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Geometry of spin-field coupling on the worldline

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We derive a geometric representation of couplings between spin degrees of freedom and gauge fields within the worldline approach to quantum field theory. We combine the string-inspired methods of the worldline formalism with elements of the loop-space approach to gauge theory. In particular, we employ the loop (or area) derivative operator on the space of all holonomies which can immediately be applied to the worldline representation of the effective action. This results in a spin factor that associates the information about spin with zigzag motion of the fluctuating field. Concentrating on the case of quantum electrodynamics in external fields, we obtain a purely geometric representation of the Pauli term. To one-loop order, we confirm our formalism by rederiving the Heisenberg-Euler effective action. Furthermore, we give closed-form worldline representations for the all-loop order effective action to lowest nontrivial order in a small-$N_{\rm f}$ expansion.

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I. INTRODUCTION

The mapping of quantum field theoretic problems onto the language of quantum mechanics of point particles in the form of the worldline formalism [1] has become a powerful computational tool in recent years. The worldline approach, which can also be viewed as the field theoretic limit of string theory [2–7], establishes a direct connection between a ''second-quantized'' and a ''first-quantized'' formalism. Particularly for correlators in background fields, computations simplify drastically with worldline techniques [8,9].

The relation between field theory and quantum particle mechanics can best be illustrated by the worldline representation of a scalar field's propagator in Euclidean spacetime,

$$G(x_2, x_1) = \int_0^\infty dT e^{-m^2T} \mathcal{N} \int_{x(0)=x_1}^{x(T)=x_2} \mathcal{D}x e^{-(1/4)\int_0^T d\tau \dot{x}^2(\tau)}, \tag{1}$$

where the integration parameter T is called propertime, and the path integral runs over all paths with fixed end points distributed by a Gaussian velocity weight. The resulting ensemble of paths can be viewed as the set of possible trajectories of the quantum field. This associates virtual fluctuations of a field with particle worldlines in coordinate space, which constitutes a highly intuitive picture for the nature of quantum fluctuations. Incidentally, the path integral with Gaussian velocity weight can also be represented by a sum over trajectories of a random walker [10].

The standard route to worldline representations of propagators for higher-spin fields proceeds with the aid of Grassmann-valued path integrals that encode the spin degrees of freedom as well as the corresponding algebra [11]. Though technically elegant and computationally powerful, this approach goes along with a loss of intuition: trajectories in Grassmann-space are difficult to be visualized.

An alternative approach has been suggested in [12,13] for $D = 2, 3$ dimensions, where the information about fermionic spin can be encoded in terms of the ''Polyakov spin factor.'' This spin factor acts as an insertion in the path integrand and depends solely on the worldline itself; for instance, in $D = 2$, it can be represented by the trace of a path-ordered exponential,

$$\begin{aligned} \Phi_{\rm Pol}[x] &= {\rm tr}_\gamma\, \mathcal{P} e^{(i/2)\int_0^T d\tau \sigma \omega_{\rm Pol}}, \\ \omega_{{\rm Pol},\mu\nu} &= \frac{1}{4}(\dot{x}_\mu \ddot{x}_\nu - \ddot{x}_\mu \dot{x}_\nu), \qquad \dot{x}^2 = 1, \end{aligned} \tag{2}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. Note that this representation holds for propertime-parametrized worldlines, $\dot{x}^2 = 1$. The latter property arises naturally in the so-called ''first-order'' formalism for fermions in which the Dirac operator acts linearly on spinor states. The Polyakov spin factor is not only a purely geometric quantity; it also has a topological meaning for closed worldlines. In $D = 2$, it equals $(-1)^n$, with n counting the number of twists of a closed loop. Generalizations of the Polyakov spin factor to higher dimensions reveal interesting relations to geometric quantities [13–15], such as torsion of the worldline in $D = 3$, Berry phases or the notion of a Wess-Zumino term for a bosonic worldline path integral.

However, worldline calculations with fermions are most conveniently performed in the ''second-order'' formalism in which Dirac-algebra valued expressions are rewritten such that the Dirac operator always acts quadratically on spinor states [8,9].[1] The main advantage is that (at least for the symmetric part of the spectrum) spinorial properties always occur in the form of explicit spin-field couplings, such as the Pauli term $\sim\sigma_{\mu\nu}F_{\mu\nu}$ in QED. The natural question arises as to whether the second-order formalism can also be supplemented with a spin-factor calculus,

[1] For a detailed calculation in the first-order formalism, see, e.g., [16].

whether such a spin factor also has a topological meaning and whether it opens the door to new calculational strategies. Guided by the idea that gauge-field information can solely be covered by a description in terms of holonomies (Wilson loops), the existence of a spin factor can be anticipated.

In the present work, we derive such a spin-factor representation for the second-order formalism, employing the loop-space approach to gauge theory [17–19] (this approach has recently witnessed a revival as an alternative strategy for quantizing gravity [20]). Concentrating on QED, we are able to rewrite the Pauli term as a geometric quantity, i.e., an insertion term that depends solely on the worldline of the fluctuating particle itself. We develop a spin-factor calculus for practical computations; as a concrete example, we rederive the famous Heisenberg-Euler effective action of QED [21].

In fact, we have not been able to identify a topological content similar to that of the first-order formalism for our spin factor. But a new geometric conclusion emerges from our formalism: it is the continuous but nondifferentiable nature of the random worldlines that gives rise to the coupling between spin and external fields. By contrast, smooth worldlines, i.e., smooth trajectories of a virtual fluctuation, would not support any coupling between spinorial degrees of freedom and an external field. Particularly the "zigzag" motion of quantum mechanical worldlines mediates spin; smooth worldlines are indeed a set of measure zero for a quantum particle.

The search for a spin-factor representation in the second-order formalism was initiated and advanced in a series of works [22,23]. Therein, it was argued that the resulting spin factor has the same form as the Polyakov spin factor in $D = 2$, cf. Eq. (2). For concrete computations, an *ad hoc* regularization procedure was proposed to deal with possibly arising singularities [23] and was shown to work in a variety of nontrivial examples. As our results show unambiguously, the spin factor in the second-order formalism is not of the form of the classic Polyakov spin factor. It is particularly the singularity structure of our new spin factor in combination with that of the worldlines that dominates in the second-order formalism and gives rise to the new geometric interpretation.

Apart from intrinsic reasons for a spin-factor formalism, our work is also motivated by the recent development of *worldline numerics* [24,25], which combines the string-inspired worldline formalism with Monte Carlo techniques; the result is a powerful and efficient algorithm for computing quantum amplitudes in general background fields that has found a variety of applications [26–28]. Since Monte Carlo methods for computing path integrals rely on the positivity of the action, the representation of spin by Grassmann-valued integrals is of no use for worldline numerics. Even though fermionic worldline algorithms can be based on the conventional Pauli-term representations [26], the chiral limit becomes computationally demanding. Therefore, we expect that a spin-factor representation offers a new route to treating massless fermions with worldline numerics.

In this work, we develop the spin-factor formalism by considering the fermionic determinant, which is part of any perturbatively renormalizable gauge-field theory with charged fermions. For simplicity, it suffices to deal with Abelian gauge fields, which keeps the presentation more transparent. In this case, the fermionic determinant corresponds to the one-loop effective action for photons, i.e., the Heisenberg-Euler effective action. In Sec. II, we derive the spin-factor representation within the second-order formalism for spinor QED. We elucidate the single steps in some detail, paying particular attention to subtleties induced by the nonanalyticity of generic worldlines. In Sec. III, we first develop a spin-factor calculus for performing efficient computations with the new spin factor. We apply this calculus to a rederivation of the Heisenberg-Euler action for constant fields. Furthermore, we combine our spin factor with a representation of the Dirac algebra in terms of Grassmann-valued path integrals. Finally, a nonperturbative application is given by deriving a worldline representation for the effective action of Heisenberg-Euler type to leading nontrivial order in N_{f} (quenched approximation). We summarize our conclusions in Sec. IV.

II. SPIN-FACTOR REPRESENTATION IN QED

A. QED effective action on the worldline

Let us begin with the Euclidean one-loop effective action of QED, corresponding to the fermionic determinant [29],

$$\Gamma_{\mathrm{eff}}^{1} = -\ln\det(-i\not{D} - m) = -\frac{1}{2}\ln\det(\not{D}^2 + m^2), \quad (3)$$

where we have assumed the absence of a spectral asymmetry of the Dirac operator in the second step;[2] the two representations of the determinant distinguish between first-order and second-order formalism. Using Schwinger's propertime method [31] together with a path-integral representation of the propertime transition amplitude, the second-order determinant transforms into the worldline representation,

$$\Gamma_{\mathrm{eff}}^{1} = \frac{1}{2}\frac{1}{(4\pi)^{D/2}}\int_0^\infty \frac{dT}{T^{1+D/2}} e^{-m^2 T}\langle W_{\mathrm{spin}}[A]\rangle, \quad (4)$$

where the brackets denote the expectation value with respect to a path integral over closed worldlines,

[2]In QED, this holds for parity-invariant formulations which exist in any dimension. In general, our formalism holds for the symmetric part of the Dirac spectrum; worldline representations for the asymmetric part have been discussed in [30].

$$\langle\cdots\rangle = \int_{x(0)=x(T)} \mathcal{D}x \ldots e^{-(1/4)\int_0^T d\tau \dot{x}^2(\tau)}. \quad (5)$$

We emphasize that the path integral is normalized such that $\langle 1\rangle = 1$. In Eq. (4), we have introduced the ''spinorial'' Wilson loop,

$$W_{\rm spin}[A] = \exp\left[-i\oint dx_\mu A_\mu(x)\right] \times \mathrm{tr}_\gamma\, \mathcal{P}\exp\left[\frac{1}{2}\int_0^T d\tau \sigma_{\mu\nu}F_{\mu\nu}\right], \quad (6)$$

where $\mathcal{P}$ denotes path ordering with respect to the propertime. The first term is the standard Wilson loop, which can be viewed as the representation of an abstract loop operator. The last term is the spin-field coupling with the Pauli term, which is at the center of interest in the present work. Contrary to the standard Wilson loop, this last term is not worldline reparametrization invariant in the present formulation. Even though this is not essential for our investigation, a reparametrization invariant formulation can be constructed with the aid of an *einbein* formalism [32]; also our results given below can straightforwardly be generalized to such an invariant formalism.

B. Loop derivative

The spin-field coupling can be rewritten with the aid of the coordinate-space representation of the loop derivative (also called the area derivative) [17–19],

$$\frac{\delta}{\delta s_{\mu\nu}(\tau)} = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho \frac{\delta^2}{\delta x_\mu(\tau+\frac{\rho}{2})\delta x_\nu(\tau-\frac{\rho}{2})}, \quad (7)$$

which is analogous to the curvature tensor in the loop representation of gauge theory.

Let us start with an identity that is well known in the loop-space formulation of gauge theories [19], involving analytic functions $A_\mu(x)$ and $F_{\mu\nu}(x)$,

$$e^{-i\oint dxA(x)}\mathrm{tr}_\gamma\,\mathcal{P}e^{(1/2)\int_0^T d\tau\sigma F} = \mathrm{tr}_\gamma\,\mathcal{P}e^{(i/2)\int_0^T d\tau\sigma\{\delta/[\delta s(\tau)]\}}e^{-i\oint dxA(x)}. \quad (8)$$

It is important to stress that this relation is defined on the set of holonomy-equivalence classes of loops in coordinate space, such that it holds also for continuous but nondifferentiable loops; Eq. (8) can furthermore be represented with discretized worldlines with the gauge potentials sitting on the links. More comments are in order: a crucial ingredient of the loop derivative is given by the ϵ limit. A nonzero contribution arises only if the worldline derivatives produce a specific singularity structure $\sim\dot{\delta}(\rho)$, such that $\int d\rho\rho\dot{\delta}(\rho) = -1$. Weaker singularities or smooth ρ dependencies vanish in the limit $\epsilon\to 0$. In Eq. (8), the required singularity structure is provided by the worldline derivatives acting on the gauge field and the line-integral measure. The fact that the loop derivative can be exponentiated rests on a property of the Wilson loop, namely,

$$\left[\frac{\delta}{\delta s},\frac{\delta}{\delta s}\right]e^{-i\oint dxA(x)} = 0, \quad (9)$$

which holds only for the class of so-called Stokes-type functionals, as introduced in [18].

Finally, the proof of Eq. (8) [as well as that of Eq. (9)] requires a few smoothness assumptions for worldline-dependent expressions. Whether or not they are satisfied is *a priori* far from obvious, since we need these identities within the worldline integral, but the worldlines are generically continuous but nondifferentiable. This question can most suitably be analyzed with the aid of the worldline Green's function, which reads [8,9]:

$$\langle x_\mu(\tau_2)x_\nu(\tau_1)\rangle \equiv -\delta_{\mu\nu}G(\tau_2,\tau_1), \quad \text{with } G(\tau_2,\tau_1) = |\tau_2-\tau_1| - \frac{(\tau_2-\tau_1)^2}{T}. \quad (10)$$

The nonanalyticity of the worldlines becomes visible in the first term of the Green's function, involving the modulus. By Wick contraction, the worldline integral over general functionals of $x(\tau)$ can be reduced to (a series of) monomials of the Green's function and its derivatives. The following derivative is of particular importance:

$$\langle\ddot{x}_\mu(\tau_2)\dot{x}_\nu(\tau_1)\rangle = 2\dot{\delta}(\tau_2-\tau_1)\delta_{\mu\nu}, \quad (11)$$

since the singularity structure $\sim\dot{\delta}$ suitable for the loop derivative occurs. Therefore, the proof that Eq. (8) also holds under the worldline integral can be completed by the observation that all other terms occurring during the calculation do not involve Wick contractions of the type (11). The same statement applies to the proof of Eq. (9).

Let us proceed with the spin-factor derivation by performing an infinite series of partial integrations that shifts the loop derivatives from the Wilson loop to the worldline kinetic term, yielding

$$\langle W_{\rm spin}[A]\rangle = \int \mathcal{D}x(\tau)e^{-i\oint dxA(x)} \times[\mathrm{tr}_\gamma\,\mathcal{P}e^{(i/2)\int_0^T d\tau\sigma\{\delta/[\delta s(\tau)]\}}e^{-\int_0^T d\tau\{[\dot{x}^2(\tau)]/4\}}]. \quad (12)$$

No surface terms appear, since the worldlines, if stretched to infinity, have infinite kinetic action. Now the evaluation of the derivatives has to be performed with great care. We begin with the leading order of the exponential series,

$$\left(\frac{i}{2}\int_0^T d\tau\sigma\frac{\delta}{\delta s(\tau)}\right)(e^{-\int_0^T d\tau\{[\dot{x}^2(\tau)]/4\}}) = \left(\frac{i}{2}\int_0^T d\tau\sigma\omega(\tau)\right)(e^{-\int_0^T d\tau\{[\dot{x}^2(\tau)]/4\}}), \quad (13)$$

where we have defined

$$\omega_{\mu\nu}(\tau) := \frac{1}{4}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho \rho \ddot{x}_\mu\left(\tau+\frac{\rho}{2}\right)\ddot{x}_\nu\left(\tau-\frac{\rho}{2}\right). \quad (14)$$

It is this ω tensor that carries the information previously encoded in the field-strength tensor. The ω tensor is significantly different from that of Polyakov's spin factor $\omega_{\mathrm{Pol},\mu\nu} \sim (\ddot{x}_\mu \dot{x}_\nu - \ddot{x}_\nu \dot{x}_\mu)$, arising in the first-order formalism [cf. Eq. (2)]. For instance, $\omega_{\mu\nu}(\tau)=0$ for any smooth loop by virtue of the ϵ limit, whereas $\omega_{\mathrm{Pol},\mu\nu}$ is generally nonzero in this case.

It is instructive to also study the second order in the loop derivative explicitly:[3]

$$\left(\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}\right)^2 e^{-(1/4)\int_0^T d\tau \dot{x}^2} = \left\{\left(\frac{i}{2}\int d\tau\sigma\omega\right)^2 - \frac{1}{4}\int d\tau_2 d\tau_1 \sigma_{\lambda\kappa}\sigma_{\mu\nu} \times\left[\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta s_{\lambda\kappa}(\tau_2)} + \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\eta\eta\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta x_\kappa(\tau_2-\frac{\eta}{2})}\ddot{x}_\lambda\left(\tau_2+\frac{\eta}{2}\right)\right]\right\} e^{-(1/4)\int_0^T d\tau\dot{x}^2}. \quad (15)$$

Apart from the desired first term $\sim\omega^2$, we observe the appearance of derivatives of ω. The latter correspond to a nonvanishing right-hand side (rhs) of the commutator $[\delta/\delta s, \delta/\delta s]$ acting on the kinetic action. This is in contrast to Eq. (9), and reveals that the kinetic action does not belong to the class of Stokes-type functionals. Proceeding to higher orders in the loop derivative, the result can be represented as

$$\mathrm{tr}_\gamma\, \mathcal{P}e^{(i/2)\int_0^T d\tau\sigma\{\delta/[\delta s(\tau)]\}} e^{-\int_0^T d\tau\{[\dot{x}^2(\tau)]/4\}} = [\mathrm{tr}_\gamma\, \mathcal{P}e^{(i/2)\int_0^T d\tau\sigma\omega} + D[\omega]] e^{-\int_0^T d\tau\{[\dot{x}^2(\tau)]/4\}}, \quad (16)$$

where $D[\omega]$ is a functional of ω that collects all terms with at least one functional derivative of ω. This functional can formally be defined by

$$D[\omega] := e^{\int_0^T d\tau\{[\dot{x}^2(\tau)]/4\}}\mathrm{tr}_\gamma\, \mathcal{P}e^{(i/2)\int_0^T d\tau\sigma\{\delta/[\delta s(\tau)]\}} \times e^{-\int_0^T d\tau\{[\dot{x}^2(\tau)]/4\}} - \mathrm{tr}_\gamma\, \mathcal{P}e^{(i/2)\int_0^T d\tau\sigma\omega}, \quad (17)$$

with the last term simply subtracting the no-derivative terms. An explicit representation of $D[\omega]$ can be computed order by order in a series expansion in ω; the first term, for instance, is given by the second term in the braces in Eq. (15). We would like to stress that $D[\omega]$ has been missed in the literature so far, e.g., see [22]. However, this functional is absolutely crucial for rendering the spin-factor representation well-defined, as will be discussed in the next section.

C. Spin factor

The representation of the spin information derived in Eq. (16) seems highly problematic. Let us recall from the definition of ω in Eq. (14) that $\omega \sim \ddot{x}\ddot{x}$. Upon insertion into the worldline integrand, Wick contractions of the form $\langle\ddot{x}\ddot{x}\rangle$ carrying a strong singularity structure $\sim\ddot{\delta}$ will necessarily appear, cf. Eq. (11). Such singularities can survive the ϵ limits and potentially render the expressions ill-defined.

In fact, we will now prove that all singularities of the type $\sim\ddot{\delta}$ cancel exactly against the functional $D[\omega]$ occurring in Eq. (16). This can straightforwardly be derived from the zero-field limit of Eq. (12) for which the Wilson-loop expectation value is normalized to 1,

$$\begin{aligned} 1 &= \langle W_{\mathrm{spin}}[A=0]\rangle \\ &= \int \mathcal{D}x(\tau)\,\mathrm{tr}_\gamma\, \mathcal{P}e^{(i/2)\int_0^T d\tau\sigma\{\delta/[\delta s(\tau)]\}} e^{-\int_0^T d\tau\{[\dot{x}^2(\tau)]/4\}} \\ &= \int \mathcal{D}x(\tau)[\mathrm{tr}_\gamma\, \mathcal{P}e^{(i/2)\int_0^T d\tau\sigma\omega} + D[\omega]] e^{-\int_0^T d\tau\{[\dot{x}^2(\tau)]/4\}}, \end{aligned} \quad (18)$$

where we have used Eqs. (16) and (17) in the last step. Even without reference to the zero-field limit, we could have straightforwardly proven this identity by noting that

$$\int \mathcal{D}x(\tau)\left(\frac{\delta}{\delta x_\mu(\tau)}\right)^n e^{-\int_0^T d\tau\{[\dot{x}^2(\tau)]/4\}} = 0, \qquad n\geq 1 \quad (19)$$

vanishes as a total derivative; we recall that the pure Gaussian velocity integral is normalized to 1.

In the language of Wick contractions, we make the important observation from Eq. (18) that $\langle D[\omega]\rangle$ corresponds to the self-contractions of the ω exponential:[4]

$$\langle D[\omega]\rangle = 1 - \langle \mathrm{tr}_\gamma\, \mathcal{P}e^{(i/2)\int_0^T d\tau\sigma\omega}\rangle.$$

Representing the worldline operators $x(\tau)$ in Fourier space by Fock-space creation and annihilation operators of Fourier modes [cf. Eq. (27) below], the removal of self-contractions of any expression can be implemented by normal ordering of the Fock-space operators; thus, we arrive at

$$1 = \langle \mathrm{tr}_\gamma\, \mathcal{P}e^{(i/2)\int_0^T d\tau\sigma\omega} + D[\omega]\rangle \equiv \langle \mathrm{tr}_\gamma\, \mathcal{P}{:}e^{(i/2)\int_0^T d\tau\sigma\omega}{:}\rangle, \quad (20)$$

[3]We suppress the path-ordering symbol here; it can easily be reinstated at the end of the calculation.

[4]To a given order, this can algebraically be confirmed by direct computation; for an explicit second-order calculation, see Appendix A and [33].

where the colons denote the normal-ordering prescription. This concludes our search for a spin-factor representation in the fermionic second-order formalism of the worldline approach. Upon insertion into Eq. (4), we obtain a representation of the one-loop contribution to the effective action for spinor QED, involving the purely geometrical spin factor,

$$\Gamma^1_{\rm eff}[A] = \frac{1}{2}\frac{1}{(4\pi)^{D/2}}\int_0^T \frac{dT}{T^{(1+D/2)}} e^{-m^2 T} \times \int \mathcal{D}x(\tau) e^{-\int_0^T d\tau\{[\dot{x}^2(\tau)]/4\}} e^{-i\oint dx A(x)}\Phi[x],$$

$$\text{with } \Phi[x] := \mathrm{tr}_\gamma\, \mathcal{P}{:}e^{(i/2)\int_0^T d\tau\sigma\omega(\tau)}{:}, \tag{21}$$

$$\text{and} \quad \omega_{\mu\nu}(\tau) = \frac{1}{4}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\,\rho\,\ddot{x}_\mu\Big(\tau+\frac{\rho}{2}\Big)\ddot{x}_\nu\Big(\tau-\frac{\rho}{2}\Big). \tag{22}$$

An obvious advantage of this representation consists in the fact that the dependence on the external gauge field occurs solely in the form of a Wilson loop (holonomy). An explicit spin-field coupling no longer appears, but spin information is extracted from the geometric properties of the worldlines themselves. Let us emphasize once more that a nonzero spin contribution is generated only by specific singularity structures, arising from the continuous but nondifferentiable nature of generic worldlines.

III. APPLICATION OF THE SPIN FACTOR

A. Spin-factor calculus

Next we explore the applicability of the new spin factor in concrete QED examples. At first glance, the representation of the effective action (21) seems to be disadvantageous; in particular, concrete computations may be plagued by technical difficulties associated with normal ordering. Moreover, even perturbative amplitudes to finite order in A_μ seemingly receive contributions from terms with arbitrarily high products of worldline monomials: expanding the spin-factor and Wilson-loop exponentials, we find, for instance, terms of the form $\langle \omega^n \dot{x} A(x)\rangle \sim \langle (\ddot{x}\ddot{x})^n \dot{x}A(x)\rangle$, n arbitrary.

Nevertheless, it can be shown that many of these apparent high-order contributions cancel each other and that practical calculations actually boil down to roughly the same amount of technical work as in the standard formalism. In view of the variety of possible worldline monomials arising from the expansion of the Wilson loop, the spin factor and the corresponding self-contractions (hidden behind the normal ordering), we do not attempt to give a full account of all possible structures and cancellation mechanisms. Instead, we will pick out all those terms that, upon Wick contraction, lead us back to the full result for the effective action in standard representation. As a result, all possible other terms ultimately have to cancel each other.

Let us start with a new operational symbol $\{\cdots\}_\omega^{\oint A}$ that characterizes a subclass of Wick contractions: the $\{\cdots\}_\omega^{\oint A}$ bracket denotes the restriction that, among the manifold contractions arising from Wick's theorem, only those terms have to be accounted for which are *complete* contractions of one $\sigma\omega$ with *one and the same* $\oint dxA(x)$ factor. This already excludes many Wick contractions, in particular, those where the two $\ddot{x}$'s out of one $\omega_{\mu\nu}$ are either self-contracted or contracted with two different objects (be it gauge fields or other ω's). It turns out that all these terms of the latter type cancel each other or vanish by the ϵ limit. Using the Schwinger-Fock gauge,

$$A_\alpha(x(\tau)) = \frac{1}{2}x_\lambda(\tau)F_{\lambda\alpha}(0) + \frac{1}{3}x_\lambda(\tau)x_\sigma(\tau)\partial_\sigma F_{\lambda\alpha}(0) + \cdots = \sum_{n=0}^{\infty}\frac{x^\lambda x^{\nu_1}\cdots x^{\nu_n}}{n!(n+2)}\partial_{\nu_1}\cdots\partial_{\nu_n}F_{\lambda\alpha}, \tag{23}$$

the subclass of $\{\cdots\}_\omega^{\oint A}$ contractions of the Wilson-loop exponential with an ω term can straightforwardly be computed order by order in the series (23). The resulting series is identical to the Taylor series of the field-strength tensor, which can be summed up to yield

$$\Big\{\frac{i}{2}\int d\tau\sigma\omega(-i)\int d\tau\dot{x}_\nu(\tau)A_\nu(x(\tau))\Big\}_\omega^{\oint A} = \frac{1}{2}\int d\tau\sigma_{\mu\nu}F^{\mu\nu}(x(\tau)). \tag{24}$$

Since the operation of Wick contractions of bosonic fields satisfies the elementary rules of a derivation, the same holds for the $\{\cdots\}_\omega^{\oint A}$ symbol. With this observation (or with straightforward combinatorics), it follows that

$$\{\mathrm{tr}_\gamma\,\mathcal{P}e^{(i/2)\int_0^T d\tau\sigma\omega(\tau)}e^{-i\int_0^T d\tau\dot{x}A(x)}\}_\omega^{\oint A} = e^{-i\int_0^T d\tau\dot{x}A(x)}\mathrm{tr}_\gamma\,\mathcal{P}e^{(1/2)\int_0^T d\tau\sigma F}. \tag{25}$$

This tells us immediately that it is the subclass of Wick contractions described by the $\{\cdots\}_\omega^{\oint A}$ symbol which already gives us back the full result for the Pauli term. The resulting recipe is: the spin factor can only contribute if a factor $\sim\int_0^T\sigma\omega(\tau)$ is completely Wick contracted with a factor $\sim\oint dxA(x)$ from the Wilson loop. Since the ω-independent Wick contractions still have to be performed, the expectation value of the spinorial Wilson loop can finally be written as

$$\langle W_{\rm spin}\rangle = \langle\{\mathrm{tr}_\gamma\,\mathcal{P}e^{(i/2)\int_0^T d\tau\sigma\omega(\tau)}e^{-i\int_0^T d\tau\dot{x}A(x)}\}_\omega^{\oint A}\rangle. \tag{26}$$

Note that this recipe also dispenses with a consideration of normal ordering or a detailed analysis of the self-contraction terms, since these do not contribute to the $\{\cdots\}_\omega^{\oint A}$ bracket by construction. Beyond its definition via partial Wick contractions, the $\{\cdots\}_{\omega}^{\oint A}$ symbol can more abstractly be used as a projector that removes all terms generated by self-contractions of ω or mixed contractions as specified above. As such, the $\{\cdots\}_{\omega}^{\oint A}$ symbol is a linear operator that can formally be interchanged with the (regularized) worldline integral. This viewpoint will be exploited below.

The spin-factor calculus developed here has a physical interpretation: the spin factor is only operating at those space-time points where the fluctuating particle interacts with the external field. The spin of the fluctuation does not generate self-interactions of the fluctuation with its own worldline, nor does spin interact nonlocally with the external field at two different space-time points simultaneously. In the following section we demonstrate the applicability of the spin-factor calculus by rederiving the classic Heisenberg-Euler effective action with this new formalism.

B. Heisenberg-Euler action

As a concrete example, let us compute the one-loop effective action for a constant background field, i.e., the Heisenberg-Euler effective action for soft photons. We describe the background field, which is constant in space and time but otherwise arbitrary, by the gauge potential $A_\mu = -(1/2)F_{\mu\nu}x_\nu$. As a first simplification, we note that path ordering is irrelevant for a constant field. Furthermore, we observe that the path integral becomes Gaussian, since both Wilson-loop as well as spin-factor exponents depend quadratically on x. The propertime derivatives become diagonal in Fourier space where the worldlines are represented as

$$x_\mu(\tau) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{T}} a_{n\mu} e^{[(2\pi i n\tau)/T]}. \tag{27}$$

The fact that $x_\mu \in \mathbb{R}^D$ translates into the reality condition $a^*_{-n\mu} = a_{n\mu}$. In terms of the $a_{n\mu}$ variables, the worldline integral becomes

$$\langle W_{\rm spin}\rangle = \int \mathcal{D}a\, {\rm tr}_\gamma \{e^{-(1/2)\sum_n a^*_{\mu n}\{(1/2)[(2\pi)/T]^2 n^2 \delta_{\mu\nu} - [(2\pi n)/T]F_{\mu\nu} + (1/2)[(2\pi n)/T]^2 \sigma_{\mu\nu} g_n(\epsilon)\} a_{n\nu}}\}_{\omega}^{\oint A}, \tag{28}$$

with

$$g_n(\epsilon) = \left(\frac{2\pi n}{T}\epsilon\right)\cos\left(\frac{2\pi n}{T}\epsilon\right) - \sin\left(\frac{2\pi n}{T}\epsilon\right), \tag{29}$$

arising from the Fourier transform of the spin factor. Here and in the following, the limit $\epsilon \to 0$ is implicitly understood. In Eq. (28), we can separate off the Fourier zero mode $n = 0$, i.e., the worldline center of mass, corresponding to the space-time integration of the effective action. We obtain

$$\begin{aligned}\langle W_{\rm spin}\rangle &= \int d^D x \int \mathcal{D}a\, {\rm tr}_\gamma \{e^{-(1/2)\sum'_n a^*_{\mu n} M_{\mu\nu} a_{\nu n}}\}_{\omega}^{\oint A} \\ &= \int d^D x\, {\rm tr}_\gamma \left\{ {\rm det}'^{-(1/2)}\left[\frac{M}{M_0}\right]\right\}_{\omega}^{\oint A},\end{aligned}$$

where M denotes the quadratic fluctuation operator in the exponent of Eq. (28). The operator M_0 abbreviates M in the limit $F_{\mu\nu} \to 0$ and the formal limit[5] $g_n(\epsilon) \to 0$; the appearance of M_0 implements the correct normalization of the path integral. The prime indicates the absence of the $n = 0$ zero mode. Exponentiating the determinant results in

$$\begin{aligned}\left\{{\rm det}'^{-(1/2)}\frac{M}{M_0}\right\}_{\omega}^{\oint A} &= \exp\left[-\frac{1}{2}\left\{\sum_n{}' {\rm tr}_{\rm L} \ln\left(\mathbb{1} - 2\left(\frac{T}{2\pi n}\right)F + \sigma g_n(\epsilon)\right)\right\}_{\omega}^{\oint A}\right] \\ &= \exp\left[\sum_{n=1}^{\infty} {\rm tr}_{\rm L} \sum_{m=1}^{\infty} \frac{1}{2m}\left\{\left(2\left(\frac{T}{2\pi n}\right)F - \sigma g_n(\epsilon)\right)^{2m}\right\}_{\omega}^{\oint A}\right],\end{aligned} \tag{30}$$

where we have expanded the logarithm in the last step. Now we use the binomial sum for the term in the $\{\cdots\}_{\omega}^{\oint A}$ symbol,

$$\begin{aligned}\{\%\}_{\omega}^{\oint A} &:= \left\{\left(2\left(\frac{T}{2\pi n}\right)F - \sigma g_n(\epsilon)\right)^{2m}\right\}_{\omega}^{\oint A} \\ &= \sum_{k=0}^{m}\binom{2m}{k}\left(2\left(\frac{T}{2\pi n}\right)F\right)^{2m-k}(\sigma g_n(\epsilon))^k.\end{aligned}$$

Here, the $\{\cdots\}_{\omega}^{\oint A}$ symbol has by definition removed all those terms for which at least one $\sigma g_n(\epsilon)$ term cannot be paired with an F term. This reduces the upper limit of the sum from $2m$ to m. Furthermore, we have used that in the constant field case $[F, \sigma] = 0$; therefore, F and σ can be arranged in arbitrary order. We decompose this sum further by separating off the $k = 0$ and $k = m$ terms,

[5]These terms would vanish anyway because of the $\{\cdots\}_{\omega}^{\oint A}$ symbol.

$$\{\%\}^{\oint A}_{\omega} = \underbrace{\left(\frac{TF}{\pi n}\right)^{2m}}_{\text{(I)}} + \underbrace{\binom{2m}{m}\left(\frac{T}{\pi n} F\sigma g_n(\epsilon)\right)^m}_{\text{(II)}} + \underbrace{\sum_{k=1}^{m-1}\binom{2m}{k}\left(\frac{TF}{\pi n}\right)^{2m-k}(\sigma g_n(\epsilon))^k}_{\text{(III)}}. \quad (31)$$

The first term (I) carries no spin information; this scalar part obviously corresponds to the contribution that we would equally encounter in scalar QED. The second term (II) represents a perfect pairing of spin factor and field-strength contribution; it will turn out to contain the entire spinorial information. The remaining sum (III) has always at least one unpaired F term, even for $k = m - 1$. As will be demonstrated below, this sum vanishes completely in the ϵ limit, owing to its too-weak singularity structure. Let us now compute the various pieces of Eq. (31) separately.

Let us first consider the scalar part, substituting the first term (I) of Eq. (31) into Eq. (30); we take over the result of this standard calculation from [8,9],

$$\text{(I)}: \quad \exp\left[\sum_{n=1}^{\infty} \mathrm{tr}_L \sum_{m=1}^{\infty} \frac{1}{2m}\left(\frac{TF}{\pi n}\right)^{2m}\right] = \det{}^{-(1/2)}\left(\frac{\sin(FT)}{FT}\right),$$

where the remaining determinant refers to the Lorentz structure. For instance, for the constant B field case, this reduces to $BT/\sinh BT$.

Next, we consider the spinor contributions in some detail; this part of the spin-factor-based Heisenberg-Euler calculation is genuinely new. The spinor part induced by substitution of the second term (II) of Eq. (31) into Eq. (30) can be written as

$$\text{(II)}: \quad \exp\left[\sum_{m=1}^{\infty} \frac{(2m-1)!}{(m!)^2}\left(\frac{T}{\pi}\right)^m \mathrm{tr}_L (F\sigma)^m \sum_{n=1}^{\infty} \frac{g_n(\epsilon)}{n^m}\right]. \quad (32)$$

Let us discuss the Fourier sum for different values of m, using the definition of $g_n(\epsilon)$ in Eq. (29),

$$S_m := \lim_{\epsilon\to 0} \sum_{n=1}^{\infty} \frac{g_n(\epsilon)}{n^m} = \lim_{\epsilon\to 0} \sum_{n=1}^{\infty} \frac{\left((\frac{2\pi n\epsilon}{T}) \cos\frac{2\pi n}{T}\epsilon - \sin\frac{2\pi n}{T}\epsilon\right)}{n^m}. \quad (33)$$

For $m = 1$, we have

$$S_1 = \lim_{\epsilon\to 0}\left[\sum_{n=1}^{\infty} \epsilon \frac{d \sin\frac{2\pi n}{T}\epsilon}{d\epsilon\, n} - \sum_{n=1}^{\infty} \frac{\sin\frac{2\pi n}{T}\epsilon}{n}\right] = \lim_{\epsilon\to 0}\left[\epsilon\frac{d}{d\epsilon}\left(\frac{\pi - \frac{2\pi}{T}\epsilon}{2}\right) - \left(\frac{\pi - \frac{2\pi}{T}\epsilon}{2}\right)\right] = -\frac{\pi}{2}. \quad (34)$$

Let us stress that this nonzero contribution survives the ϵ limit, since the Fourier sum results in a nonanalytic function (resembling a sawtooth profile). This agrees with our general observation that the spin information is encoded in the nonanalytic behavior of the worldline trajectory in space-time.

In fact, the $m = 1$ contribution is the only nonvanishing term; all S_m for $m > 1$ as well as all contributions arising from term (III) in Eq. (31) are zero in the limit $\epsilon \to 0$, as is shown in Appendix B. The whole spinor contribution is that of Eq. (32), boiling down to $\exp(-(T/2)\,\mathrm{tr}_L[F\sigma])$. The spinorial Wilson-loop expectation value thus becomes

$$\langle W_{\text{spin}}\rangle = \int d^D x \det{}^{-(1/2)}\left(\frac{\sin(FT)}{FT}\right) \mathrm{tr}_\gamma e^{-(T/2)\mathrm{tr}_L[F\sigma]} = 4\int d^D x \det{}^{-(1/2)}\left(\frac{\tan(FT)}{FT}\right), \quad (35)$$

where the Dirac trace has been taken in the last step. For instance, for a constant B field, the last line reads $4\int d^D x BT/\tanh BT$. Inserting this final result into Eq. (4), we arrive at the (unrenormalized) Heisenberg-Euler action [21,31],

$$\Gamma^1_{\text{eff}}[A] = \frac{2}{(4\pi)^{D/2}} \int_0^\infty \frac{dT}{T^{(1+D/2)}} e^{-m^2 T} \det{}^{-1/2}\left(\frac{\tan FT}{FT}\right). \quad (36)$$

We would like to stress that the present derivation of this well-known result is independent of other standard calculational techniques, as far as the spinor part is concerned. The spinor contribution arises from the subtle interplay between the purely geometric spin factor and the Wilson loop. Nonzero contributions arise only from terms with a particular singularity structure. Since these singularities cannot arise from smooth worldlines, we conclude that the random zigzag course of the worldlines is an essential ingredient for the coupling between spin and fields.

C. Spin factor with Grassmann variables

In the standard approaches to describing fermionic degrees of freedom, spin information is encoded in additional Grassmann-valued path integrals. One motivation for the spin-factor representation has been to find a purely bosonic description devoid of both an explicit spin-field coupling and additional Grassmann variables.

But since the latter two criteria are independent of each other, we can combine our spin-factor representation with Grassmann variables, in order to make use of the elegant formulation of the Dirac algebra and the path ordering by means of anticommuting worldline variables.

For instance, the standard worldline formulation for the one-loop effective action of QED in terms of a Grassmannian path integral is given by [8,9]

$$\Gamma^1_{\text{eff}}[A] = \frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{\text{p.}} \mathcal{D}x \int_{\text{a.p.}} \mathcal{D}\psi e^{-\int_0^T d\tau L_{\text{spin}}}, \quad (37)$$

with

$$L_{\mathrm{spin}} = \frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi_\mu \dot{\psi}^\mu + i\dot{x}^\mu A_\mu - i\dot{\psi}^\mu F_{\mu\nu}\psi^\mu. \tag{38}$$

The path integrals satisfy either periodic (p.) or antiperiodic (a.p.) boundary conditions, depending on their statistics. Starting from this representation, our line of reasoning can immediately be applied, resulting in the following new expression for the QED action:

$$\Gamma^1_{\mathrm{eff}}[A] = -\frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{\mathrm{p.}} \mathcal{D}x \int_{\mathrm{a.p.}} \mathcal{D}\psi e^{-\int d\tau (\dot{x}^2/4)} e^{-i\oint dx A} e^{-\int d\tau [(\psi\dot{\psi})/2]} {:}e^{-\int d\tau \dot{\psi}\omega\psi}{:}. \tag{39}$$

Normal ordering takes care of the removal of ω self-contractions of the spin factor, whereas the path ordering is automatically guaranteed by the Grassmann integral. An interesting question of this representation concerns the fate of supersymmetry. Whereas the standard representation has a worldline supersymmetry, the supersymmetry is not manifest in the present formulation [the Wilson-loop exponent and the Pauli term are supersymmetric partners in Eq. (38)].

D. Nonperturbative worldline dynamics

The derivation of nonperturbative worldline expressions is an application where our spin-factor representation becomes highly advantageous. So far, we have considered perturbative diagrams involving one charged fermion loop, but no photon fluctuations. Promoting the fermions to a Dirac spinor with N_{f} flavor components, the functional integral over photon fluctuations becomes Gaussian in leading nontrivial order in a small-N_{f} expansion. In scalar QED, this gauge-field integral can be done straightforwardly, since the worldline–gauge-field coupling occurs simply in the form of the Wilson loop, which is a bosonic-current interaction. In the literature, the leading-order N_{f} expansion has already been used in early works on worldline techniques for scalar QED [1,34]. For instance, the nonperturbative effective action of Heisenberg-Euler type in this approximation reads for scalar QED,

$$\Gamma^{\mathrm{Scalar\ QED}}_{\mathrm{QA}}[A_\mu] = \int_x \frac{1}{4e^2} F_{\mu\nu}F_{\mu\nu} - \frac{N_{\mathrm{f}}}{(4\pi)^{D/2}} \int_0^\infty \frac{dT}{T^{1+D/2}} \times \langle e^{-i\oint dx\cdot A} e^{-(e^2/2)\int_0^T d\tau_1 d\tau_2 \dot{x}_{1\mu}\Delta_{\mu\nu}(x_1,x_2)\dot{x}_{2\nu}}\rangle, \tag{40}$$

where $\langle \cdots \rangle$ again represents the worldline average as defined in Eq. (5). For a detailed derivation of Eq. (40), see [35]. The subscript "QA" refers to the leading-order N_{f} expansion as the "quenched approximation," since diagrams with further charged loops are neglected. In Eq. (40), we have abbreviated $x_{1,2} \equiv x(\tau_{1,2})$ and employed the photon propagator,

$$\Delta_{\mu\nu}(x_1, x_2) = \frac{\Gamma(\frac{D-2}{2})}{4\pi^{D/2}}\left[\frac{1+\alpha}{2}\frac{1}{|x_1 - x_2|^{D-2}} + \left(\frac{D}{2} - 1\right) \times (1-\alpha)\frac{(x_1 - x_2)_\mu (x_1 - x_2)_\nu}{|x_1 - x_2|^D}\right], \tag{41}$$

in D dimensions with gauge parameter α. The additional insertion term involving the photon propagator in the worldline average corresponds to all possible internal photon lines in the charged loop and carries the nonperturbative contribution. It can be shown that the quenched approximation is reliable for weak external fields, but for arbitrary values of the coupling.[6]

Applying the strategy of the quenched approximation to spinor QED, a further technical complication arises from the Pauli term. Even though the photon integral remains Gaussian, the worldline current becomes Dirac-algebra valued which has to be treated with greater care [36], see, e.g., [37] for Grassmann-valued representations. At this point, our spin-factor approach becomes elegant, since the worldline–gauge-field coupling is reduced to the Wilson loop. The derivation of the corresponding nonperturbative representations in spinor QED becomes identical to scalar QED. We can immediately write down the effective action to leading order in N_{f}:

$$\Gamma^{\mathrm{Spinor\ QED}}_{\mathrm{QA}}[A_\mu] = \int_x \frac{1}{4e^2} F_{\mu\nu}F_{\mu\nu} + \frac{N_{\mathrm{f}}}{2}\frac{1}{(4\pi)^{D/2}} \int_0^\infty \frac{dT}{T^{1+D/2}} \langle e^{-i\oint dx\cdot A} e^{-(e^2/2)\int_0^T d\tau_1 d\tau_2 \dot{x}_{1\mu}\Delta_{\mu\nu}\dot{x}_{2\nu}} \mathrm{tr}_\gamma\, \mathcal{P}{:}e^{(i/2)\int_0^T d\tau \sigma\omega}{:}\rangle. \tag{42}$$

This representation can now serve as the basis for nonperturbative studies of strong-coupling QED [38] in quenched approximation along the lines proposed in [35]. Further interesting versions of this nonperturbative formula may be obtained by trading the spin factor backwards for loop derivatives, acting now on the Wilson loop as well as the photon insertion; this will be the subject of future work.

IV. CONCLUSIONS

In this work, we have used the worldline approach to quantum field theory for a study of couplings between spinors and external gauge fields. Guided by the idea that

[6]In non-Abelian gauge theories with N_{c} colors, the quenched approximation has also been shown to hold to leading order in a large-N_{c} expansion [12].

gauge-field information can solely be covered by holonomies (Wilson loops), we have investigated a reformulation of the familiar Pauli term in spinorial QED. In this instance, we have shown that the Pauli term can be reexpressed in terms of a spin factor which is a purely geometric quantity in the sense that it depends only on the worldline trajectory. Our final representation of the fermionic fluctuation determinant, i.e., the one-loop effective action for QED, has the following form:

$$\Gamma^1_{\text{eff}}[A] = \frac{1}{2}\frac{1}{(4\pi)^{(D/2)}}\int_0^T \frac{dT}{T^{1+(D/2)}} e^{-m^2 T} \times \int \mathcal{D}x(\tau) e^{-\int d\tau\{[\dot{x}^2(\tau)]/4\}} \times e^{-i\oint dx A} \text{tr}_\gamma\, \mathcal{P}{:}e^{(i/2)\int d\tau \sigma\omega}{:}. \quad (43)$$

The last factor represents the spin factor in the fermionic second-order formalism with $\omega = \omega[x]$ defined in Eq. (22). Loosely speaking, the exponent $\sigma_{\mu\nu}\omega_{\mu\nu}[x]$ replaces the spin-field coupling $\sim\sigma_{\mu\nu}F_{\mu\nu}$ of the standard representation of the fermionic effective action.

The spin factor deviates in a number of aspects from the Polyakov spin factor, occurring in the first-order formalism. These differences, which have been missed so far in the literature [22,23], are rooted in the fact that the worldlines in the two formalisms obey different velocity distributions: in the first-order formalism, the worldlines are propertime-parametrized, $|\dot{x}| = 1$, whereas their velocity is Gaussian-distributed in the second-order formalism. A consequence for the spin factors is, for instance, that smooth differentiable worldlines give zero contribution to our spin-factor exponent; ω has only nonzero support for worldlines of zigzag shape, inducing a particular singularity structure. By contrast, the Polyakov spin factor is not sensitive to the analytic properties of the worldlines; on the contrary, it has not only a geometric but also a topological meaning (e.g., counting the twists of a worldline in $D = 2$). We have not been ably to identify a topological meaning for our spin factor. Even if there was one, its relevance would be unclear, since the spin factor enters the worldline integrand with a normal-ordering prescription. As a consequence, the spin factor in itself does not appear to have a particular meaning; only contractions of the spin factor with other observables such as the Wilson loop in the integrand become meaningful.

For practical perturbative calculations, we have developed a spin-factor calculus that reduces the amount of analytical computational steps to roughly the same amount as in the standard approach. The main advantage of our formulation consists in the fact that the dependence on the external gauge field occurs solely in the form of the Wilson loop. Particularly in computer-algebraic realizations of high-order amplitude calculations, this may lead to algorithmic simplifications compared to the standard approach. On the other hand, we have to mention that the isolation of all those terms with the required singularity structure for the ϵ limit might lead to algorithmic complications. We have demonstrated all these aspects in the concrete example of the classic Heisenberg-Euler effective action.

Our spin-factor formalism becomes truly advantageous for the analysis of nonperturbative worldline dynamics based on the small-N_{f} expansion, i.e., quenched approximation. Here, the spin factor dispenses with all complications for the photon-fluctuation integrations induced by direct spin-field couplings. We have presented a closed-form worldline expression for the leading-order-N_{f} nonperturbative effective action of Heisenberg-Euler–type that can serve as a starting point for strong-coupling investigations.

We believe that our work paves the way to further studies of spin-factor representations. Our techniques are, for instance, directly applicable to diagrams with open fermionic lines, such as propagators, etc. We expect that our approach will become particularly powerful in the case of non-Abelian gauge fields, since the gluonic spin-field coupling can also be traded for a spin factor. Non-Abelian gauge-field dependencies will then be described only in terms of holonomies. In this sense, our work can be viewed as a bottom-up approach to a loop-space formulation of gauge theories [17–19].

Since our work was also motivated by the development of worldline numerics [24,25], we have to face the problem of a numerical implementation of our formalism. An immediate numerical realization seems inhibited by the normal-ordering prescription. This requires the study of possible alternatives. If the nature of our spin factor turns out to be topological, it might be possible to classify the worldlines in terms of their topological properties. This would facilitate the implementation of an algorithm that performs a Monte Carlo sampling for each individual topological sector separately.

To summarize, we have performed a first detailed analysis of the spin factor in the second-order formalism of QED. We believe that this opens the door to many further studies of the interrelation between spin and external fields in a geometric language.

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APPENDIX A: SINGULARITY CANCELLATIONS: AN EXPLICIT EXAMPLE

Here, we demonstrate by an explicit calculation to second order that the Wick self-contractions of the spin factor cancel against the $D[\omega]$ term defined in Eq. (17). This cancellation also guarantees the absence of severe singularities. To be precise, we show explicitly that

$$1 = \langle \mathrm{tr}_\gamma \mathcal{P} e^{(i/2)\int_0^T d\tau\sigma\omega} + D[\omega]\rangle \tag{A1}$$

holds to second order (the counting of orders can formally be defined by the number of $\sigma_{\mu\nu}$ matrices involved). First, we observe that the zeroth order on the rhs trivially reproduces the left-hand side (lhs). The first order vanishes by virtue of the Dirac trace. The second-order calculation requires to show that [cf. Equation (15)]

$$\left\langle \mathcal{P}\left(\int d\tau\sigma\omega\right)^2 + \mathcal{P}\int d\tau_2 d\tau_1 \sigma_{\lambda\kappa}\sigma_{\mu\nu}\left[\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta s_{\lambda\kappa}(\tau_2)} + \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\eta\,\eta\,\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta x_\kappa(\tau_2-\frac{\eta}{2})}\ddot{x}_\lambda\left(\tau_2+\frac{\eta}{2}\right)\right]\right\rangle = 0. \tag{A2}$$

Since the cancellation will turn out to hold already for the τ_1, τ_2 integrands, we can suppress the path-ordering symbol in the following. Let us first compute the derivatives of ω, beginning with

$$\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta s_{\lambda\kappa}(\tau_2)} = \frac{1}{2}\lim_{\epsilon_1,\epsilon_2\to 0}\int_{-\epsilon_1}^{\epsilon_1}\int_{-\epsilon_2}^{\epsilon_2} d\eta d\rho\,\rho\,\eta\,\delta_{\mu\lambda}\delta_{\nu\kappa} \times \ddot{\delta}\left[\tau_1+\frac{\rho}{2}-\left(\tau_2+\frac{\eta}{2}\right)\right] \times \ddot{\delta}\left[\tau_1-\frac{\rho}{2}-\left(\tau_2-\frac{\eta}{2}\right)\right], \tag{A3}$$

where we have already used the antisymmetry properties of ω. Furthermore, we encounter

$$\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta x_\kappa(\tau_2-\frac{\eta}{2})} = \frac{1}{2}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\,\rho\,\delta_{\mu\kappa}\ddot{x}_\nu\left(\tau_1-\frac{\rho}{2}\right) \times \ddot{\delta}\left[\tau_1+\frac{\rho}{2}-\left(\tau_2-\frac{\eta}{2}\right)\right]. \tag{A4}$$

In order to carry out the Wick contractions, we need the worldline propagator,

$$\langle x_\mu(\tau_1)x_\nu(\tau_2)\rangle = -\delta_{\mu\nu}|\tau_2-\tau_1| + \delta_{\mu\nu}\frac{(\tau_2-\tau_1)^2}{T}, \tag{A5}$$

and, in particular, its propertime derivative of the form

$$\langle \ddot{x}_\mu(\tau_1)\ddot{x}_\nu(\tau_2)\rangle = -2\ddot{\delta}(\tau_1-\tau_2)\delta_{\mu\nu}. \tag{A6}$$

Finally, we have to compute the contraction of the first term of Eq. (A2), which involves

$$\begin{aligned}\frac{1}{4}&\left\langle \ddot{x}_\mu\left(\tau_1+\frac{\rho}{2}\right)\ddot{x}_\nu\left(\tau_1-\frac{\rho}{2}\right)\ddot{x}_\lambda\left(\tau_2+\frac{\eta}{2}\right)\ddot{x}_\kappa\left(\tau_2-\frac{\eta}{2}\right)\right\rangle \\ &= \delta_{\mu\nu}\delta_{\lambda\kappa}\ddot{\delta}\left[\tau_1+\frac{\rho}{2}-\left(\tau_1-\frac{\rho}{2}\right)\right]\ddot{\delta}\left[\tau_2+\frac{\eta}{2}-\left(\tau_2-\frac{\eta}{2}\right)\right] \\ &\quad + \delta_{\nu\lambda}\delta_{\mu\kappa}\ddot{\delta}\left[\tau_1-\frac{\rho}{2}-\left(\tau_2-\frac{\eta}{2}\right)\right]\ddot{\delta}\left[\tau_1+\frac{\rho}{2}-\left(\tau_2-\frac{\eta}{2}\right)\right] \\ &\quad + \delta_{\mu\lambda}\delta_{\nu\kappa}\ddot{\delta}\left[\tau_1+\frac{\rho}{2}-\left(\tau_2+\frac{\eta}{2}\right)\right]\ddot{\delta}\left[\tau_1-\frac{\rho}{2}-\left(\tau_2-\frac{\eta}{2}\right)\right].\end{aligned} \tag{A7}$$

Now, inserting Eqs. (A3) and (A4) into the lhs of Eq. (A2) and performing all Wick contractions with the aid of Eq. (A6) and (A7), it is straightforward to observe that Eq. (A2) holds as an identity. Some terms vanish because of the contraction of $\delta_{\mu\nu}$ with $\sigma_{\mu\nu}$, such as the first term on the rhs of Eq. (A7); all remaining terms cancel each other exactly under the parameter integrals. This verifies the identity (A1) to second order which has been proved to all orders in Sec. II C.

APPENDIX B: EXPLICIT CALCULATIONS OF SPINOR PARTS

In the following, we show that possible further spinor parts, occurring during the calculation of the Heisenberg-Euler action, vanish, since they do not support a sufficient nonanalyticity.

Let us first consider the cases of $m > 1$ of the sum S_m, defined in Eq. (33) and appearing in the computation of term (II) in Eq. (32). For this, we use an integral representation of the function $g_n(\epsilon)$ which is defined in Eq. (29),

$$\begin{aligned} S_m &= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\frac{g_n(\epsilon)}{n^m} \\ &= -i\lim_{\epsilon\to 0}\frac{2\pi^2}{T^2}\int_{-\epsilon}^{\epsilon} d\rho\,\rho\sum_{n=1}^{\infty}\frac{e^{-[(2i\pi\rho)/T]n}}{n^{m-2}} \\ &= -\frac{\pi}{T}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\sum_{n=1}^{\infty}\frac{e^{-[(2i\pi\rho)/T]n}}{n^{m-1}}, \end{aligned}$$

where we have integrated by parts in the last step. In the $\epsilon \to 0$ limit, any nonzero contribution requires the n sum to exhibit a $\delta(\rho)$ singularity. As shown in the main text, this is exactly the case for the $m = 1$ term. For $m \geq 3$, the n sum corresponds to a poly-logarithm of degree $m - 1 \geq 2$, which is an analytic function for $\rho \to 0$. Hence all $m \geq 3$ terms vanish. The $m = 2$ term is more subtle. Here we encounter

$$\sum_{n=1}^{\infty}\frac{e^{-[(2i\pi\rho)/T]n}}{n^1} = \sum_{n=1}^{\infty}\frac{\cos(\frac{2\pi\rho}{T})}{n} + i\sum_{n=1}^{\infty}\frac{\sin(\frac{2\pi\rho}{T})}{n}.$$

The second sum is $\sim \frac{\pi-\rho}{2}$ and vanishes by the ϵ limit. The first sum can be carried out:

$$\sum_{n=1}^{\infty} \frac{\cos(n\frac{2\pi\rho}{T})}{n} = \frac{1}{2}\ln\left(\frac{1}{2(1-\cos\frac{2\pi\rho}{T})}\right).$$

Therefore the ρ integral becomes

$$-\frac{\pi}{2T}\int_{-\epsilon}^{\epsilon} d\rho \ln\frac{1}{2(1-\cos\rho)} \approx \frac{\pi}{T}\int_{-\epsilon}^{\epsilon} d\rho \ln\rho \to 0.$$

Even though there is a nonanalyticity, the singular structure of the integrand is not sufficient, and the integral vanishes in the $\epsilon \to 0$ limit. This proves our first statement in the main text that S_m contributes to the effective action only in the case of $m = 1$.

Finally, we discuss the remaining sum (III) of Eq. (31). Similarly to the preceding discussion, a nonzero contribution can only arise if the result of Fourier sum over n is sufficiently singular. Concentrating on the n dependence, the terms of the Fourier sum are of the form

$$\frac{1}{n^{2m-k}} g_n^k(\epsilon) \sim \int_{-\epsilon}^{\epsilon} d\rho\rho \frac{e^{in\rho}}{n^{2m-k}}, \quad k = 1, \ldots, m-1, \quad m > 1.$$

For all $k < m$, we end up with Fourier sums of the same type as discussed before in this appendix; all go to zero in the $\epsilon \to 0$ limit. Hence, the whole part (III) of Eq. (31) makes no contribution to the effective action, as claimed in the main text.

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