# Improved measurement of $2\beta + \gamma$

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We propose to measure the Cabibbo-Kobayashi-Maskawa parameter $2\beta + \gamma$ using $B^0$ decays involving several intermediate states, and describe a general formalism that applies to a broad class of decays. The main advantage of this method is that the ratios between the interfering amplitudes can be measured without requiring external input. In addition, discrete ambiguities are resolved.

## I. INTRODUCTION

*CP* violation is one of the most important topics in current particle physics research. In the standard model, *CP* violation arises due to a single complex phase in the Cabibbo-Kobayashi-Maskawa (CKM) matrix $V$ [1]. A major goal of $B$ meson physics is to measure the angles and sides of the CKM unitarity triangle. Theoretically clean measurement methods are crucial for obtaining these parameters accurately. The *BABAR* [2] and Belle [3] measurements of the parameter $\sin(2\beta)$, where $\beta = \arg(-V_{cd}V_{cb}^*/V_{td}V_{tb}^*)$, confirm the standard model to within the precision of the experiments, and increased precision is expected in the coming years.

Crucial studies of the CKM mechanism and constraints on new physics can be obtained by measuring the CKM angle $\gamma = \arg(-V_{ud}V_{ub}^*/V_{cd}V_{cb}^*)$. The greatest challenges presented by these measurements is that they require very large data samples and are subject to discrete ambiguities. It is therefore important to use every possible mode and method for measuring $\gamma$, and to devise methods that help resolve the ambiguities.

An important class of measurements makes use of decays such as $B \to D^-\pi^+$ to measure $2\beta + \gamma$. Proposed initially by Dunietz [4], the first attempts to measure time-dependent *CP* asymmetries proportional to $\sin(2\beta + \gamma)$ and $\cos(2\beta + \gamma)$ have been conducted by *BABAR* [5] and Belle [6] using the modes $B \to D^{(*)-}\pi^+$ and $B \to D^-\rho^+$. While these measurements are currently statistically limited, their precision will become significant as more data are accumulated [7]. At that stage, the greatest difficulty in extracting $2\beta + \gamma$ from these results will be the lack of precise knowledge of the ratio between the interfering amplitudes, defined as $r \equiv |A(\bar{B}^0 \to D^{(*)-}h^+)/A(B^0 \to D^{(*)-}h^+)|$, where $h^+$ indicates the light hadron $\pi^+$ or $\rho^+$.

In principle, $r$ may be obtained from the difference between the magnitudes of two terms with different time dependences in the decay rate. These terms are $(1 + r^2)$ and $(1 - r^2)\cos(\Delta mt)$, where $\Delta m$ is the $B^0 - \bar{B}^0$ oscillation frequency. However, with $r \sim \mathcal{O}(1 - 2\%)$, extracting it from the $\mathcal{O}(r^2)$ difference between these $\mathcal{O}(1)$ terms

requires prohibitively large data sets and an excellent understanding of the $B$ flavor mistag rates. Thus, the time-dependent measurement has negligible sensitivity to the value of $r$, which must therefore be obtained by assuming factorization and SU(3) symmetry to make use of the ratio of branching fractions $\mathcal{B}(\bar{B}^0 \to D_s^{(*)-}h^+)/\mathcal{B}(B^0 \to D^{(*)-}h^+)$. This approximation ignores the contribution of annihilation diagrams and some SU(3) breaking effects, and is taken to have a theoretical error of roughly 30% [5].

In $B \to D^{*-}\rho^+$, the single parameter $r$ is replaced by a matrix $\rho_{mn}$ of ratios between the magnitudes of the $b \to u\bar{c}d$ and $b \to c\bar{u}d$ contributions of the of three different helicity amplitudes contributing to the decay. It has been shown [8] that $\rho_{mn}$ may be obtained using only first-order $\mathcal{O}(\rho_{mn})$ terms. Not having to rely on small second-order terms or external input regarding amplitude ratios, this provides a much improved, theoretically clean measurement of $2\beta + \gamma$.

In this paper we generalize and extend that method to other decays that proceed through more than one intermediate state. Examples include $B \to D^-\rho^+$, which can interfere with $B \to D^-\rho^+(1450)$ and nonresonant $B \to D^-\pi^+\pi^0$; $B \to D^-a_1^+$, where nonresonant contributions are expected under the $a_1$ peak; and $B \to D^{**-}\pi^+$, where interference between several excited charmed mesons may be realized in the decays $D^{**-} \to D\pi$ and $D^{**-} \to D^*\pi$, in addition to possible contributions from nonresonant decays.

In all these cases, the interfering contributions have overlapping yet different distributions in relevant analysis variables. The first of these variables is the invariant mass squared $s$ of the final state of the resonance. The second variable $s'$ typically describes an angular distribution that is fully determined by the spin of the resonance. In the case of $B \to D^-a_1^+$, $s'$ corresponds to the two variables of the Dalitz plot of the $a_1$ decay.

Our method applies equally well to modes with higher excitations, such as $B \to D^{*-}\rho^+$, $B \to D^{*-}a_1^+$, $B \to D^{**-}\rho^+$, and $B \to D^{**-}a_1^+$, in which $s'$ corresponds to several angular and mass-related variables. In addition to

the interference between several resonances and nonresonant contributions, these decays involve several helicity amplitudes, which are treated as different intermediate states in our method.

## II. MEASURING $2\beta + \gamma$

Let us consider a decay of the type described above, involving the interference of $N$ intermediate states. We denote the final state by $f$ ($\bar{f}$) if it contains a $\bar{c}$ ($c$) quark. The four decay amplitudes of interest are

$$A(B^0 \to f) = A(\bar{B}^0 \to \bar{f}) = \sum_{m=1}^{N} A_m g_m(s, s') e^{i\Delta_m},$$

$$A(B^0 \to \bar{f}) = \sum_{m=1}^{N} a_m g_m(s, s') e^{i(\delta_m + \gamma)}, \tag{1}$$

$$A(\bar{B}^0 \to f) = \sum_{m=1}^{N} a_m g_m(s, s') e^{i(\delta_m - \gamma)},$$

where $\Delta_m$ ($\delta_m$) is the $CP$-conserving phase and $A_m$ ($a_m$) is the magnitudes of the $b \to c\bar{u}d$ ($b \to u\bar{c}d$) decay amplitude proceeding via intermediate state $m$, and $g_m(s, s')$ is a known function of the final state variables $s$ and $s'$ that depends on the nature of the intermediate state $m$. For example, for $f = D^- \pi^+ \pi^0$ and $m$ being the index of the $D^- \rho^+$ intermediate state, $g_m(s, s') = R(s)s'$, where $s$ is the square of the $\pi\pi^0$ invariant mass, $R(s)$ is a Breit-Wigner function, and $s'$ is the cosine of the angle between the momenta of the $B$ and of one of the pions, calculated in the $\pi^+ \pi^0$ rest frame (the "helicity" angle). Vector-vector intermediate states, such as $D^{*-}\rho^+$, must be further divided into the different helicity amplitude, each of which has a different $s'$ dependence.

With the above equations, the time-dependent decay rates for $B^0(t) \to f$ and $B^0(t) \to \bar{f}$ become

$$\Gamma(B^0(t) \to f) = e^{-\Gamma t} \sum_{m,n} [I_{mn} + C_{mn} \cos(\Delta m t)$$
$$- S_{mn}^- \sin(\Delta m t)], \tag{2a}$$

$$\Gamma(B^0(t) \to \bar{f}) = e^{-\Gamma t} \sum_{m,n} [I_{mn} - C_{mn} \cos(\Delta m t)$$
$$- S_{mn}^+ \sin(\Delta m t)], \tag{2b}$$

where for convenience we define the symbols

$$I_{mn} \equiv \frac{1}{2}\{g_m g_n^*(A_m A_n e^{-i(\Delta_n - \Delta_m)} + a_m a_n e^{-i(\delta_n - \delta_m)})\},$$

$$C_{mn} \equiv \frac{1}{2}\{g_m g_n^*(A_m A_n e^{-i(\Delta_n - \Delta_m)} - a_m a_n e^{-i(\delta_n - \delta_m)})\},$$

$$S_{mn}^- \equiv \text{Im}\{g_m g_n^* A_n a_m e^{i(\delta_m - \Delta_n)} e^{i\phi}\}, \tag{3}$$

$$S_{mn}^+ \equiv \text{Im}\{g_m g_n^* A_m a_n e^{-i(\delta_n - \Delta_m)} e^{i\phi}\},$$

$$\phi \equiv -(2\beta + \gamma).$$

The decay rates for $\bar{B}^0$ decays are obtained from the $B^0$

rates by inverting the sign of the $\cos(\Delta m t)$ and $\sin(\Delta m t)$ terms. They double the statistics but do not yield additional information.

Next, we determine the conditions under which all the unknown parameters of Eqs. (2) can be obtained from the measurement, and show that these conditions are satisfied in the typical case of interfering Breit-Wigner resonances and a possible nonresonant contribution.

The three terms of Eq. (2a) are distinguishable based on their different time dependences, thus determining their coefficients. The relative differences between $I_{mn}$ and $C_{mn}$ are of order $(a_m a_n)/(A_m A_n) \sim r^2 \sim 10^{-4}$, which is practically unobservable. As a result, these terms yield the parameters $A_m$ and $\Delta_m$, while $a_m$ and $\delta_m$ are measured from the coefficients of the $\sin(\Delta m t)$ terms, as described later. To study the conditions for obtaining $A_m$ and $\Delta_m$, we expand

$$\sum_{m,n} A_m A_n \{g_m g_n^* e^{-i(\Delta_n - \Delta_m)}\}$$

$$= \sum_m |g_m|^2 A_m^2 + 2\sum_{m<n} \text{Re}(g_m g_n^*) A_m A_n \cos(\Delta_n - \Delta_m)$$

$$+ 2\sum_{m<n} \text{Im}(g_m g_n^*) A_m A_n \sin(\Delta_n - \Delta_m). \tag{4}$$

If $|g_m|^2$, $\text{Re}(g_m g_n^*)$, and $\text{Im}(g_m g_n^*)$ all have different $s$ and/or $s'$ dependences, Eq. (4) yields $N^2$ unique observables, which is more than enough to determine the $2N - 1$ unknowns $A_m$ and $\Delta_m$ (one of the $\Delta_m$ phases is a global phase and can be chosen arbitrarily) for $N \geq 2$. This uniqueness condition is satisfied when all the $g_m$ are Breit-Wigner functions,

$$g_m(s) = \frac{M_m \Gamma_m}{s - M_m^2 + iM_m \Gamma_m}, \tag{5}$$

even when all contributions have the same $s'$ dependence. A nonresonant contribution $g_1 = 1$ introduces $N - 1$ degenerate relations:

$$\text{Im}(g_1 g_m^*) = |g_m|^2, \tag{6}$$

where $g_m(m > 1)$ is a Breit-Wigner function. In this case, the number of observables is reduced to $N^2 - (N - 1)$. However, a solution still exists for $N \geq 2$, and this solution is unambiguous when the nonresonant contribution is small enough relative to the resonant contributions. In addition, most practical cases involve resonances with total spins different from 0, and hence $s'$ dependences that distinguish them from a nonresonant $s$-wave contribution. This guarantees a unique solution of Eq. (4) in terms of $A_m$ and $\Delta_m$.

We note that these conclusions do not depend on the assumption that the $a_m a_n$ terms in $I_{mn}$ and $C_{mn}$ are negligible. In fact, they apply equally well to the $a_m a_n \{g_m g_n^* e^{-i(\delta_n - \delta_m)}\}$ terms in Eq. (3).

We now show how the coefficients of the $\sin(\Delta mt)$ terms in Eq. (2) yield the values of the remaining $2N + 1$ unknowns, namely, $a_m$, $\delta_m$, and $\phi$. The coefficients are

$$\sum_{mn} S_{mn}^{\mp} = \sum_m A_m a_m |g_m|^2 \sin(\phi \pm \delta_{mm}) + \sum_{m<n} \text{Im}(g_m g_n^*)$$
$$\times [\mp A_m a_n \cos(\phi \pm \delta_{nm}) \pm A_n a_m \cos(\phi \pm \delta_{mn})]$$
$$+ \sum_{m<n} \text{Re}(g_m g_n^*)[A_m a_n \sin(\phi \pm \delta_{nm})$$
$$+ A_n a_m \sin(\phi \pm \delta_{mn})] \qquad (7)$$

where $\delta_{nm} \equiv \delta_n - \Delta_m$. If $|g_m|^2$, $\text{Re}(g_m g_n^*)$, and $\text{Im}(g_m g_n^*)$ are all different, Eq. (7) yields $N^2$ observables for $S_{mn}^-$ and $N^2$ for $S_{mn}^+$. It is therefore possible to obtain all the unknowns for $N \geq 2$.

## III. DISCRETE AMBIGUITIES

In the $N = 1$ case, only the first term in Eq. (7) is nonvanishing. The measurement of $\phi$ then suffers from an eight-fold ambiguity, due to the invariance of the observable $\sin(\phi \pm \delta_{mn})$ under the three symmetry operations [9]:

$$S_{\pi/2} \equiv \phi \to \delta_{mm} + \pi/2, \quad \delta_{mm} \to \phi - \pi/2,$$
$$S'_{\pi} \equiv \phi \to \delta_{mm} + \pi, \quad \delta_{mm} \to \phi + \pi, \qquad (8)$$
$$S'_{\pm} \equiv \phi \to \pi - \phi, \quad \delta_{mm} \to -\delta_{mm}.$$

In the typical $N > 1$ case, $S_{\pi/2}$ and $S'_{\pm}$ are no longer good symmetries, since they are broken by the $\cos(\phi \pm \delta_{mn})$ terms, which are distinguishable from the $\sin(\phi \pm \delta_{mn})$ terms by virtue of the different $s$ and/or $s'$ dependences of their coefficients. This further improves the measurement of $\phi$.

## IV. DISCUSSION AND SUMMARY

In this paper, we have outlined the formalism for measuring $2\beta + \gamma$ with neutral $B$ meson decays involving interference between several intermediate states. We have shown that, despite involving a more complicated analysis, these decays have distinct advantages over $B \to D^{(*)-} \pi^+$, once our formalism is applied to their analysis, thus enhancing the overall precision with which $2\beta + \gamma$ is known.

First, as already noted for the special case of $B \to D^{*-} \rho^+$ decays [8], our method is sensitive to $2\beta + \gamma$ using only first-order terms in the ratios $a_m/A_n$ between the $b \to u\bar{c}d$ and $b \to c\bar{u}d$ amplitudes. By contrast, in $B \to D^{(*)-} \pi^+$, or in the analysis of other decay modes that ignores the contribution of multiple intermediate states, one needs to extract $r = a_1/A_1$ from $\mathcal{O}(r^2)$ terms, or rely on external measurements and incur a large theoretical uncertainty. Since $r$ is as small as $1 - 2\%$, this advantage is realized in our method even when the ampli-

tude of one of the interfering intermediate states is much greater than the others.

Second, the $B \to D^{(*)-} \pi^+$ measurement is subject to an eight-fold ambiguity, while in our method, the ambiguity is only two-fold.

Our method is not completely model-independent, since one has to assume specific forms for the $g_m$ functions, such as a Breit-Wigner for the resonances. However, this model dependence is much smaller than the 30% theoretical error estimated for $r$. Most resonances are well understood, and their shapes can be studied with the terms of Eqs. (4). In addition, the number of observables in Eq. (7) is greater than the number of unknowns when there are more than two intermediate states. The additional constraints may be used to further reduce the model dependence associated with some $g_m$ functions.

We emphasize that these conclusions and the formalism presented here do not depend on a specific final state, but apply whenever enough is known about the $g_m(s, s')$ functions for a solution to be obtainable, which in practice holds for a majority of the cases.

It is interesting to note some similarities and differences between the method we present here and methods developed for measuring $\gamma$ in $B \to DK$. Multibody final state $B \to DK$ decays (such as $B^- \to DK^- \pi^0$ [10], $B^- \to D^{**} K^-$ [11], $B^- \to D^* K^{*-}$ [12], or $B \to DK$ with multibody $D$ decays [13]) have been shown to improve the measurement of $\gamma$. This improvement is mostly due to the resolution of ambiguities and the ability to make use of new $B$ and $D$ modes that are not accessible with two-body $\gamma$-measurement methods. As we have shown here, similar advantages are realized by interference between intermediate states in the measurement of $2\beta + \gamma$ with multibody $B \to D^- \pi^+$-like modes. But in addition, these measurements benefit mostly from the fact that they do not depend on the very small $r^2$ terms. By contrast, $B \to DK$ decays are governed by the amplitude ratio $r_B = |A(B^- \to \bar{D}^0 K^-)/A(B^- \to D^0 K^-)| \sim 10 - 20\%$, which is about an order of magnitude larger than $r$. Therefore, the sensitivity advantage brought about by interference between intermediate states is much greater in $B \to D^- \pi^+$ than in $B \to DK$.

Finally, it is worthwhile to mention the possibility of measuring $2\beta + \gamma$ with modes in which the $b \to c\bar{u}d$ diagram is strongly suppressed, resulting in $r \sim \mathcal{O}(1)$ [14]. Unfortunately, due to the small branching fractions, one expects a low signal to background ratio in these modes, severely limiting their sensitivity [15].

[1] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[2] B. Aubert *et al.* (*BABAR* Collaboration), Phys. Rev. D **66**, 032003 (2002).

[3] K. Abe *et al.* (Belle Collaboration), Phys. Rev. D **66**, 071102 (2002).

[4] I. Dunietz, Phys. Lett. B **427**, 179 (1998).

[5] B. Aubert *et al.* (*BABAR* Collaboration), Phys. Rev. Lett. **92**, 251802 (2004); B. Aubert *et al.* (*BABAR* Collaboration), Phys. Rev. Lett. **92**, 251801 (2004); B. Aubert *et al.* (*BABAR* Collaboration), hep-ex/0408059; B. Aubert *et al.* (*BABAR* Collaboration), hep-ex/0504035.

[6] K. Abe *et al.* (BELLE Collaboration), Phys. Rev. Lett. **93**, 031802 (2004); [**93**, 059901 (2004)]; K. Abe *et al.* (BELLE Collaboration), Phys. Lett. B **624**, 11 (2005).

[7] For an overview, see Chap. 3, J. Hewett *et al.*, hep-ph/0503261.

[8] D. London, N. Sinha, and R. Sinha, Phys. Rev. Lett. **85**, 1807 (2000).

[9] J. P. Silva, A. Soffer, L. Wolfenstein, and F. Wu, Phys. Rev. D **67**, 036004 (2003).

[10] R. Aleksan, T. C. Petersen, and A. Soffer, Phys. Rev. D **67**, 096002 (2003).

[11] N. Sinha, Phys. Rev. D **70**, 097501 (2004).

[12] N. Sinha and R. Sinha, Phys. Rev. Lett. **80**, 3706 (1998).

[13] A. Giri, Y. Grossman, A. Soffer, and J. Zupan, Phys. Rev. D **68**, 054018 (2003).

[14] M. Diehl and G. Hiller, Phys. Lett. B **517**, 125 (2001).

[15] A. Soffer, in *Proceedings of the APS/DPF/DPB Summer Study on the Future of Particle Physics, Snowmass, 2001)* edited N. Graf, eConf C010630, E208 (2001).