BPS objects in $N = 2$ supersymmetric gauge theories

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(Received 5 August 2005; published 30 September 2005)

We explore BPS soliton configurations in $N = 2$ supersymmetric Yang-Mills theory with matter fields arising from parallel D3 branes on D7 branes. Especially we focus on a two parameter family of $1/8$ BPS equations, dyonic objects, and $1/8$ BPS objects and raise a possibility of the absence of BPS vortices when the number of D3 branes is larger than that of D7 branes.

DOI: 10.1103/PhysRevD.72.065023 PACS numbers: 11.27.+d, 11.25.−w, 11.30.Pb

I. INTRODUCTION

Recently there has been considerable interest in BPS solitons in the Higgs phase of supersymmetric Yang-Mills theories with eight supercharges [1–6]. Almost all known BPS objects, like magnetic flux vortices [7–9], magnetic monopoles [10], domain walls [11,12], and instantons [4,13–15], have appeared here, sometimes with a bit of a twist. These theories can allow many degenerate vacua which can be interpolated by domain walls. With broken U(1) gauge theories, one can have magnetic flux vortex. One of the most interesting features has been that there can be magnetic monopoles which appear as beads on vortex strings [16].

These BPS objects can be interpreted in a simple manner from a D-brane point of view [17]. A simple but rich picture appears with N parallel D3 branes and N_f D7 branes. In this setting, one can have $N = 2$ supersymmetric (SUSY) U(N) gauge theory with N_f matter hypermultiplets in the fundamental representation and a single adjoint hypermultiplet. One can add Fayet-Iliopoulos (FI) terms and a mass term for the matter hypermultiplet without breaking the supersymmetry. There is considerable work done along this line to represent the configurations in the brane picture [18].

In this work, we focus on BPS equations, dyonic $1/4$ BPS, and $1/8$ BPS solutions. In addition, we explore BPS vortex equations when $N = 2$, $N_f = 1$ and find the cases where there are no vortex solutions of unit or double vorticity.

By studying the known bosonic BPS equations, we found that there are two parameter families of $1/8$ BPS equations in $3 + 1$ dimension modulo spatial rotation and $SU(2)_R \times U(1)_R$. The FI term breaks $SU(2)_R$ to U(1) and the mass terms for matter hypermultiplet breaks $U(1)_R$ completely. One would expect more general BPS configurations in this setting.

Dyonic objects mean objects carrying "electric" charge. Of course there will be no isolated electric charge in the Higgs phase due to screening. Electrically charged solitons could be interpreted as composites of soliton with fundamental strings whose ends carry electric charge. In the Higgs phase the electric charge is neutralized by an electric charge carried by the Higgs field. As the Higgs fields carry global flavor charge, the conserved flavor charge instead of the total electric flux would appear in the BPS energy formula. Besides dyonic monopoles, we show that dyonic domain walls as well as dyonic composites of domain wall-monopole-vortex are also possible. When parallel D7 branes are not lying on a single line in their transverse space, dyonic BPS configurations which make weblike structures are also possible. These dyonic solutions could be interpreted as the excitations in phase moduli of BPS objects and they belong to $1/4$ BPS states.

We also look for BPS solutions preserving $1/8$ of eight supersymmetries. By exploring a small perturbation of a homogeneous $1/4$ BPS configuration in $3 + 1$ dimensional theories, we argue that there may be no $1/8$ BPS configurations satisfying the BPS equations. However we find easily $1/8$ BPS configurations in a theory with product gauge group $U(1) \times U(1)$ with bifundamental and fundamental matter fields. In this analysis, the recently discovered [5] bound states of monopoles and domain walls play some role.

The key aspect here is that the FI parameters break the $SU(2)_R$ symmetry of the $5 + 1$ dimensional theory with eight supercharges. For a single U(1) gauge group, one can use the broken R symmetry to choose a single direction in $SU(2)_R$ space. However with product gauge groups, the FI parameters cannot be rotated to a single direction in general. This is what allows the presence of $1/8$ BPS configurations to be possible.

As there are multi-BPS vortex string configurations in U(1) theory with $N_f = 1$, we may expect there are BPS vortex string configurations in U(2) theory with $N_f = 1$. While there exist degenerate supersymmetric vacua, we will show that classically there exists no BPS vortex configuration with unit and double magnetic flux. We argue that this may imply that there exists no BPS vortex solitons of finite magnetic flux in the theory.

One interesting direction to explore further is the interaction between domain walls and monopoles. (See also a recent work by Sakai and Tong [5].) In the string picture,

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parallel D1 and D3 branes are attracted to each other. This is not apparent from the energy argument of a BPS monopole-vortex-domain composition. The moduli space of domain wall-monopole separation should be analyzed carefully to resolve the question.

Another direction is to study the moduli space dynamics of magnetic monopoles and domain walls when some of the non-Abelian gauge symmetry is restored. It would be interesting to see whether there exists a similar restoration of symmetry in the moduli space dynamics.

Finally, all BPS solutions we study here have extended structures with infinite energy. There may be finite action BPS solitons in the theory. Especially it may be possible to have finite energy (dyonic) instantons in $R^3 \times S^1$ (noncommutative) space, which do not have diverging gauge flux [15].

The plan of this paper is as follows. In Sec. II, we describe $5+1$ dimensional supersymmetric Yang-Mills theories and find supersymmetric Lagrangian and its vacuum structure. In Sec. III, we find two parametered BPS equations, especially $1/8$ BPS equations. In Sec. IV, we study dyonic solutions. In Sec. V, we study $1/8$ BPS configurations and find BPS configurations with product gauge group. In Sec. VI, we show that there exists no BPS vortex solitons of unit and double magnetic flux when $N=2$ and $N_f=1$.

II. SIX DIMENSIONAL CASE

The vector multiplet of super Yang-Mills theory of the $U(N)$ gauge group with eight supersymmetries in six dimensions is made of A_M, $\lambda_i (i=1,2)$, $\mathbf{D}^a$, which are Hermitian $N\times N$ matrix valued fields. The gaugino field λ_i, $i=1,2$ is made of two eight component spinors satisfying both chirality and symplectic Majorana conditions

$$\Gamma^6\lambda_i = \lambda_i (i=1,2), \qquad \lambda_i = (i\sigma^2)_{ij} B(\lambda_j^\dagger)^T, \tag{2.1}$$

where B is a matrix such that $B\Gamma^M B^{-1} = (\Gamma^M)^*$. Because of this constraint, there are only four physical degrees of freedom in gaugino spinor. Our choices of six dimensional gamma matrices are

$$\begin{aligned}
&\Gamma^0 = 1_2 \otimes i\sigma^3 \otimes \sigma^1,\\
&\Gamma^a = \sigma^a \otimes \sigma^1 \otimes \sigma^1 \quad (a=1,2,3),\\
&\Gamma^4 = 1_2 \otimes \sigma^2 \otimes \sigma^1, \qquad \Gamma^5 = 1_2 \otimes 1_2 \otimes \sigma^2.
\end{aligned} \tag{2.2}$$

In addition, $\Gamma^6 = \Gamma^0\Gamma^1 \cdots \Gamma^5 = 1_2 \otimes 1_2 \otimes \sigma^3$. With the above choice,

$$B = -i\sigma^2 \otimes 1_2 \otimes \sigma^3. \tag{2.3}$$

The Lagrangian for the gauge multiplet is

$$\mathcal{L}_1 = \mathrm{tr}\left(-\frac{1}{4}F_{MN}F^{MN} - \frac{i}{2}\bar{\lambda}_i\Gamma^M D_M\lambda_i + \frac{1}{2}(\mathbf{D}^a)^2\right). \tag{2.4}$$

The supersymmetric transformation becomes

$$\delta A_M = i\bar{\lambda}_i\Gamma_M\epsilon_i, \tag{2.5}$$

$$\delta\lambda_i = \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon_i + i\mathbf{D}^a\sigma^a_{ij}\epsilon_j, \tag{2.6}$$

$$\delta\mathbf{D}^a = \bar{\epsilon}_i\sigma^a_{ij}\Gamma^I D_I\lambda_j, \tag{2.7}$$

where the supersymmetric parameter ϵ_i is also a chiral spinor and satisfies the symplectic Majorana condition. The Lagrangian and supersymmetric transformation are compatible with the symplectic Majorana condition. The above Lagrangian is invariant under $SU(2)_R$ transformation, under which λ_i and $\mathbf{D}^a$ belong to the fundamental and adjoint representations, respectively.

The Lagrangian for an adjoint hypermultiplet $y_i (i=1,2)$, χ where the matter spinor is antichiral $\Gamma^6\chi = -\chi$, is

$$\mathcal{L}_2 = \mathrm{tr}\left(-\frac{1}{2}D_M\bar{y}_i D^M y_i + \frac{1}{2}\mathbf{D}^a\sigma^a_{ij}[\bar{y}_j, y_i] - i\bar{\chi}\Gamma^M D_M\chi + \bar{\lambda}_i[\bar{y}_i, \chi] - \bar{\chi}[y_i, \lambda_i]\right), \tag{2.8}$$

where $D_M y_i = \partial_M y_i - i[A_M, y_i]$. Here $y_i (i=1,2)$ is a doublet under $SU(2)_R$ and χ is a singlet. The supersymmetric transformation is

$$\delta\bar{y}_i = 2i\bar{\chi}\epsilon_i, \qquad \delta\chi = D_M y_i\Gamma^M\epsilon_i. \tag{2.9}$$

The matter hypermultiplets q_{fi}, ψ_f with flavor index $f=1,\ldots,N_f$ belong to the fundamental representation $\bar{N}$ of the gauge group $U(N)$. As in the adjoint hypermultiplet, the matter spinor field is antichiral. The Lagrangian for the matter multiplet is

$$\mathcal{L}_3 = \mathrm{tr}\left(-\frac{1}{2}D_M\bar{q}_{fi}D^M q_{fi} + \frac{1}{2}\mathbf{D}^a\sigma^a_{ij}\bar{q}_{fj}q_{fi} - i\bar{\psi}_f\Gamma^M D_M\psi_f + \bar{\lambda}_i\bar{q}_{fi}\psi_f - \bar{\psi}_f q_{fi}\lambda_i\right), \tag{2.10}$$

where $D_M q_{fi} = \partial_M q_{fi} + iq_{fi}A_M$. The supersymmetric transformation is

$$\delta\bar{q}_{fi} = 2i\bar{\psi}_f\epsilon_i, \qquad \delta\psi_f = D_M q_{fi}\Gamma^M\epsilon_i. \tag{2.11}$$

The above Lagrangians are invariant under the $SU(2)_R$ symmetry. For a theory with the Abelian gauge group, one can add the Fayet-Iliopoulos term

$$\mathcal{L}_{\mathrm{FI}} = \frac{1}{2}\mathrm{tr}(\zeta^a\mathbf{D}^a). \tag{2.12}$$

If the gauge group is a product group, there would be FI terms for each independent $U(1)$ theory. The FI parameter ζ^a breaks the $SU(2)_R$ symmetry explicitly and so one can use $SU(2)_R$ symmetry to rotate them to be

$$\zeta^1 = 0, \qquad \zeta^2 = 0, \qquad \zeta^3 = v^2, \tag{2.13}$$

with $v \geq 0$. We will use both ζ^a and parameter v. The $\mathbf{D}^a$

field is not dynamical and its field equation leads to

$$\mathbf{D}^a = \frac{e^2}{2}\{\zeta^a - \sigma^a_{ij}([\bar{y}_j, y_i] + \bar{q}_{fj}q_{fi})\}. \tag{2.14}$$

The dimensional reduction to $3 + 1$ dimension induces additional $U(1)_R$ symmetry which is a rotation under two reduced spaces. The dimensional reduction with the Scherk-Schwartz mechanism induces two mass parameters m_f, m_f' for each flavor matter multiplet along the reduced space. If x^4, x^5 are reduced, then

$$D_4 q_{fi} = iq_{fi}(A_4 - m_f), \qquad D_5 q_{fi} = iq_{fi}(A_5 - m_f'). \tag{2.15}$$

This theory with the $U(N)$ gauge group has a simple D-brane interpretation. It is a Yang-Mills theory on N parallel D3 branes near N_f D7 branes whose transverse location at x^4, x^5 is given by the mass parameter. The location of the D3 branes along the x^4, x^5 direction is given by the vacuum expectation value of adjoint scalars A_4, A_5. The location of D3 branes along transverse 4 directions in D7 branes would be decided by the expectation value of y_i. The dimensional reduction to $4 + 1$ dimension is a bit simpler with only one mass parameter and no additional R symmetry. The D-brane interpretation could be a D4-D8 system.

One of the vacuum conditions $\mathbf{D}^a = 0$ is the ADHM condition of N instantons on the $U(N_f)$ gauge theory of noncommutative four space. The scalar fields are denoting the separation and size of instantons. As D3 branes act as instantons on D7 branes, one can see that the vacuum moduli space modulo gauge transformation is the moduli space of instantons when the mass parameters are turned off. With the mass parameters turned on, every D3 brane should lie on some D7 brane at the ground state. Thus, every eigenvalue pair of expectation values of (A_4, A_5), which is diagonal at the vacuum, should coincide with (m_f, m_f') for some f.

One of the simplest vacua appears when $N = N_f$ and all the eigenvalue pairs of A_4, A_5 are distinct, such that there is only one D3 brane for each D7 brane. It is the so-called color-flavor locking phase, where the matter field will have a Higgs condensation $\langle q_{f1}\rangle_{\text{vacuum}} = v$ and the gauge symmetry plus the flavor symmetry is spontaneously broken down to unbroken $U(1)^N$ global symmetry.

When $N = 2$, $N_f = 1$, the vacuum moduli space would be that of two U(1) instantons on noncommutative four space [19], which is the so-called Eguchi-Hanson space. In this case y_i does have intrinsic non-Abelian components and the gauge group U(2) is spontaneously broken to global U(1) symmetry.

III. BPS EQUATIONS

Classically a BPS field configuration is a bosonic field configuration which leaves some of the supersymmetry invariant. We consider now the supersymmetric transformation to obtain the BPS equations. Inspired by the bosonic BPS equations, we rewrite the supersymmetric transformation of the gaugino field as

$$\begin{aligned}\delta\lambda_i = {} & \Gamma^{12}((F_{12} - F_{34}\Gamma^{1234})\epsilon_i - i\mathbf{D}^3\Gamma^{12}\sigma^3_{ij}\epsilon_j)\\ & + \Gamma^{23}((F_{23} - F_{14}\Gamma^{1234})\epsilon_i - i\mathbf{D}^1\Gamma^{23}\sigma^1_{ij}\epsilon_j)\\ & + \Gamma^{31}((F_{31} - F_{24}\Gamma^{1234})\epsilon_i - i\mathbf{D}^2\Gamma^{31}\sigma^2_{ij}\epsilon_j)\\ & + \Gamma^{\mu 0}(F_{\mu 0} - F_{\mu 5}\Gamma^{05})\epsilon_i + F_{05}\Gamma^{05}\epsilon_i.\end{aligned} \tag{3.1}$$

As $\Gamma^4\epsilon_i = -\Gamma^{123}\Gamma^{05}\epsilon_i$, the adjoint spinor transformation is written as

$$\begin{aligned}\delta\chi = {} & -\Gamma^{123}(D_1 y_i\Gamma^{23} + D_2 y_i\Gamma^{31} + D_3 y_i\Gamma^{12}\\ & + D_4 y_i\Gamma^{05})\epsilon_i + \Gamma^0(D_0 y_i - D_5 y_i\Gamma^{05})\epsilon_i.\end{aligned} \tag{3.2}$$

The spinor in fundamental hypermultiplet transforms as

$$\begin{aligned}\delta\psi_f = {} & -\Gamma^{123}(D_1 q_{fi}\Gamma^{23} + D_2 q_{fi}\Gamma^{31} + D_3 q_{fi}\Gamma^{12}\\ & + D_4 q_{fi}\Gamma^{05})\epsilon_i + \Gamma^0(D_0 q_{fi} - D_5 q_{fi}\Gamma^{05})\epsilon_i.\end{aligned} \tag{3.3}$$

We want to find some supersymmetric parameter ϵ_i such that $\delta\lambda_i$, $\delta\chi$, and $\delta\psi_f$ remain zero. On eight independent parameters of spinor ϵ_i, we impose three independent conditions (in the case of the $N = 2$ nonlinear sigma model, see [20]):

$$\begin{gathered}\Gamma^{05}\epsilon_i = \eta\epsilon_i, \qquad \Gamma^{12}\sigma^3_{ij}\epsilon_j = i\alpha\epsilon_i,\\ \Gamma^{23}\sigma^1_{ij}\epsilon_j = i\beta\epsilon_i,\end{gathered} \tag{3.4}$$

where α, β, and η take ± 1 independently. Since $\Gamma^0\Gamma^1\cdots\Gamma^5 = 1$ for chiral ϵ_i, these conditions imply that

$$\Gamma^{31}\sigma^2_{ij}\epsilon_i = -i\alpha\beta\epsilon_i, \qquad \Gamma^{1234}\epsilon_i = \eta\epsilon_i. \tag{3.5}$$

These are conditions on eight independent Majorana parameters in the spinor ϵ_i, as they are compatible with the symplectic Majorana condition. If we impose any one of the conditions, the number of independent SUSY parameters would be reduced by $1/2$ to four of the original value. If we impose any two of them, the number of independent SUSY parameters is reduced to two or $1/4$ of the original one. If we impose all three of them, the number of independent parameters is reduced to one, $1/8$ of the original value.

One can obtain different conditions by six dimensional Lorentz transformations and $SU(2)_R$ transformations. In reduction to $3 + 1$ dimensions, only nontrivial ones modulo remaining symmetries is the rotation between the remaining coordinates and the reduced coordinates. In the reduction to $3 + 1$ dimensions of coordinates x^0, x^1, x^2, and x^3, the above condition can be generalized to new spinor conditions with two parameters:

$$\Gamma^0(\Gamma^5\cos\theta+\Gamma^3\sin\theta)\epsilon_i=\eta\epsilon_i,$$
$$\Gamma^1(\Gamma^2\cos\varphi+\Gamma^4\sin\varphi)\sigma^3_{ij}\epsilon_{ij}=i\alpha\epsilon_i,$$
$$(\Gamma^2\cos\varphi+\Gamma^4\sin\varphi)(\Gamma^3\cos\theta-\Gamma^5\sin\theta)\sigma^1_{ij}\epsilon_j=i\beta\epsilon_i. \tag{3.6}$$

This implies that

$$(\Gamma^3\cos\theta-\Gamma^5\sin\theta)\Gamma^1\sigma^2_{ij}\epsilon_j=-i\alpha\beta\epsilon_i,$$
$$\Gamma^{124}(-\Gamma^3\cos\theta+\Gamma^5\sin\theta)\epsilon_i=\eta\epsilon_i. \tag{3.7}$$

Note also $D_4q_{fi}=iq_{fi}(A_4-m_f)$ and $D_5q_{fi}=iq_{fi}(A_5-m'_f)$. In the reduction to $4+1$, we can put $\varphi=0$ as it is a part of the four dimensional spatial rotation.

We use the generalized spinor condition (3.6) to find the BPS equations satisfied by the bosonic configurations for the minimum amount $1/8$ of the original supersymmetries. For any vector with spatial indices, we introduce barred indices so that

$$V_{\bar 1}=V_1,\qquad V_{\bar 2}=V_2\cos\varphi+V_4\sin\varphi,$$
$$V_{\bar 3}=V_3\cos\theta-V_5\sin\theta,\qquad V_{\bar 4}=V_4\cos\varphi-V_2\sin\varphi,$$
$$V_{\bar 5}=V_5\cos\theta+V_3\sin\theta. \tag{3.8}$$

From $\delta\lambda_i=0$, we get the gauge field part of the BPS equations,

$$F_{0\bar 5}=0,\qquad F_{\bar\mu 0}-\eta F_{\bar\mu\bar 5}=0\quad(\mu=1,\ldots,4),$$
$$F_{1\bar 2}-\eta F_{\bar 3\bar 4}+\alpha\mathbf{D}^3=0,\qquad F_{\bar 2\bar 3}-\eta F_{1\bar 4}+\beta\mathbf{D}^1=0,$$
$$F_{\bar 3 1}-\eta F_{\bar 2\bar 4}-\alpha\beta\mathbf{D}^2=0. \tag{3.9}$$

From $\delta\chi=0$ and $\delta\psi_f=0$, we also obtain

$$\beta D_1y_j\sigma^1_{ji}-\alpha\beta D_{\bar 2}y_j\sigma^2_{ji}+\alpha D_{\bar 3}y_j\sigma^3_{ji}-i\eta D_{\bar 4}y_i=0,$$
$$D_0y_i-\eta D_{\bar 5}y_i=0,\qquad D_0q_{fi}-\eta D_{\bar 5}q_{fi}=0,$$
$$\beta D_1q_{fj}\sigma^1_{ji}-\alpha\beta D_{\bar 2}q_{fj}\sigma^2_{ji}+\alpha D_{\bar 3}q_{fj}\sigma^3_{ji}-i\eta D_{\bar 4}q_{fi}=0. \tag{3.10}$$

These are the BPS equations for $1/8$ BPS configurations. The BPS equations preserving more supersymmetry can be obtained by imposing additional conditions to the above BPS equations. For example, $1/4$ BPS configurations satisfy two sets of $1/8$ BPS equations with, say, both $\alpha=1$ and $\alpha=-1$. There is also a Gauss law constraint for the BPS configurations,

$$-\frac{1}{e^2}\sum_{\mu=0}^{5}D_{\bar\mu}F_{\bar\mu 0}-\frac{i}{2}([\bar y_i,D_0y_i]-[D_0\bar y_i,y_i])$$
$$-\frac{i}{2}(\bar q_{fi}D_0q_{fi}-D_0\bar q_{fi}q_{fi})=0. \tag{3.11}$$

Using the BPS equation, the central charge [21] for the BPS energy bound can be found to be

$$Z=\frac{1}{2}\int d^3x\,\mathrm{tr}\Big(\frac{\eta}{2e^2}F_{\bar\mu\bar\nu}\tilde F_{\bar\mu\bar\nu}-\alpha\zeta^3(F_{1\bar 2}-\eta F_{\bar 3\bar 4})$$
$$-\beta\zeta^1(F_{\bar 2\bar 3}-\eta F_{1\bar 4})+\alpha\beta\zeta^2(F_{\bar 3 1}-\eta F_{\bar 2\bar 4})\Big)$$
$$+\eta m_fQ_f\cos\theta+\eta T_{03}\sin\theta+Z', \tag{3.12}$$

where $\mu,\nu=1,2,3,4$ and $\tilde F_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$. After dimensional reduction to $(3+1)$ dimensions, $D_4y_i=-i[A_4,y_i]$ and $D_4q_{fi}=iq_{fi}(A_4-m_f)$, and so $F_{14}=D_1A_4$ and $F_{45}=-i[A_4,A_5]$. The charge Q_f is the one carried by the fth-flavor matter field,

$$Q_f=\frac{i}{2}\int d^3x\,\mathrm{tr}(\bar q_{fi}D_0q_{fi}-D_0\bar q_{fi}q_{fi}), \tag{3.13}$$

and T_{03} is the linear momentum along the x^3 direction,

$$T_{03}=\frac{1}{2}\int d^3x\,\mathrm{tr}\Big(\frac{1}{e^2}\sum_{\mu=1,2,4,5}F_{\mu 0}F_{\mu 3}+(D_0\bar y_iD_3y_i$$
$$+D_3\bar y_iD_0y_i)+(D_0\bar q_{fi}D_3q_{fi}+D_3\bar q_{fi}D_0q_{fi})\Big). \tag{3.14}$$

The boundary term Z' is given by

$$Z'=\int d^3x(\eta\partial_i\,\mathrm{tr}(F_{i0}A_5)\cos\theta+\cdots), \tag{3.15}$$

where $\cdots$ indicates the terms quadratic in matter fields and is expected to have a zero boundary contribution in both Coulomb and Higgs phases. The first part would have a nontrivial contribution in the Coulomb phase where there would be a nontrivial electric field.

The above BPS equations and the energy bound are complicated functions of two parameters φ and θ. For example, a complication arises as

$$F_{\bar 3\bar 4}=F_{34}\cos\theta\cos\varphi-F_{32}\cos\theta\sin\varphi-F_{54}\sin\theta\cos\varphi$$
$$+F_{52}\sin\theta\sin\varphi. \tag{3.16}$$

Using the unbarred coordinate indices, we note that the first term of the above expression can be expressed as

$$F_{\bar\mu\bar\nu}\tilde F_{\bar\mu\bar\nu}=F_{\mu\nu}\tilde F_{\mu\nu}\cos\theta+4(F_{12}F_{45}+F_{24}F_{15}$$
$$+F_{41}F_{25})\sin\theta. \tag{3.17}$$

There are also the boundary terms depending on quark fields, which are supposed to make vanishing contributions in almost all cases.

Once we fix $\zeta^a=v^2\delta^a_3$, which is possible for the theories of the U(N) gauge group but not for those with the product gauge group like $\mathrm{U}(1)\times\mathrm{U}(1)$, the BPS energy does not depend on the choice of the parameter β. This means that $1/4$ BPS configurations defined by α and η parameters could have $1/8$ BPS excitations without generating additional energy, which is strange. Indeed we see that this is impossible in some simple case studied in Sec. V.

We can choose two parameters θ and φ to be arbitrary. If we fix ζ^a, we no longer have the freedom of $SU(2)_R$ transformation, and the parameters θ and φ become physically meaningful. One typical case of BPS equations would be when $\theta = \varphi = 0$. In this case, the barred spacial indices become the unbarred ones and $\partial_4 = \partial_5 = 0$. The other extreme may be when $\theta = \varphi = \pi/2$. In this case the time dependent part becomes $F_{03} = 0$ $(D_0 - \eta D_3)$ any field $= 0$, and

$$\begin{aligned}
&\eta F_{12} + i[A_4, A_5] + \beta \mathbf{D}^1 = 0,\\
&D_1 A_4 + \eta D_2 A_5 + \alpha \mathbf{D}^3 = 0,\\
&D_1 A_5 - \eta D_2 A_4 - \alpha\beta \mathbf{D}^2 = 0,\\
&(\beta D_1 y_j \sigma^1_{ji} + i\eta D_2 y_i + i\alpha\beta [A_4, y_j]\sigma^2_{ji}\\
&\quad + i\alpha [A_5, y_j]\sigma^3_{ji}) = 0,\\
&(\beta D_1 q_{fj}\sigma^1_{ji} + i\eta D_2 q_{fi} - i\alpha\beta q_{fj}(A_4 - m_f)\sigma^2_{ji}\\
&\quad - i\alpha q_{fj}(A_5 - m'_f)\sigma^3_{ji}) = 0.
\end{aligned} \tag{3.18}$$

We know quite a bit of the topological objects of the theories in $\theta = \varphi = 0$. The simplest object is a $1/2$ BPS vortex soliton along the x^3 direction in U(1) theory with $N_f = 1$ [9]. It satisfies the BPS equation with $\beta = -1$,

$$2F_{12} = v^2 - |q_1|^2, \qquad (D_1 - iD_2)q_1 = 0, \tag{3.19}$$

where $y_i = 0$, $q_2 = 0$, dropping the flavor index. Especially a unit flux vortex has a vortex tension $T_v = \pi v^2$. This could be regarded as a D1 string on a single D3 brane in a single D7 brane. The next simplest object is a $1/2$ BPS domain wall parallel to the (x^1, x^2) plane [1–3,22]. With $N = 1$ and $N_f = 2$ with two different m_f along the x^4 direction, the $1/2$ BPS equations with $\alpha\beta = 1$ become

$$2\partial_3 A_4 = v^2 - \sum_{f=1}^{2} |q_{f1}|^2, \qquad \partial_3 q_{f1} = -q_{f1}(A_4 - m_f), \tag{3.20}$$

where $y_i = 0$, $q_{f2} = 0$, $m_1 < m_2$, and $A_5 = 0$. The A_4 interpolates between m_1 and m_2. It describes the D3 brane on the first D7 brane interpolating to the second D7 brane. The wall tension is $T_{12} = \pi v^2 (m_2 - m_1)$.

A more complicated object is a $1/4$ BPS configuration made of magnetic monopole beads in a vortex flux tube [16]. With $N = N_f = 2$ and in the color-flavor locking phase with $m_1 < m_2$ and $A_5 = m'_f = 0$, the D1 string on the first D3-D7 branes can interpolate to the second D3-D7 branes. The D1 string connecting two D3 branes appears as a magnetic monopole. In the Higgs phase, the magnetic flux is confined to a flux string and so the $1/4$ BPS object is made of two vortices emerging opposite to the magnetic monopole, where two U(1)'s of U(2) flux are carried to opposite directions. The composite has the energy of a simple sum of vortex tension and monopole mass.

The most complicated $1/4$ BPS object is a composite made of vortex and domain walls, which also allows some magnetic monopoles [5,23,24]. With $\beta = -1$, $\alpha = -1$, from the BPS energy one notices that with positive $\mathrm{tr}F_{12}$ and $\mathrm{tr}F_{34}$, which means positive vortex flux and domain wall charge where A_4 is increasing, there is negative instanton energy, or monopole energy. This is the so-called bound energy of the vortex-domain wall [5]. If a vortex terminates at the domain wall, the wall shape gets deformed far from the contact point. The detail has been also studied recently [24]. Of course one can add additional monopole kink to this vortex-domain wall junction, which carries the positive monopole energy. In some cases, the magnetic monopole can pass the domain wall. When a vortex penetrating a domain wall is deformed so that the contact points at both sides of the domain wall do not coincide to the same point, the monopole could not pass the domain wall due to the energy consideration, which means that there could be repulsive potential at the domain wall. It would be interesting to find whether our conjecture is true.

A typical solution of the BPS equations of $\theta = \varphi = \pi/2$ would be the $1/4$ BPS domain wall junction [6,25,26] with $N = 1$, $N_f = 3$. Suppose that the three complex masses $m_f + im'_f$ lie on vertices of an equitriangle so that $m_f + im'_f = me^{2\pi i f/3}$ with $f = 1, 2, 3$. The BPS equation would be given by (3.18) with $\zeta^3 = v^2$ and the wall junction would lie on the x^1, x^2 plane with x^3 translation invariance. The ansatz is that $y_i = 0$, $q_{f2} = 0$, $A_1 = A_2 = A_3 = 0$, $\partial_3 = 0$, and the BPS equation becomes

$$\begin{aligned}
&\partial_1 A_4 + \eta \partial_2 A_5 = -\frac{\alpha}{2}(v^2 - |q_{f1}|^2),\\
&\partial_1 A_5 - \eta \partial_2 A_4 = 0,
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
&\partial_1 q_{f1} - \alpha(A_4 - m_f)q_{f1} = 0,\\
&\eta \partial_2 q_{f1} - \alpha(A_5 - m'_f)q_{f1} = 0.
\end{aligned} \tag{3.22}$$

The web of wall solutions of this type in a bit more complicated setting has also been studied recently [6].

IV. LORENTZ BOOSTED, OR DYONIC SOLUTIONS

For the BPS configurations, the time dependent part can be solved with

$$\begin{aligned}
&A_0 = \eta(A_5 \cos\theta + A_3 \sin\theta),\\
&\partial_0 q_{fi} - \eta(\partial_3 \sin\theta - im'_f \cos\theta) q_{fi} = 0,
\end{aligned} \tag{4.1}$$

while $(\partial_0 - \partial_3 \sin\theta) = 0$ for any field in the adjoint representation. One can see that it is a Lorentz boost along the x^3 axis with velocity $v = \sin\theta$ when $|\theta| < \pi/2$. However, the $\theta = \pi/2$ case is still physically distinct as it cannot be obtained through finite boost. The Gauss law is also an equivalently Lorentz boosted version. This matches with the energy being increased with $T_{03}v = \mathcal{O}(v^2)$ for small v as T_{03} itself is linear in v for small v. For the domain wall junctions, $T_{03} = 0$ with $\theta = \pi/2$ due to the x^3 translation invariance of the configuration. Thus one cannot boost them along x^3, but may be able to put some massless wave along x^3 without breaking the supersymmetry further.

When $\theta = 0$, $A_0 = \eta A_5$ and all the adjoint fields are time independent and $\partial_0 q_{fi} + i\eta m'_f q_{fi} = 0$. The fth-flavor charge becomes

$$Q_f = \eta \int d^4x \, \mathrm{tr}((m'_f - A_5)\bar{q}_{fi} q_{fi}). \tag{4.2}$$

As the total electric charge vanishes in the Higgs phase, we put the constraint $\sum_f Q_f = 0$. Here we consider the fundamental string connecting D3 branes with net $\mathrm{U}(1) = \mathrm{tr}(\mathrm{U}(N))$ charge vanishes in the Higgs phase. The energy carried by the flavor charge becomes

$$E_Q = \eta \sum_f m'_f Q_f = \sum_f \int d^3x m'_f \, \mathrm{tr}((m'_f - A_5)\bar{q}_{fi} q_{fi}). \tag{4.3}$$

For most of the BPS objects considered here, they have a moduli space parameter corresponding to a global phase rotation. The excitation along this direction would lead to the dyonic solutions. The Gauss law would give the equation for A_5 which is exactly the zero mode equation satisfied by the phase moduli coordinate in the background gauge of solutions without A_5 and m'_f included. The parameters m'_f serve as coefficients of the excited phase moduli direction vector of the dyonic solution.

Consider a vortex-monopole composite with $N = N_f = 2$ in a color-flavor locking phase with $m_1 < m_2$. One can impose an additional BPS condition on the electric charge section without breaking any additional supersymmetry. One has to solve the above Gauss law which can be solved in principle in this monopole-vortex background. The result describes a composite of D1-fundamental strings connecting D3 branes, which means that the monopole carries electric charge. However, the A_5 would approach an exponentially vacuum expectation value away from the monopole region, implying that the electric charge is shielded by the Higgs field. For the two flavor case, one can choose $A_5 \sim A_4$ up to constant shift as (m_f, m'_f) lies along a line. Note that $E_Q \sim (\Delta m')^2$ and $Q_2 - Q_1 \sim \Delta m'$, and so the relative flavor charge fixes $m'_2 - m'_1$ as in the dyons in the Coulomb phase.

When $N_f \geq 3$, D7 branes do not need to lie on a line as three points given by the mass parameters do not lie along a line in general. In this case one could have a web of D1, F1, and (p, q) strings [27]. For example consider $N = N_f = 3$ in the color-flavor locking phase. If the D7 branes are separated from each other and lie on an almost straight line, one can imagine a D1 string interpolating two D3 branes at the end. When we introduce the fundamental strings connecting, say first and second D3 branes, the resulting configuration would be a vortex string where there are two fundamental monopoles attracted to each other, but the Coulomb repulsion due to the electric charge in short distance keeps them away from each other. This is quite similar to the corresponding configuration in the Coulomb phase. The key difference would be that in the Higgs phase there may be no upper bound on F1 string numbers as the electric repulsion would be shielded in large separation.

It is straightforward to extend this to situations of multiple domain walls. Consider $N = 2$, $N_f = 3$ with two domain walls interpolating m_1, m_2 by first D3 and m_2, m_3 by second D3 ($m_1 < m_2 < m_3$). If we turn on m'_2 slightly, these two domain walls are attracted, and it is balanced by giving them electric charges proportional to m'_2 distributed on their world volume. This would be a weblike structure of D3 branes and a sheet of fundamental strings, attached to D7 branes.

Another dyonic BPS configuration is possible. Start with a $1/2$ BPS domain wall of a single D3 brane, interpolating two D7 branes in position. Fundamental strings connecting two D7 branes at the wall generate the electric dipole on the D3 brane. Two ends of the dipole are shielded by the Higgs field of different flavor, and so the configuration has the Higgs charge. One needs to solve the Gauss law in the domain wall background. From the domain wall world sheet point of view, the fundamental F1 string appears as a charge of phase or magnetic flux on effective $2 + 1$ dimensional theory. Uniform charge configuration corresponds to uniform magnetic flux configuration on effective $2 + 1$ dimensional theory.

In our BPS equation there is an additional parameter φ. To see its role in $N = 1$, $N_f = 2$ with $\zeta^a = v^2\delta_{a3}$, $A_5 = m'_f = 0$, let us consider the domain wall solution with 2, 4 directions mixed. With only dependence on x^1 and x^3 and $A_1 = A_2 = A_3 = 0$, $\eta = \alpha = -1$, $\theta = 0$, the BPS equations (3.9) and (3.10) for A_4 become

$$\begin{aligned}
(\partial_3 \cos\varphi - \partial_1 \sin\varphi)A_4 &= \frac{1}{2}\Big(v^2 - \sum_f |q_{f1}|^2\Big),\\
(\partial_3 \cos\varphi - \partial_1 \sin\varphi)q_{f1} + q_{f1}(A_4 - m_f) &= 0,\\
(\partial_3 \sin\varphi + \partial_1 \cos\varphi)A_4 &= 0,\\
(\partial_3 \sin\varphi + \partial_1 \cos\varphi)q_{f1} &= 0.
\end{aligned} \tag{4.4}$$

This corresponds to a spatial rotation in the (x^1, x^3) plane. The origin of this fact can be traced back to the correlation between (x^2, x^4) and (x^1, x^3) in the spinor projection conditions.

V. 1/8 BPS OBJECTS IN THEORIES WITH PRODUCT GAUGE GROUPS

While we found $1/8$ BPS equations which seem to be general up to six dimensional Lorentz boost and $SU(2)_R$ symmetry, it is not clear whether $1/8$ BPS configurations are allowed. After the dimensional reduction to $3+1$ dimensions with two general angle parameters, one cannot make an arbitrary six dimensional rotation, especially $F_{45} = 0$ in U(1) theory. While we are interested in the general characteristics of $1/8$ BPS configurations, if any exist, it seems very hard to solve the BPS equations.

Let us start with a theory with a simple gauge group, say, U(N). To find out what the characteristics of $1/8$ BPS configurations are, let us start with a BPS configuration of constant field strength with zero matter expectation value. From BPS equations for the gauge fields (3.9) for the constant field strength, we can make $SU(2)_R$ rotation to put the FI parameter to the 3rd direction and SU(2) spatial rotation in x^1, $x^{\bar{2}}$, and $x^{\bar{3}}$, which rotates both ϵ_i and the gauge field strength $F_{\bar{\mu}\bar{\nu}}$ with $\mu, \nu = 1, 2, 3, 4$. From this one can see that the constant field configuration is at most $1/4$ BPS configuration.

An inhomogeneous BPS field configuration can be obtained by extracting magnetic fluxes from the system. To see whether $1/8$ BPS configurations are possible when the field configuration is inhomogeneous in space, we ask whether $1/8$ BPS perturbation arises in $1/4$ BPS homogeneous background [28].

Let us start with a U(1) gauge theory on $3+1$ dimension with single flavor. Let us start with a $1/4$ BPS configuration which is homogeneous in space and time with $A_0 = A_5$ and $\eta = \alpha = -1$ with $\theta = \varphi = 0$. The FI term becomes $\mathbf{D}^a = e^2 v^2/2\delta_{a3}$ and we choose the constant $1/4$ BPS field strengths to be

$$F_{12} = \frac{e^2 v^2}{2} a, \qquad F_{34} = \frac{e^2 v^2}{2}(1-a), \qquad F_{23} = \frac{e^2 v^2}{2} b, \qquad F_{14} = -\frac{e^2 v^2}{2} b \tag{5.1}$$

with constants a, b. This is a generalization of many previously known homogenous solutions. The homogeneous BPS configuration in the U(1) Higgs model with a single Higgs field represents the uniform distribution of vortices on plane, which has the critical total magnetic flux [28]. In SU(2) gauge theory, one could have a magnetic monopole sheet or homogenous field configuration with uniform instanton density. The energy density is then

$$\mathcal{E} = \frac{e^2 v^4}{4}(1 + b^2 - a(1-a)). \tag{5.2}$$

In four dimensions, the contribution from the intersection of F_{12} and F_{34} can decrease the tension when $0 < a < 1$ and can be regarded as an anti-self-dual instanton part with the negative energy, which can be regarded as a bound energy of two uniform magnetic fluxes. Note that the minimum energy is positive.

In $3+1$ dimensions, it represents the bound energy of a domain wall and infinite number of vortex strings penetrating domain walls. The number of flavors does not play any role. For $b > 0$ and a not in this interval induces self-dual instanton density which contributes positive energy. Note that there are critical total fluxes $e^2 v^2/2$ in our unit. From the brane point of view, the above BPS solution induces D3 branes with homogeneous field on its world sheet, tilted with respect to D7 branes.

We want to see whether there is any $1/8$ BPS deformation of this homogeneous configuration. The BPS equation implies that there should be nonzero q_i, $i = 1, 2$ for $1/8$ BPS configurations, which we regard as a small perturbation. (Here we drop the flavor index as there is only one flavor.) We solve the $1/8$ BPS equation by the perturbation expansion with $\beta = -1$. To first order we first solve the matter BPS equation in the uniform background,

$$D_1 q_j \sigma^1_{ji} + D_2 q_j \sigma^2_{ji} + D_3 q_j \sigma^3_{ji} - i D_4 q_i = 0. \tag{5.3}$$

We choose the gauge

$$A_1 = 0, \qquad A_2 = \frac{e^2 v^2}{2}(a x^1 - b x^3), \qquad A_3 = 0, \qquad A_4 = \frac{e^2 v^2}{2}((1-a)x^3 - b x^1). \tag{5.4}$$

The above equation is satisfied if

$$\partial_1 q_i + \frac{e^2 v^2}{2} x^1 q_j (a\sigma^3 - b\sigma^1)_{ji} = 0, \qquad \partial_3 q_i + \frac{e^2 v^2}{2} x^3 q_j (b\sigma^1 + (1-a)\sigma^3)_{ji} = 0. \tag{5.5}$$

One can convince one's self that only q_1 becomes normalizable along both x^1 and x^3 directions for $b = 0$ and $0 < a < 1$ while q_2 is not normalizable at all. For $1/8$ BPS deviation, we need both normalizable q_1 and q_2 modes to start the perturbative approach and so there is no $1/8$ BPS deviation from the $1/4$ BPS configuration. The BPS equation for the gauge fields indicates the second order effect of the q_1 perturbation reducing the total magnetic flux and instanton or monopole number. Thus one can guess that the above homogeneous configuration, while remaining $1/4$ BPS, is continuously connected to the two intersecting flux sheets along $x^1 - x^2$ and $x^3 - x^4$ planes with finite magnetic monopole charge and negative bound energy. In the brane picture, the end result would be the intersection of the D3 brane domain wall and the D1 string.

While the above analysis does not provide a clear picture about the existence of $1/8$ BPS configurations in 8 supersymmetric U(N) gauge theories, it suggests that $1/8$ BPS configurations are unlikely.

Now consider a theory with $U(1) \times U(1)$ gauge group with fundamental matter fields in each gauge group and also many bifundamental matter fields of charge $(+1, -1)$. Let assume that two FI parameters are not parallel and so, say, $\zeta^{(1)a} = \delta_{a3}$ and $\zeta^{(2)a} = \delta_{a1}$. (Here we put the proportional numbers and electric charges to be 1 for simplicity.) If there are no bifundamental matter fields, two theories are not interacting and so it is obvious that there can be $1/8$ BPS configurations. They can be made of $1/4$ BPS configurations of each gauge group but they are not aligned and so break the supersymmetry further to $1/8$. Even when bifundamental fields exist, such $1/8$ BPS configurations are possible if the bifundamental field has zero expectation value.

To see whether the bifundamental matter field can develop any nontrivial expectation value, let us start with $1/8$ BPS homogeneous configuration in this theory of two product gauge groups,

$$F^{(1)}_{12} = a, \qquad F^{(1)}_{34} = 1 - a, \tag{5.6}$$

$$F^{(2)}_{23} = b, \qquad F^{(2)}_{14} = 1 - b. \tag{5.7}$$

The energy density of the configuration becomes

$$\mathcal{E} = \frac{1}{4}(2 - a(1 - a) - b(1 - b)). \tag{5.8}$$

With the gauge

$$\begin{aligned} A^{(1)}_2 &= ax^1, \qquad A^{(1)} = (1 - a)x^3, \\ A^{(2)}_2 &= -bx^3, \qquad A^{(2)}_4 = (1 - b)x^1. \end{aligned} \tag{5.9}$$

The interesting question is whether there exists a nonzero mode for the bifundamental field q_i, whose BPS equation is satisfied if

$$\partial_1 q_i + x^1 q_j (a\sigma^3 - (1 - b)\sigma^1)_{ji} = 0, \tag{5.10}$$

$$\partial_3 q_i + x^3 q_j ((1 - a)\sigma^3 - b\sigma^1)_{ji}. \tag{5.11}$$

The normalizable solution along the x^1, x^3 direction is possible if $a = b = 1/2$, in which case two matrices are proportional to each other and so can be exponentiated easily. Once we found this normalizable zero mode, we fed it to the BPS equation for the gauge field, which leads to the second order perturbation, which reduces the sum of the magnetic fluxes. Of course there will be also nontrivial BPS deformation of the fundamental matter field for each gauge group. One can imagine the continuous deflation of the total flux would lead to some sort of intersecting U(1) magnetic vortex sheets, while remaining $1/8$ BPS. From the $1/4$ BPS case, one can see that the first U(1) vortex line along the x^3 direction meets a first U(1) domain parallel to the $1 - 2$ plane. The second U(1) vortex line along the x^1 direction meets a second U(1) domain wall parallel to the $2 - 3$ plane. Together they would remain $1/8$ BPS. In addition, there would be nontrivial bifundamental matter field in this $1/8$ BPS configuration, making two configurations to be connected together.

VI. NONEXISTENCE OF BPS VORTICES

Most of the analyses on solitons so far have been done when $N_f \geq N$. Especially there would be no supersymmetric vacua if $N_f < N$ without an adjoint hypermultiplet. When $N_f < N$, the adjoint hypermultiplet plays a crucial role for supersymmetric vacua to exist. When $N = 2$, $N_f = 1$, the explicit vacuum solution modulo local gauge transformations is known [19]. At the vacuum the scalars in vector multiplet $(A_4, A_5) = (m_1, m'_1)$, proportional to the identity matrix. With the adjoint hypermultiplet, the vacuum equation $\mathbf{D}^a = 0$ is the ADHM condition on two instantons in noncommutative U(1) theory, and the moduli space metric becomes the Eguchi-Hanson space. It is depending on eight parameters, four of which are the position of the center of the mass of two D3 branes in D7 branes, and so is flat and does not affect our analysis. There are an additional four parameters which indicate the relative distance and phase between two D3 branes in D7 branes. Because of the FI term, there would be Higgs condensation on D3 branes. Explicitly,

$$\begin{aligned} \langle \bar{y}_1 \rangle &= w_1 + \frac{z_1}{2}\begin{pmatrix} 1 & \sqrt{\frac{2b}{a}} \\ 0 & -1 \end{pmatrix}, \\ \langle y_2 \rangle &= w_2 + \frac{z_2}{2}\begin{pmatrix} 1 & \sqrt{\frac{2b}{a}} \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{6.1}$$

$$\langle \bar{q}_1 \rangle = v\begin{pmatrix} \sqrt{1 - b} \\ \sqrt{1 + b} \end{pmatrix}, \qquad \langle q_2 \rangle = 0, \tag{6.2}$$

where $a = (|z_1|^2 + |z_2|^2)/(2v^2)$ and $b = \sqrt{a^2 + 1} - a$. The vacuum moduli space is characterized by four complex parameters w_i, z_i. The parameter w_i denotes the location of the center of mass points of two D3 branes on D7 background and the parameter z_i denotes the relative position.

We know there are BPS multivortex solutions when $N = N_f = 1$. The question is whether any BPS vortex solitons exist when $N = 2$, $N_f = 1$. Suppose we put a single D1 string on one of the D3 branes when two D3 branes are in infinite separation. Clearly it is BPS. As we change vacuum moduli parameters so that two D3 branes are almost on top of each other, we may expect that there would be $1/2$ BPS vortex solutions. To see whether this is true, we look at a consistent ansatz.

Rather the surprise appears when two D3 branes are on top of each other, or when the vacuum moduli is at minimum two sphere of Eguch-Hanson space. In this case the consistent ansatz becomes

$$\begin{aligned} \bar{y}_1 = \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}, \qquad y_2 = 0, \qquad \bar{q}_1 = \begin{pmatrix} 0, \\ Q_2 \end{pmatrix}, \\ q_2 = 0, \qquad A_1 + iA_2 = \mathrm{diag}(A, B). \end{aligned} \tag{6.3}$$

The BPS equation get simplified to $[\partial = \frac{1}{2}(\partial_1 - i\partial_2)]$

$$\partial Z - i(A - B)Z = 0, \qquad \partial Q_2 - iBQ_2 = 0, \qquad (6.4)$$

$$-i(\bar{\partial} A - \partial \bar{A}) = v^2 - |Z|^2, \qquad (6.5)$$

$$-i(\bar{\partial} B - \partial \bar{B}) = v^2 + |Z|^2 - |Q|^2. \qquad (6.6)$$

Asymptotic values of $|Z|^2$ and $|Q|^2$ are v^2 and $2v^2$, respectively. The above BPS equations can be combined to

$$-\partial_i^2 \ln|Z/Q|^2 = v^2 - |Z|^2, \qquad (6.7)$$

$$-\partial_i^2 \ln|Q|^2 = v^2 + |Z|^2 - |Q|^2. \qquad (6.8)$$

The BPS energy is determined by $\frac{v^2}{2}(F_A + F_B) = 2v^2 - |Z|^2$. The vorticities of Z and Q_2 are l_1, l_2, then the flux $\int d^2x(F_A - F_B) = 2\pi l_1 > 0$ and $\int d^2xF_B = 2\pi l_2 > 0$. The $\int d^2xF_A = 2\pi(l_1 + l_2)$ and the energy is $\pi v^2(l_1 + l_2)$. From examining the above equations, one can easily draw the fact that there is no solution with $l_1 = 0$, $l_2 > 0$ or $l_1 > 0$, $l_2 = 0$ or $l_1 = l_2 > 0$. The only possibility is $l_1 - 1 \geq l_2 \geq 1$. As we move the D3 branes apart, it suggests that there are no BPS configurations possible for vortices with vorticity 1 or 2 and even D3 branes are apart. Assuming the continuity of the BPS configurations here as we do not see any critical separation between D3 branes matter, there seems to be only one logical conclusion, that is, that two D3 branes with any parallel D1 string on them become *repulsive*. That means there is no BPS configuration with any vorticity and finite separation. This seems to be the only consistent result. It would be interesting to verify this conjecture.

Note added.—In the early stage of the draft of our paper, we came to know that the authors of Ref. [29] worked on the classification of $1/8$ BPS equations of a similar model we considered.

ACKNOWLEDGMENTS

We thank Muneto Nitta for correspondence about his work, and Norisuke Sakai and Piljin Yi for helpful discussions. K. L. is supported in part by KOSEF R01-2003-000-10319-0 and KOSEF through CQUeST at Sogang University. H. U. Y is supported by Grant No. R01-2003-000-10391-0 from the Basic Research Program of the Korea Science and Engineering Foundation.

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