Bosonic P-brane and Arnowitt-Deser-Misner decomposition

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A master action for the bosonic p-brane, interpolating between the Nambu-Goto and Polyakov formalisms, is discussed. The fundamental arbitrariness of extended structures (p-brane) embedded in spacetime manifold has been exploited to build an independent metric in the brane world volume. The cosmological term for the generic case follows naturally in the scheme. The dynamics of the structure leads to a natural emergence of the A-D-M like split of this world volume. The role of the gauge symmetries vis-à-vis reparametrization symmetries is analyzed by a constrained Hamiltonian approach.

I. INTRODUCTION

String theory is considered as the candidate theory to unite all fundamental interactions including gravity. For uniqueness and consistency of different perturbative string theories, extended structures like membranes [1] are required to be introduced with an independent physical status [2]. A generic element of this class is a p-dimensional spatially extended structure called the p-brane. Apart from the symmetries of the spacetime in which the brane is embedded it also carries various symmetries associated with reparametrization of the world volume. These symmetries have been discussed extensively in the literature. However, all these studies are based on either the Nambu-Goto (N-G) [1,3–6] or the Polyakov-type [7] actions and a systematic analysis of the mutual correspondence between the different formalisms is lacking. Aspects of symmetries continue to be of fundamental importance in the study of dynamics of the brane and it is indeed crucial to understand these issues from a unified point of view. The present paper will be devoted to the analysis of the basic world volume symmetries of a bosonic p-brane from such a unified approach. Implications and consequences of the symmetries will also be discussed.

The p-brane sweeps out a $p + 1$ dimensional world volume in the embedding spacetime. The dynamics of the brane can be analyzed from different action formalisms. In the N-G description the physical action is prescribed solely in terms of the spacetime coordinates of the brane, taken as independent fields. Alternatively, in the Polyakov action formalism the metric in the world volume is considered as a collection of independent fields in addition to the usual spacetime coordinates. The equivalence of these two approaches is usually established by starting from the Polyakov action and solving out the independent metric in favor of the spacetime coordinates. In the present paper, on the contrary, we address the reverse problem by demonstrating how the independent metric can be generated by exploiting the reparametrization symmetry of the NG action for the p-brane. An intermediate step is the construction of an interpolating action. Such actions were earlier introduced for discussing various aspects of symmetries and noncommutativity in the case of strings and membranes [8–10]. However the methods used were specific to the particular choice of $p = 1$(strings), $p = 2$(membranes) only, which do not admit a generalization to the arbitrary p-case that is essential for the present analysis. The interpolating action is based on the first-class constraints of the NG theory. The independent metric will be generated from the corresponding Lagrange multipliers enforcing these constraints. This reveals a deep connection of the metric components with the gauge symmetries of the brane. The mismatch between the number of independent gauge degrees of freedom and the number of independent metric elements brings out the arbitrariness, inherent in the Polyakov formulation, explicit in our construction. Fixing the arbitrariness in terms of the embedding makes the transition to the Polyakov form complete. Notably, the cosmological term emerges as a logical consequence of our analysis.

The process of introducing the independent metric in the world volume through the interpolating action formalism has a very interesting outcome. First-class constraints of the N-G theory generate temporal development and also shifts in the space like directions. The independent metric constructed with the help of the Lagrange multipliers enforcing these constraints naturally emerge with a decomposition of the $(p + 1)$-dimensional metric into the (p)-dimensional spatial part plus the multipliers which are the analogues of the lapse and shift variables of general relativity. Indeed, the metric generated in our formalism appears in a canonical form which is shown to be identical with the famous Arnowitt-Deser-Misner (A-D-M) representation in general relativity [11]. In other words our analysis provides a genesis of the A-D-M representation from a string theoretic perspective.

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The interpolating action formalism is based on the gauge symmetries of the theory. These gauge degrees of freedom actually correspond to the invariance under reparametrisation of the brane world volume. Considering the pivotal role played by the gauge symmetries in our analysis, it is only natural to undertake a thorough investigation of these symmetries vis-à-vis the reparametrisation symmetries. In this paper we address this problem using a Hamiltonian method of abstracting the independent gauge parameters introduced earlier in the literature [12].

The organisation of the paper is as follows. In Sec. II we introduce the interpolating action for the generic p-brane and discuss its passage to the N-G and the Polyakov forms. Specifically, the method of obtaining the independent metric components from the fields of the interpolating theory has been elaborated. Various consistency conditions have been deduced which are used in the sequel. In Sec. III the emergence of the A-D-M decomposition of the p-brane world volume from our construction has been indicated. Sec. IV discusses a comprehensive analysis of the gauge symmetries of the interpolating action and its parallel with the reparametrisation invariance of the theory. We conclude in Sec. V.

II. INTERPOLATING ACTION FOR THE BOSONIC P-BRANE

The p-brane is a p dimentional object which sweeps out a $(p + 1)$ dimensional world volume parametrized by τ and σ_a. The index a run from 1 to p. Henceforth these parameters are collectively referred as $\xi_i(\xi_0 = \tau, \xi_a = \sigma_a)$. The N-G action of bosonic p-brane is the integrated proper area of this world volume:

$$S_{\text{NG}} = -\int d^{p+1}\xi\sqrt{-h} \tag{1}$$

where h is the determinant of the induced metric

$$h_{ij} = \partial_i X^\mu \partial_j X_\mu \tag{2}$$

Note that we have kept the p-brane tension implicit. The canonical momenta conjugate to X_μ are

$$\Pi_\mu = \frac{\bar{h}}{\sqrt{-h}}\{\partial_0 X_\mu - \partial_a X_\mu \bar{h}^{ab} h_{0b}\} \tag{3}$$

where $\bar{h}$ is the determinant of the matrix h_{ab}. Also $\bar{h}^{ab}$ [1] is the inverse of h_{ab}. The primary constraints following from (3) are,

$$\Omega_0 = \frac{1}{2}(\Pi^2 + \bar{h}) \approx 0 \; ; \quad \Omega_a = \Pi_\mu \partial_a X^\mu \approx 0 \tag{4}$$

The nontrivial Poission's bracket of the theory are given by

$$\{X^\mu(\tau, \xi), \Pi_\nu(\tau, \xi')\} = \eta^\mu_\nu \delta(\xi - \xi') \tag{5}$$

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[1]Note that $\bar{h}^{ab}$ is different from the space part of h^{ij}.

Using these Poisson brackets it is easy to work out the algebra of the constraints

$$\{\Omega_0(\xi), \Omega_0(\xi')\} = 4[\bar{h}(\xi)\bar{h}^{ab}(\xi)\Omega_b(\xi)$$
$$+ \bar{h}(\xi')\bar{h}^{ab}(\xi')\Omega_b(\xi')]\partial_a(\delta(\xi - \xi'))$$

$$\{\Omega_a(\xi), \Omega_0(\xi')\} = [\Omega_0(\xi) + \Omega_0(\xi')]\partial_a(\delta(\xi - \xi'))$$

$$\{\Omega_a(\xi), \Omega_b(\xi')\} = [\Omega_b(\xi)\partial_a(\delta(\xi - \xi'))$$
$$- \Omega_a(\xi')\partial'_b(\delta(\xi - \xi'))] \tag{6}$$

Clearly, the Poisson brackets between the constraints (6) are weakly involutive so that the set (4) is first class. Since the p-brane action (1) possesses reparametrization invariance, the canonical Hamiltonian following from the action vanishes. Thus the total Hamiltonian is only a linear combination of the constraints (4):

$$\mathcal{H}_T = -\frac{\lambda_0}{2}(\Pi^2 + \bar{h}) - \lambda^a \Pi_\mu \partial_a X^\mu \tag{7}$$

In the above expression λ_0 and λ^a are the Lagrange multipliers.

The Polyakov action for the p-brane is given by,

$$S_{\text{P}} = -\frac{1}{2}\int d^{p+1}\xi\sqrt{-g}\{g^{ij}\partial_i X^\mu \partial_j X_\mu - (p - 1)\} \tag{8}$$

The metric g_{ij} are now considered as independent fields. The equations of motion for g_{ij} are

$$g_{ij} = h_{ij} \tag{9}$$

Substituting these in (8) we retrieve the N-G form (1). Note the cosmological term $\sqrt{-g}(p - 1)$ in the action. For $p = 1$ this term vanishes. We thus observe that the presence of the cosmological term is characteristic of the higher branes as opposed to the strings. The reason for this difference is the Weyl invariance of the string which is not shared by the higher branes. In our action level construction this cosmological term will emerge systematically.

We now come to the construction of an interpolating action for the p-brane. The first step is to consider the Lagrange multipliers as independent fields and write an alternative first order Lagrangian for the p-brane

$$\mathcal{L}_I = \Pi_\mu \dot{X}^\mu - \mathcal{H}_T \tag{10}$$

The equation of motion for Π_μ following from the Lagrangian (10) is

$$\Pi_\mu = -\frac{\dot{X}_\mu + \lambda^a \partial_a X_\mu}{\lambda_0} \tag{11}$$

Substituting Π_μ in (10) we get the interpolating Lagrangian

$$\mathcal{L}_I = -\frac{1}{2\lambda_0}[\dot{X}^\mu \dot{X}_\mu + 2\lambda^a \dot{X}_\mu \partial_a X^\mu + \lambda^a \lambda^b \partial_a X^\mu \partial_b X_\mu]$$
$$+ \frac{\lambda_0}{2}\bar{h} \tag{12}$$

for the p-brane.

The Lagrangian (12) has been referred to as the interpolating Lagrangian because this can be reduced to either the N-G or the Polyakov form of the p-brane action. Let us first discuss the passage to the N-G form. From the interpolating Lagrangian it is easy to write down the equations of motion for λ_0 and λ^a:

$$\lambda_0^2 = \frac{-h}{\bar{h}^2}; \quad \lambda^a = -h_{0b}\bar{h}^{ba} \tag{13}$$

From the first equation of (13) λ_0 is determined modulo a sign. This can be fixed by demanding the consistency of (3) with (11), the equation of motion for Π_μ following from the first order Lagrangian (10). Thus we have

$$\lambda_0 = -\frac{\sqrt{-h}}{\bar{h}} \tag{14}$$

Substituting λ^a and λ_0 in (12) we retrieve the Nambu–Goto action (1).

The reduction of the interpolating Lagrangian to the Polyakov form of the p-brane action is nontrivial. In deriving the interpolating Lagrangean from the N-G theory we have promoted the $(p + 1)$ Lagrange multipliers as independent fields. Note that in the Polyakov action the extra degrees of freedom is more than this number. The precise size of the mismatch is $(p)(p + 1)/2$. We thus observe that the interpolating action is a less redundant description than the Polyakov action. So to make the transition from the interpolating Lagrangean to the Polyakov form we require to introduce just as many independent fields. This can be done by including an arbitrary spatial part G^{ab} in $\mathcal{L}_I$, which has the right number of independent components. We therefore modify the interpolating Lagrangian (12) for the p-brane in the following way

$$\mathcal{L}_I = -\frac{1}{2\lambda_0}[\dot{X}^\mu \dot{X}_\mu + 2\lambda^a \dot{X}_\mu \partial_a X^\mu + (\lambda^a \lambda^b \partial_a X^\mu \partial_b X_\mu$$
$$- \lambda_0^2 \bar{G} G^{ab} \partial_a X_\mu \partial_b X^\mu)] - \frac{\lambda_0}{2}(\bar{G} G^{ab} \partial_a X_\mu \partial_b X^\mu - \bar{h}) \tag{15}$$

where $\bar{G}$ is the determinant of G_{ab} which is the inverse of the arbitrary matrix G^{ab}, $(a, b = 1, 2, \ldots p)$. This specific choice of the arbitrary part will be convenient in the subsequent calculation. Observe that (15) can be cast as

$$\mathcal{L}_I = -\frac{1}{2}\sqrt{-g}g^{ij}\partial_i X_\mu \partial_j X^\mu - \frac{\lambda_0}{2}(\bar{G} G^{ab} \partial_a X_\mu \partial_b X^\mu - \bar{h}) \tag{16}$$

where

$$g^{ij} = (-g)^{-1/2}\begin{pmatrix} \frac{1}{\lambda_0} & \frac{\lambda^a}{\lambda_0} \\ \frac{\lambda^b}{\lambda_0} & \frac{\lambda^a \lambda^b - \lambda_0^2 \bar{G} G^{ab}}{\lambda_0} \end{pmatrix} \tag{17}$$

Here g is the determinant of g_{ij} which is the inverse of g^{ij}. This imposes stringent constraints on the construction (17). So its consistency must explicitly be examined. Observe that by exploiting the dynamics of the p-brane we are able to generate an independent metric on the world volume of the brane. The arbitrary function G_{ab} signifies a fundamental elasticity in the spatial part of the metric. The Lagrangean (16) is almost in the required Polyakov form except for the omission of the cosmological constant. Also there is an additional term which is not there in the Polyakov Lagrangean. It is precisely the consistency requirement of the construction (17) which identifies this extra piece in (16) with the cosmological constant, provided we fix the elasticity in the embedding. The validity of these assertions will be demonstrated now.

From the identification (17) we get after a straightforward calculation that

$$\det g^{ij} = (-1)^p \frac{\lambda_0^{(p-1)}}{(\sqrt{-g})^{(p+1)}} \det(\bar{G} G^{ab}) \tag{18}$$

But we require $\det g^{ij} = g^{-1}$. Comparing, we get the condition

$$\lambda_0^{(p-1)} = (-1)^{(1-p)}\left(\frac{\sqrt{-g}}{\bar{G}}\right)^{(p-1)} \tag{19}$$

Starting from our construction (17) one can solve for λ_0 and λ^a as

$$\frac{1}{\lambda_0} = \sqrt{-g}g^{00}, \quad \lambda^a = \frac{g^{0a}}{g^{00}} \tag{20}$$

Using (20) in (17), we get after a few steps

$$G^{ab} = \frac{g}{\bar{G}}(g^{ab}g^{00} - g^{0a}g^{0b}) \tag{21}$$

Inverting G^{ab} we arrive at

$$G_{ab} = \left(\frac{gg^{00}}{\bar{G}}\right)g_{ab} \tag{22}$$

From (21) we obtain after some calculations

$$\det G^{ab} = \left(\frac{g}{\bar{G}}\right)^p \det g^{ij}(g^{00})^{p-1} \tag{23}$$

But, by definition, $\det G^{ab} = 1/\bar{G}$. Using this in (23) we find,

$$\bar{G}^{(p-1)} = (gg^{00})^{(p-1)} \tag{24}$$

There is an apparent ambiguity of sign in determining $\bar{G}$ from (24) when p is odd. For now we take the positive solution for all p. Then from (22)

$$G_{ab} = g_{ab} \qquad (25)$$

The consistency requirment thus restricts the arbitrariness of G_{ab} through (25). We use (19) and (25) to express the $\sqrt{-g}$ factor in terms of the $\bar{G}$ and λ_0 as [2]

$$\sqrt{-g} = -\lambda_0 \bar{G} = -\lambda_0 \det g_{ab} \qquad (26)$$

Finally, we discuss the reduction of the interpolating action to the Polyakov form. Note that the spatial part of the metric g^{ij} is still remaining arbitrary. Also no attention has so far been paid to the background spacetime in which the brane is embedded. We now propose the rigid structure

$$g_{ab} = h_{ab} \qquad (27)$$

In this connection it may be observed that this is just the spatial part of (9) which is required to demonstrate the equivalence of the Polyakov form with the N-G. Now Eq. (27), along with (25), imposes

$$G_{ab} = h_{ab} \qquad (28)$$

Plugging it in the Lagrangean (16) and using (26) we find that the last term of (16) is precisely equal to the cosmological constant occurring in the Polyakov action (8). This completes the reduction of the interpolating Lagrangian to the Polyakov form. The connection (27) fixes the brane in its embedding. A couple of interesting observations also follow from this. First, we can understand now the nature of the ambiguities of sign encountered above for odd p more clearly. If we chose the opposite sign in (25) then we would have $G_{ab} = -h_{ab}$ and λ_0 should then be expressed from (19) as $\lambda_0 = \frac{\sqrt{-g}}{\bar{G}}$. Otherwise there would be contradiction with (14). Next, for $p = 1$ we find that the imposed rigidity admits a residual scale transformation. This is the well known Weyl invariance of the string.

III. EMERGENCE OF A-D-M DECOMPOSITION FROM THE BRANE DYNAMICS

The interpolating action formalism enables us to introduce an independent metric in the world volume swept out by the N-G brane. The process depends crucially on the first-class constraints of the theory. We have also clearly identified the arbitrariness in the spatial part of the metric. Our method thus introduces the metric in a very special way such that the world volume is decomposed into the p dimensional spatial part along with the multipliers which generate temporal evolution with shifts in the spacelike direction. This decomposition of the metric is reminiscent of the A-D-M decomposition in geometrodynamics [11]. Indeed, the connection with A-D-M decomposition was noticed earlier in the special cases of string and membrane [9]. We are now in a position to show how the A-D-M

representation follows from the dynamics of the generic p-brane. To see this we have to use (26) to first express the metric in terms of its arbitrary spatial part and the Lagrange multipliers only. The construction (17) then reduces to

$$g^{ij} = \begin{pmatrix} -\frac{1}{\lambda_0^2 \det g_{ab}} & -\frac{\lambda^a}{\lambda_0^2 \det g_{ab}} \\ -\frac{\lambda^b}{\lambda_0^2 \det g_{ab}} & (\bar{g}^{ab} - \frac{\lambda^a \lambda^b}{\lambda_0^2 \det g_{ab}}) \end{pmatrix} \qquad (29)$$

where $\bar{g}^{ab}$ is the inverse of the spatial metric g_{ab}[3]. In the A-D-M construction the metric g^{ij} of physical spacetime is represented as

$$g^{ij} = \begin{pmatrix} -\frac{1}{(N)^2} & \frac{N^a}{(N)^2} \\ \frac{N^b}{(N)^2} & (\bar{g}^{ab} - \frac{N^a N^b}{(N)^2}) \end{pmatrix} \qquad (30)$$

where N and N^a are, respectively, the lapse and shift variables and $\bar{g}^{ab}$ is the inverse of the "metric" g_{ab} on the spatial hypersurface. Using the correspondence

$$(N^a) \mapsto -\lambda^a, \quad \text{and} \quad N \mapsto \lambda_0 \sqrt{\det g_{ab}} \qquad (31)$$

it is easy to convince oneself that the A-D-M decomposition of the brane volume emerges from our analysis. Note that in the correspondences (31) apart from the Lagrange multipliers only the space part of the metric g_{ij} is involved. The flexibility in g_{ab} is apparent in our Eq. (25). Modulo this arbitrariness the lapse and shift variables are the fields λ_0 and λ^a in our interpolating Lagrangean (16). They in turn owe their existence to the constraints (4) which are nothing but the superhamiltonian and supermomentum of the theory. Our interpolating Lagrangean (12) can thus be considered as the brane analog of the A-D-M formulation of geometrodynamics.

IV. CONSTRAINT STRUCTURE AND GAUGE SYMMETRY

The interpolating action formalism offers a composite scenario for discussing alternative actions of the p-brane. A thorough understanding of its gauge symmetries will thus be very much appropriate to our context. In this section we will discuss the gauge symmetries of the different actions and find their exact correspondence with the reparametrization invariances. Since our discussion will be centered on the interpolating action (12) let us begin with an analysis of its constraint structure. The independent fields in (12) are X_μ, λ_0 and λ^a. Let the corresponding momenta be denoted by Π_μ, Π_{λ_0} and Π_{λ^a} respectively. By definition

$$\Pi_\mu = -\frac{1}{\lambda_0}(\dot{X}_\mu + \lambda^a \partial_a X_\mu) ; \quad \Pi_{\lambda_0} = 0 ; \qquad \Pi_{\lambda^a} = 0$$

$$(32)$$

In addition to the Poisson brackets similar to (5) we now

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[2]Note that for odd p another sign ambiguity appears here. This is actually related with the corresponding uncertainty about sign stated above. We shall explore the connection subsequently.

[3]Note that as in the case of $\bar{h}^{ab}$, $\bar{g}^{ab}$ is also different from the spatial part of the identification matrix g^{ij}.

have

$$\{\lambda_0(\tau, \xi), \Pi_{\lambda_0}(\tau, \xi')\} = \delta(\xi - \xi')$$
$$\{\lambda^a(\tau, \xi), \Pi_{\lambda^b}(\tau, \xi')\} = \delta(\xi - \xi')\delta^a_b \qquad (33)$$

The canonical Hamiltonian following from (12) is

$$\mathcal{H}_c = -\lambda^a \Pi_\mu \partial_a X^\mu - \frac{\lambda_0}{2}(\Pi^2 + X'^2) \qquad (34)$$

which reproduces the total Hamiltonian of the NG action. From the definition of the canonical momenta we can easily identify the primary constraints

$$\Pi_{\lambda_0} \approx 0 \; ; \quad \Pi_{\lambda^a} \approx 0 \qquad (35)$$

Conserving these primary constraints we find that two new secondary constraints emerge. These are the constraints of Eq. (4), as expected. The primary constraints of the NG action appear as secondary constraints in this formalism. No more secondary constraints are obtained. The system of constraints for the Interpolating Lagrangean thus comprises of the set (4) and (35). The Poisson brackets of the constraints of (35) vanish within themselves. Also the PB of these with (4) vanish. All the constraints are first class and therefore generate gauge transformations on $\mathcal{L}_I$ but the number of independent gauge parameters is equal to the number of independent primary first-class constraints. In the following analysis we will apply a systematic procedure [12] of abstracting the most general local symmetry transformations of the Lagrangean. A brief review of the procedure will thus be appropriate.

Consider a theory with first-class constraints only. The set of constraints Ω_a is assumed to be classified as

$$[\Omega_a] = [\Omega_{a_1}; \Omega_{a_2}] \qquad (36)$$

where a_1 belong to the set of primary and a_2 to the set of secondary constraints. The total Hamiltonian is

$$H_T = H_c + \Sigma \lambda^{a_1} \Omega_{a_1} \qquad (37)$$

where H_c is the canonical Hamiltonian and λ^{a_1} are Lagrange multipliers enforcing the primary constraints. The most general expression for the generator of gauge transformations is obtained according to the Dirac conjecture as

$$G = \Sigma \epsilon^a \Omega_a \qquad (38)$$

where ϵ^a are the gauge parameters, only a_1 of which are independent. By demanding the commutation of an arbitrary gauge variation with the total time derivative, (i.e. $\frac{d}{dt}(\delta q) = \delta(\frac{d}{dt} q)$) we arrive at the following Eqs. [12,13]

$$\delta \lambda^{a_1} = \frac{d\epsilon^{a_1}}{dt} - \epsilon^a (V^{a_1}_a + \lambda^{b_1} C^{a_1}_{b_1 a}) \qquad (39)$$

$$0 = \frac{d\epsilon^{a_2}}{dt} - \epsilon^a (V^{a_2}_a + \lambda^{b_1} C^{a_2}_{b_1 a}) \qquad (40)$$

Here the coefficients $V^{a_1}_a$ and $C^{a_1}_{b_1 a}$ are the structure functions of the involutive algebra, defined as

$$\{H_c, \Omega_a\} = V^b_a \Omega_b \; ; \quad \{\Omega_a, \Omega_b\} = C^c_{ab} \Omega_c \qquad (41)$$

Solving (40) it is possible to choose a_1 independent gauge parameters from the set ϵ^a and express G of (38) entirely in terms of them. The other set (39) gives the gauge variations of the Lagrange multipliers. It can be shown that these equations are not independent conditions but appear as internal consistency conditions. In fact the conditions (39) follow from (40) [12].

We begin the analysis with the interpolating action (12). Here the fields are X^μ, λ_0 and λ^a. The set of constraints are given by (35) along with (4). All these constraints are first class. Denoting these by the set $\{\Psi_k\}$ we write the generator of the gauge transformations of (12) as

$$G = \int d\xi \alpha_k \Psi_k \qquad (42)$$

where α_k are the gauge parameters. We could proceed from this and construct the generator of gauge transformations from (42) by including the whole set of first-class constraints (4) and (35). Using (40) the dependent gauge parameters can be eliminated. After finding the gauge generator in terms of the independent gauge parameters, the variations of the fields X^μ, λ_0 and λ^a can be worked out. However, looking at the intermediate first order form (10) we understand that the variations of the fields λ_0 and λ^a can be calculated alternatively, (using (39)) from the NG theory where they appear as Lagrange multipliers. We adopt this alternative procedure. The generator of gauge transformations has already been given in (42) where Ω_i now stands for the first-class constraints of the N-G theory, i.e. (4). The variations of λ_i are obtained from (39)

$$\delta \lambda_i(\xi) = -\dot{\alpha}_i - \int d\xi' d\xi'' C^i_{kj}(\xi', \xi'', \xi) \lambda_k(\xi') \alpha_j(\xi'')$$
$$(43)$$

where $C^i_{kj}(\xi', \xi'', \xi)$ are given by

$$\{\Omega_\alpha(\xi), \Omega_\beta(\xi')\} = \int d\xi'' C_{\alpha\beta}{}^\gamma(\xi, \xi', \xi'') \Omega_\gamma(\xi'') \quad (44)$$

Observe that the structure function $V_a{}^b$ does not appear in (43) since $H_c = 0$ for the NG theory. The nontrivial structure functions $C_{\alpha\beta}{}^\gamma(\xi, \xi', \xi'')$ are obtained from the N-G constraint algebra (6).

$$C^b_{00} = 4[\bar{h}\bar{h}^{ab}\delta(\xi - \xi') + \bar{h}\bar{h}^{ab}\delta(\xi' - \xi'')]\partial_a\{\delta(\xi - \xi')\}$$
$$(45)$$

$$C^0_{a0} = [\delta(\xi - \xi'') + \delta(\xi' - \xi'')]\partial_a\{\delta(\xi - \xi')\} \quad (46)$$

$$C^c_{ab} = [\delta(\xi - \xi'')\partial_a\{\delta(\xi - \xi')\}\delta^c_b$$
$$+ \delta(\xi' - \xi'')\partial_b\{\delta(\xi - \xi')\}\delta^c_a] \qquad (47)$$

and using the structure functions (45)–(47) we calculate the required gauge variations by applying Eq. (43)

$$\delta\lambda^a = -\dot{\alpha}_a + (\alpha_b\partial_b\lambda^a - \lambda^b\partial_b\alpha_a)$$
$$+ 4\bar{h}\bar{h}^{ab}(\alpha_0\partial_b\lambda_0 - \lambda_0\partial_b\alpha_0)$$
$$\delta\lambda_0 = -\dot{\alpha}_0 + (\alpha_0\partial_a\lambda^a - \lambda^a\partial_a\alpha_0)$$
$$+ (\alpha_a\partial_a\lambda_0 - \lambda_0\partial_a\alpha_a) \qquad (48)$$

Now we will systematically investigate and explicitly establish the parallel between gauge symmetry and reparametrization symmetry of the p-brane actions. To start with we require to find a correspondence between the transformation parameters in both the cases. A particular usefulness of the interpolating Lagrangean formalism can be appreciated now. It is not required to find the gauge variations of g_{ij} from scratch. The identification (17) allows us to find the required gauge variations from the corresponding transformations of λ_0 and λ^a (48). The Polyakov action offers the most appropriate platform to test this proposition. Indeed, the complete equivalence between the two concepts (i.e. gauge variation and reparametrization) can be demonstrated from the Polyakov action by devising an exact mapping between the reparametrization parameters and gauge parameters by comparing the changes of λ_0 and λ^a from the alternative approaches using the identification (17).

Under infinitesimal reparametrization of the world volume coordinates

$$\xi'_i = \xi_i - \Lambda_i \qquad (49)$$

where Λ_i are arbitrary functions of ξ_i, the variations of the fields X_μ and g_{ij} are

$$\delta X^\mu = \Lambda^i\partial_iX^\mu = \Lambda^0\dot{X}^\mu + \Lambda^a\partial_aX^\mu \qquad (50)$$

$$\delta g_{ij} = D_i\Lambda_j + D_j\Lambda_i \qquad (51)$$

where the covariant derivative is defined as usual,

$$D_i\Lambda_j = \partial_i\Lambda_j - \Gamma_{ij}^k\Lambda_k \qquad (52)$$

with $\Gamma_{ij}^{\ k}$ being the Christoffel symbols [14].

The infinitesimal parameters Λ^i characterizing reparametrization correspond to infinitesimal gauge transformations and will ultimately be related with the gauge parameters α_k introduced earlier such that the symmetry transformations on X^μ agree from both the approaches. Since the metric g_{ij} is associated with λ_0 and λ^a by the correspondence (17), Eq. (51) will enable us to establish the complete equivalence between the variations due to the gauge transformation and the reparametrizations.

To compare variations of X^μ from these alternative approaches we proceed as follows. From the Lagrangean corresponding to (8) we find

$$\Pi^\mu = -\sqrt{-g}g^{00}\dot{X}^\mu - \sqrt{-g}g^{0a}\partial_aX^\mu \qquad (53)$$

Substituting $\dot{X}^\mu$ from (53) in (50) we get after some calculation

$$\delta X^\mu = \Lambda^0\frac{\sqrt{-g}}{gg^{00}}\Pi^\mu + \left(\Lambda^a - \frac{g^{0a}}{g^{00}}\Lambda^0\right)\partial_aX^\mu \qquad (54)$$

Now the variation of X^μ in (12) under (42) is

$$\delta X_\mu = \{X_\mu, G\} = (\alpha_a\partial_aX_\mu + 2\alpha_0\Pi_\mu) \qquad (55)$$

Comparing the above expression of δX^μ with that of (54) we find the mapping

$$\Lambda^0 = -\frac{\alpha_0}{\lambda_0} \ ; \quad \Lambda^a = \alpha_a - \frac{\lambda^a\alpha_0}{\lambda_0} \qquad (56)$$

With this mapping the gauge transformation on X_μ in both the formalism agree. The complete equivalence between the transformations can now be established by computing $\delta\lambda^a$ and $\delta\lambda_0$ from the alternative approach. The mapping (17) yields,

$$\lambda^a = \frac{g^{0a}}{g^{00}} \qquad (57)$$

We require to express these in terms of g_{ij}. To this end we start from the identity

$$g^{ij}g_{jk} = \delta^i_{\ k} \qquad (58)$$

and obtain the following equations for λ^a

$$\lambda^ag_{ab} = -g_{0b} \qquad (59)$$

which gives

$$\lambda^a = -\bar{g}^{ab}g_{0b} \qquad (60)$$

Taking variation on both sides of (59) we get

$$\delta\lambda^ag_{ab} = -\delta g_{0b} - \lambda^a\delta g_{ab} \qquad (61)$$

Rearranging the terms conveniently we write

$$\delta\lambda^a = -\delta g_{0b}\bar{g}^{ab} - \lambda^c\delta g_{cb}\bar{g}^{ba} \qquad (62)$$

Now using (51) we compute δg_{0b} and δg_{ab} and simplify using relation (63) and the fact that $\bar{g}^{ab}$ is the inverse of the spatial metric g_{ab}. This gives the variations of λ^a under reparametrization as

$$\delta\lambda^a = -\partial_0\Lambda^a + \lambda^a\partial_0\Lambda^0 - \lambda^b\partial_b\Lambda^a + \lambda^a\lambda^b\partial_b\Lambda^0$$
$$+ \Lambda^k\partial_k\lambda^a + (g_{00} - g_{c0}\bar{g}^{cb}g_{0b})\bar{g}^{ad}\partial_d\Lambda^0 \qquad (63)$$

We further simplify the last term using a relation[4]

$$\frac{1}{\lambda_0^2}[h_{00} - h_{0c}\bar{h}^{cb}h_{0b}] + \bar{h} = 0 \qquad (64)$$

and (9) to get

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[4]This relation is obtained by substituting for λ_0 and λ^a from (13) in the interpolating Lagrangean (12)

$$\delta\lambda^a = -\partial_0\Lambda^a + \lambda^a\partial_0\Lambda^0 - \lambda^b\partial_b\Lambda^a + \lambda^a\lambda^b\partial_b\Lambda^0$$
$$+ \Lambda^k\partial_k\lambda^a + \lambda_0^2\bar{h}\bar{h}^{ad}\partial_d\Lambda^0 \qquad (65)$$

Now introducing the mapping (56) in (65) we find that the variations of λ^a are identical with their gauge variations in (48). Finally we compute the variation of λ_0. This can be conveniently done by taking the expression of λ_0 in (26) and using the variations (51). We get the expression of $\delta\lambda_0$ in terms of the reparametrization parameters Λ_i as

$$\delta\lambda_0 = \lambda_0\partial_i\Lambda^i + \Lambda^k\partial_k\lambda_0 - 2\lambda_0\partial_a\Lambda^a + 2\lambda_0\lambda^a\partial_a\Lambda^0 \quad (66)$$

Again using the mapping (56) we substitute Λ_i by α_i and the resulting expression for $\delta\lambda_0$ agrees with that given in (48). The complete matching, thus obtained, illustrates the equivalence of reparametrization symmetry with gauge symmetry for the generic p-brane.

V. CONCLUSION

We have discussed in the context of the bosonic p-brane how an independent metric can be generically introduced in the world volume of the brane. This has been done with the help of an interpolating action based on the first-class constraints. The specific method adopted here leads to the introduction of the metric in a very special way, namely, we have achieved a segregation of the $(p + 1)$ dimentional world volume in the p dimentional spatial part and the Lagrange multipliers analogous to the lapse and shift variables of classical gravity. Using this correspondence we have shown that the Arnowitt-Deser-Misner like decomposition of the brane world volume emerges from our analysis. A comprehensive analysis of the gauge symmetries of the interpolating action has been elaborated and equivalence of gauge and reparametrisation invariances has been established.

[1] P. A. Collins and R. W. Tucker, Nucl. Phys. **B112**, 150 (1976).
[2] For a review of the theory of membranes, see: J. Hoppe, hep- th/ 0206192; W. Taylor, Rev. Mod. Phys. **73**, 419 (2001).
[3] E. G. Floratos and J. Iliopoulos, Phys. Lett. B **B201**, 237 (1988).
[4] R. P. Zaikov, Phys. Lett. B **B213**, 422 (1988).
[5] S. K. Bose and S. A. Bruce, Phys. Lett. B **B225**, 331 (1989).
[6] P. L. FitzGerald, Phys. Lett. B **318**, 433 (1993).
[7] K. Fujikawa and K. Okuyama, Phys. Lett. B **411**, 261 (1997).
[8] R. Banerjee, B. Chakraborty, and S. Ghosh, Phys. Lett. B **537**, 340 (2002).
[9] R. Banerjee, P. Mukherjee, and A. Saha, Phys. Rev. D **70**, 026006 (2004).
[10] R. Banerjee, B. Chakraborty, and K. Kumar, Nucl. Phys. **B668**, 179 (2003).
[11] R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **117**, 1595 (1960); gr-qc/0405109.
[12] R. Banerjee, H. J. Rothe, and K. D. Rothe, Phys. Lett. B **479**, 429 (2000); R. Banerjee, H. J. Rothe, and K. D. Rothe, Phys. Lett. B**463**, 248 (1999).
[13] M. Henneaux, C. Teitelboim, and J. Zanelli, Nucl. Phys. **B332**, 169 (1990).
[14] J. W. van Holten, Lect. Notes Phys. **659**, 99 (2005).