \sum_{m=1}^\infty e^{-Lm} m^{\nu-1}. \end{aligned} \quad (3.9)$$

The series on the rhs of the latter equality coincides with the definition of the Lerch transcendental function [48] $\Phi(z, \nu', i)$ at $\nu' = \nu - 1 < 0$ for $i = 1$, i.e.,

$$\sum_{m=1}^\infty \frac{z^m}{m^{1-\nu}} = z\Phi(z, 1 - \nu, 1). \quad (3.10)$$

The analytic continuation of $\Phi(z, s, 1)$ in the variables z, s , adopting the notation of Batemann and Erdélyi [48], determines Φ as an analytic function of the variable z in the plane with a cut along $(1, \infty)$ for *any* fixed s (see for more details in Appendix A.³). Finally, $\tilde{\mathcal{A}}_\nu$ in Eq. (3.9) can be rewritten in the form of an analytic function with respect to both variables ν and L ; viz.,

$$\mathcal{A}_\nu(L) = \frac{1}{L^\nu} - \frac{e^{-L}}{\Gamma(\nu)} \Phi(e^{-L}, 1 - \nu, 1). \quad (3.11)$$

We state here and prove in Appendix A that \mathcal{A}_ν is an entire function in ν .

B. Analytic properties

To assess the analytic properties of Eq. (3.11), it is useful to recast the Lerch function $\Phi(z, \nu, 1)$ via (see [48], Eq. (1.10.14) and also [49], Chap. 8)

$$z\Phi(z, \nu, 1) \equiv F(z, \nu) \quad (3.12)$$

entailing

$$\mathcal{A}_\nu(L) = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1 - \nu)}{\Gamma(\nu)}, \quad (3.13)$$

where the first term in Eq. (3.13) corresponds to the standard PT, while the second one expresses the pole remover.

³The transcendental Lerch function $\Phi(z, s, 1)$ is included in the widespread programs ‘‘MATHEMATICA 5’’ and ‘‘MAPLE 7’’.

Note that for a positive integer index, $\nu = m \geq 2$, one has the relation [48]

$$F(z, 1 - m) = (-1)^m F\left(\frac{1}{z}, 1 - m\right), \quad (3.14)$$

so that substituting Eq. (3.14) in (3.13), one arrives at

$$\mathcal{A}_m(L) = (-1)^m \mathcal{A}_m(-L) \quad (3.15)$$

that confirms the specific symmetry relations worked out in [4]. From relation (3.15) and Eq. (A6) one obtains the explicit asymptotic expression for $\mathcal{A}_m(L)$ at $L \rightarrow -\infty$

$$\begin{aligned} \mathcal{A}_m(L \rightarrow -\infty) &= (-1)^m \mathcal{A}_m(|L| \rightarrow \infty) \\ &= (-1)^m / |L|^m + \mathcal{O}(1/|L|^m). \end{aligned} \quad (3.16)$$

This estimate can be extended to any real value $\nu > 1$ of the index m . To make the content of Eq. (3.13) more transparent, we display in Fig. 1(a) the graphs of the analytic coupling for indices from -3 to 0 and values of L in the range -3 to 3 . Appealing to Eqs. (3.17) and (3.18) for negative values of the index ν , one makes sure that, for $L = 1$ (red thick dotted line above the zero line), this function is equal to unity for all integer values of the index $\nu = -m$. On the other hand, for $L < -1$, the value of $\mathcal{A}_{-m}(L)$ depends on whether or not m is even or odd. For even values, it is positive, whereas for odd values it is negative, therefore giving rise to oscillations shown in Fig. 1(a). Note also that for values of $L \geq 1$, the oscillatory behavior of the graphs for $\mathcal{A}_\nu(L)$ starts to be much less pronounced (red thick broken lines) because L^m is positive for all positive values of m . The opposite behavior is exhibited for $L < 0$, as one sees from the blue broken lines.

From Fig. 1(b), we observe that, in the region where $\mathcal{A}_\nu(L)$ is smaller than unity (as explicitly indicated in the figure), this function is monotonic in ν for $\nu \leq 2$. On the other hand, in the region where $\mathcal{A}_\nu(L) > 1$ —possible only for $\nu < 1$ and $L < 0$ —this function starts to be non-monotonic in ν , so that there are two different points ν_1 and ν_2 , both corresponding to the same value of $\mathcal{A}_\nu(L)$.

Focusing on the values of $\mathcal{A}_\nu(L)$ for $L > 0$, we see that all curves are monotonic in L and are bounded by an envelope represented by $\mathcal{A}_\nu(0)$ [blue thick solid line in Fig. 1(b)]. If we consider only the interval of $\nu \in (0, 1)$, then the monotonicity property extends also to the negative values of L .

Contrary to that case, the coupling $\mathcal{A}_m(L)$ oscillates in L [4] for higher values of $m > 2$. These oscillations are not visible in Fig. 1(b) because of the smallness of the corresponding amplitudes. They appear due to rather general reasons: (i) the asymptotic conditions given by Eq. (3.16): $\mathcal{A}_m(-\infty) = \mathcal{A}_m(\infty) = 0$ for $m \geq 2$; (ii) the differential relation between \mathcal{A}_m and \mathcal{A}_1 , expressed in Eq. (3.1). Therefore, \mathcal{A}_{m+2} has m zeros in the vicinity of the former “Landau pole” ($L = 0$) [4]—see Fig. 2. This property is rather unexpected from the point of view of the standard power-series perturbation theory and will be discussed below in connection with Eq. (3.22). This oscillation property of the coupling extends to $\mathcal{A}_\nu(L)$ for all real values of the index $\nu \geq 2$.

To reveal the relevance of this representation for physical applications, let us now consider \mathcal{A} for some particular values of the index ν . For the case of a negative index, the $\mathcal{A}_{-\nu}$ play the role of the “inverse powers” of \mathcal{A}_1 that may be considered as the images of $a_s^{-\nu}$. Then, expression (3.13) can be rewritten in the form

$$\mathcal{A}_{-\nu}(L) = L^\nu - \frac{\mathbf{Li}_{\nu+1}(e^{-L})}{\Gamma(-\nu)}, \quad (3.17)$$

$$\mathcal{A}_{-m}(L) = \lim_{\varepsilon \rightarrow 0} \mathcal{A}_{-m+\varepsilon}(L) = L^m, \quad \text{at } m = 0, 1, 2, \dots, \quad (3.18)$$

where we have taken into account that for $\nu \geq 0$

$$F(z, \nu) = \mathbf{Li}_\nu(z), \quad (3.19)$$

with \mathbf{Li}_ν being the well-known polylogarithm function. It is worth remarking here that the “inverse powers” $\mathcal{A}_{-m}(L) = L^m$ coincide with the inverse powers of the original running coupling $a^{-m}(L) = L^m$.

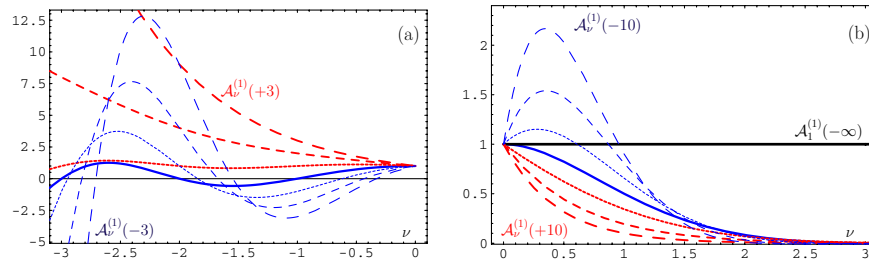


FIG. 1 (color online). (a) Comparison of different curves for $\mathcal{A}_\nu^{(1)}(L)$ as functions of the index $\nu \leq 0$, corresponding to various values of L , ranging from $L = -3$ to $L = +3$. The blue broken lines correspond to $L < 0$, whereas the red thick broken lines correspond to $L > 0$. The two dotted lines denote the results for $L = \pm 1$, with the line associated with the value $L = 1$ being closer to the zero line. (b) The same comparison for $\nu \geq 0$ using the same line designations, but for values of L , ranging from $L = -10$ to $L = +10$ (dashed lines). The dotted lines here correspond to $L = \pm 2$, whereas the short-dashed lines represent the result for $L = \pm 5$. The blue solid lines in both panels show $\mathcal{A}_\nu^{(1)}(0)$.

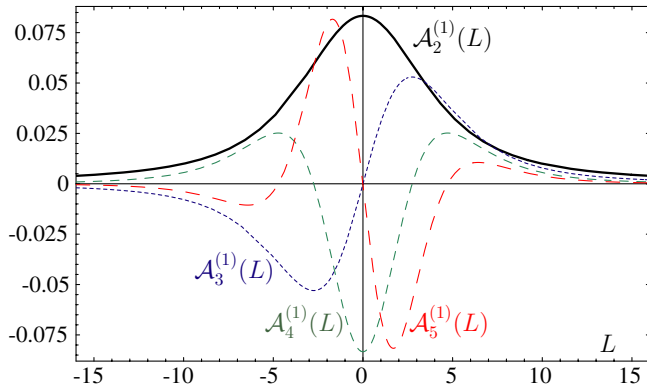


FIG. 2 (color online). Comparison of different curves for $\mathcal{A}_m^{(1)}(L)$ as functions of L , corresponding to various values of the index m , ranging from $m = 2$ to $m = 5$. To show all the details in a single plot, higher couplings are multiplied by numerical factors in order to normalize them to the scale of $\mathcal{A}_2^{(1)}(L)$. The solid line shows $\mathcal{A}_2^{(1)}(L)$, the blue dotted line corresponds to $8 \cdot \mathcal{A}_3^{(1)}(L)$, the green short-dashed line to $60 \cdot \mathcal{A}_4^{(1)}(L)$, and the red long-dashed line to $480 \cdot \mathcal{A}_5^{(1)}(L)$.

To make explicit the properties of Eq. (3.13), we convert this equation into a series representation, using Eq. (A3), to obtain

$$F(z, 1 - \nu) = \Gamma(\nu) \left[\ln\left(\frac{1}{z}\right) \right]^{-\nu} + \sum_{r=0}^{\infty} \zeta(1 - \nu - r) \frac{\ln^r(z)}{r!} \quad (3.20)$$

for $|\ln(z)| < 2\pi$, where $\zeta(\nu)$ is the Riemann ζ function. Now we are in the position to express \mathcal{A}_ν in the form of a series, i.e.,

$$\mathcal{A}_\nu(L) = -\frac{1}{\Gamma(\nu)} \sum_{r=0}^{\infty} \zeta(1 - \nu - r) \frac{(-L)^r}{r!} \quad \text{for } |L| < 2\pi \quad (3.21)$$

because the ‘‘standard logarithms,’’ contained in both parts of expression (3.13), mutually cancel, as one verifies by substituting Eq. (3.20) into Eq. (3.13). Then, we can state the following important corollaries:

- (1) $\mathcal{A}_\nu(0) = -(\zeta(1 - \nu))/\Gamma(\nu)$ is an entire function of ν . Of particular importance are the following values:

$$\begin{aligned} \mathcal{A}_1(0) &= \frac{1}{2}, & \mathcal{A}_2(0) &= \frac{1}{12}, & \mathcal{A}_3(0) &= 0, \\ \mathcal{A}_4(0) &= -\frac{1}{720}, & \mathcal{A}_5(0) &= 0 \end{aligned} \quad (3.22)$$

that coincide with the results provided in [4]. Note that $\mathcal{A}_{2n+1}(0) = 0$ for $n \geq 1$ is due to the property $\zeta(-2n) = 0$, while the set of $\mathcal{A}_{2n}(0)$ is alternating in sign [48]. These properties illustrate the details of the coupling oscillations in the vicinity of $L = 0$ for index values $m > 2$. A convenient series representa-

tion of \mathcal{A}_m for an integer index m is presented and discussed in Appendix A (item 3).

- (2) Taking into account the relation $-\lim_{\epsilon \rightarrow 0} (\zeta(1 - \epsilon - r)/\Gamma(\epsilon)) = \delta_{0r}$ [see, e.g., Eq. (A7)], one can take the limit

$$\mathcal{A}_0(L) = \lim_{\nu \rightarrow 0} \mathcal{A}_\nu(L) = 1, \quad (3.23)$$

dispensing with the constraint $|L| < 2\pi$ and proving assertion (2.8).

- (3) Equation (3.18) can be rederived from representation (3.21) in a way similar to that described in the previous item.

The upshot of these considerations is that the linear space $\{\mathcal{A}_n\}$ is now completed via the inclusion of the elements \mathcal{A}_ν for any real values of the index ν , so that one can take derivatives with respect to this continuous variable—dubbed ‘‘index derivative.’’

C. Analytic images of products of coupling powers and logarithms

To this point, we have considered only powers of the running coupling, adopting the viewpoint of the Shirkov-Solovtsov APT. Now we are going to consider more complicated expressions, like $(a_{(l)})^\nu L^m$, where the power ν is a real number and the power m is an integer, following the broader analytization principle of KS [38,39].

To compute this image, we have first to determine the image of $a^\nu \ln(a)$, which can be rewritten as the derivative of a^ν with respect to ν ; viz.,

$$a^\nu \ln(a) = \frac{d}{d\nu} a^\nu. \quad (3.24)$$

Because of the linearity of the differential operator, this derivative can be directly applied to any element of the completed space $\{\mathcal{A}_n\}$ to generate the corresponding image, $[a^\nu \ln(a)]_{\text{an}}$, and define

$$\left[\frac{d}{d\nu} a^\nu \right]_{\text{an}} \stackrel{\text{def}}{=} \frac{d}{d\nu} \mathcal{A}_\nu. \quad (3.25)$$

In the following, we shall employ for the sake of simplicity a special notation for the derivatives with respect to the index of the nonpower expansion and define

$$\mathcal{D}^k \mathcal{A}_\nu \equiv \frac{d^k}{d\nu^k} \mathcal{A}_\nu. \quad (3.26)$$

From Eqs. (3.13) and (3.25), we obtain

$$[a^\nu \ln(a)]_{\text{an}} = -\frac{\ln L}{L^\nu} - \frac{d}{d\nu} \left(\frac{F(e^{-L}, 1 - \nu)}{\Gamma(\nu)} \right) \quad (3.27)$$

and taking multiple derivatives on both sides of Eq. (3.25), we compute the image of $a^\nu L^m$, as in Eq. (3.27). This procedure applies to any desired degree m of such terms.

The extension to higher loops makes use of the APT expansion of higher-loop quantities in terms of one-loop

ones. Before doing that, we consider first the image of $a_{(2)}$ on the basis of the perturbation expansion, given in Eq. (B5) in conjunction with Eq. (3.26), to obtain the element \mathcal{A}_1 at the two-loop level:

$$\mathcal{A}_1^{(2)}(L) = \mathcal{A}_1^{(1)} + c_1 \mathcal{D} \mathcal{A}_{\nu=2}^{(1)} + c_1^2 (\mathcal{D}^2 + \mathcal{D}^1 - 1) \mathcal{A}_{\nu=3}^{(1)} + \mathcal{O}(\mathcal{D}^3 \mathcal{A}_{\nu=4}^{(1)}). \quad (3.28)$$

This formula can be readily generalized to any index ν :

$$\mathcal{A}_\nu^{(2)}(L) = \mathcal{A}_\nu^{(1)} + c_1 \nu \mathcal{D} \mathcal{A}_{\nu+1}^{(1)} + c_1^2 \nu \left[\frac{(1+\nu)}{2} \mathcal{D}^2 + \mathcal{D}^1 - 1 \right] \mathcal{A}_{\nu+2}^{(1)} + \mathcal{O}(\mathcal{D}^3 \mathcal{A}_{\nu+3}^{(1)}), \quad (3.29)$$

where $c_1 = b_1/b_0^2$ is an auxiliary expansion parameter. The quality of the two-loop approximation for the lowest index [cf. Eq. (3.28)] and higher indices [cf. Eq. (3.29)] will be analyzed numerically in the next section. Here it suffices to mention that the achieved accuracy is of the order of about 1% down to $L = 0$.

To construct the image of $(a_{(2)})^\nu L$, cf. Eq. (2.10), we first perform the analytization of Eq. (B7) and then use Eq. (3.25) to arrive at the final expression

$$\begin{aligned} [(a_{(2)}(L))^\nu L]_{\text{an}} &\equiv \mathcal{L}_{\nu,1}^{(2)}(L) \\ &= \mathcal{A}_{\nu-1}^{(2)} + c_1 \mathcal{D} \mathcal{A}_\nu^{(2)} + \mathcal{O}(\mathcal{A}_{\nu+1}^{(2)}), \end{aligned} \quad (3.30)$$

which can be recast, by means of the one-loop analytic coupling and with the aid of Eq. (3.27), in the form

$$\mathcal{L}_{\nu,1}^{(2)}(L) = \mathcal{A}_{\nu-1}^{(1)} - c_1 \nu \left[\frac{\ln(L) - \psi(\nu)}{L^\nu} + \psi(\nu) \mathcal{A}_\nu^{(1)} + \frac{\mathcal{D}F(e^{-L}, 1-\nu)}{\Gamma(\nu)} \right] + \mathcal{O}(\mathcal{D}^2 \mathcal{A}_{\nu+1}^{(1)}). \quad (3.31)$$

The analytization of $(a_{(2)}(L))^\nu \cdot L^2$, expressed in terms of Eq. (B9), can be performed in an analogous way with the result

$$\mathcal{L}_{\nu,2}^{(2)}(L) = \mathcal{A}_{\nu-2}^{(2)} + 2c_1 \mathcal{D} \mathcal{A}_{\nu-1}^{(2)} + c_1^2 \mathcal{D}^2 \mathcal{A}_\nu^{(2)} - 2c_1^2 \mathcal{A}_\nu^{(2)} + \mathcal{O}(\mathcal{D} \mathcal{A}_{\nu+1}^{(2)}). \quad (3.32)$$

The one-loop approximation of this two-loop expression is given by

$$\begin{aligned} \mathcal{L}_{\nu,2}^{(2)}(L) &= \mathcal{A}_{\nu-2}^{(1)} + c_1 \nu \mathcal{D} \mathcal{A}_{\nu-1}^{(1)} \\ &+ c_1^2 \frac{\nu^2 - \nu + 4}{2} \mathcal{D}^2 \mathcal{A}_\nu^{(1)} + \mathcal{O}(\mathcal{D} \mathcal{A}_{\nu+1}^{(1)}). \end{aligned} \quad (3.33)$$

A compilation of the required formulas to achieve the analytization of powers of the coupling in conjunction with logarithms at the two-loop order, is provided in Appendix B.

TABLE I. Comparison of the standard PT, APT, and FAPT with $L = \ln(Q^2/\Lambda^2)$.

Theory	Space	Series expansion	Inverse powers	Multiplication	Index derivative
PT	$\{a_{(l)}^\nu\}_{\nu \in \mathbb{R}}$	$F(L) = \sum_m f_m a_{(l)}^m(L)$	$(a_{(l)}(L))^{-m}$	$a_{(l)}^\mu a_{(l)}^\nu = a_{(l)}^{\mu+\nu}$	$a_{(l)}^m \ln^k a_{(l)}$
APT	$\{\mathcal{A}_m^{(l)}\}_{m \in \mathbb{N}}$	$\mathcal{F}(L) = \sum_m f_m \mathcal{A}_m^{(l)}(L)$	No	No	No
FAPT	$\{\mathcal{A}_\nu^{(l)}\}_{\nu \in \mathbb{R}}$	$\mathcal{F}(L) = \sum_m f_m \mathcal{A}_m^{(l)}(L)$	$\mathcal{A}_{-m}^{(l)}(L) = L^m$	No	$\mathcal{D}^k \mathcal{A}_m^{(l)}$

TABLE II. Computational rules for FAPT with $L = \ln(Q^2/\Lambda^2)$, $m \in \mathbb{N}$, and $\nu \in \mathbb{R}$.

Standard QCD PT	QCD FAPT
$a_{(1)}^1(L) = 1/L$	$\mathcal{A}_1^{(1)}(L) = \frac{1}{L} - (1/(e^L - 1))$
$a_{(1)}^\nu(L) = 1/L^\nu$	$\mathcal{A}_\nu^{(1)}(L) = \frac{1}{L^\nu} - [F(e^{-L}, 1-\nu)/\Gamma(\nu)]$
$a_{(l)}^\nu(L) \ln^m[a_{(l)}(L)]$	$\mathcal{D}^m \mathcal{A}_\nu^{(l)}(L) \equiv (d^m/d\nu^m)[\mathcal{A}_\nu^{(l)}(L)]$
$a_{(2)}^\nu(L)$	$\mathcal{A}_\nu^{(2)}(L) = \mathcal{A}_\nu^{(1)}(L) + c_1 \nu \mathcal{D} \mathcal{A}_{\nu+1}^{(1)}(L) + c_1^2 \nu [((\nu+1)/2) \mathcal{D}^2 + \mathcal{D}^1 - 1] \mathcal{A}_{\nu+2}^{(1)}(L) + \mathcal{O}(\mathcal{D}^3 \mathcal{A}_{\nu+3}^{(1)})^a$
$a_{(2)}^\nu(L)L$	$\mathcal{L}_{\nu,1}^{(2)}(L) = \mathcal{A}_{\nu-1}^{(2)}(L) + c_1 \mathcal{D} \mathcal{A}_\nu^{(2)}(L) + \mathcal{O}(\mathcal{D}^2 \mathcal{A}_{\nu+1}^{(2)})^b$ $\approx \mathcal{A}_{\nu-1}^{(1)}(L) - c_1 \nu [(\ln(L) - \psi(\nu))/L^\nu] + \psi(\nu) \mathcal{A}_\nu^{(1)}(L) + ((\mathcal{D}F(e^{-L}, 1-\nu))/\Gamma(\nu))$
$a_{(2)}^\nu(L)L^2$	$\mathcal{L}_{\nu,2}^{(2)}(L) = \mathcal{A}_{\nu-2}^{(2)}(L) + 2c_1 \mathcal{D} \mathcal{A}_{\nu-1}^{(2)}(L) + c_1^2 \mathcal{D}^2 \mathcal{A}_\nu^{(2)}(L) - 2c_1^2 \mathcal{A}_\nu^{(2)}(L) + \mathcal{O}(\mathcal{D} \mathcal{A}_{\nu+1}^{(2)})^b$ $\approx \mathcal{A}_{\nu-2}^{(1)}(L) + c_1 \nu \mathcal{D} \mathcal{A}_{\nu-1}^{(1)}(L) + c_1^2 [(\nu^2 - \nu + 4)/2] \mathcal{D}^2 \mathcal{A}_\nu^{(1)}(L)$
$\exp[-xa(L)]$	$e^{-x/L} + \sqrt{x} \sum_{m=1} e^{-mL} [J_1(2\sqrt{xm})/\sqrt{m}]$ for $L > 0^c$

^aNote that in evaluating this expression in the next line we use Eqs. (B6) and (3.13).

^bNote that in evaluating this expression in the next line we use Eqs. (B8), (B9), and (3.13).

^cFor the derivation of this expression, we refer to Appendix C.

Up to now we have studied expressions appearing in fixed-order perturbation theory. But similar considerations apply also to resummed perturbation theory. Indeed, first attempts to apply the analytization procedure of Shirkov-Solovtsov APT were already presented in [36,37,39]. The crucial point here is how to deal with the requirement of analyticity when performing a Sudakov resummation. Because of the nonpower series character of APT, resummation of (soft-gluon) logarithms does not lead to exponentiation. The latter can be retained only in the case of the so-called *naive* analytization [2], proposed in [36,37]. The exact expression for the Sudakov factor is too complicated and too specific to be discussed in the present analysis. We, therefore, consider in Appendix C a simplified version of a ‘‘toy Sudakov’’ factor that, nevertheless, bears the key characteristics pertaining to resummation under the assertion of analyticity. For clarity, we compare the basic ingredients of FAPT in Table I with their counterparts in conventional perturbation theory and APT. More detailed expressions are shown in Table II in the next section.

IV. VALIDATION OF THE NEW SCHEME

A. Analytic verification of the one-loop spectral density

An alternative way to derive Eq. (2.15) for the spectral density ρ_ν , is to compare two different representations for \mathcal{A}_ν : one given by the dispersion relation, Eq. (2.3), and the other provided by the Laplace representation, Eq. (3.3). Then, we get

$$\mathcal{A}_\nu(L) = \int_0^\infty \frac{\rho_\nu(\sigma)}{\sigma + Q^2} d\sigma = \int_0^\infty e^{-Lt} \tilde{\mathcal{A}}_\nu(t) dt. \quad (4.1)$$

Next, we make a double Borel transformation of both representations, the Laplace one and that of the dispersion integral, the aim being to extract $\rho_\nu(\sigma)$. This is done by applying first $M^2 \hat{B}_{(M^2 \rightarrow Q^2)}$ on both sides of Eq. (4.1) and then employing

$$\begin{aligned} M^2 \hat{B}_{(M^2 \rightarrow Q^2)} \left(\frac{1}{\sigma + Q^2} \right) &= \exp(-\sigma/M^2), \\ M^2 \hat{B}_{(M^2 \rightarrow Q^2)} \left(\frac{\Lambda^2}{Q^2} \right)^t &= \frac{M^2}{\Gamma(t)} \left(\frac{\Lambda^2}{M^2} \right)^t. \end{aligned} \quad (4.2)$$

$$\Delta_3^{\text{FAPT}}(L) = 1 - \frac{\mathcal{A}_1^{(1)}(L) + c_1 \mathcal{D} \mathcal{A}_{\nu=2}^{(1)}(L) + c_1^2 (\mathcal{D}^2 + \mathcal{D}^1 - 1) \mathcal{A}_{\nu=3}^{(1)}(L)}{\mathcal{A}_1^{(2)}(L)}. \quad (4.7)$$

For the corresponding quantities within the standard QCD perturbation theory, we use Eq. (B5) to obtain

(i) NLO, i.e., retaining terms of order c_1

$$\Delta_2^{\text{PT}}(L) = 1 - \frac{a_{(1)}(L) + c_1 a_{(1)}^2(L) \ln a_{(1)}(L)}{a_{(2)}(L)} \quad (4.8)$$

(ii) NNLO, i.e., retaining terms up to order c_1^2

In the second step, we carry out one more Borel transformation, $\hat{B}_{(1/\sigma \rightarrow 1/M^2)}$, to obtain

$$\rho_\nu(\sigma) = \int_0^\infty \left(\frac{\Lambda^2}{\sigma} \right)^t \frac{\sin(\pi t)}{\pi} \tilde{\mathcal{A}}_\nu(t) dt. \quad (4.3)$$

The final step is to substitute in Eq. (4.3) the expression for $\tilde{\mathcal{A}}_\nu(t)$, given by Eq. (3.7), to arrive at the final result

$$\begin{aligned} \rho_\nu(\sigma) &= \frac{1}{(\pi^2 + L^2(\sigma))^{\nu/2}} \sin \left[\nu \arccos \left(\frac{L(\sigma)}{\sqrt{L^2(\sigma) + \pi^2}} \right) \right] \\ &= \frac{1}{(\pi^2 + L^2(\sigma))^{\nu/2}} \sin \left[\nu \arctan \left(\frac{\pi}{L(\sigma)} \right) \right] \quad \text{for } L(\sigma) > 0, \end{aligned} \quad (4.4)$$

where $L(\sigma) = \ln(\sigma/\Lambda^2)$. To gain a more complete understanding of the role of the Landau pole remover in $\tilde{\mathcal{A}}_\nu$, it is important to remark that it does not contribute to the spectral density, the reason being that this part is not altering the nature of the discontinuity. The latter is solely determined by the term $1/L$. One appreciates that expressions (4.5) and (2.15) coincide, as they should, hence establishing the equivalence between the two alternative extensions of the analytization procedure to fractional indices. The two-loop approximate expression for the spectral density is given in Appendix B.

B. Verification of the two-loop approximations

Now look specifically at the quality of the two-loop expansion in FAPT. In doing so, we define the following quantities with the help of an auxiliary parameter c_1 and the index derivative \mathcal{D} , [as in Eq. (3.28)]:

(i) NLO, i.e., retaining terms of order c_1

$$\Delta_2^{\text{FAPT}}(L) = 1 - \frac{\mathcal{A}_1^{(1)}(L) + c_1 \mathcal{D} \mathcal{A}_{\nu=2}^{(1)}(L)}{\mathcal{A}_1^{(2)}(L)} \quad (4.6)$$

(ii) NNLO, i.e., retaining terms up to order c_1^2

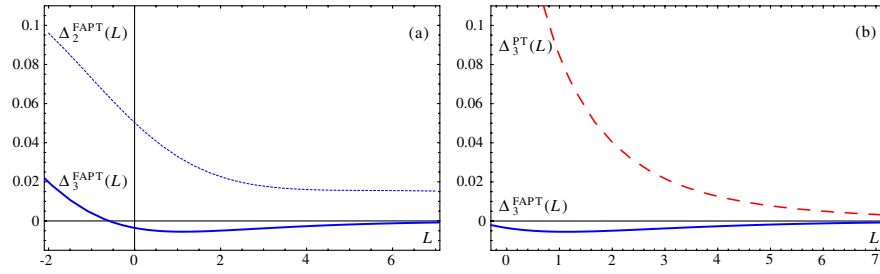


FIG. 3 (color online). (a) The dotted line corresponds to $\Delta_2^{\text{FAPT}}(L)$ and the solid line to $\Delta_3^{\text{FAPT}}(L)$. (b) The dashed line corresponds to $\Delta_3^{\text{PT}}(L)$ and the solid line to $\Delta_3^{\text{FAPT}}(L)$.

$$\Delta_3^{\text{PT}}(L) = 1 - \frac{a_{(1)}(L) + c_1 a_{(1)}^2(L) \ln a_{(1)}(L) + c_1^2 a_{(1)}^3(L) (\ln^2 a_{(1)}(L) + \ln a_{(1)}(L) - 1)}{a_{(2)}(L)}. \quad (4.9)$$

First, let us compare the transition from the NLO [cf. Eq. (4.6)] to the NNLO [cf. Eq. (4.7)] in FAPT [see Fig. 3(a)]. One appreciates that by taking into account the NNLO terms, a significant improvement of the convergence quality of the FAPT series is achieved. Indeed, even at $Q^2 = \Lambda^2$, which corresponds to $L = 0$, the error of truncating the FAPT series at the NLO is about 5%, while by taking into account the NNLO correction this error becomes even smaller than 0.5%. In Fig. 3(b) we show the relative quality of these approximations concerning the loop expansion between the standard perturbation theory and FAPT, as quantified by Eqs. (4.7) and (4.9). One appreciates the strong suppression of $\Delta_3^{\text{FAPT}}(L)$ relative to its conventional analogue in the small L region, say, below approximately $L = 2$.

The same comparison can be realized for $\mathcal{A}_2^{(2)}$, using Eq. (3.29). Indeed, we demonstrate in Fig. 4(a) the quality of this FAPT expansion in comparison with the results of the numerical integration of the NLO spectral density ρ_2 (for more details, we refer to Appendix B and [5]) in the dispersion-integral representation, provided by Eq. (2.3). In this graphics, we also display the results obtained numerically by Magradze in [23]. The message from Fig. 4(a)

is quite clear. Our analytic (solid line) and our numerical calculation (dashed line) are in mutual support, while the results of [23] differ considerably with respect to both the magnitude and the trend of the negative values of L . The good convergence of the proposed series for $\mathcal{A}_{1,2}^{(2)}$ [Eq. (3.29)], that had been demonstrated above, can be traced to the basis of APT. Indeed, this nonpower expansion of the quantities $\mathcal{A}_{1,2}^{(2)}$ in terms of $\mathcal{A}_m^{(1)}$ has a finite radius of convergence, the reasons being discussed in Ref. [50].

To give the reader an impression of the dependence on L of $\mathcal{A}_\nu^{(2)}(L)$, we show in Fig. 4(b) a comparison of this quantity with its counterpart in standard QCD, namely, $[a_{(2)}(L)]^\nu$. For the purpose of illustration, we select the value $\nu = \gamma_2/(2b_0) \approx 0.62$, which corresponds to the 1-loop evolution exponent of the nonsinglet quark operator of index 2, entering a number of applications in DIS and also various exclusive reactions [43,51,52].

In support of our two-loop approximation (within FAPT), we display in Fig. 5 results for the analytic images $\mathcal{A}_\nu^{(2)}$ with $\nu = k/4$ and $k = 0, 1, \dots, 8$. We observe the same monotonic pattern, i.e., no crossing, of curves, found

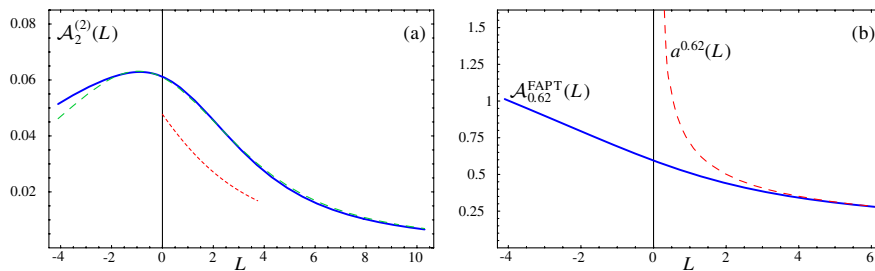


FIG. 4 (color online). (a) Comparison of different results for $\mathcal{A}_2^{(2)}(L)$. The solid line corresponds to $\mathcal{A}_2^{(2);\text{FAPT}}(L)$, computed analytically via Eq. (3.29); $\mathcal{A}_2^{(2);\text{num}}(L)$ (dashed line) is derived by means of a numerical integration. The dotted line represents the available results of the numerical procedure of Magradze in [23]. (b) Comparison of FAPT and standard QCD PT with respect to the fractional index (power) of the coupling. The solid line represents $\mathcal{A}_{0.62}^{(2);\text{FAPT}}(L)$, computed analytically via Eq. (3.29), while the dashed line stands for $a_{(2)}^{0.62}(L)$.

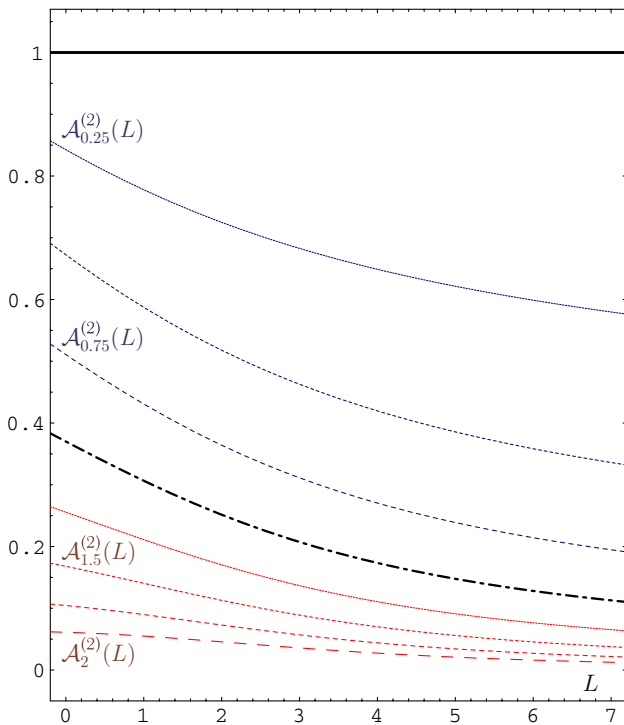


FIG. 5 (color online). Analytic images $\mathcal{A}_\nu^{(2)}(L)$ at the two-loop level for some characteristic index values, calculated with NNLO FAPT. The solid line corresponds to $\nu = 0$, the dash-dotted one to $\nu = 1$, and the lowest line to $\nu = 2$. All other lines correspond to $\nu = k/4$ with $k = 1, \dots, 7$.

already for the one-loop case, shown in Fig. 1(b). We investigated numerically the range of negative values of L and found that crossing appears only for indices $\nu < 1$, again in close analogy to the one-loop case.

The pivotal results of this paper are collected in Table II, where we provide the reader with explicit calculational rules to connect the standard QCD perturbation theory with FAPT. We stress that the presented algorithm has broad applications in phenomenology and can play a major role in the perturbative analysis of observables both in inclusive and exclusive QCD reactions.

V. CONCLUSIONS

With hindsight we can say that the requirement of analyticity at the amplitude level of hadronic quantities in QCD, expressed by Karanikas and Stefanis [38], is instrumental in improving perturbation-theory calculations. First, as shown in an accompanying paper by two of us (A. P. B. and N. G. S.) together with Karanikas [40], it enables one to minimize both the sensitivity on the renormalization scheme and scale setting and also the dependence on the factorization scale. The reason for this latter advantage is that it includes into the analytization procedure not only the powers of the running coupling, but also logarithms (or exponentials) that may contain the momen-

tum scale with respect to which analyticity is required. Second, starting at this point, we have shown in the present work that invoking this analyticity principle gives rise to a generalization of the original APT to fractional powers of the coupling. As a bonus, this approach improves the convergence of the perturbative expansion significantly—see Fig. 3.

Our main goal in this analysis was to work out in mathematical detail the procedure for determining analytic expressions for any real power of the running coupling and delineate the main results. To keep our presentation as general as possible, we have purposefully refrained from considering specific examples and concentrated instead on generic features and expressions. In this vein, we have discussed products of the running coupling (or its powers) with (powers of) logarithms, which are typical for contributions encountered in higher-order corrections of QCD perturbation theory, or when taking into account evolution effects via the renormalization-group equation. Similar logarithmic terms also appear in calculations employing light-cone sum rules [42,43]. In an analogous way, we have discussed the resummation of nonpower series in the analytic images in order to capture the key features of Sudakov resummation of soft-gluon effects (see Appendix C). All these elements of FATP, required for further applications of this formalism to improve the calculation of any hadronic amplitude at the two-loop level, are collected in Table II. In this context, we mention that we have developed approximate expressions for the two-loop analytic images in terms of one-loop quantities that can facilitate practical computations significantly.

In conclusion, this report has emphasized rigorous methods rather than specific applications. A first example of the present framework is discussed in [40], focusing on the topic of the renormalization-scheme and factorization-scale independence of the electromagnetic pion form factor, relative to its treatment within the standard QCD perturbation theory or original APT. The methods presented here are intended to be used in the low-to-medium momentum range where the standard perturbative approach faces the problem of the Landau pole and in processes or under circumstances where the original APT is insufficient because it is tied to integer powers of the coupling. We believe that our assortment of analytic expressions for a variety of expressions ranging from any real powers of the coupling to more complicated products containing logarithms, provides sufficient evidence for the usefulness of the approach for higher-order perturbative calculations.

ACKNOWLEDGMENTS

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APPENDIX A: ANALYTIC PROPERTIES OF $\Phi(z, -\nu, 1)$ AND \mathcal{A}_ν

1. The function $\Phi(z, -\nu, i)$ can be determined by means of the analytic continuation of the series [48]

$$\Phi(z, s, i) = \sum_{m=0}^{\infty} \frac{z^m}{(m+i)^s} \quad \text{for } |z| < 1, s \neq 1, 2, 3, \dots \quad (\text{A1})$$

in both variables z and s . This analytic continuation for every fixed s , which is not a positive integer, determines Φ as an analytic function of z , regular in the plane with a cut along the axis $(1, \infty)$, and for every fixed z in the cut plane as an analytic function of s being regular, except possibly at the points $s = 1, 2, 3, \dots$, as was mentioned in [48]. We can improve this statement for $s \geq 1$ using Eq. (3.19) to obtain

$$\begin{aligned} \Phi(z, s, i) &= \frac{1}{z^i} \left[z\Phi(z, s, 1) - \sum_{m \geq 1}^{i-1} \frac{z^m}{m^s} \right] \\ &= \frac{1}{z^i} \left[\text{Li}_s(z) - \sum_{m \geq 1}^{i-1} \frac{z^m}{m^s} \right]. \end{aligned} \quad (\text{A2})$$

One appreciates that there are no singularities for any positive integer values of s . Hence, we can conclude that $\Phi(z, s, 1)$ is an analytic function in s at any fixed z on the cut plane. Moreover, the function $\mathcal{A}_\nu(L)$, see expression (3.11),

$$\mathcal{A}_\nu(L) = \frac{1}{L^\nu} - \frac{e^{-L}\Phi(e^{-L}, 1-\nu, 1)}{\Gamma(\nu)},$$

has no poles in ν and is, therefore, an entire function in ν .

2. There is a useful series representation for $\Phi(z, s, 1)$ [48] [cf. (3.20)]; viz.,

$$\Phi(z, s, 1) = \frac{1}{z} \left[\Gamma(1-s) \left[\ln\left(\frac{1}{z}\right) \right]^{s-1} + \sum_{r=0}^{\infty} \zeta(s-r) \frac{\ln^r(z)}{r!} \right] \quad (\text{A3})$$

for $|\ln(z)| < 2\pi, s \neq 1, 2, 3, \dots$ that allows one to continue $\Phi(z, s, 1)$ for integer positive $s = m$ values by means of the limit $s = m + \varepsilon, \varepsilon \rightarrow 0$ in Eq. (A3). To take this limit, one should expand in ε the first term in the square brackets in expression (A3), which is proportional to $\Gamma(1-m-\varepsilon)$. The other singular term appears in the sum and is proportional to $\zeta(1+\varepsilon) = 1/\varepsilon - \psi(1) + \mathcal{O}(\varepsilon)$. The singularities, contained in both these parts, mutually cancel.

3. The expansion of $\mathcal{A}_\nu(L)$, Eq. (3.21), is simplified for an integer index $\nu = m \geq 1$ to read

$$\mathcal{A}_m(L) = \frac{1}{(m-1)!} \sum_{r=0}^{\infty} \frac{B_{m+r}}{(m+r)r!} (-L)^r \quad \text{for } |L| < 2\pi, \quad (\text{A4})$$

where B_m are the Bernoulli numbers. From the property $B_{2n+1} = 0$ it follows that $\mathcal{A}_{2n}(L)$ is an even function of its argument, while $\mathcal{A}_{2n+1}(L)$ is an odd one. Note here that the pole remover $F(e^{-L}, 1-m)$ in expression (3.13) reduces to elementary functions for the case of an integer index. Indeed, according to Eq. (3.1), the operator $-d/dL$ shifts the second argument of the function F by unity, i.e., $m \rightarrow m+1$, to get

$$-\frac{d}{dL} F(e^{-L}, -m) = F(e^{-L}, -(m+1)).$$

Taking into account that $F(e^{-L}, 0) = (e^L - 1)^{-1}$ and applying the previous relation m times we arrive at

$$F(e^{-L}, -m) = \left(-\frac{d}{dL}\right)^m F(e^{-L}, 0) = \left(-\frac{d}{dL}\right)^m \frac{1}{e^L - 1}. \quad (\text{A5})$$

This representation leads to an exponentially suppressed asymptotic limit for the function $F(e^{-L}, -m)$; viz.,

$$F(e^{-L}, -m)|_{L \rightarrow \pm\infty} \sim e^{-|L|}. \quad (\text{A6})$$

4. We supply here the Lindelöf formula [53]

$$\zeta(\nu) = \frac{1}{2} + \frac{1}{\nu-1} + \int_0^\infty \frac{\sin[\nu \arctan(t)] dt}{(1+t^2)^{\nu/2} (e^{2\pi t} - 1)}, \quad (\text{A7})$$

which fixes $\zeta(\nu)$ as an analytic function with a simple pole at $\nu = 1$. This representation for $\zeta(\nu)$ has been used in Sec. III B.

5. Now we are in the position to supply also the analytic images of the coupling in the timelike regime for $L(s) \equiv \log(s/\Lambda^2) \geq 0$, employing the notation of [6,54]:

$$\begin{aligned} \mathfrak{A}_\nu(s) &= \int_s^\infty \frac{d\sigma}{\sigma} \rho_\nu(\sigma) \\ &= \frac{1}{\pi} \int_{L(s)}^\infty dL \frac{\sin[\nu \arctan(\pi/L)]}{(\pi^2 + L^2)^{\nu/2}}. \end{aligned} \quad (\text{A8})$$

This integral can be evaluated to provide a result analogous to $\mathcal{A}_\nu(L)$ for the spacelike regime; namely,

$$\mathfrak{A}_\nu(s) = \frac{\sin[(\nu-1) \arctan(\pi/L(s))]}{\pi(\nu-1)(\pi^2 + L(s)^2)^{(\nu-1)/2}}. \quad (\text{A9})$$

A similar expression for the timelike coupling has been obtained before in [55] using the ‘‘contour-improved resummation technique.’’

APPENDIX B: ANALYTIZATION OF POWERS OF THE COUPLING MULTIPLIED BY LOGARITHMS

1. The expansion of the β function in the NLO approximation is given by

$$\frac{d}{dL} \left(\frac{\alpha_s(L)}{4\pi} \right) = -b_0 \left(\frac{\alpha_s(L)}{4\pi} \right)^2 - b_1 \left(\frac{\alpha_s(L)}{4\pi} \right)^3, \quad (\text{B1})$$

where $L = \ln(\mu^2/\Lambda^2)$ and

$$\begin{aligned} b_0 &= \frac{11}{3} C_A - \frac{4}{3} T_R N_f, \\ b_1 &= \frac{34}{3} C_A^2 - \left(4C_F + \frac{20}{3} C_A \right) T_R N_f \end{aligned} \quad (\text{B2})$$

with $C_F = (N_c^2 - 1)/2N_c = 4/3$, $C_A = N_c = 3$, $T_R = 1/2$, and N_f denoting the number of flavors. Then, the corresponding two-loop equation for our coupling $a = b_0 \alpha / (4\pi)$ looks like

$$\frac{da_{(2)}}{dL} = -a_{(2)}^2 [1 + c_1 a_{(2)}] \quad \text{with } c_1 \equiv \frac{b_1}{b_0^2}. \quad (\text{B3})$$

The renormalization-group solution of this equation assumes the form

$$\frac{1}{a_{(2)}} + c_1 \ln \left[\frac{a_{(2)}}{1 + c_1 a_{(2)}} \right] = L. \quad (\text{B4})$$

Then, for the expansion of $a_{(2)}(L)$ in terms of $a_{(1)}(L) = 1/L$ we have, retaining terms of the order $a_{(1)}^3$,

$$\begin{aligned} a_{(2)} &= a_{(1)} + c_1 a_{(1)}^2 \ln a_{(1)} + c_1^2 a_{(1)}^3 (\ln^2 a_{(1)} + \ln a_{(1)} - 1) \\ &\quad + \mathcal{O}(a_{(1)}^4 \ln^3 a_{(1)}). \end{aligned} \quad (\text{B5})$$

2. Now, for the product $[a_{(2)}]^\nu L$, we obtain from (B4)

$$(a_{(2)})^\nu L = (a_{(2)})^{\nu-1} + (a_{(2)})^\nu c_1 \ln \left[\frac{a_{(2)}}{1 + c_1 a_{(2)}} \right]. \quad (\text{B6})$$

Expanding the logarithmic term $\ln[1 + c_1 a_{(2)}]$, while retaining terms of order $a_{(2)}^{\nu-1}$, $a_{(2)}^\nu \ln(a_{(2)})$; viz.,

$$(a_{(2)})^\nu L = (a_{(2)})^{\nu-1} + c_1 (a_{(2)})^\nu \ln(a_{(2)}) - \mathcal{O}(a_{(2)}^{\nu+1}) \quad (\text{B7})$$

and, finally, expanding the coupling $a_{(2)}$ in terms $a = a_{(1)}$, we find

$$(a_{(2)})^\nu L = a^{\nu-1} + \nu a^\nu c_1 \ln(a) + \mathcal{O}(a^{\nu+1} \ln^2(a)). \quad (\text{B8})$$

Calculating $(a_{(2)})^\nu L^2$ in an analogous way, we derive

$$\begin{aligned} (a_{(2)})^\nu L^2 &= (a_{(2)})^{\nu-2} [1 + c_1 a_{(2)} \ln(a_{(2)})]^2 - 2c_1^2 a_{(2)}^\nu \\ &\quad - \mathcal{O}(a_{(2)}^{\nu+1} \ln(a_{(2)})) \\ &= a^{\nu-2} + \nu a^{\nu-1} c_1 \ln(a) \\ &\quad + \left(\frac{\nu^2 - \nu + 4}{2} \right) a^\nu c_1^2 \ln^2(a) + \mathcal{O}(a^\nu \ln(a)). \end{aligned} \quad (\text{B9})$$

3. We consider here the spectral density $\rho_\nu(\sigma)$ beyond the leading-order approximation. At the l -loop level, $\rho_\nu^{(l)}(\sigma)$ can always be presented in the same form as for the leading order, given in Eq. (2.15), i.e.

$$\rho_\nu^{(l)}(\sigma) = \frac{1}{\pi} \text{Im}[a_{(l)}^\nu(-\sigma)] = \frac{1}{\pi} \frac{\sin[\nu \varphi_{(l)}(\sigma)]}{(R_{(l)}(\sigma))^\nu}, \quad (\text{B10})$$

but keeping in mind that the phase $\varphi_{(l)}$ and the radial part $R_{(l)}$ have a multiloop content. At the two-loop level, one should, strictly speaking, deal with the imaginary part of the Lambert function W_{-1} (see [20]) because the exact solution of Eq. (B4) can be realized in terms of the Lambert function. Instead of following this procedure, we can alternatively take the well-known first-iteration solution of Eq. (B4) that provides us with sufficient accuracy the following result:

$$\frac{1}{a_{(2)}(L)} \rightarrow \frac{1}{a_{(2)}^{\text{iter}}(L)} = L + c_1 \ln(L + c_1). \quad (\text{B11})$$

For this approximate solution $a_{(2)}^{\text{iter}}$, we have

$$\begin{aligned} (R_{(2)}(\sigma))^2 &= (L(\sigma) + c_1 \ln(\sqrt{(L(\sigma) + c_1)^2 + \pi^2}))^2 \\ &\quad + (\pi + c_1 \phi(L(\sigma)))^2, \end{aligned} \quad (\text{B12})$$

$$\varphi_{(2)}(\sigma) = \pi + c_1 \phi(L(\sigma)), \quad (\text{B13})$$

$$\phi(L(\sigma)) = \arccos \left[\frac{L(\sigma) + c_1}{\sqrt{(L(\sigma) + c_1)^2 + \pi^2}} \right], \quad (\text{B14})$$

with $L(\sigma) = \ln(\sigma/\Lambda^2)$. The spectral density $\rho_{\nu=1}^{(2)\text{-iter}}(\sigma)$ with the phase and the radial part from Eqs. (B12)–(B14) appears to be very close to the numerical, but exact result for $\rho_1^{(2)}(\sigma)$, based on W_{-1} —see, e.g., [5].

APPENDIX C: ANALYTIZATION OF THE TOY MODEL FOR SUDAKOV RESUMMATION

Here we discuss the analytic image of expression $F_S(x, L) \equiv \exp[-x a_{(l)}(L)]$, which originates as a part of the procedure of the Sudakov resummation, where x is a free parameter. We consider the following example, serving as a “toy model” for this resummation:

$$\begin{aligned} \{F_S(x, L)\}_{\text{an}} &\equiv \left\{ 1 + \sum_{m=1} \frac{[-x a_{(l)}(L)]^m}{m!} \right\}_{\text{an}} \\ &= 1 + \sum_{m=1} (-x)^m \frac{\mathcal{A}_m^{(l)}(L)}{m!}. \end{aligned} \quad (\text{C1})$$

One can verify that for the asymptotic limits of L , Eq. (C1) reduces to the evident forms:

$$1 + \sum_{m=1} (-x)^m \frac{\mathcal{A}_m^{(l)}(L)}{m!} = \begin{cases} 1 - x & \text{for } L \rightarrow -\infty, \\ 1 & \text{for } L \rightarrow +\infty. \end{cases} \quad (\text{C2})$$

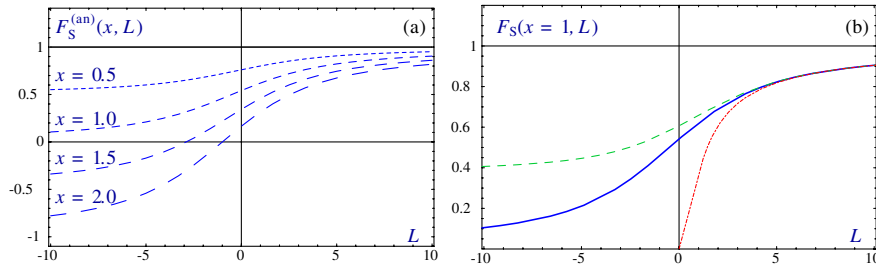


FIG. 6 (color online). (a) Functions $F_S^{(\text{an})}(x, L)$ calculated for different values of x using the one-loop FAPT. The horizontal solid line corresponds to $x = 0$, while the broken lines represent the specific x values shown above them. (b) Comparison of the toy Sudakov function, calculated within the present framework (solid line), with the result of the *naive* analytization [2,36,37] (dashed line), and that from conventional QCD perturbation theory (dotted line). Note the different scale for the ordinate relative to panel (a).

The first asymptote on the rhs of Eq. (C2) appears due to the equality $\mathcal{A}_m^{(l)}(-\infty) = \delta_{1m}$, see, for instance, [4]. The second asymptote is due to the property that in the UV regime APT reduces to the standard perturbation theory. Both properties are illustrated in Fig. 6(a). In Fig. 6(b) we compare different versions of the toy Sudakov function, obtained within the present framework (solid line), using the *naive* analytization [2,36,37] (dashed line), and conventional QCD perturbation theory (dotted line).

On the other hand, restricting the loop order to $l = 1$, one can derive an explicit expression for Eq. (C1) in the region of $L > 0$, which is based on the Laplace represen-

tation given by Eq. (3.3) and expression (3.8) for $\tilde{\mathcal{A}}_1$, namely,

$$F_S^{(\text{an})}(x, L) = e^{-x/L} + \sqrt{x} \sum_{m=1} e^{-Lm} \frac{J_1(2\sqrt{xm})}{\sqrt{m}}, \quad L > 0. \quad (\text{C3})$$

The perturbative part of $\tilde{\mathcal{A}}_m$ in (C1) reproduces exactly the asymptotic expression $\exp[-xa_{(1)}(L)]$ on the rhs of Eq. (C3), while the pole remover generates the sum of the exponents in $-L$, weighted by the Bessel functions, J_1 , exhibiting how the large L behavior is violated.

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