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From free fields to AdS. III

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In previous work, we have shown that large N field theory amplitudes, in Schwinger parametrized form, can be organized into integrals over the stringy moduli space $\mathcal{M}_{g,n} \times R_+^n$. Here we flesh this out into a concrete implementation of open-closed string duality. In particular, we propose that the closed string world sheet is reconstructed from the unique Strebel quadratic differential that can be associated to (the dual of) a field theory skeleton graph. We are led, in the process, to identify the inverse Schwinger proper times ($\sigma_i = 1/\tau_i$) with the lengths of edges of the critical graph of the Strebel differential. Kontsevich's matrix model derivation of the intersection numbers in moduli space provides a concrete example of this identification. It also exhibits how closed string correlators emerge very naturally from the Schwinger parameter integrals. Finally, to illustrate the utility of our approach to open-closed string duality, we outline a method by which a world sheet operator product expansion can be directly extracted from the field theory expressions. Limits of the Strebel differential for the four punctured sphere play a key role.

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I. INTRODUCTION

The emergence of a closed string world sheet from large N gauge theory diagrams has been one of those insights that we have struggled to make precise in the past 30 years. 't Hooft's double line representation of Feynman graphs [1] had made it pictorially plausible that a two-dimensional surface underlay these diagrams. More recently with the anti-de Sitter/conformal field theory (AdS/CFT) conjecture [2–4] and other examples, it became clearer that open-closed string duality was the basic mechanism by which a closed string theory emerged from the gauge theory. However, despite some improved understanding in specific examples [5–7], we do not yet have a complete and concrete proposal of how the closed string world sheet (and the CFT living on it) arises for general field theory amplitudes.

In earlier papers in this series [8,9], we have developed an approach to open-closed string duality for general field theory amplitudes starting with the free field limit. (See [10] for an overview. See also [11–13].) The intuition [8] was that the Schwinger parametrized representation of field theory correlators is the right starting point to try and see the closed string world sheet emerge from the field theory. This is because the Schwinger parameters are really the moduli of the world line graphs of the field theory (i.e. the open string picture), and it should be these same moduli that should parametrize the shape of the emergent closed string world sheet, on correctly implementing open-closed string duality.[1]

In Ref. [9] this intuition was largely borne out. An appropriate reorganization of free field theory n-point amplitudes (of fixed genus g in the 't Hooft sense) was made in terms of so-called skeleton graphs. The integral over the effective Schwinger parameters together with the sum over inequivalent skeleton graphs was argued to be actually an integral over the *stringy* moduli space $\mathcal{M}_{g,n} \times R_+^n$. Here $\mathcal{M}_{g,n}$ is the moduli space of genus g surfaces with n *punctures*—exactly what one would expect for the closed string dual. This identification crucially used the fact that the space of these skeleton graphs (or, rather, their dual), together with a positive length assignment (the Schwinger time) for each edge, gives a natural cell decomposition of $\mathcal{M}_{g,n} \times R_+^n$. This cell decomposition, which is originally due to Mumford, Harer, Penner, Kontsevich [16–18] and other mathematicians, is also natural from the point of view of cubic open string field theory [19,20]. That the space of Schwinger parameters of skeleton graphs is isomorphic to a stringy moduli space is a very encouraging sign that we are implementing open-closed string duality. It is evidence that a closed string theory *can* very naturally, and rather generally, emerge from the gauge theory amplitudes.

We would, of course, like to do more. The identification of the space of Schwinger parameters and skeleton graphs with the stringy moduli space implies that the field theory expression for the *integrand* over the Schwinger parameter space is to be identified with a correlator of closed string vertex operators. We would, therefore, like to extract from these expressions the properties of the putative closed string world sheet CFT. For this we would need a dictionary between the Schwinger parameters of the field theory skeleton graphs and the usual complex coordinates on $\mathcal{M}_{g,n}$ in terms of which, for instance, the holomorphic properties of CFT correlators are made manifest. Strictly speaking, we would use such a dictionary to first *check* whether the integrand in the Schwinger parameter space indeed satisfies all the requirements for a world sheet CFT correlator. In any case, arriving at such a dictionary entails making a definite identification of the Schwinger parameters with parameters specifying the closed string world

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[1]Another approach to seeing the world sheet of the string in the light cone framework is that of Thorn and collaborators [14]. See also [15].

sheet. It is therefore the ingredient that would make our proposal for implementing the *geometry* of open-closed string duality complete.

One of the goals of the present work is to arrive at this precise dictionary. We will, in fact, propose a concrete method to reconstruct the particular closed string world sheet corresponding to a given point in the Schwinger parameter space of skeleton graphs. The identification makes use of the mathematical correspondence which underlies the cell decomposition of $\mathcal{M}_{g,n} \times R_+^n$ that was mentioned above. This correspondence proceeds via the construction of certain unique quadratic differentials, known as Strebel differentials, on a Riemann surface. The so-called critical graphs of these Strebel differentials will be identified with the dual of the field theory skeleton graphs. The crucial ingredient will be the dictionary between the Schwinger proper times (τ_r) and the lengths (l_r) of the edges of the critical graph. The l_r are important since they give a unique parametrization of the closed string world sheet. In particular, they are determined in terms of the complex coordinates on $\mathcal{M}_{g,n}$ denoted collectively by z_a, together with additional data of the R_+^n. We will argue that the relation between the Schwinger times and the Strebel lengths is simply

$$\sigma_r \equiv \frac{1}{\tau_r} = l_r(z_a). \tag{1.1}$$

We will see that this identification is natural from many points of view. The general picture of open-closed string duality that then emerges is very much in line with one's intuition of field theory Wick contraction lines being glued up to form the closed string world sheet. It is also in line with various bit pictures of the closed string world sheet. Interactions in the field theory are also readily incorporated into this picture, since they correspond to insertions of additional closed string vertex operators.

We will also revisit Kontsevich's classic derivation [18] of Witten's conjecture [21] on the intersection numbers in moduli space. It will illustrate for us, in a concrete way, how the Schwinger parametrization provides the natural passage to the dual closed string. Moreover, it will also exhibit how integrands in the Schwinger parameter space become correlators on $\mathcal{M}_{g,n}$.

All this adds up to a satisfying picture of the way the closed string world sheet emerges from the field theory. But, as mentioned above, one of the aims of getting a precise dictionary is to read off expressions for world sheet CFT correlators in terms of the usual complex parameters z_a (such as for the locations of punctures). Thus, we would like to reexpress the field theory integrand, which can be written in terms of σ_r, in terms of the z_a using (1.1). In general, (1.1) implies a complicated transcendental relation between the σ_r and the z_a. This is because, as we shall review, the relation between the Strebel lengths l_r and the z_a, while precisely defined, can be analytically involved in general.

We will therefore take the strategy of looking for simplifications at the boundary of moduli space. The instance of the four point function on the sphere is the simplest nontrivial case to consider. We will focus on the limiting Strebel differentials that arise when two punctures come together on $\mathcal{M}_{0,4}$. From the field theory point of view, we can zoom in on this region by considering particular UV limits of the free field spacetime four point correlator. Basically, the idea is that, as we take two points in spacetime close to each other, the field theory integrand gets all its contribution from the corresponding proper time interval $\tau \to 0$. This translates into limiting behavior that we expect for the Strebel differential near boundaries of $\mathcal{M}_{0,4} \times R_+^4$.

In other words, *the world sheet operator product expansion originates from the spacetime operator product expansion.* Indeed, we can systematically write the short distance expansion of spacetime correlators in a Schwinger parametrization. As mentioned, this will be an expansion in the proper times τ that go to zero in this limit. The dictionary (1.1) together with the characterization of the limiting Strebel differential in this region of $\mathcal{M}_{0,4}$ gives a well-defined method to convert the latter expansion into a regular world sheet expansion in terms of a usual complex coordinate $z \to 0$. This provides us the precise setting to check whether the Schwinger integrand satisfies all the requirements of a world sheet CFT correlator. Though we will reserve a detailed study of this issue for later, here we will use the scaling behavior of the limiting Strebel differential to deduce that the world sheet expansion is actually in powers of $|z|^{1/2}$. This is very encouraging, as it might be a signature of an underlying *fermionic* string.

An outline of the organization of the paper: In the next section, we briefly review the reorganization [9] of Schwinger parametrized field theory amplitudes into skeleton graphs. In Sec. III, we review the notion of a Strebel differential, its critical graph, and Strebel lengths, as well as some of the important results regarding these objects. What will play an important role for us is how the Strebel differentials give rise to a natural cell decomposition of $\mathcal{M}_{g,n} \times R_+^n$. In Sec. IV, we detail the connection of this cell decomposition to the space of field theory skeleton graphs with its Schwinger parameters. We give a couple of arguments for the dictionary (1.1). We also describe why the resulting picture implements open-closed string duality as expected. We revisit Kontsevich's derivation in Sec. V. We also see here how closed string correlators naturally arise from the Schwinger integrand. In Sec. VI, we take the first steps to using our open-closed dictionary. We first explain the mathematical steps in going from the Strebel parametrization of moduli space to the usual holomorphic one. Applying this to the four point function, we see how to go to the boundary of moduli space by taking various UV limits of the spacetime correlator. We study the Strebel differential in these limits and show by scaling arguments that the resultant world sheet expansion hints at an underlying fermionic string theory. In the appendix, we collect some useful explicit expressions for the Schwinger parametrization of certain four point functions.

II. SKELETON GRAPHS AND FIELD THEORY AMPLITUDES

The idea behind reorganizing field theory diagrams into skeleton graphs [9] is quite simple. Any perturbative gauge invariant correlator can be written in terms of double lines (assuming only fields in the adjoint representation) with some number of vertices as well as wick contractions between the fields at each vertex. These vertices could be either internal or external. We will assume that the graphs have already been organized according to their 't Hooft genus. The large N limit gives a way to separate out the contributions of different genera via a natural small expansion parameter [1].

At the graphical level, the skeleton graph associated to any such gauge theory diagram is simply the graph obtained by merging together all the homotopically equivalent contractions between any two pairs of vertices. (See Fig. 1.) By homotopically equivalent, we mean those double line contractions that can be deformed into each other without crossing any other line or vertex. The net result is that we will not have any faces which are bounded by only two edges (contractions). Each face of the skeleton graph will have at least three edges. In fact, the generic situation for correlators of composite operators with enough elementary fields will be to have all faces triangular. Thus, the generic skeleton graph will be a triangulation of the genus g surface with as many vertices as there are internal and external ones.

This partial gluing up of the field theory contractions is something one would expect from open-closed string duality. It is the first stage of closing up all the holes of the open string world sheet. In Ref. [9] we saw that this gluing is reflected very nicely in the Schwinger parametrized expressions for correlators. Let us briefly review the logic. For definiteness, and simplicity, we considered a free field

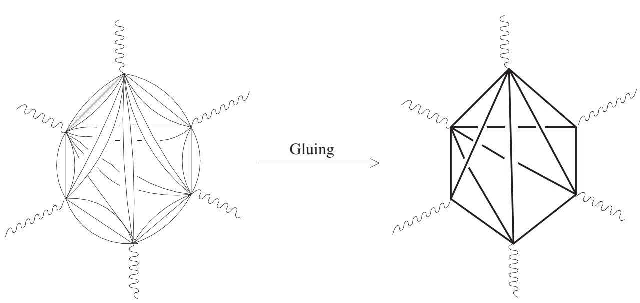

FIG. 1. Gluing up of a planar six point function into a skeleton graph.

n-point function built out of adjoint scalars

$$G_g^{\{J_i\}}(k_1, k_2, \dots k_n) = \left\langle \prod_{i=1}^{n} \mathrm{Tr}\Phi^{J_i}(k_i) \right\rangle_g . \quad (2.1)$$

The subscript indicates that we are considering the contributions of genus g.

In terms of Schwinger times $\tilde{\tau}$ for each propagator, and after carrying out the integral over the d-dimensional internal momenta, one obtained an expression of the form

$$G_g^{\{J_i\}}(k_1, k_2, \dots k_n) = \sum_{\text{graphs}} \int_0^\infty \frac{[d\tilde{\tau}]}{\Delta(\tilde{\tau})^{d/2}} \exp[-P(\tilde{\tau}, k)]. \quad (2.2)$$

There are definite graph theoretic expressions for $P(\tilde{\tau}, k)$ and $\Delta(\tilde{\tau})$ whose details need not concern us at present. The main point is that, though this is an expression depending on as many Schwinger times $\tilde{\tau}$ as there are internal propagators to the graph, there is actually a vast simplification. The above expression, for any given graph, can be written purely in terms of the skeleton graph associated to that graph, where one assigns the effective Schwinger time

$$\frac{1}{\tau_r} = \sum_{\mu_r=1}^{m_r} \frac{1}{\tilde{\tau}_{r\mu_r}}. \quad (2.3)$$

Here r labels an edge of the skeleton graph obtained by merging the homotopic set of m_r edges, indexed by μ_r. After making the change of variables (2.3) to the contributions in (2.2), we get

$$\int_0^\infty \frac{\prod_{r,\mu_r} d\tilde{\tau}_{r\mu_r}}{\Delta(\tilde{\tau})^{d/2}} e^{-P(\tilde{\tau},k)} = C^{\{m_r\}} \int_0^\infty \prod_r \left(\frac{d\tau_r}{\tau_r^{(m_r-1)[(d/2)-1]}} \right) \times \frac{e^{-P_{\text{skel}}(\tau,k)}}{\Delta_{\text{skel}}(\tau)^{d/2}}. \quad (2.4)$$

$P_{\text{skel}}(\tau, k)$ and $\Delta_{\text{skel}}(\tau)$ are given in terms of the graph connectivity of the skeleton graph with the assignment $\{\tau_r\}$ to its edges. The $C^{\{m_r\}}$ is a numerical factor coming from the Jacobian of the change of variables and is computed in Ref. [9].

As a result, the entire expression for the n-point function can be expressed completely as a sum over all the skeleton graphs that contribute to the amplitude.

$$G_g^{\{J_i\}}(k_1, k_2, \dots k_n) = \sum_{\substack{\text{skel}\\ \text{graphs}}} \int_0^\infty \frac{\prod_r d\tau_r f^{\{J_i\}}(\tau)}{\Delta_{\text{skel}}(\tau)^{d/2}} e^{-P_{\text{skel}}(\tau,k)}. \quad (2.5)$$

The $f^{\{J_i\}}(\tau)$ come from carrying out the sum over the multiplicities m_r that are compatible with the same skeleton graph and the net number of fields $\{J_i\}$. All the J_i dependence resides in this term. Its explicit form is again available in Ref. [9]. The sum in Eq. (2.5) is then over the various inequivalent (i.e. with inequivalent connectivity) skeleton graphs of genus g with n vertices.

This partial gluing up has accomplished a big simplification of the Schwinger parametrized representation. In particular, an important point to note is that the number of edges (and, thus, effective Schwinger times τ) in the skeleton graph depends only on n and g (and *not* the J_i). In fact, for a generic triangulation, the number of edges is $6g - 6 + 3n$. The universality of this representation, so to say, is the reason why the moduli space of skeleton graphs is the more natural object to consider. It is the space which will provide the cell decomposition of $\mathcal{M}_{g,n} \times R_+^n$, to which we now turn.

III. STREBEL DIFFERENTIALS AND THE CELL DECOMPOSITION OF $\mathcal{M}_{g,n} \times R_+^n$

To make the connection between the field theory skeleton graphs and $\mathcal{M}_{g,n} \times R_+^n$, we will need to take a small excursion into the topic of quadratic differentials [22] on Riemann surfaces. For a very nice and clear exposition of much of the material in this section, see [23]. See also [24] for a physicist's review of some general facts about quadratic and Strebel differentials.

A. Quadratic differentials

On any Riemann surface, we will be interested in considering meromorphic quadratic differentials. These take the form $\phi(z)dz^2$ in any complex coordinate chart (parametrized by z), with $\phi(z)$ being a meromorphic function of z in that chart. Under a holomorphic change of coordinates to $w = w(z)$, we have

$$\phi(z)dz^2 = \tilde{\phi}(w)dw^2 \Rightarrow \tilde{\phi}(w) = \phi(z(w))\left(\frac{dz}{dw}\right)^2. \quad (3.1)$$

Something that will be important for us is that, given a meromorphic quadratic differential, we can use it to define a (locally) flat metric on the Riemann surface. This is simply given by the line element

$$ds^2 = |\phi(z)|dzd\bar{z}. \quad (3.2)$$

This is well defined away from the zeros and poles of $\phi(z)$.

Another crucial notion is that of horizontal and vertical trajectories. Consider a curve $z(t)$ in a coordinate chart, where t is a parameter along the curve taking values in some interval of the real line. A curve $z_H(t)$ is called horizontal with respect to a given quadratic differential $\phi(z)$ if

$$\phi(z_H(t))\left(\frac{dz_H}{dt}\right)^2 > 0, \quad (3.3)$$

for all t in the interval. Similarly, a curve $z_V(t)$ is vertical if

$$\phi(z_V(t))\left(\frac{dz_V}{dt}\right)^2 < 0, \quad (3.4)$$

again for all t in the relevant interval.

The terminology is motivated by the simplest such differential on C, namely, dz^2. Then it is obvious that the horizontal trajectories are simply all horizontal lines (i.e. parallel to the real axis), and the vertical trajectories are all vertical lines, parallel to the imaginary axis. In fact, at a *regular* point on the Riemann surface, by a suitable change of coordinates, we can always put a general quadratic differential to be of the form dz^2 in the infinitesimal vicinity of that point. Then the horizontal and vertical trajectories in the vicinity of the point form a rectangular grid like in C.

Near a zero or a pole, however, the behavior is quite different. Any quadratic differential in the vicinity of a zero or pole can be put in the form

$$\phi(z)dz^2 = z^m dz^2. \quad (3.5)$$

Here m is an integer, positive for a zero and negative for a pole. However, a double pole, i.e. $m = -2$, will have to be treated specially. It is easy to verify (for $m \neq -2$) that the radial half lines

$$z_H(t) = t\exp\left(\frac{2\pi ik}{m+2}\right); \qquad t > 0, \qquad (k = 0\ldots m+1) \quad (3.6)$$

are horizontal trajectories and that

$$z_V(t) = t\exp\left(\frac{\pi i(2k+1)}{m+2}\right); \quad t > 0, \qquad (k = 0\ldots m+1) \quad (3.7)$$

are vertical trajectories. Note that, near a simple zero ($m = 1$), we have three horizontal (as well as three vertical) trajectories meeting at the location of the zero. A double zero has four horizontal trajectories intersecting, etc.

The case of a double pole, $m = -2$, is special. Consider a quadratic differential in the neighborhood of such a pole and taking the form

$$\phi(z)dz^2 = -\frac{p^2}{(2\pi)^2}\frac{dz^2}{z^2}. \quad (3.8)$$

Notice that by a single valued change of coordinates we cannot change the coefficient multiplying the double pole. We will denote this invariant coefficient [in this case $-[p^2/(2\pi)^2](dz^2/z^2)$] as the residue of the quadratic differential at the double pole. The reason for the above parametrization of the residue will become clear very soon.

We will, in fact, need to consider only p to be real and positive. In this case, it is easy to work out that the horizontal trajectories are concentric circles about the pole, i.e.

$$z_H(t) = re^{it}; \qquad t \in (0, 2\pi) \quad (3.9)$$

with r an arbitrary constant. Meanwhile, *any* radial line emanating from $z = 0$

$$z_H(t) = te^{i\theta}; \qquad t > 0, \tag{3.10}$$

with θ fixed is a vertical trajectory.

With respect to the flat metric defined by (3.2), we see that, near a double pole, the circular horizontal trajectories in (3.9) all have equal circumference p. This explains the parametrization of the residue in (3.8). Notice that the pole itself is at an infinite distance from any finite point. The geometry of the Riemann surface near a double pole is thus that of a semi-infinite cylinder of circumference p in which the horizontal trajectories are the circular cross sections, while the vertical trajectories are parallel to the axis of the cylinder.

We will be interested, in what follows, in quadratic differentials with only double poles.

B. Strebel differentials

So far, we have discussed the *local* structure of horizontal and vertical trajectories of a general meromorphic quadratic differential. The *global* structure of these trajectories is actually very interesting.

Consider a Riemann surface $\Sigma_{g,n}$ of genus g with n marked points (which we will identify with the punctures). Generically, a horizontal trajectory of a quadratic differential on $\Sigma_{g,n}$ will wander around the surface without closing on itself. However, there are *special* quadratic differentials on $\Sigma_{g,n}$ for which essentially all horizontal trajectories are closed curves. These special differentials have the following properties: They have *only* double poles which are all at the locations of the marked points. The residues at these double poles, as defined in (3.8), have p real and positive. The set of closed horizontal trajectories around each such pole foliate a punctured disc about this pole (a so-called maximal ring domain). The boundary of the disc contains a certain number of zeros of the quadratic differential. In fact, the boundary is a union of the *nonclosed* horizontal trajectories whose end points are the zeros. Moreover, the entire Riemann surface is a union of the n ring domains about each marked point (double pole) together with the nonclosed horizontal trajectories which comprise the boundary of these discs. Because of a theorem of Strebel stated below, we will call such a special quadratic differential a Strebel differential. A ring domain of a Strebel differential near a double pole is shown in Fig. 2.

The result of Strebel [22] states that, for every Riemann surface $\Sigma_{g,n}$ (with $n > 0$ and $2g + n > 2$) *and* any n specified positive numbers $(p_1, p_2, \ldots p_n)$, there exists a *unique* Strebel differential, namely, a quadratic differential that is holomorphic everywhere on $\Sigma_{g,n}$ except for the n marked points, where it has double poles with residue determined by p_i at the ith pole. It has the property that the ring domains about the poles, foliated by the closed horizontal trajectories, cover the entire surface. The measure zero set of boundaries of the ring domains are comprised of the nonclosed horizontal trajectories that begin and end on zeros.[2]

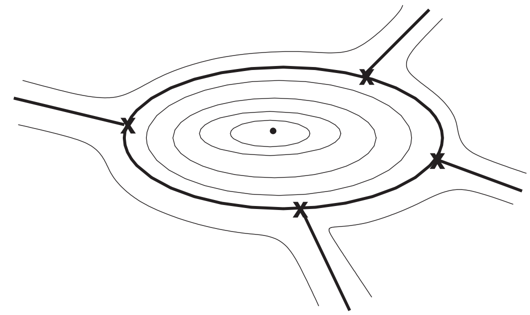

FIG. 2. A characteristic ring domain in the vicinity of a double pole (marked with a dot). The nonclosed horizontal trajectories are shown by thick lines. These begin and end at zeros marked by a cross.

C. The cell decomposition of $\mathcal{M}_{g,n} \times R_+^n$

What we have just stated is that associated with every point on the extended (or ‘‘decorated’’) moduli space $\mathcal{M}_{g,n} \times R_+^n$ is a unique Strebel differential. This result gives rise to a nice cell decomposition of $\mathcal{M}_{g,n} \times R_+^n$, which will then connect up with our field theory discussion.

First, notice that the set of nonclosed horizontal trajectories of a Strebel differential (i.e. the boundaries of the ring domains), on $\Sigma_{g,n}$, forms a graph which is embedded into the Riemann surface. By imagining the edges of the graph to be thickened (a double line or ‘‘ribbon’’ graph), we see that we can, as usual, associate a genus g to this graph. This is called the *critical graph* of the corresponding Strebel differential. Since, as we mentioned, the edges of this graph connect various zeros of the Strebel differential, the vertices where they meet are generically trivalent (corresponding to simple zeros). Because the ring domains are all punctured discs and are n in number, the resulting graph has n (polygonal) faces. A simple application of $V - E + F = 2 - 2g$ for the generic trivalent graph implies that the number of edges of the critical graph is $E = 6g - 6 + 3n$. The number of vertices, which is also the number of (simple) zeros, is $V = 4g - 4 + 2n$.

We can also assign a length to each edge of the critical graph. For an edge connecting zeros z_i and z_j, the length is

[2]This theorem of Strebel was used by Refs. [25,26] to construct the higher order contact vertices in *closed* string field theory. In that context, the residues were fixed to be all equal to 2π. Moreover, since the contact vertices were supposed to cover only the part of moduli space missed by the Feynman diagrams from lower order vertices, certain inequalities were also imposed on the lengths of closed cycles. This description of closed string field theory has been used to study quartic interactions [27–30].

given by

$$l_{ij} = \int_{z_i}^{z_j} \sqrt{\phi(z)}dz, \tag{3.11}$$

where the contour is chosen so that it is homotopic to the nonclosed horizontal trajectory connecting z_i with z_j. Note that the value of the integral is then independent of the particular contour and is actually real positive since it is real positive along the horizontal trajectory connecting the two zeros. Given these length assignments, we can define a combinatorial space $\mathcal{M}_{g,n}^{\text{comb}}$, following Kontsevich. It is the space of all ribbon graphs of genus g with n marked faces, with all vertices at least trivalent and a length assigned to each edge. Each inequivalent ribbon graph with trivalent vertices fills out a top dimensional cell in $\mathcal{M}_{g,n}^{\text{comb}}$. These different cells connect with each other when one edge collapses to zero length in the s channel, so to say. One can go to the adjacent cell by expanding out the collapsed vertex but now in the t channel.

It turns out that $\mathcal{M}_{g,n}^{\text{comb}}$ gives a cell decomposition of $\mathcal{M}_{g,n} \times R_+^n$. Let us describe the isomorphism between $\mathcal{M}_{g,n}^{\text{comb}}$ and the extended moduli space $\mathcal{M}_{g,n} \times R_+^n$. Given a point in $\mathcal{M}_{g,n} \times R_+^n$, we can immediately associate a point in $\mathcal{M}_{g,n}^{\text{comb}}$ via the result of Strebel. We just construct the unique Strebel differential on the corresponding surface $\Sigma_{g,n}$ (with specified residues at the n poles) and find its critical graph. This is a ribbon graph of genus g with n marked faces, and we can assign to the edges of this graph the Strebel lengths defined in (3.11). This gives us a unique point in $\mathcal{M}_{g,n}^{\text{comb}}$.

The reverse map is more interesting for us. Given a point in $\mathcal{M}_{g,n}^{\text{comb}}$ (i.e. a ribbon graph with specified lengths of the edges), there is a canonical way to construct a Riemann surface $\Sigma_{g,n}$. The geometric picture is that $\Sigma_{g,n}$ consists of semi-infinite flat cylinders glued onto each face of the given graph (in an oriented way). The circumference of the cylinder on the ith face is given by the sum of all the lengths of the edges bordering that face.

There is actually a systematic way of constructing $\Sigma_{g,n}$ from the ribbon graph, which will be important for open-closed string duality. We will not go into the details of the construction here, which can be found, for instance, in Ref. [23], but rather sketch the main elements involved. For each edge of the ribbon graph of assigned length l_{ij}, we construct an infinite strip of uniform width l_{ij} aligned parallel to the imaginary axis in C. On this strip we have the differential dz^2 in terms of the natural coordinate. How do we glue all these strips into a surface? When three or more edges meet at a vertex, we patch the corresponding strips together in a way familiar from open string field theory. For instance, for a trivalent vertex, in some neighborhood of the vertex we use a new coordinate $w \propto z^{2/3}$. This maps the part of each strip in the vicinity of the vertex into a wedge of angle $\frac{2\pi}{3}$. The three wedges are glued together in the w plane (see Fig. 3). The individual differentials dz^2 for each strip are now transformed into a differential of the form wdw^2 in the w plane which smoothly overlaps between the different strips in the vicinity of a vertex. For a general k-valent vertex of the ribbon graph, we use a $z^{2/k}$ map for gluing the strips together.

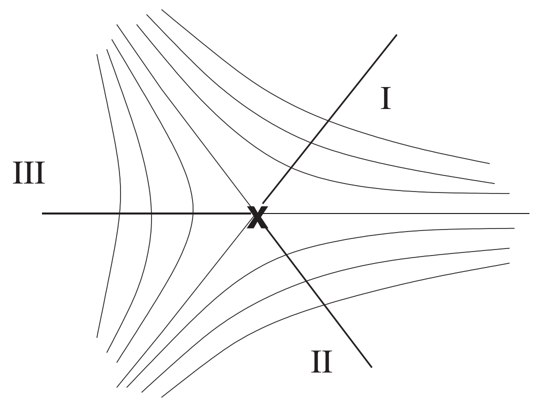

FIG. 3. Three strips glued in the vicinity of a trivalent vertex. The thick lines are the *horizontal* trajectories in the middle of each strip. The thin lines are representative *vertical* trajectories.

Similarly, when we have several edges bounding a face, we glue halves of the corresponding strips so that they form a cylinder (see Fig. 4). This can be done by an exponential map $u(z) \propto \exp(2\pi iz/p_i)$, which transforms the differentials dz^2 in the individual strips into a differential of the form

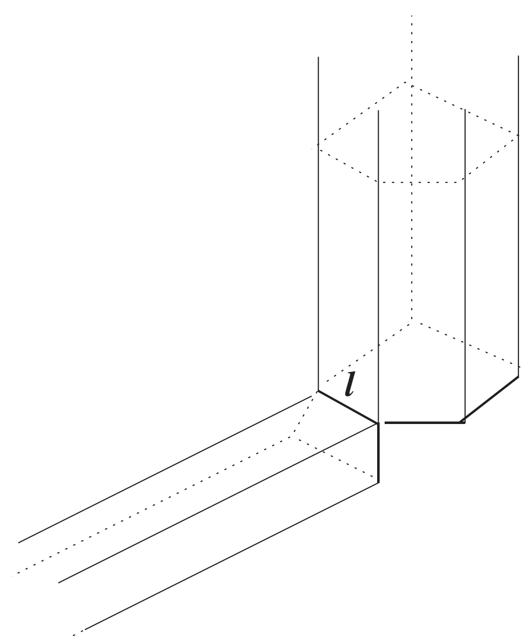

FIG. 4. Half strips glued together into cylinders. Only one full strip of width l is shown. Figure adapted from Ref. [32].

$$\tilde{\phi}(u)du^2 = -\frac{p_i^2}{(2\pi)^2}\frac{du^2}{u^2}.$$

Here p_i is the sum of the lengths of all the edges bounding the ith face, and the map is such that, as the argument of u goes over $(0, 2\pi)$, one goes over all the edges in the corresponding order. For explicit details of these maps, see [23].

Having specified the complex coordinates in each strip and the gluing rules at vertices and the center of faces, one can show that one has constructed a unique Riemann surface $\Sigma_{g,n}$. Moreover, the quadratic differential obtained by thus gluing together the ones in each chart is the unique Strebel differential for this surface with residues given by $\{p_i\}$. This Strebel differential, therefore, captures all the information about the point in $\mathcal{M}_{g,n} \times R_+^n$ that we obtain from this construction. Combined with the previous map from $\mathcal{M}_{g,n} \times R_+^n$ to $\mathcal{M}_{g,n}^{\rm comb}$, we have described an isomorphism between the two spaces.

What is of primary interest to us is that we have a definite mathematical procedure to build up a closed string surface of genus g with n punctures entirely out of flat strips glued together, starting from the data in the ribbon graph. In the next section, we will argue that this construction is exactly how open-closed string duality is implemented.

IV. SCHWINGER TIMES AND STREBEL LENGTHS

In Ref. [9] it was observed that the space of inequivalent skeleton graphs (with a Schwinger time associated to each edge) is isomorphic to the space $\mathcal{M}_{g,n}^{\rm comb}$ described above. This is because one can consider the space of dual graphs to the skeleton graphs considered in Sec. II. The dual graphs also have genus g but n faces. The generically triangular faces of the skeleton graph go over to generically trivalent vertices in the dual graph. These are exactly the set of ribbon graphs that appear in $\mathcal{M}_{g,n}^{\rm comb}$. Since the original edges were assigned a parameter τ_r, and since there is a dual edge for each edge, we also have a length assignment for each edge of the dual graph. Thus, we have an isomorphism from the moduli space of skeleton graphs to $\mathcal{M}_{g,n}^{\rm comb}$.

Using the further isomorphism between $\mathcal{M}_{g,n}^{\rm comb}$ and $\mathcal{M}_{g,n} \times R_+^n$ described in the previous section, it was argued in Ref. [9] that the field theory expression (2.5) was actually therefore an integral over $\mathcal{M}_{g,n} \times R_+^n$. While this argument is adequate to understand how field theory diagrams can reorganize into string amplitudes, it is not sufficient for extracting useful information about the dual string theory.

For that, one has to have a precise dictionary between the Schwinger times and the string moduli. Since the isomorphism between the Schwinger parameter space and the string moduli space was established via $\mathcal{M}_{g,n}^{\rm comb}$, we basically need a relation between the Schwinger times and the Strebel lengths which parametrize $\mathcal{M}_{g,n}^{\rm comb}$.

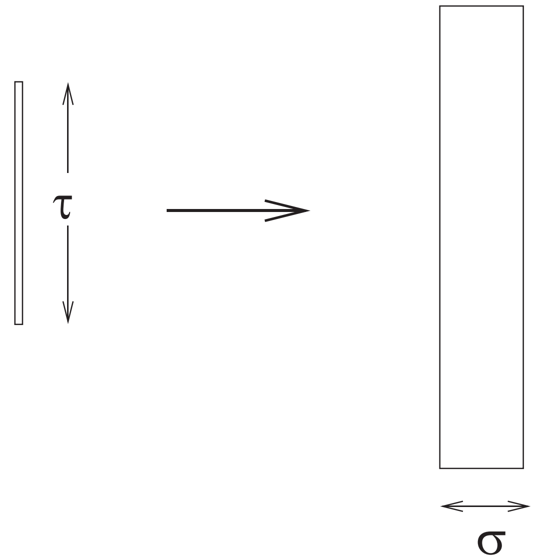

FIG. 5. A world line of length τ is equivalent to an infinite strip of width $\sigma = \frac{1}{\tau}$.

We will now argue that this is simply given by (1.1), i.e. we identify the inverse Schwinger times σ_r with the Strebel lengths l_r.[3] An heuristic argument for this identification is as follows. View the field theory world line as an open string world sheet whose width is going to zero. In fact, let us regularize the width to be ϵ with the understanding that it is to be eventually taken to zero. The length of this line is the Schwinger time τ. By means of a uniform conformal transformation on the world sheet, we map this to a flat world sheet of width $\sigma = \frac{1}{\tau}$ and length $\frac{1}{\epsilon}$ (see Fig. 5). In the limit as $\epsilon \to 0$, this is an infinite strip of fixed width σ. *We will identify these flat infinite strips with the ones that appeared in the gluing construction of the previous section.* These formed the building blocks for the closed Riemann surface. There we saw that the widths of the strips were the Strebel lengths of the closed string Riemann surface. By our identification of the open string world sheet with these strips, we see that $l = \sigma = \frac{1}{\tau}$.

Another, somewhat complementary, way to arrive at this identification of σ with the Strebel lengths is to notice that in the field theory, when we partially glued together homotopically equivalent lines, the effective Schwinger parameters were given by (2.3). In terms of the "conductances," this means simply that the σ's are *additive* on gluing. Now notice that the Strebel lengths are also additive when we glue two parallel strips with each other. In

[3]Note that, mathematically, there is no real restriction on the functional relation between l and σ. Any monotonic function f such as $f(0) = 0$ and $f(\infty) = \infty$ would fit the bill as far as the isomorphism between the moduli space of skeleton graphs and $\mathcal{M}_{g,n}^{\rm comb}$ is concerned.

constructing the Riemann surface, we simply place two such strips parallel to each other.

Therefore, since the functional relation between the σ and the l respects this additive property, the two have to be a multiple of each other. We can always set that multiple to one by a change of scale. This again suggests the dictionary (1.1). In addition to both these arguments, we will also see in the next section that this identification is consistent with Kontsevich's derivation.

Note that, having made the physical identification between the Schwinger times with the Strebel lengths, the field theory integrand, such as in (2.5), is now expressible directly in terms of parameters of the closed string. The parametrization in terms of Strebel lengths is perhaps an unfamiliar parametrization of $\mathcal{M}_{g,n}$, though it suggests a natural string field theory origin. In any case, as mentioned in the introduction, to put the integrand into familiar form, we should construct the map between the l_r and the usual complex moduli z_a of $\mathcal{M}_{g,n}$. This is a well-defined map since it is constructed via the Strebel differential. What is to be stressed is that the construction of this map is a purely mathematical question. We will return to it in Sec. VI.

The arguments of this section add up to a coherent picture of how open-closed string duality is being implemented. The original double line wick contractions in the field theory, we have seen, are equivalent to infinite open string strips with width $\tilde{\sigma} = 1/\tilde{\tau}$. Homotopically equivalent wick contractions are easily glued onto each other, and the width (σ_r) of the resultant strip is just the sum of the individual widths ($\tilde{\sigma}_{r\mu_r}$). This is the stage where we obtain the skeleton graph with effective Schwinger parameters. The strips corresponding to the skeleton graph are now glued together as described in the previous section. We now also see why the dual to the skeleton graph played a role in Ref. [9]. The open string strips are parallel to the *vertical* trajectories of the Strebel differential of the closed string world sheet. It is, therefore, the dual graph to the skeleton graph which is in correspondence with the (non-closed) *horizontal* trajectories which form the critical graph of the Strebel differential.

That strips of varying sizes are getting glued up also goes well with a bit picture in which bit world lines combine to form the closed string world sheet. It would be nice to make this more precise, perhaps in a lightcone gauge or as in the recent work of Alday *et al.* [31].

Though we had discussed mainly correlators in the context of free fields, it is clear that the process of open-closed duality we have described here is more general. We can consider an arbitrary field theory diagram in perturbation theory and carry out the procedure described in Sec. II and over here. All that changes is that the internal vertices give rise to additional marked points or punctures on the Riemann surface. These punctures would have closed string vertex operator insertions just as with the external punctures. We will need to exponentiate these contributions to obtain the string theory background for a finite 't Hooft coupling. In other words, spacetime perturbation theory leads to a world sheet perturbation theory.

In the next section, we see how many of these elements are concretely realized in Kontsevich's matrix model derivation of Witten's conjecture.

V. THE KONTSEVICH MODEL

Kontsevich proposed a one matrix model whose free energy served as the generating function of the closed string correlators of topological gravity. We will see that his derivation of this result can be viewed as a special case of our general approach. Being very explicit, it has the virtue of illustrating several features. In particular, he connects the flat measure over the space of Schwinger parameters to a top form on $\mathcal{M}_{g,n}$, which shows how the closed string correlators emerge from the integrand of the Schwinger parameter space.

To illustrate our points, we will actually reverse the logic of Kontsevich. Let us start from the Kontsevich Hermitian matrix model

$$Z(\Lambda) = c(\Lambda)^{-1} \int [DM] e^{-1/2\,\mathrm{Tr}[\Lambda M^2 - i(M^3/3)]}, \qquad (5.1)$$

where

$$c(\Lambda) = \int [DM] e^{-1/2\,\mathrm{Tr}(\Lambda M^2)}. \qquad (5.2)$$

The constant matrix Λ contains the couplings of the theory via the dictionary

$$t_i = -(2i-1)!!\,\mathrm{Tr}\Lambda^{-(2i+1)}. \qquad (5.3)$$

We can choose Λ to be a diagonal matrix with entries $\Lambda_{ab} = \Lambda_a \delta_{ab}$. What is of interest is the free energy of this matrix model which is a sum over the connected vacuum diagrams.

The vertices are all cubic, and the propagator is given by

$$\langle M_{ab} M_{cd} \rangle = \frac{2\delta_{ad}\delta_{bc}}{\Lambda_a + \Lambda_b}. \qquad (5.4)$$

So the Feynman diagrams are a sum over all double line graphs Γ with cubic vertices, which can be written as

$$\begin{aligned} F(t_0, t_1, \ldots) &= \sum_{\Gamma} \frac{(\frac{i}{2})^{V_\Gamma}}{|\Gamma|} \sum_{(r(a),r(b))} \prod_r \frac{2}{\Lambda_{r(a)} + \Lambda_{r(b)}} \\ &= \sum_{\Gamma} \frac{(\frac{i}{2})^{V_\Gamma} 2^{E_\Gamma}}{|\Gamma|} \sum_{(r(a),r(b))} \int \prod_r dl_r e^{-\sum_r l_r(\Lambda_{r(a)} + \Lambda_{r(b)})} \\ &= \sum_{g,n} \frac{1}{n!} \sum_{\Gamma_{g,n}} \sum_{a_i} \frac{(-1)^n 2^{2g-2+n}}{|\Gamma_{g,n}|} \\ &\quad \times \int \prod_r dl_r e^{-\sum_i \Lambda_{a_i} p_i}. \end{aligned} \qquad (5.5)$$

Notice that the l_r that have been introduced are Schwinger parameters for the propagator. It will be important for the derivation that these are later identified with the Strebel lengths. In the first line, the $r(a)$, $r(b)$ denote the two color indices associated with the double line for edge r. These are to be summed over, subject to the constraint that in every closed loop a single color flows. In fact, in the third line, this constraint has been explicitly taken into account. The sum is now only over the independent color indices a_i, where i labels the n different faces of the graph. (Here we have also organized the graphs Γ by genus g, with n marked faces n.) In the exponent, we have gathered the Λ's with the same color index, and, therefore, it multiplies the circumference

$$p_i = \sum_{r_i=1}^{m_i} l_{r_i}, \tag{5.6}$$

i.e. sum over the lengths of all the m_i edges that appear in the ith loop. The symmetry factors of the graphs have been denoted by $|\Gamma|$. (There is an extra symmetry factor of $n!$ in going to graphs $\Gamma_{g,n}$ with n marked faces.)

At this stage, we have expressed the matrix model free energy as an integral over the space $\mathcal{M}_{g,n}^{\mathrm{comb}}$ of ribbon graphs with Schwinger parameters for each edge. One of the important steps in Kontsevich's derivation is the conversion of the flat measure $\prod_r dl_r$ on $\mathcal{M}_{g,n}^{\mathrm{comb}}$ to one on $\mathcal{M}_{g,n} \times R_+^n$. Kontsevich obtains the following relation:

$$2^{5g-5+2n} \prod_{r=1}^{6g-6+3n} dl_r = \prod_{i=1}^{n} dp_i \times \frac{\Omega^d}{d!}, \tag{5.7}$$

where the p_i are the circumferences of the loops (5.6) and parametrize the R_+^n, while $\Omega^d/d!$ will be identified with a top form ($d = 3g - 3 + n$) on $\mathcal{M}_{g,n}$. In fact,

$$\Omega = \sum_{i=1}^{n} p_i^2 \omega_i; \qquad \omega_i = \sum_{1 \le r_i < r_i' < m_i - 1} d\left(\frac{l_{r_i}}{p_i}\right) \wedge d\left(\frac{l_{r_i'}}{p_i}\right). \tag{5.8}$$

Kontsevich identifies the ω_i above with $c_1(\mathcal{L}_i)$, the first Chern class of the cotangent line bundle $T^*\Sigma_{g,n}|_{z_i}$ at the puncture z_i. (See, for instance, [32] for a detailed explanation of this identification.) This step requires that the l_r be the Strebel lengths, i.e. the widths of the strips making up the closed string cylinders.

With this important step, Kontsevich has now converted the integral over the Schwinger parameters l_r into one over $\mathcal{M}_{g,n} \times R_+^n$, with the Jacobian giving a very natural top form on $\mathcal{M}_{g,n}$. Integrating $\Omega^d/d!$ over $\mathcal{M}_{g,n}$ gives various intersection numbers of the classes ω_i. These are exactly the correlators of topological gravity according to Witten [33]. So we have not just any integral over moduli space, rather, a very natural one from the point of view of the closed string world sheet.

Kontsevich then goes on to perform the integral over the p_i and, thus, obtain a generating function for the intersection numbers (i.e. closed string correlators).

$$\begin{aligned} F(t_0, t_1, \ldots) &= \sum_{g,n} \frac{(-1)^n}{n!} \sum_{a_i} \sum_{\{d_i\}} \int_0^\infty \prod_{i=1}^{n} dp_i \frac{p_i^{2d_i}}{2^{d_i} d_i!} e^{-\Lambda_{a_i} p_i} \\ &\quad \times \int_{\mathcal{M}_{g,n}} \prod_{i=1}^{n} \omega_i^{2d_i} \\ &= \sum_{g,n} \frac{(-1)^n}{n!} \sum_{a_i} \sum_{\{d_i\}} \prod_i \frac{(2d_i)!}{2^{d_i} d_i!} \Lambda_{a_i}^{-(2d_i+1)} \\ &\quad \times \int_{\mathcal{M}_{g,n}} \prod_{i=1}^{n} \omega_i^{2d_i} \\ &= \sum_{g,n} \frac{1}{n!} \sum_{\{d_i\}} \prod_i t_{d_i} \left\langle \prod_i V_{d_i} \right\rangle. \end{aligned} \tag{5.9}$$

In the summation over d_i, it is understood that $\sum_i d_i = d$. The last line has used the definition (5.3) to write the final expression in the desired form of generating function of closed string correlators. In fact, this could be exponentiated and interpreted as a closed string theory in a background specified by the couplings t_i.

We can take away a few lessons for our general approach from this derivation. First, the identification of the Schwinger parameters with the Strebel lengths is crucial here. It allows the identification of the ω_i defined in (5.8) with the tautological classes $c_1(\mathcal{L}_i)$ on moduli space. Second, the change of measure from the flat one for Schwinger parameters to one on $\mathcal{M}_{g,n} \times R_+^n$ gives rise to a top form on $\mathcal{M}_{g,n}$. Notice that this change of variables is common to all Schwinger integrals in field theory, which means that this particular piece of the closed string correlators would be *common* to all the putative string duals to field theory. This is a nice feature since these are the basic $2d$-gravity correlators [33–36] which one might expect are present in all closed string correlators.

It would be nice to understand the Chern-Simons topological string duality [5] in such terms. Like the Chern-Simons theory, the Kontsevich model has also been interpreted as an open string field theory [7] and used to illustrate open-closed string duality. The matrix model variables have been understood in terms of strings stretching between Fateev-Zamolodchikov-Zamolodchikov-Teschner branes [7,37]. For generalizations of this system, see [38].

VI. THE FOUR POINT FUNCTION

How can we use this approach to open-closed string duality to further our understanding of the closed string duals to field theories? The dictionary (1.1) applied to field theory expressions such as (2.5) gives, in principle, a candidate closed string correlator on moduli space. We would like, for instance, to check for a general field theory whether this is indeed a correlator of a consistent world sheet CFT. The only way to do this, at present, seems to require us to know the correlator as a function of the holomorphic coordinates on moduli space. In these coordinates, various properties of the CFT would be manifest.

To do this, we should write the Strebel lengths l_r in terms of the complex coordinates z_a on moduli space (and the residues p_i). This is a mathematically well posed problem. By constructing the unique Strebel differential on $\mathcal{M}_{g,n} \times R^n_+$, we obtain the relation between the l_r and the z_a. The problem for us is that the explicit construction of the Strebel differential at an arbitrary point in moduli space is not an easy task. As we will see, one can easily write down the general quadratic differential having double poles at n specified points. This is actually a vector space of complex dimension $3g - 3 + 2n$. Fixing the residues at the n double poles cuts the dimension down by n. Nevertheless, we have a $3g - 3 + n$-dimensional space of quadratic differentials. To find the unique Strebel differential in this space, we need to further impose the reality of various Strebel lengths (3.11). This condition is generally an implicit one for the parameters of the quadratic differential. To solve for this condition and fix the Strebel differential is, therefore, not easy. This is what makes the task of expressing the l_r as an explicit function of the z_a difficult.

Let us illustrate this by looking at the simplest nontrivial case, namely, that of the four punctured sphere. We will use $SL(2, C)$ invariance to place the punctures at $(1, \pm t, \infty)$. The general quadratic differential with double poles at $(1, \pm t, \infty)$ is of the form

$$\phi(z)dz^2 = -\left(\frac{a}{2\pi}\right)^2 \frac{\prod_{i=1}^{4}(z - z_i)dz^2}{(z-1)^2(z^2 - t^2)^2}. \tag{6.1}$$

The numerator is fixed to be a polynomial of degree four because of the requirement of a double pole at ∞. We thus have a vector space of five complex parameters (a, z_i). Fixing the four real residues $(p_1, p_\infty, p_\pm)$ gives four complex conditions on the (a, z_i). Note that these are all algebraic conditions. But we still have one complex or two real parameters which are not fixed thus far. As mentioned above, these are fixed by demanding reality of two of the Strebel lengths (3.11) between the zeros z_i.[4] We see that the condition

$$\mathrm{Im} \int_{z_i}^{z_j} \sqrt{\phi(z)}dz = 0 \tag{6.2}$$

is a transcendental condition on the parameters (a, z_i). For $\phi(z)$ as in (6.1), the integral in (6.2) is an elliptic integral and is not easy, in general, to solve.[5] This is the obstruction to explicitly obtaining the Strebel differential and, thus, the Strebel lengths/Schwinger parameters

$$l_{ij} = \int_{z_i}^{z_j} \sqrt{\phi(z)}dz. \tag{6.3}$$

Since the string amplitudes are integrals over the entire moduli space, it might seem that we need the change of variables at a generic point on the moduli space. We would thus be up against a technical roadblock in extracting the physical CFT correlators from the Schwinger parametrized expressions of the field theory.

But, fortunately, this is not quite true. As we will now explain, by going to appropriate kinematical limits of the spacetime correlator, we can focus our attention on *limiting regions of moduli space*. In these limits, we will see that there are some simplifications.

The kinematical limits mentioned above are different UV limits in the field theory correlator. The dominant contribution to the Schwinger integrand in these limits will actually come from the region of the moduli space $\mathcal{M}_{0,4}$ where two of the punctures come together. In other words, we will really need only the relation between the Strebel lengths and the complex parameter t [in the notation of (6.1)] in the regime where $t \to 0$. Note that this is precisely the region where we expect the integrand to satisfy a *world sheet* operator product expansion. Having an explicit relation between the Schwinger parameters and t in this limit will give us a way to check whether this is actually so, and, if so, we can, in principle, extract out the world sheet operator product expansion (OPE) coefficients and thus get some insight into the world sheet CFT. We hope to carry this through fully later. Here we will only set up the limiting forms of the Strebel differential and show that its scaling implies the existence of a world sheet expansion in powers of $|t|^{1/2}$.

A. UV limits of the field theory

For concreteness, let us stick to a simple free field planar four point correlator of the form

$$G^{(4)}(k_1, k_2, k_3, k_4) = \langle \mathrm{Tr}\Phi^3(k_1)\mathrm{Tr}\Phi^3(k_2)\mathrm{Tr}\Phi^3(k_3)\mathrm{Tr}\Phi^3(k_4)\rangle. \tag{6.4}$$

There are a handful of graphs which contribute to the connected piece of this correlator. We will focus on the one which involves the maximal number of effective Schwinger parameters. This is the tetrahedral graph shown in Fig. 6. There is no partial gluing to be done here, and, therefore, it is also the skeleton graph. As shown in the

[4]Since we have already imposed the conditions on residues, which are sums of Strebel lengths, there are only two independent reality conditions that we need to impose on the Strebel lengths. The rest are automatically real once these are imposed.

[5]See [28,29] for a combination of numerical and analytic approaches to solve the Strebel conditions in the restricted case, arising in closed string field theory, where the residues are all equal.

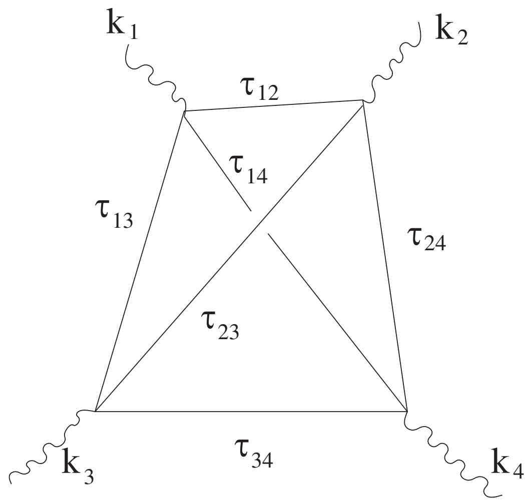

FIG. 6. Tetrahedral graph for four point function with Schwinger times for the six edges.

figure, it has six Schwinger parameters τ_{ij}, the correct number to parametrize an open set in $\mathcal{M}_{0,4} \times R_+^4$. This is the reason we chose this correlator: It has the minimal number of fields which yields a tetrahedral skeleton graph. The other graphs that contribute to this correlator have skeleton graphs which have fewer edges. They will be special cases (some of the $\tau_{ij} \to \infty$) of the general case considered in the tetrahedral graph.

We are interested in taking the short distance or UV limit of this correlator. The Schwinger parametrized form of this tetrahedral graph contribution to the four point function is

$$G^{(4)}(k_1, k_2, k_3, k_4) = \int_0^\infty \frac{\prod_{(ij)} d\tau_{ij}}{\Delta(\tau)^{d/2}} \exp\{-P(\tau, k)\}. \quad (6.5)$$

We can use the explicit graph theoretic expressions for $\Delta(\tau)$ and $P(\tau, k)$ given in Ref. [9]. These are written out for this graph in various forms in the appendix. For our purpose of taking UV limits, a useful hybrid form is in terms of τ_{12}, τ_{34}, and σ_{ij} for the rest of the edges. This is given in Eqs. (A5)–(A8). Let us first take the limit $k_{12}^2 \propto (k_1 - k_2)^2 \to \infty$ (keeping other momenta finite). This corresponds in position space to taking the separation $|x_1 - x_2|^2 \to 0$. As can be seen from Eq. (A8), the nonzero contributions in this short distance limit come from a region of the Schwinger parameter space where $\tau_{12}\sigma \propto 1/k_{12}^2$, i.e. the ratio $\sigma/\sigma_{12} \to 0$. If k_{34} is finite, then we also see from (A8) that σ/σ_{34} is finite in this limit. So we essentially have $\sigma_{12} \to \infty$ while all other Schwinger parameters are finite. This is physically reasonable, since we expect a UV limit to correspond to a proper time (in this case, τ_{12}) going to zero.

In fact, the entire spacetime OPE expansion can be generated by making an appropriate expansion of the Schwinger integrand (6.5) [using expressions (A6) and (A8)] in powers of τ_{12}. We keep the exponential term $e^{-\tau_{12}k_{12}^2}$, but expand all other terms in $P(\tau, \sigma, k)$ and $\Delta(\tau, \sigma)$ containing τ_{12}, in a power series. Since $\tau_{12} \propto 1/k_{12}^2$, carrying out the τ_{12} integral essentially gives more and more inverse powers of k_{12}. These are the different terms in the spacetime OPE. For instance, the leading term in the τ_{12} expansion corresponds to the integrand of the three point function $\langle \mathrm{Tr}\Phi^4(k_1 + k_2)\mathrm{Tr}\Phi^3(k_3)\mathrm{Tr}\Phi^3(k_4)\rangle$ coming from the leading connected contraction. The higher terms in the τ_{12} expansion contain both descendants as well as other conformal primary operators (of the spacetime field theory). Note that the presence of terms linear in k_{12} in the exponent does not affect the expansion. They are actually necessary to reproduce the appropriate tensor structure in an expansion about large k_{12}. So what we are seeing here is how the Schwinger parametric representation implements the *spacetime* OPE. This is not too much of a surprise. The interesting thing for us is to identify this expansion in powers of τ_{12} with a *world sheet* expansion.

Because of our identification of Schwinger parameters σ with Strebel lengths, we see that $\tau_{12} \to 0$ corresponds to taking a particular limit in the space $\mathcal{M}_{0,4} \times R_+^4$. Effectively, therefore, only this limiting region of moduli space is relevant in this kinematic regime. What is this region? We will see from a consideration of the Strebel differential that the limit where one of the Strebel lengths is much larger than the others corresponds to two punctures coming together, i.e. $t \to 0$. Actually, physical reasoning leads one to guess this conclusion. After all, a UV limit in the field theory corresponds to an IR limit in the closed string dual, which in turn must come from a UV limit on the world sheet. Thus, we expect to see the short distance structure of the world sheet theory in this limit. In particular, if there is a closed string dual, then the integrand in (6.5) must exhibit a world sheet OPE consistent with being a CFT.

To extract this world sheet OPE, we therefore need just find the change of variables from the Strebel lengths to t, *in the vicinity of* $t = 0$.[6] In other words, it is sufficient to consider a *tangent space* approximation to the moduli space $\mathcal{M}_{0,4}$ around the singular (stable) curve $t = 0$.

For the present, we will only construct the scaling behavior of the limiting Strebel differential. The change of variables following from this construction, as well as its use in obtaining a systematic expansion in powers of $|t|$, will be postponed for later.

Another interesting UV limit in the field theory is when *both* $k_{12}^2 \to 0$ and $k_{34}^2 \to 0$. In many ways, this is more symmetric and natural in a spacetime conformal field theory.[7] In this particular limit, we see that the contributions come from the vicinity of $\tau_{12}, \tau_{34} \to 0$ keeping the

[6]The $\mathcal{M}_{0,4}$ is the part where the change of variables is nontrivial. The R_+^4 part is trivially given by the sum of the Strebel lengths/Schwinger parameters around each face of the tetrahedron.

[7]We thank A. Sen for this remark.

rest of the σ_{ij} finite. This limit of the Strebel lengths also corresponds to bringing the two punctures together, as might be expected from the previous physical reasoning. The difference is that the limit in the R^4_+ part of $\mathcal{M}_{0,4} \times R^4_+$ is not the same as in the earlier case. In the next subsection, we will construct the limiting Strebel differential in this as well as the previous case.

B. Limiting Strebel differentials on $\mathcal{M}_{0,4}$

Let us consider general quadratic differentials of the form (6.1) in the vicinity of $t = |t|e^{i\theta} \to 0$. The Strebel differential is uniquely determined given t and residues specified by $\{p_a\} = (p_1, p_\infty, p_\pm)$ at the double poles $(1, \infty, \pm t)$, respectively. Therefore, we would like to obtain $z_i = z_i(t, p_a)$ and $a = a(t, p_a)$ for the Strebel differential. Actually, we immediately see from the residue at ∞ that

$$a = p_\infty. \tag{6.6}$$

This is an exact relation, valid for any t.

Let us now look at the behavior of the differential as $t \to 0$. We will look at the two cases mentioned in the previous subsection, namely, one where one of the Strebel lengths is very large compared to the others, or, equivalently, when all but one length is scaling to zero, with the relative ratios of these finite in the limit.[8] The other case is when two lengths on opposite sides of the tetrahedron [the duals to the edges (12) and (34)] are kept finite while others scale uniformly to zero.

In the first case, we will take the Strebel length of the edge separating the poles at $z = 1, \infty$ finite while all others scale to zero. In the second, we will take the length of the edge separating the poles $\pm t$ finite as well.

In these limits, we would like to look at the behavior of the zeros $z_i(t)$. We will often suppress the dependence on the p_a when it does not create confusion. Let us make the scaling ansatz

$$z_i(t) = |t|^{\alpha_i}\tilde{z}_i(t), \tag{6.7}$$

with $\tilde{z}_i(t) \to \tilde{z}_i$ a finite limit as $t \to 0$.

In either limiting case, since the length of the edge separating $z = 1, \infty$ is finite, we must have $p_1 = p_\infty$ finite in the limit $t \to 0$. Looking at the residue at $z = 1$ then immediately implies that $\alpha_i > 0$. So the quadratic differential looks to leading order like

$$\phi(z)dz^2 = -\left(\frac{a}{2\pi}\right)^2 \frac{(z - |t|^{\alpha_1}\tilde{z}_1)(z - |t|^{\alpha_2}\tilde{z}_2)(z - |t|^{\alpha_3}\tilde{z}_3)(z - |t|^{\alpha_4}\tilde{z}_4)dz^2}{(z-1)^2(z^2-t^2)^2}. \tag{6.8}$$

In the limit as $t \to 0$, we will have the limiting differential taking the simple form

$$\mathrm{Lim}_{t\to 0}\phi(z)dz^2 = -\left(\frac{a}{2\pi}\right)^2 \frac{dz^2}{(z-1)^2}. \tag{6.9}$$

In this form, we see only two of the poles at $z = 1, \infty$. In fact, in these coordinates, the coincident point of the punctures ($z = 0$) is a regular point.

To look at the degenerating Riemann surface more closely, we must use the familiar plumbing fixture construction to go to new coordinates $w = \frac{|t|}{z}$. This will enable us to zoom in on the behavior near the colliding poles. Using the rule for transformation for quadratic differentials, we have

$$\tilde{\phi}(w)dw^2 = -\left(\frac{a}{2\pi}\right)^2 \frac{\tilde{z}_1\tilde{z}_2\tilde{z}_3\tilde{z}_4|t|^{\alpha_1+\alpha_2+\alpha_3+\alpha_4-2}}{e^{4i\theta}} \frac{\prod_{i=1}^4(w - |t|^{1-\alpha_i}w_i)dw^2}{w^2(w-|t|)^2(w^2-e^{-2i\theta})^2}. \tag{6.10}$$

Here $w_i = 1/\tilde{z}_i$. In the w coordinates, the poles at $z = \pm t$ are now at finite location $w = \pm e^{i\theta}$. Here it is the other set of poles, at $w = 0, |t|$, that appear to be colliding, exactly as the plumbing fixture is designed to exhibit.

We can now fix the values of the α_i in either of the two different limiting cases. In the first case, where all but one edge has length going to zero with all others scaling uniformly to zero, we can argue either from symmetry or from a more careful analysis of the Strebel condition that we must have all the $\alpha_i = \alpha$. Furthermore, the residues $p_\pm$ are seen to scale as $p_\pm \propto |t|^{2\alpha-1}$. Similarly, the Strebel length between any two of the zeros can be seen to scale as $|t|^\alpha$ [rescale w in (6.10) by $|t|^{1-\alpha}$]. Requiring that these two scale in the same way (since all these Strebel lengths are supposed to go uniformly to zero) fixes $\alpha = 1$.

In the second case, we instead demand that the length of the edge separating $w = \pm e^{i\theta}$ also remains finite as $|t| \to 0$ (while others scale uniformly to zero). This immediately

[8] Recall that we have the freedom to make an overall scaling of all lengths. This does not affect the parametrization of the closed string surface.

implies that the residues $p_\pm$ are finite as well, and, therefore, from (6.10), again since all α_i are equal, we must have $4\alpha - 2 = 0$, i.e. $\alpha = \frac{1}{2}$.

Thus, we can readily fix the scaling of the zeros and, therefore, the Strebel lengths in both the limits. This is interesting, because we see, for instance, in the second case, that the Strebel lengths vanish as $|t|^{1/2}$. At the same time, the circumferences p_i are finite. In other words, τ_{12} and τ_{34} vanish as $|t|^{1/2}$. As we argued earlier, the spacetime OPE corresponds to viewing the Schwinger integrand in an expansion in powers of τ_{12}, τ_{34}. We now see that this corresponds to a world sheet expansion in powers of $|t|^{1/2}$. This is already quite nice, since we know that the general world sheet OPE in a CFT can have powers of $|t|^{1/2}$, where t is the separation of the punctures. In fact, such a fractional power is perhaps a signature of a dual fermionic string theory. Of course, locality on the world sheet would be a consistency condition on any Schwinger integrand for it to qualify as a candidate string correlator. In other words, even though we are seeing half integral powers here, the sum of all contributions to an amplitude must be single valued as a function of t. This would be quite a nontrivial check from the point of view of the field theory.

In any case, in the above we have set up the limiting Strebel differentials for a systematic study in an expansion in powers of $|t|$. To be able to extract the precise world sheet OPE, we will need to proceed further and get the exact dependence of the Strebel lengths on the p_i and $\theta(= \arg t)$, in this systematic expansion. We plan to return to this in later work.

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APPENDIX: SCHWINGER PARAMETRIZED FOUR POINT FUNCTION

We will use the general graph theoretic expressions for the functions $\Delta(\tau)$ and the Gaussian exponent $P(\tau, k)$ in the Schwinger parametrization

$$\Delta(\tau) = \sum_{T_1}\left(\prod^{l}\tau\right), \tag{A1}$$

$$P(\tau, k) = \Delta(\tau)^{-1}\sum_{T_2}\left(\prod^{l+1}\tau\right)\left(\sum k\right)^2. \tag{A2}$$

For the definition of the 1-trees T_1 and 2-trees T_2, etc., entering into this definition, we refer the reader to Ref. [9], where these are reviewed.

For the particular case of the tetrahedral graph shown in Fig. 6, the complete expressions are

$$\Delta(\tau) = (\tau_{13}\tau_{23}\tau_{24} + \tau_{14}\tau_{13}\tau_{23} + \tau_{14}\tau_{23}\tau_{24} + \tau_{14}\tau_{13}\tau_{24}) + \tau_{12}(\tau_{23} + \tau_{24})(\tau_{13} + \tau_{14}) + \tau_{34}(\tau_{14} + \tau_{24})(\tau_{13} + \tau_{23}) + \tau_{12}\tau_{34}(\tau_{13} + \tau_{14} + \tau_{23} + \tau_{24}), \tag{A3}$$

$$P(\tau, k) = \Delta(\tau)^{-1}[\tau_{13}\tau_{14}\tau_{23}\tau_{24}(k_1 + k_2)^2 + \tau_{12}\{\tau_{14}\tau_{13}(\tau_{23} + \tau_{24})k_1^2 + \tau_{23}\tau_{24}(\tau_{13} + \tau_{14})k_2^2\} + \tau_{34}\{\tau_{13}\tau_{23}(\tau_{14} + \tau_{24})k_3^2 + \tau_{14}\tau_{24}(\tau_{13} + \tau_{23})k_4^2\} + \tau_{12}\tau_{34}\{\tau_{14}\tau_{13}k_1^2 + \tau_{23}\tau_{24}k_2^2 + \tau_{13}\tau_{23}k_3^2 + \tau_{14}\tau_{24}k_4^2 + \tau_{14}\tau_{23}(k_1 + k_3)^2 + \tau_{13}\tau_{24}(k_1 + k_4)^2\}]. \tag{A4}$$

Here we have gathered together terms involving τ_{12} and τ_{34}, since we will be interested in the limiting behavior of these proper times when we take various UV limits.

We will rewrite this in hybrid form in terms of τ_{12}, τ_{34}, and $\sigma_{ij} = 1/\tau_{ij}$ for the rest of the edges.

$$P(\tau, \sigma, k) = \Delta(\tau, \sigma)^{-1}[(k_1 + k_2)^2 + \tau_{12}\{(\sigma_{23} + \sigma_{24})k_1^2 + (\sigma_{13} + \sigma_{14})k_2^2\} + \tau_{34}\{(\sigma_{14} + \sigma_{24})k_3^2 + (\sigma_{13} + \sigma_{23})k_4^2\} + \tau_{12}\tau_{34}\{\sigma_{24}\sigma_{23}k_1^2 + \sigma_{14}\sigma_{13}k_2^2 + \sigma_{14}\sigma_{24}k_3^2 + \sigma_{13}\sigma_{23}k_4^2 + \sigma_{13}\sigma_{24}(k_1 + k_3)^2 + \sigma_{14}\sigma_{23}(k_1 + k_4)^2\}], \tag{A5}$$

$$\Delta(\tau, \sigma) = (\sigma_{13} + \sigma_{14} + \sigma_{23} + \sigma_{24}) + \tau_{12}(\sigma_{23} + \sigma_{24})(\sigma_{13} + \sigma_{14}) + \tau_{34}(\sigma_{14} + \sigma_{24})(\sigma_{13} + \sigma_{23}) + \tau_{12}\tau_{34}(\sigma_{13}\sigma_{23}\sigma_{24} + \sigma_{13}\sigma_{23}\sigma_{14} + \sigma_{14}\sigma_{23}\sigma_{24} + \sigma_{13}\sigma_{14}\sigma_{24}). \tag{A6}$$

In terms of new momentum variables

$$k_s = \tfrac{1}{2}(k_1 + k_2) = -\tfrac{1}{2}(k_3 + k_4), \qquad k_{12} = \tfrac{1}{2}(k_1 - k_2), \qquad k_{34} = \tfrac{1}{2}(k_3 - k_4), \tag{A7}$$

(A5) becomes

$$\begin{aligned} P(\tau, \sigma, k) = \Delta(\tau, \sigma)^{-1}[&k_s^2 + (\tau_{12} + \tau_{34})\sigma k_s^2 + \tau_{12}\sigma k_{12}^2 + \tau_{34}\sigma k_{34}^2 + 2\tau_{12}(\sigma_{23} + \sigma_{24} - \sigma_{13} - \sigma_{14})k_s \cdot k_{12} \\ &- 2\tau_{34}(\sigma_{14} + \sigma_{24} - \sigma_{13} - \sigma_{23})k_s \cdot k_{34} + \tau_{12}\tau_{34}\{(\sigma_{24} + \sigma_{13})(\sigma_{23} + \sigma_{14})k_s^2 + (\sigma_{24} + \sigma_{14})(\sigma_{23} + \sigma_{13})k_{12}^2 \\ &+ (\sigma_{23} + \sigma_{24})(\sigma_{13} + \sigma_{14})k_{34}^2 + 2(\sigma_{13}\sigma_{24} - \sigma_{14}\sigma_{23})k_{12} \cdot k_{34} + 2(\sigma_{23}\sigma_{24} - \sigma_{13}\sigma_{14})k_s \cdot k_{12} \\ &+ 2(\sigma_{13}\sigma_{23} - \sigma_{14}\sigma_{24})k_s \cdot k_{34}\}]. \end{aligned} \tag{A8}$$

Here $\sigma \equiv \sigma_{13} + \sigma_{14} + \sigma_{23} + \sigma_{24}$.

Note that the dependence on the σ's and the τ's in this expression are quite suggestive. For example, many of the particular sums of σ's which appear are given in terms of the circumferences p_i alone of the critical graph of the closed string surface. We expect that this is a signature of a nice form for the dual string correlator once it is expressed fully in terms of (t, p_i).

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