Effective potential (in)stability and lower bounds on the scalar (Higgs) mass

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It is widely believed that the top loop corrections to the Higgs effective potential destabilize the electroweak vacuum and that, imposing stability, lower bounds on the Higgs mass can be derived. With the help of a scalar-Yukawa model, we show that this apparent instability is due to the extrapolation of the potential into a region where it is no longer valid. Stability turns out to be an intrinsic property of the theory (rather than an additional constraint to be imposed on it). However, lower bounds for the Higgs mass can still be derived with the help of a criterion dictated by the properties of the potential itself. If the scale of new physics lies in the TeV region, sizable differences with the usual bounds are found. Finally, our results exclude the alternative metastability scenario, according to which we might be living in a sufficiently long lived metastable electroweak vacuum.

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I. INTRODUCTION

The standard model (SM) of particle physics is a very successful theory which has received a great number of experimental confirmations. As is well known, however, it is not complete. Its scalar sector, in particular, poses deep (and so far unanswered) questions.

The value of the Higgs mass is not fixed by the theory; it is a free parameter. Nevertheless, in order to get information on this fundamental quantity, theorists have tried to exploit at best the properties of the scalar sector of the SM (or some of its extensions).

Through the analysis of the scalar effective potential, upper and lower bounds on the Higgs mass m_H have been obtained as a function of the physical cutoff, the scale of new physics. The upper bounds come from the triviality of the quartic coupling [1] (for an alternative point of view, see [2]), the lower ones from the requirement that the electroweak (EW) vacuum be stable (or, at least, metastable) [3–12].

For the lower bounds, the analysis is performed with the help of the renormalization group (RG)-improved effective potential $V_{\rm RGI}(\phi)$. Because of the $t\bar{t}$ loop corrections, $V_{\rm RGI}$ bends down for ϕ larger than v, the EW minimum. Depending on the value of the physical parameters, the resulting potential either can be unbounded from below up to the Planck scale or can rise up again after forming a new minimum, which is typically deeper than the EW vacuum. The latter is then said to be metastable.

As the instability occurs for sufficiently large values of the field, $V_{\rm RGI}$ is approximated by keeping only the quartic term [9]. Using standard notations:

$$V_{\rm RGI}(\phi) \sim \frac{\overline{\lambda}(\phi)}{24}\phi^4. \qquad (1)$$

In Eq. (1), the dependence of $\overline{\lambda}(\phi)$ on ϕ is essentially the same as that of the corresponding RG-improved quartic coupling constant $\lambda(\mu)$ on the running scale μ, so that the behavior of the effective potential can be read out from the $\lambda(\mu)$ flow.[1]

The bending of the potential is due to the quarks-Higgs-Yukawa couplings, namely, to the minus sign carried by the fermion loops. Practically, it is sufficient to consider only the top, as the other (much lighter) quarks give comparably negligible contributions.

The physical request that the EW vacuum be stable against quantum fluctuation is seen as an *additional phenomenological constraint* to be imposed on the effective potential. This constraint induces a relation between the physical cutoff and the Higgs mass.

The derivation of the lower bounds goes as follows. Taking a boundary value for $\lambda(\mu)$ and for the other couplings, typically at $\mu = M_Z$, the coupled RG equations are run. As μ increases, $\lambda(\mu)$ (initially) decreases. Depending on its initial value $\lambda(M_Z)$, it may happen that at a certain scale, $\mu = \Lambda$, the running coupling λ vanishes, becoming negative for higher values of μ. Requiring that the EW vacuum be stable, Λ is interpreted as the physical cutoff of the theory, the scale where new physics appears. From the matching condition, which relates m_H to $\lambda(M_Z)$ (at the tree level it is $m_H^2 = [\lambda(M_Z)/3]v^2$), a lower bound for m_H as a function of Λ is obtained. This is the stability bound.

The possibility of having a minimum deeper than the EW one is also considered. The argument is that, as far as the tunneling time between the false (EW) and the true vacuum is sufficiently large compared to the age of the Universe, we may well be living in the metastable EW vacuum. In this case, metastability bounds on m_H are found [4,6,13].

These results, however, are at odds with a property of the effective potential, $V_{\rm eff}(\phi)$, which, as is well known, is a

[1]As correctly pointed out in Ref. [12], however, $\overline{\lambda}(\phi)$ contains also terms not contained in $\lambda(\mu)$. They are really negligible only for very large values of ϕ.

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convex function of its argument [14–16]. It is also known that, when the classical potential is not convex (the phenomenologically interesting case), at any finite order of the loop expansion, $V_{\rm eff}$ does not enjoy this fundamental property. Alternative nonperturbative methods of computing the effective potential, though, such as lattice simulations [17], variational approaches [18], or suitable averages of the perturbative results [19], provide the proper convex shape. The Wilsonian RG approach also gives a nonperturbative convex approximation for $V_{\rm eff}$ [20–24].

One of the main goals of the present work is to show that $V_{\rm eff}$ is *nowhere unstable.* Its apparent instability is due to an extrapolation to values of ϕ which lie beyond its region of validity. Naively, however, the instability seems to occur in a region of ϕ where perturbation theory can be trusted [8], and this explains why previous analyses have missed this point.[2]

We also show that, despite the convexity of the potential, actually thanks to this property, lower bounds for the Higgs mass can still be derived. Nevertheless, they no longer come as a result of an additional phenomenological constraint on $V_{\rm eff}$, namely, the requirement of stability; they are already encoded in the theory. As we shall see, if the scale of new physics lies in the TeV region, the difference between our bounds and those obtained with the help of the usual stability criterion becomes sizable. The metastability scenario, on the contrary, is definitely excluded.

Finally, in order to shed more light on this (often mistreated and misunderstood) subject, we reconsider here some popular arguments [6,25], sometimes quoted as the resolution of the instability (convexity) problem, and show that they are (at least) misleading. In Sec. II we concentrate mainly on this last point, which gives a good introduction to the subject and provides further motivation for our analysis.

To understand the origin of the instability, we do not need to consider the complete SM. The group and the gauge structure of the theory are not essential for its occurrence. As it is due to the top-Higgs coupling (actually to the minus sign carried by the $t\bar{t}$ loop), the same instability occurs in the simpler model of a scalar coupled to a fermion with Yukawa coupling. To illustrate our argument, it will be sufficient to limit ourselves to consider this model. The extension of our results to the SM is immediate.

The instability of the scalar effective potential is the subject of many studies. The one-loop (or higher loops) and the RG-improved potentials are computed with the help of dimensional regularization. We also begin by computing the effective potential of our model in the modified minimal subtraction ($\overline{\rm MS}$) scheme (Sec. III). However, as will become clear in the following, dimensional regularization cannot reveal (in fact, it masks) the origin of the problem.

The flaw in the usual procedure will be uncovered with the help of more physical renormalization schemes, the momentum cutoff regularization and the Wilsonian RG method. Dimensional regularization is a very powerful scheme which directly gives the finite results of renormalized perturbation theory. These other schemes allow one to better follow the steps for the derivation of the renormalized potential from the bare one. This will help in finding the origin of the instability problem.

While completing our paper, we noted that this issue was recently considered in Refs. [26,27]. Although our conclusions look similar to those reached by these authors, we believe that their work differs from ours in some important aspects, worth being discussed. A comparison will be presented in the conclusions.

The rest of the paper is organized as follows. In Sec. II we show how the Bogoliubov criterion of dynamical instability allows one to reconcile the convexity of $\Gamma_{\rm eff}$ with the existence of a broken phase and how the broken phase Green's functions can be derived from (the convex) $\Gamma_{\rm eff}$. Moreover, we show how the dynamical instability criterion can be implemented within the framework of the Wilsonian RG method. In Sec. III we compute the $\overline{\rm MS}$ one-loop and RG-improved effective potentials for our model and see that they both are unstable. In Sec. IV the same problem is considered within the momentum cutoff regularization scheme. In Sec. V we analyze the results of the previous section and show that the instability comes from an illegal extrapolation of the renormalized potential beyond its range of validity. In addition, consistently with the stability constraint, we consider a criterion for finding the physical cutoff of the theory. In Sec. VI we apply this criterion to the SM, thus getting lower bounds on the Higgs mass as a function of the scale of new physics, and compare with previous results. In Sec. VII we reconsider the instability problem within the framework of the nonperturbative Wilsonian RG method. Section VIII is for the summary and for our conclusions.

II. BROKEN PHASE AND DYNAMICAL INSTABILITY

Before starting the detailed study of our model, in the present section we carefully analyze some popular arguments [6,25], often presented as the resolution of the instability problem, and show that they are misleading. Moreover, by combining the Bogoliubov criterion of dynamical instability with the Wilsonian RG method, we shall provide further support to our analysis.

In Refs. [6,25], the effective action $\Gamma_{\rm eff}[\phi]$ and the generating functional of the broken phase 1PI vertex functions $\Gamma_{\rm 1PI}[\phi]$ are presented as two different functionals. Actually, these authors consider the first order in the $\hbar$

[2]In addition, the use of RG techniques, which enlarge the domain of validity of perturbation theory via the resummation of leading, next to leading, ... logarithms, leads one to believe that the derivation of this instability is theoretically sound [8].

expansion of Γ_{1PI}, Γ_{1PI}^{1l} and note that it is not convex. It is then argued that, when studying the stability of the EW vacuum, the relevant quantity to consider is V_{1PI} (or, more generally, its RG-improved version V_{RGI}) rather than the convex V_{eff} and that, being that V_{1PI} is nonconvex, there is no convexity (instability) problem [6].[3]

The argument is the following. $V_{eff}(\phi)$ comes from the minimization of $\langle\psi|\hat{H}|\psi\rangle$, where $\hat{H}$ is the energy density of the system and $|\psi\rangle$ is a state which satisfies the constraint $\langle\psi|\hat{\phi}|\psi\rangle = \phi$. For a symmetry breaking classical potential, the states that correspond to values of ϕ in the region between the classical minima are not localized (more on this point later). As only localized states are of interest to us, and V_{1PI}^{1l} is supposed to correspond to localized states also in the region between the minima [19], the conclusion is that V_{1PI}^{1l} rather than V_{eff} is the appropriate potential to consider.

It is not difficult to see, however, that these lines of reasoning are misguiding. Indeed, the instability occurs for values of ϕ above v. Now, differently from those related to the region $-v \leq \phi \leq v$, the states that correspond to this range of ϕ are perfectly well localized and the above argument does not apply.

Moreover, as we briefly show in Appendix A, the broken phase zero momentum Green's functions $\Gamma_n^{(v)}$ can be obtained from the convex V_{eff} once we consider a physical procedure [28,29] based on the dynamical instability of the classical vacua (Bogoliubov criterion), and the usual loop expansion for V_{eff} can be obtained within this framework.

This will help to further clarify the relation between V_{eff} and V_{1PI}^{1l}. In any case, the potential to consider is V_{eff}, which is everywhere convex. However, as long as we are interested only in the broken theory Green's functions, i.e. in the local properties of V_{eff} at $\phi = v$, it is possible (and, from a practical point of view, even more convenient) to consider a nonconvex approximation, such as V_{1PI}^{1l} (or higher order ones), which coincides with V_{eff} in the neighborhood of v (see footnote 3 and Appendix A).

Actually, the only range of ϕ's where a significative difference between the loop approximation and the exact effective potential is expected is the internal region, $-v \leq \phi \leq v$. The reason is easy to understand. By construction, the one-loop approximation for the path integral which defines the effective potential considers the expansion of the action around a single saddle point. For values of ϕ in the internal region, however, there are two competing saddle points having the same weight [16]. Taking into account both of these contributions, we get for the effective potential the known flat (convex) shape between the classical minima (Maxwell construction). On the contrary, for $\phi \geq v$ the path integral is dominated by a single saddle point. Therefore, no significative difference can occur in this region between the one-loop (or higher loop) approximation and the exact effective potential.

A similar argument can be given within the framework of the Wilsonian RG approach, where it was shown that, differently from the unbroken phase, the path integral which defines the infinitesimal RG transformation for the Wilsonian potential in the broken phase is saturated by nontrivial saddle points [23].

As is well known, the nonperturbative RG equation for the Wilsonian effective potential, $U_k(\phi)$, in $d = 4$ dimensions can be written as [30–32]:

$$k\frac{\partial}{\partial k}U_k(\phi) = -\frac{k^4}{16\pi^2}\ln\left(\frac{k^2 + U_k''(\phi)}{k^2 + U_k''(0)}\right), \tag{2}$$

where the prime indicates derivation with respect to ϕ. Note that the classical (bare) potential is $V_{cl}(\phi) = U_\Lambda(\phi)$, while the effective potential is $V_{eff}(\phi) = U_{k=0}(\phi)$.

For a theory in the broken phase, however, Eq. (2) becomes unstable. More precisely, for values of ϕ in the internal region, this equation develops a singularity at finite critical values $k_{cr}(\phi)$ of the running scale k. Starting from $k = k_{cr}(\phi)$, Eq. (2) is no longer valid.

In Ref. [23], a new nonperturbative RG equation for ϕ in the unstable region was established:

$$U_{k-\delta k}(\phi) = \min_{\{\varrho\}}\left[k^2\varrho^2 + \frac{1}{2}\int_{-1}^{1} dx U_k(\phi + 2\varrho\cos(\pi x))\right]. \tag{3}$$

The minimum of Eq. (3), $\varrho_k(\phi)$, is the amplitude of the nontrivial saddle point which dominates the path integral defining the infinitesimal RG transformation ($k \to k - \delta k$) in the internal region. In the external region, on the contrary, the path integral is dominated by the trivial saddle point, i.e. $\varrho_k(\phi)$ vanishes.

The case of the symmetric potential (see Fig. 8 in Appendix A) was considered in Ref. [23], and the Maxwell construction for V_{eff} was established. Here we extend the analysis of Ref. [23] to the case of the potential with an explicit symmetry breaking term (see Appendix A, in particular, Fig. 9).

In Fig. 1 we show the flow of the Wilsonian potential $U_k^{(\varepsilon)}(\phi)$, starting from the critical values $k_{cr}(\phi)$. From this figure we see that, even in the asymmetric case, there is a region where the effective potential $V_{eff}^{(\varepsilon)}(\phi) = U_{k=0}^{(\varepsilon)}(\phi)$ is flat and coincides with the double tangent construction. The same considerations done for the lowest order result (see Appendix A) are valid. In particular, the tangent point is displaced to the left of v_ε and the derivatives of $V_{eff}^{(\varepsilon)}(\phi)$ at v_ε can be safely taken.

[3]Presenting $\Gamma_{1PI}[\phi]$ and $\Gamma_{eff}[\phi]$ as two different quantities is a first source of confusion. As we have already said, the convexity property of the exact Γ_{eff} cannot be recovered within the loop expansion. Γ_{1PI}^{1l}, which is the quantity considered in Refs. [6,25], is a nonconvex, $O(\hbar)$, approximation of Γ_{eff}. It correctly approximates Γ_{eff} in the neighborhood of the minima (with some warnings specified later). In the region where it is nonconvex, however, it is a bad approximation of Γ_{eff}.

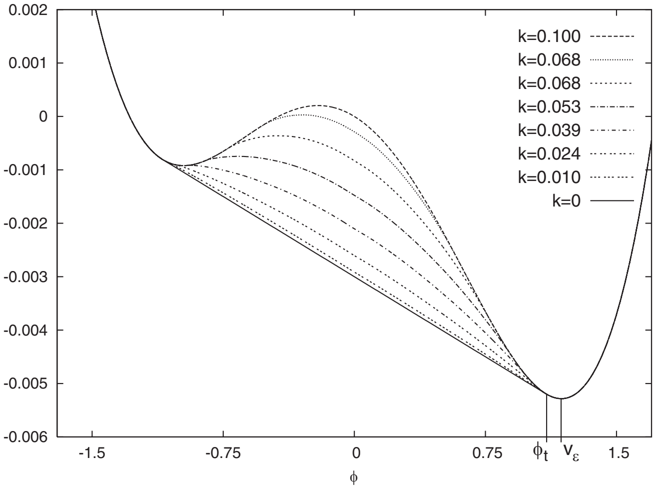


FIG. 1. RG flow for the potential of the single component scalar theory with explicit symmetry breaking term. Only the flow in the internal region is considered, i.e. the flow given by Eq. (3). The boundary values for the parameters at $k = 0.1$ are $\lambda = 5 \times 10^{-2}$, $m^2 = -10^{-2}$, and $\varepsilon = 2 \times 10^{-3}$.

The general conclusion of this analysis is that, with the help of Eqs. (2) and (3), the Wilsonian potential can be run all the way down from $k = \Lambda$ to $k = 0$. The result is a nonperturbative convex approximation for $V_{\rm eff}$, which shows the typical flat shape in the internal region [given by the running of Eq. (3)] while in the external region has the shape governed by Eq. (2).

We consider now the one-loop potential $V^{1l}(\phi; \varepsilon)$. In view of the previous discussion, it is not difficult to understand that, as far as we limit ourselves to consider a range of values of ϕ sufficiently close to the absolute minimum, $V^{1l}(\phi; \varepsilon)$ provides a good approximation for $V_{\rm eff}(\phi; \varepsilon)$. Clearly, this is true for higher order loops, too.

Before ending this section, we would like to expand, as anticipated, on the argument according to which, when studying the stability of the vacuum, the convex $V_{\rm eff}$ is not the appropriate potential to consider [6].

Let us indicate with $|v\rangle$ and $|-v\rangle$ the vacua constructed around $\phi = v$ and $\phi = -v$, respectively. The flatness of $V_{\rm eff}$ in the $-v < \phi < v$ region implies that all the linear combinations of states $\alpha|v\rangle + \beta|-v\rangle$ (with $|\alpha|^2 + |\beta|^2 = 1$) are equivalent vacua; they all have the same energy. Apart from the trivial ones ($|\alpha| = 1$, $\beta = 0$ and $\alpha = 0$, $|\beta| = 1$), with any of the other nontrivial combinations we would obtain Green's functions, which violate the cluster decomposition property. Moreover, the expectation value of the field is not constant all over V, the quantization volume. In fact, for the generic state $\alpha|v\rangle + \beta|-v\rangle$, the expectation value $\langle\phi\rangle$ is given by $(|\alpha|^2 - |\beta|^2)v$, and V contains a fraction $|\alpha|^2$ of $\langle\phi\rangle = v$ and a fraction $|\beta|^2$ of $\langle\phi\rangle = -v$. Clearly, these states are not localized.

The above considerations are viewed as an indication that the convex $V_{\rm eff}$ is not the appropriate potential with which to deal. Although correct, these observations have nothing to do with the instability problem. As we have just seen, the nonlocalized states correspond to values of ϕ in the internal region. The instability problem, however, occurs in the external region, where the states are perfectly well localized. Moreover, with the help of the Bogoliubov criterion, we have seen how the degeneracy in the internal region is lifted and (in the infinite volume limit) only one vacuum is selected.

III. ONE-LOOP AND RGI POTENTIALS. $\overline{\rm MS}$ SCHEME

We compute now the one-loop effective potential V^{1l} for our model in the $\overline{\rm MS}$ scheme and the corresponding RG-improved potential $V_{\rm RGI}$.

The model consists of a single scalar field plus a single fermion field with scalar quartic interaction and Yukawa coupling, i.e.:

$$\mathcal{L}(\phi, \psi, \overline{\psi}) = \int d^4x \bigg(\frac{1}{2} \partial_\mu \phi \partial_\mu \phi + i\overline{\psi}\gamma_\mu \partial_\mu \psi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{2}\phi^4 + g\phi\overline{\psi}\psi \bigg). \tag{4}$$

Straightforward application of the $\overline{\rm MS}$ prescriptions gives

$$V^{1l}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4 + \frac{1}{64\pi^2}\Big(m^2 + \frac{\lambda}{2}\phi^2\Big)^2 \times \bigg(\ln\bigg(\frac{m^2 + \frac{\lambda}{2}\phi^2}{\mu^2}\bigg) - \frac{3}{2}\bigg) - \frac{g^4\phi^4}{16\pi^2}\bigg(\ln\frac{g^2\phi^2}{\mu^2} - \frac{3}{2}\bigg), \tag{5}$$

where m^2, λ, and g depend on the renormalization scale μ:

$$m^2 = m^2(\mu), \qquad \lambda = \lambda(\mu), \qquad g = g(\mu). \tag{6}$$

In the right-hand side of Eq. (5), the fermionic contribution comes with a negative sign. Therefore, we can easily find values of λ and g (with $g^4 > \lambda$), together with a range of values of ϕ, which satisfy the perturbative conditions

$$\lambda < 1, \qquad g < 1, \quad \text{and} \quad \frac{g^4}{16\pi^2}\ln\frac{g^2\phi^2}{\mu^2} < 1, \tag{7}$$

so that $V^{1l}(\phi)$ bends down and becomes lower than $V^{1l}(v)$ (see Fig. 2). This is the instability problem for our one-loop potential.

As is well known, we can improve on this result with the help of renormalization group techniques. Let us consider the one-loop RG functions for λ, g, m^2, and for the vacuum

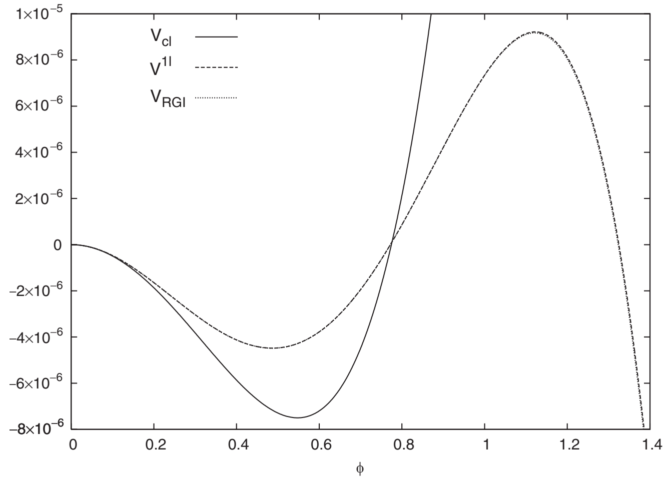


FIG. 2. Together with the classical potential $V_{\rm cl}$ of Eq. (4), here we plot the one-loop V^{1l} and the RG-improved $V_{\rm RGI}$ effective potentials. The parameters are chosen at the scale $\mu = 1.1 \times 10^{-1}$ and are $\lambda = 2 \times 10^{-3}$, $m^2 = -10^{-4}$, $g = 3 \times 10^{-1}$. The instability of V^{1l} and $V_{\rm RGI}$ is immediately evident. Moreover, in this region, they are very close.

energy[4] Ω (for simplicity, we omit the wave-function renormalization):

$$\beta_\lambda = \frac{3\lambda^2}{16\pi^2} - \frac{3g^4}{\pi^2}; \qquad \beta_g = \frac{g^3}{8\pi^2}, \qquad \beta_\Omega = \frac{\lambda m^4}{32\pi^2}; \qquad \gamma_{m^2} = \frac{\lambda}{16\pi^2}. \tag{8}$$

The largest logarithmic correction in the right-hand side of Eq. (5) comes from the last term (the fermion). According to the RG-improvement logic, we now choose the running variable t so that we get rid of this term in the improved potential: $t = \frac{1}{2}\ln(g^2\phi^2/\mu^2) - \frac{3}{4}$. As usual, the running functions $\overline{\lambda}(t)$, $\overline{g}(t)$, $\overline{m}^2(t)$, and $\overline{\Omega}(t)$ are defined as the solutions of the differential equations:

$$\frac{d\overline{\lambda}}{dt} = \beta_\lambda(\overline{\lambda}, \overline{g}, \overline{\Omega}, \overline{m}^2); \qquad \frac{d\overline{g}}{dt} = \beta_g(\overline{\lambda}, \overline{g}, \overline{\Omega}, \overline{m}^2), \qquad \frac{d\overline{m}^2}{dt} = \gamma_{m^2}(\overline{\lambda}, \overline{g}, \overline{\Omega}, \overline{m}^2); \qquad \frac{d\overline{\Omega}}{dt} = \beta_\Omega(\overline{\lambda}, \overline{g}, \overline{\Omega}, \overline{m}^2), \tag{9}$$

with boundary conditions:

$$\overline{\lambda}(t=0) = \lambda; \qquad \overline{g}(t=0) = g; \qquad \overline{\Omega}(t=0) = 0; \qquad \overline{m}^2(t=0) = m^2. \tag{10}$$

It is not difficult to see that the differential equations (9) can be solved analytically. For $\overline{g}(t)$ and $\overline{\lambda}(t)$, for instance, we have

$$\overline{g}(t) = g\left(1 - \frac{g^2 t}{4\pi^2}\right)^{-1/2}, \qquad \overline{\lambda}(t) = \frac{2}{3}\overline{g}^2(t)\left(1 - \alpha + 2\alpha\left[1 + \left(\frac{\overline{g}(t)^2}{g^2}\right)^\alpha \times \frac{2g^2(\alpha+1) - 3\lambda}{2g^2(\alpha-1) + 3\lambda}\right]^{-1}\right), \tag{11}$$

with $\alpha = \sqrt{37}$.

Finally, the one-loop RG-improved potential is

$$V_{\rm RGI} = \frac{1}{2}\overline{m}^2(t)\phi^2 + \frac{\overline{\lambda}(t)}{24}\phi^4 + \overline{\Omega}(t) + \left(\frac{\bar{m}^2(t) + \frac{\bar{\lambda}(t)}{2}\phi^2}{64\pi^2}\right)^2 \ln\frac{\bar{m}^2(t) + \frac{\bar{\lambda}(t)}{2}\phi^2}{\overline{g}^2(t)\phi^2}. \tag{12}$$

In Figs. 2 and 3 we plot $V_{\rm RGI}$ together with the one-loop and the classical potentials for a particular choice of the renormalized parameters. A simple inspection of this figure shows that $V_{\rm RGI}$ (as well as V^{1l}) is unstable.

Before ending this section, we note that, due to the competition between the λ^2 and the g^4 terms in β_λ [first of Eqs. (8)], $\lambda(\mu)$, after decreasing for a certain range of energy, finally increases (toward the Landau pole). This generates a second minimum in the effective potential, typically lower than the first one.

Now, for certain values of m_t and m_H, which are compatible with the current experimental determinations and limits, the Higgs effective potential of the SM shows such a behavior already below the Planck scale. As the tunneling time between the false (EW) and the true vacuum appears to be sufficiently large (as compared to the age of the Universe), the alternative scenario of a metastable EW

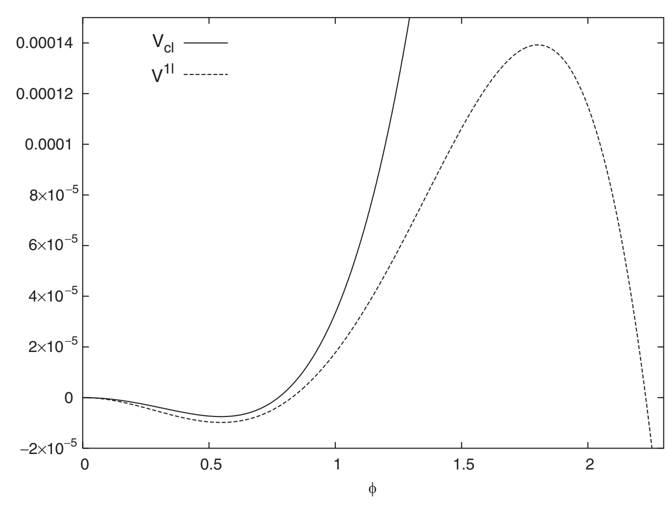


FIG. 3. Differently from Fig. 2, here we have implemented the RG conditions so that the location of the minimum and the curvature of V^{1l} at this point are the same as for $V_{\rm cl}$ (see Appendix B). The parameters are chosen as in Fig. 2.

[4]When considering the RG improvement, the cosmological constant term has to be taken into account even if it was originally absent.

vacuum is also considered and lower metastability bounds on the Higgs mass are derived [6,13]. As we have anticipated, however, the proper treatment of the problem will show that effective potential is nowhere unstable. As a consequence, this scenario will be excluded.

IV. MOMENTUM CUTOFF SCHEME

In this section, we show how the one-loop renormalized effective potential of Eq. (5) is obtained by considering the theory defined with a momentum cutoff. To prepare the discussion of the next section, we follow the computation in some detail.

The parameters of the Lagrangian are now the bare ones. Therefore, in Eq. (4) we replace m^2, λ, and g with m_Λ^2, λ_Λ, and g_Λ, respectively. As in the previous section, for the sake of simplicity, we neglect the wave function renormalization.[5] A straightforward application of perturbation theory gives

$$V^{1l}(\phi) = \frac{m_\Lambda^2}{2}\phi^2 + \frac{\lambda_\Lambda}{24}\phi^4 + \frac{1}{64\pi^2}\left\{\Lambda^4 \ln\left(\frac{\Lambda^2 + m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2}{\Lambda^2}\right) + \left(m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2\right)\Lambda^2 - \left(m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2\right)^2 \times \ln\left(\frac{\Lambda^2 + m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2}{m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2}\right)\right\} - \frac{1}{16\pi^2}\left\{\Lambda^4 \ln\left(1 + \frac{g_\Lambda^2\phi^2}{\Lambda^2}\right) + g_\Lambda^2\phi^2\Lambda^2 - g_\Lambda^4\phi^4 \ln\left(\frac{\Lambda^2 + g_\Lambda^2\phi^2}{g_\Lambda^2\phi^2}\right)\right\}. \tag{13}$$

Considering only values of ϕ small compared to the cutoff,

$$\frac{\phi}{\Lambda} < 1, \tag{14}$$

expanding the right-hand side of Eq. (13) in powers of $\frac{\phi}{\Lambda}$, and neglecting terms which are suppressed by negative powers of Λ, we get

$$V^{1l}(\phi) = \frac{m_\Lambda^2}{2}\phi^2 + \frac{\lambda_\Lambda}{24}\phi^4 - \frac{1}{16\pi^2}\left\{2g_\Lambda^2\phi^2\Lambda^2 - g_\Lambda^4\phi^4\left[\ln\left(\frac{\Lambda^2}{g_\Lambda^2\phi^2}\right) + \frac{1}{2}\right]\right\} + \frac{1}{64\pi^2}\left\{2\left(m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2\right)\Lambda^2 - \left(m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2\right)^2\left[\ln\left(\frac{\Lambda^2}{m_\Lambda^2 + \frac{\lambda_\Lambda}{2}\phi^2}\right) + \frac{1}{2}\right]\right\}. \tag{15}$$

We now move from bare to renormalized perturbation theory. After performing the splitting of the bare parameters in the usual way:

$$m_\Lambda^2 = m^2 + \delta m^2, \qquad \lambda_\Lambda = \lambda + \delta\lambda, \qquad g_\Lambda = g + \delta g, \tag{16}$$

we insert Eq. (16) in Eq. (15), neglecting the higher order terms, i.e. removing δm^2, $\delta\lambda$, and δg from the quantum fluctuation contribution. Finally, the counterterms are determined so as to cancel the quadratic and logarithmic divergences of V^{1l}.

There is an arbitrariness in the determination of the counterterms (different renormalization conditions) which is reflected in an arbitrariness in the finite parameters of the renormalized potential. By choosing

$$\delta m^2 = \delta m_{\rm bos}^2 + \delta m_{\rm fer}^2, \qquad \delta\lambda = \delta\lambda_{\rm bos} + \delta\lambda_{\rm fer}, \tag{17}$$

with (μ is an arbitrary low energy scale)

$$\delta m_{\rm bos}^2 = -\frac{\lambda\Lambda^2}{32\pi^2} + \frac{\lambda m^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{\mu^2}\right) - 1\right];$$
$$\delta m_{\rm fer}^2 = \frac{g^2\Lambda^2}{4\pi^2}, \qquad \delta\lambda_{\rm bos} = \frac{3\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{\mu^2}\right) - 1\right]; \tag{18}$$
$$\delta\lambda_{\rm fer} = -\frac{3g^4}{2\pi^2}\left[\ln\left(\frac{\Lambda^2}{\mu^2}\right) - 1\right],$$

we get

$$V^{1l}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4 + \frac{1}{64\pi^2}\left(m^2 + \frac{\lambda}{2}\phi^2\right)^2 \times\left(\ln\left(\frac{m^2 + \frac{\lambda}{2}\phi^2}{\mu^2}\right) - \frac{3}{2}\right) - \frac{g^4\phi^4}{16\pi^2}\left(\ln\frac{g^2\phi^2}{\mu^2} - \frac{3}{2}\right), \tag{19}$$

that is, the one-loop potential of Eq. (5).

As for Eq. (5), the renormalized parameters that appear in Eq. (19) are defined at the scale μ. Now, repeating the same steps of the previous section, we obtain from Eq. (19) the RG-improved potential of Eq. (12).

V. STABILITY OF THE EFFECTIVE POTENTIAL

We show now that the effective potential is nowhere unstable, the claimed (apparent) instability being due to the

[5]When, in Sec. VI, we shall be interested in the derivation of lower bounds on the Higgs mass, the anomalous dimension will be appropriately taken into account.

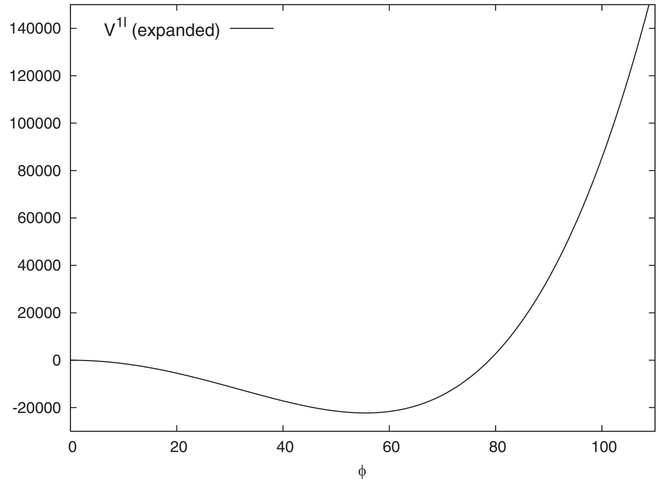


FIG. 4. The one-loop effective potential of Eq. (15) (before the subtraction of the quadratic divergences) for $\lambda_\Lambda = 5 \times 10^{-2}$, $m_\Lambda^2 = -10^{-2}$, $g_\Lambda = 0.35$, and $\Lambda = 100$. Neglecting, as explained in the text, the internal region, we see that beyond the minimum the potential is convex.

extrapolation of V^{1l} (V_{RGI}) into a region where it is no longer valid.

Before turning our attention to the renormalized potential, we begin by considering the bare theory as defined by the one-loop potential of Eq. (15). For a certain region in the (m_Λ^2, λ_Λ, g_Λ)-parameter space, this potential, as the classical one, has two minima (Higgs phase). As we have already explained, the loop expansion is inadequate for the region between the minima. In the following, we ignore this region and concentrate our attention only on the external one, where the loop expansion is expected to hold (as we know, in the internal region the convexity is restored via the Maxwell construction).

A careful analysis of Eq. (15) shows that, in the external region and within the range of ϕ where the potential is defined, i.e. for $\frac{|\phi|}{\Lambda} < 1$, the bare effective potential is convex (in agreement with exact theorems). Therefore, it does not present any instability. In Fig. 4 we show a plot of V^{1l}, Eq. (15), for a particular choice of the parameters.

We now subtract from the bare potential of Eq. (15) the quadratically divergent terms.[6] As illustrated in Fig. 5 for a specific choice of the parameters, again the resulting potential turns out to be convex (once more, only the external region has to be considered). The bare potential, even after the subtraction of the quadratic divergences, does not show any sign of instability.

As an aside remark, we note that, as they describe different degrees of freedom, the potentials of Figs. 4 and 5 actually belong to two different effective theories (with or without the Λ^2 terms). From the point of view of the phenomenological applications in particle physics, however, we are typically interested in the theory where the quadratically divergent terms are subtracted.

We have just seen that the bare potential (before and after the subtraction of the quadratic divergences) is everywhere stable. How can the renormalized potential show an instability? To answer this question, let us consider again Eq. (15) after the subtraction of the quadratically divergent terms.

As we already know, the instability occurs because the quantum fluctuations due to the fermions can compensate and then overwhelm the classical ϕ^4 term. Therefore, with no loss of generality, we can now neglect the bosonic contribution, as well as other unimportant finite terms, and limit ourselves to write

$$V^{1l}(\phi) = \frac{1}{2} m_\Lambda^2 \phi^2 + \frac{\lambda_\Lambda}{24}\phi^4 + \frac{g_\Lambda^4 \phi^4}{16\pi^2} \ln\frac{\Lambda^2}{g_\Lambda^2 \phi^2}. \qquad (20)$$

At a lower scale μ ($< \Lambda$) we have

$$V^{1l}(\phi) = \frac{1}{2} m_\mu^2 \phi^2 + \frac{\lambda_\mu}{24}\phi^4 + \frac{g_\mu^4 \phi^4}{16\pi^2} \ln\frac{\mu^2}{g_\mu^2 \phi^2}, \qquad (21)$$

which is the same potential of Eq. (20) written in terms of the renormalized parameters m_μ^2, λ_μ, and g_μ.

Clearly, if Eq. (20) does not show any instability, the same is true for Eq. (21). However, let us pretend (for the moment) that we have not made this observation and move to consider the usual phenomenological application of Eq. (21), which could have been obtained (Sec. III) within the $\overline{\mathrm{MS}}$ scheme.

From the first two terms of Eq. (21), the classical vacuum

$$v^2 = -\frac{6 m_\mu^2}{\lambda_\mu} \qquad (22)$$

is obtained. The last term can destabilize this vacuum if it becomes too large and negative. Strictly speaking, the presence of the last term also modifies the position of the classical minimum, but this is not a complication. In fact, although this is not a necessary step, we can slightly modify the above expressions by adopting renormalization conditions that keep the position of the minimum unchanged. With this choice, Eq. (21) is replaced by (see

[6]As is well known, a well defined physical meaning can be attached to this operation. In a nonsupersymmetric scenario, this cancellation is interpreted as the result of the conspiracy between unknown degrees of freedom, which live above the cutoff, and the quantum fluctuations of the fields below the cutoff. This way, the scalar (Higgs) mass is protected from getting too large corrections from the quantum fluctuations (this interpretation, however, poses the problem of the fine-tuning required for the cancellation, the naturalness problem). In a supersymmetric scenario, on the contrary, this cancellation is obtained in a more "natural" way. It is due to the presence of additional degrees of freedom (fields) below the cutoff.

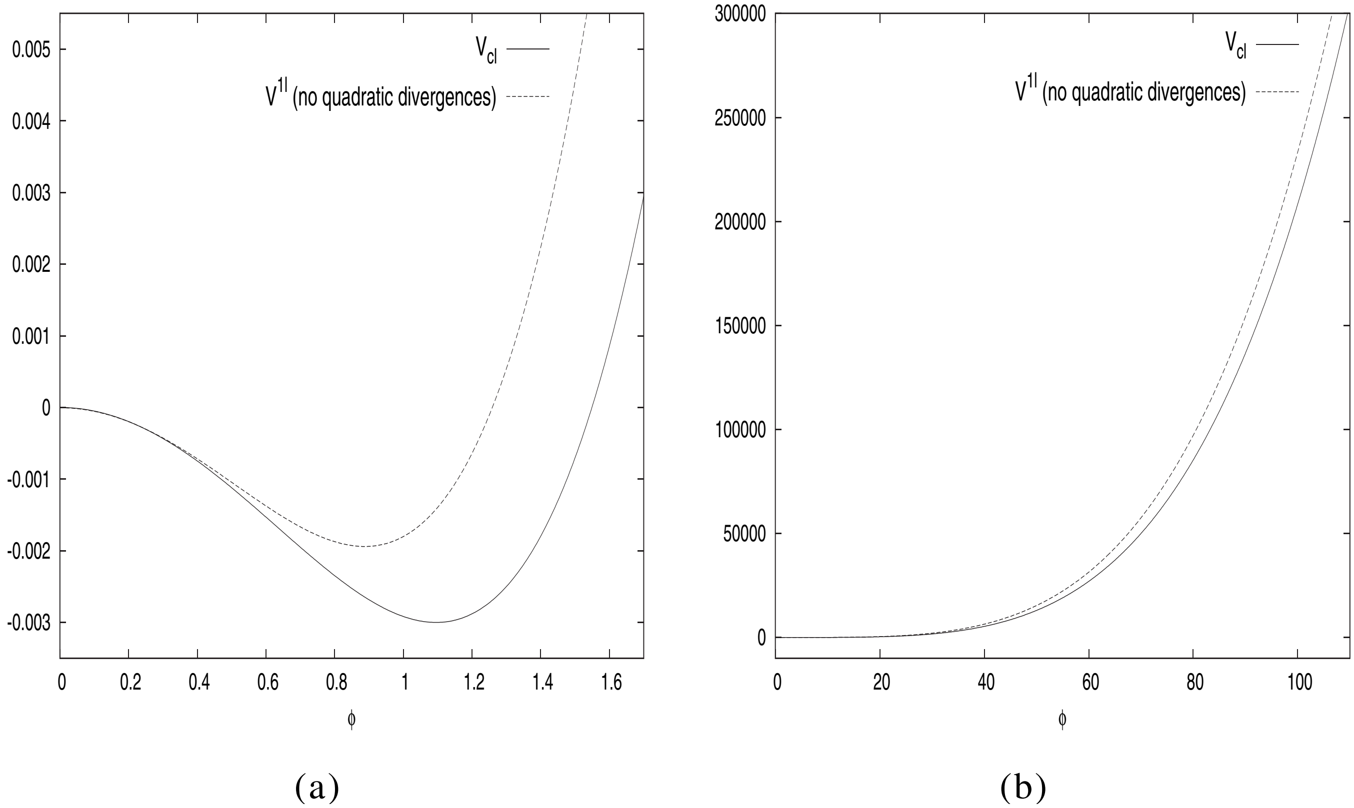


FIG. 5. The bare potential together with the one-loop potential after subtraction of the quadratic divergences. The bare parameters are chosen as in Fig. 4. (a) We zoom on a small region of ϕ, close to the classical minimum. (b) Here we see that the effective potential, up to the cutoff scale, is stable.

Appendix A):

$$V^{1l}(\phi) = \frac{1}{2} m_v^2 \phi^2 + \frac{\lambda_v}{24}\phi^4 - \frac{g_v^4 \phi^4}{16\pi^2}\left(\ln\frac{\phi^2}{v^2} - \frac{3}{2}\right) - \frac{g_v^4 v^2}{8\pi^2}\phi^2. \tag{23}$$

In Eq. (23) we have defined the parameters m_v^2, λ_v, and g_v at the IR scale v, the classical (and quantum) minimum, which is now given by:

$$v^2 = -\frac{6m_v^2}{\lambda_v}. \tag{24}$$

Correspondingly, Eq. (20) is replaced by:

$$V^{1l}(\phi) = \frac{1}{2} m_\Lambda^2 \phi^2 + \frac{\lambda_\Lambda}{24}\phi^4 - \frac{g_\Lambda^4 \phi^4}{16\pi^2}\left(\ln\frac{\phi^2}{\Lambda^2} - \frac{3}{2}\right) - \frac{g_\Lambda^4 v^2}{8\pi^2}\phi^2. \tag{25}$$

Going back to Eq. (23), we now look for values of λ_v, g_v, and ϕ such that this equation is (expected to be) valid and, at the same time, gives

$$V^{1l}(\phi) < V^{1l}(v). \tag{26}$$

The usual requirements for the validity of Eq. (23) are that the renormalized coupling constants λ_v and g_v, as well as the quantum correction $(g_v^4/16\pi^2)\ln(\phi^2/v^2)$, be perturbative, i.e.:

$$\lambda_v < 1, \qquad g_v < 1, \tag{27}$$

and

$$\left|\frac{g_v^4}{16\pi^2}\ln\frac{\phi^2}{v^2}\right| < 1 \tag{28}$$

[note that Eqs. (27) and (28) are nothing but the perturbative conditions of Eq. (7) adapted to our current choices].

In the following, we show that, contrary to the common expectation, Eqs. (27) and (28) are not sufficient to guarantee that Eq. (23) can be trusted. An additional condition has to be considered. As we shall see, the apparent instability of the potential is due to the neglect of this condition.

Let us choose λ_v and g_v so that these couplings, in addition to Eq. (27), also satisfy the relation:

$$\lambda_v = \frac{3g_v^4}{4\pi^2}. \tag{29}$$

Moreover, let us consider $\overline{\phi}$ such that:

$$\ln\frac{\overline{\phi}^2}{v^2} = 2. \tag{30}$$

Being that $g_v < 1$, it is a trivial exercise to see that, by virtue of Eq. (30), Eq. (28) holds for $\overline{\phi}$. Moreover, inserting Eqs. (29) and (30) in Eq. (23), we find:

$$V^{1l}(\overline{\phi}) < V^{1l}(v). \tag{31}$$

We would conclude that, in the range of ϕ given by $v < \phi < \overline{\phi}$, the renormalized potential of Eq. (23) can be trusted and its instability (see Fig. 2) is theoretically well established. In fact, this is what is usually stated [8]. In order to avoid any misunderstanding, it is worth stressing that the RG improvement cannot change this conclusion. In the range of ϕ that we consider here, the condition (28) holds so that, in this region, V^{1l} and V_{RGI} are very close one to the other.

As solid as they can seem, however, the above conclusions are incorrect.

To understand why, let us first simplify (without any loss of generality) the discussion by neglecting in the following the running of m^2 and g. From Eqs. (23) and (25), we have then:

$$\frac{\lambda_\Lambda}{24}\phi^4 + \frac{g^4\phi^4}{16\pi^2}\ln\frac{\Lambda^2}{\phi^2} = \frac{\lambda_v}{24}\phi^4 + \frac{g^4\phi^4}{16\pi^2}\ln\frac{v^2}{\phi^2}, \tag{32}$$

which immediately gives

$$\lambda_\Lambda = \lambda_v - \frac{3g^4}{2\pi^2}\ln\frac{\Lambda^2}{v^2}. \tag{33}$$

Inserting now Eqs. (29) and (30) in Eq. (32), we find:

$$\frac{\lambda_\Lambda}{24} + \frac{g^4}{16\pi^2}\ln\frac{\Lambda^2}{\overline{\phi}^2} = \frac{\lambda_v}{24} + \frac{g^4}{16\pi^2}\ln\frac{v^2}{\overline{\phi}^2} < 0. \tag{34}$$

Naturally, for the theory to be defined, it is $\lambda_\Lambda > 0$. Therefore, in order for Eq. (34) to be valid, we should have

$$\frac{\Lambda^2}{\overline{\phi}^2} \leq 1. \tag{35}$$

Equation (35) shows that, contrary to our naive expectation, $\overline{\phi}$ lies beyond the range of validity of V^{1l} (V_{RGI}).

We now understand the origin of the apparent instability of the renormalized potential. If, in order to decide whether a certain value of ϕ belongs to the region where V^{1l} can be trusted, we consider only Eqs. (27) and (28), *we lose the information contained in the additional independent condition*:

$$\frac{\lambda_v}{24}\phi^4 + \frac{g^4\phi^4}{16\pi^2}\ln\frac{v^2}{\phi^2} > 0. \tag{36}$$

When, on the contrary, this condition is taken into account, the effective potential does not present any instability. In other words, the instability occurs in a region of ϕ's where Eq. (23) for V^{1l} is no longer valid.

Naturally, these same conclusions could have been reached by looking at the problem the other way around. In fact, coming back to the observation that we have put aside before, we note that, due to the condition $\phi^2 < \Lambda^2$, the combination $(\lambda_\Lambda/24) + (g^4/16\pi^2)\ln(\Lambda^2/\phi^2)$ cannot be negative. Therefore, Eq. (34) cannot be fulfilled and no instability can occur.

The above longer discussion, however, is motivated by the common belief that, in order to ascertain the validity of the result for V^{1l}, Eqs. (27) and (28) are the *only conditions* to be verified. Actually, this is the reason why it is still believed that the instability of V^{1l} (and V_{RGI}) is a genuine effect due to the quantum corrections.

We can now deepen our analysis by noting that, as an elementary exercise shows, the point beyond the minimum where the effective potential ceases to be convex, i.e. the inflection point in the external region, ϕ_{inf}, is such that:

$$\phi_{\mathrm{inf}} \geq \Lambda. \tag{37}$$

Equation (37) is important for two reasons. On the one hand, it shows that the effective potential is convex wherever it is defined. On the other hand, it provides a criterion for the derivation of lower bounds on the scalar (Higgs) mass.

To better understand this last point, let us consider the usual approach, where a bound on the renormalized λ is obtained from (the equivalent of) Eq. (33). At first, it is noted that the instability occurs if $V^{1l}(\phi_0) = V^{1l}(v)$ at a certain ϕ_0 and $V^{1l}(\phi) < V^{1l}(v)$ for $\phi > \phi_0$. Then it is shown that ϕ_0 (almost) corresponds to the value of the running scale where $\lambda(\mu)$ vanishes [see Eq. (1) and footnote 1]. Finally, a vanishing λ_Λ is taken in Eq. (33) so that the highest possible physical cutoff Λ, corresponding to a given value of the renormalized coupling λ_v, is derived.

Instead, our analysis suggests that the upper bound for the range of ϕ's, which is also the highest self-consistent value for the physical cutoff, should be taken at the inflection point of Eq. (37), the value of ϕ where the potential ceases to be convex.

Although up to now we have considered a simple scalar-Yukawa model, it is clear that our results are completely general. In the next section, we shall see how the above criterion can be exported into the SM to get lower bounds on the Higgs mass.

Before we move to this phenomenological application, however, it is worth stressing that in the usual approach the requirement of stability appears to be an extra phenomenological constraint to be possibly imposed on the theory; an unstable potential is considered as a legitimate one. In fact, the metastability scenario, clearly excluded by our analysis, is based on the possibility of having a second minimum of the potential lower than the EW vacuum. As we have seen, however, the stability of the effective potential is an intrinsic property of the theory. No place is left for an unstable or metastable potential.

VI. LOWER BOUNDS ON THE HIGGS MASS

Let us consider now some important phenomenological implications of our findings for the SM. Clearly, the first thing to point out is that, contrary to common belief, the Higgs effective potential does not present any instability. As for the determination of the lower bounds on m_H, we have seen that the internal consistency of the theory requires that the physical cutoff has to be taken at the location of the inflection point of the potential (in the region beyond the minimum).

Implementing this criterion for the determination of the scale of new physics, lower bounds for the Higgs mass are found. Our results will be compared with those obtained with the help of the usual instability criterion.

The well known one-loop potential of the scalar sector of the SM reads [33]:

$$V^{1l}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4 + \frac{1}{64\pi^2}\Big[\Big(m^2 + \frac{\lambda}{2}\phi^2\Big)^2\Big(\ln\Big(\frac{m^2 + \frac{\lambda}{2}\phi^2}{\mu^2}\Big) - \frac{3}{2}\Big) + 3\Big(m^2 + \frac{\lambda}{6}\phi^2\Big)^2\Big(\ln\Big(\frac{m^2 + \frac{\lambda}{6}\phi^2}{\mu^2}\Big) - \frac{3}{2}\Big) + 6\frac{{g_1}^4}{16}\phi^4\Big(\ln\Big(\frac{\frac{1}{4}{g_1}^2\phi^2}{\mu^2}\Big) - \frac{5}{6}\Big) + 3\frac{({g_1}^2 + {g_2}^2)^2}{16}\phi^4\Big(\ln\Big(\frac{\frac{1}{4}({g_1}^2 + {g_2}^2)\phi^2}{\mu^2}\Big) - \frac{5}{6}\Big) - 12g^4\phi^4\Big(\ln\frac{g^2\phi^2}{\mu^2} - \frac{3}{2}\Big)\Big], \tag{38}$$

where g_1 and g_2 are the weak interaction coupling constants, while g is the top-Yukawa coupling.

To have a well defined comparison between our criterion and the usual one, we have chosen to follow the work of Casas, Espinosa, and Quirós [11,12]. In particular, we have taken their boundary conditions for g_1, g_2, m_t, ... at the scale M_Z as well as their matching conditions for the determination of the physical Higgs and top mass (see Appendix B and Refs. [11,12] for details).

The RG-improved potential $V_{\rm RGI}$ is obtained following the same steps of Sec. III. Naturally, the appropriate beta functions to consider in the RG equations are now the SM ones. As in Refs. [11,12], we have used the two-loops beta functions [8]. Note also that, differently from our simpler model, we now have three additional RG equations, namely, for g_1, g_2, and g_S (the strong coupling), and that no analytic solution for the running of the couplings can be found. Choosing $t = \frac{1}{2}\ln(\phi^2/\mu^2)$, we get

$$V_{\rm RGI}(\phi) = m^2(t)\frac{\phi^2(t)}{2} + \lambda_{\rm eff}(t)\frac{\phi^4(t)}{24} + \Omega(t), \tag{39}$$

where $\Omega(t)$ is the scale dependent vacuum energy, $\phi(t) = \xi(t)\phi$, with $\xi(t) = \exp(-\int_0^t \gamma(t')dt')$ and $\gamma(t)$ being the Higgs anomalous dimension, and $\lambda_{\rm eff}(t)$ is given by

$$\lambda_{\rm eff}(t) = \lambda + \frac{3}{8\pi^2}\Big[6\frac{{g_1}^4}{16}\Big(\ln\Big(\frac{{g_1}^2}{4}\Big) - \frac{5}{6}\Big) - 12g^4\Big(\ln g^2 - \frac{3}{2}\Big) + 3\frac{({g_1}^2 + {g_2}^2)^2}{16}\Big(\ln\Big(\frac{{g_1}^2 + {g_2}^2}{4}\Big) - \frac{5}{6}\Big)\Big], \tag{40}$$

with $\lambda = \lambda(t)$, $g = g(t)$, $g_1 = g_1(t)$, $g_2 = g_2(t)$.

First, we have checked that, when the usual $V_{\rm RGI} = 0$ criterion is used, meaning that the scale of new physics, Λ, is determined as the value of ϕ where [12]

$$\lambda_{\rm eff} + 12\frac{m^2}{\xi^2\Lambda^2} + 24\frac{\Omega}{\xi^4\Lambda^4} = 0, \tag{41}$$

the results of Refs. [11,12] are recovered. Then we have derived the physical cutoff according to our criterion; i.e. we have looked for the location of the external inflection point of $V_{\rm RGI}$.

In Table I we summarize the results obtained with these two criteria for different values of Λ. For small cutoffs, the lower bounds on M_H given by our criterion are $\sim$10 GeV larger than the current determinations [12], while for increasing values of Λ the difference tends to disappear.

The convergence between these two methods (for large cutoffs) has a simple explanation. Let us neglect, for a moment, the convexity constraint. As M_H increases, the location of the inflection point moves to higher and higher values of ϕ. The same, obviously, is true for the point where the potential vanishes. In this region, $V_{\rm RGI}$ is very well approximated by Eq. (1) and $\overline{\lambda}(\phi)$ changes very slowly with ϕ. Therefore, the two criteria practically give one and the same value for Λ.

TABLE I. Lower bounds on the Higgs mass as a function of the physical cutoff. The values of the physical parameters are chosen according to Refs. [11,12] (see also Appendix B). The second and third columns contain the bounds obtained with the convexity and instability criterion, respectively.

Λ (TeV)	$M_H^{\rm inf}$ (GeV)	M_H (GeV)	ΔM_H (GeV)
1	66	55.5	10.5
5	88	81	7
10	94.5	88.5	6
100	108.5	105.5	3
1000	117	115	2
10^{16}	137.5	137.5	0

TABLE II. Lower bounds on the Higgs mass as a function of the physical cutoff. Differently from Table I, the physical parameters have been chosen according to their most recent experimental determinations (see text). As for Table I, the second and third columns contain the bounds obtained with the convexity and instability criterion, respectively.

Λ (TeV)	$M_H^{\rm inf}$ (GeV)	M_H (GeV)	ΔM_H (GeV)
1	68.5	57.5	11
5	91.5	84	7.5
10	98	92	6
100	113	109.5	3.5
1000	122	120	2
10^{16}	143.5	143.5	0

The scope of Table I is to provide a comparison between the two different methods for the determination of lower bounds on m_H. To this end, the values of the physical parameters have been chosen according to Refs. [11,12] (see Appendix B) rather than to their more recent measured values. The reader can easily verify that the results we have found with the usual criterion (reported in the third column of Table I) agree with those of Refs. [11,12].

Now, considering the updated values: $M_Z = 91.2$ GeV, $M_W = 80.4$ GeV, $\alpha_s = 0.119$ [34], and $M_t = 178$ GeV [35], we find for the lower bounds on M_H the results reported in Table II. Note that, taking into account the present experimental uncertainty on M_t [35], $M_t = 178 \pm 4.3$, we get $M_H = 68.5^{+3}_{-3.5}$ for $\Lambda = 1$ TeV up to $M_H = 143.5 \pm 8.5$ for $\Lambda = 10^{19}$ GeV.

VII. WILSONIAN RG

In the previous section, we have considered a phenomenological application of our findings. Now, to further support our results, we come back to the simpler Higgs-Yukawa model of Eq. (4) and show that, with the help of the Wilsonian RG method, our analysis can be extended beyond perturbation theory.

For the Euclidean Wilsonian action of our model at the running scale k, $S_k[\phi, \bar{\psi}, \psi]$, we consider the following nonperturbative ansatz [36]:

$$S_k[\phi, \bar{\psi}, \psi] = \int d^4x \bigg(\frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \bar{\psi} \gamma_\mu \partial_\mu \psi + U_k(\phi, \bar{\psi}, \psi) \bigg). \tag{42}$$

As for the case of the scalar theory (see Sec. II and [23]), for the internal region we expect that from Eq. (42) a nonperturbative flow equation can be obtained which reproduces the Maxwell construction. Here, however, our scope is to investigate the possibility of having an instability of the scalar potential in the region beyond the minimum. Therefore, we consider only this region, where the running for the Wilsonian potential of our model is given by the nonperturbative RG equation [36]:

$$\begin{aligned} \frac{\partial U_k(\phi, \sigma)}{\partial k} &= -\frac{k^3}{16\pi^2} \ln\bigg(\frac{k^2 + U_k''(\phi, \sigma)}{k^2 + U_k''(0, \sigma)}\bigg) \\ &\quad + \frac{k^3}{4\pi^2} \ln\bigg(1 + \frac{\dot{U}_k{}^2(\phi, \sigma)}{k^2}\bigg) \\ &\quad - \frac{k^3}{16\pi^2} \ln\bigg(1 + \frac{2\sigma \dot{U}_k(\phi, \sigma)}{k^2 + \dot{U}_k{}^2(\phi, \sigma)} \\ &\quad \times \bigg(\ddot{U}_k(\phi, \sigma) - \frac{\dot{U}_k'{}^2(\phi, \sigma)}{k^2 + U_k''(\phi, \sigma)}\bigg)\bigg). \end{aligned} \tag{43}$$

Here $\sigma = \bar{\psi}\psi$, the prime indicates the derivative with respect to ϕ, and the dot the derivative with respect to σ.

The bare value of the potential, which is nothing but the boundary condition for the RG equation (43), is [see Eq. (4)]

$$U_\Lambda(\phi, \sigma) = \frac{1}{2} m_\Lambda^2 \phi^2 + \frac{\lambda_\Lambda}{24} \phi^4 + g_\Lambda \phi \sigma. \tag{44}$$

We now consider for $U_k(\phi, \sigma)$ the additional truncation:

$$U_k(\phi, \sigma) = V_k(\phi) + G_k(\phi)\sigma, \tag{45}$$

which means that we neglect the contributions from higher powers of $\bar{\psi}\psi$.

Inserting Eq. (45) in Eq. (43), we finally get the RG equations:

$$\begin{aligned} \frac{\partial V_k(\phi)}{\partial k} &= -\frac{k^3}{16\pi^2} \ln\bigg(\frac{k^2 + V_k''(\phi)}{k^2 + V_k''(0)}\bigg) \\ &\quad + \frac{k^3}{4\pi^2} \ln\bigg(1 + \frac{G_k{}^2(\phi)}{k^2}\bigg) \\ \frac{\partial G_k(\phi)}{\partial k} &= -\frac{k^3}{16\pi^2} \frac{1}{k^2 + V_k''(\phi)} \bigg(G_k''(\phi) - \frac{2G_k(\phi)G_k'(\phi)}{k^2 + G_k{}^2(\phi)}\bigg). \end{aligned} \tag{46}$$

From Eq. (44) is clear that the boundary conditions for V_k and G_k are

$$V_\Lambda(\phi) = \frac{1}{2} m_\Lambda^2 \phi^2 + \frac{1}{24} \lambda_\Lambda \phi^4, \qquad G_\Lambda(\phi) = g_\Lambda \phi. \tag{47}$$

Given m_Λ^2, λ_Λ, and g_Λ at $k = \Lambda$, we can run the RG equations (46) to get for the scalar effective potential $V_{\rm eff}(\phi)$ the nonperturbative approximation: $V_{\rm wil}(\phi) = V_{k=0}(\phi)$. Choosing $\lambda_\Lambda = 5 \times 10^{-2}$, $m_\Lambda^2 = -1 \times 10^{-2}$, $g_\Lambda = 5 \times 10^{-1}$ at $\Lambda = 100$, i.e. taking the same values used in Fig. 4, we get for $V_{\rm wil}$ the result plotted in Fig. 6 [we remind the reader that the RG equation (43) is valid only in the external region].

For comparison, we have also plotted the corresponding V^{1l} (which is nothing but the potential of Fig. 4). As we can

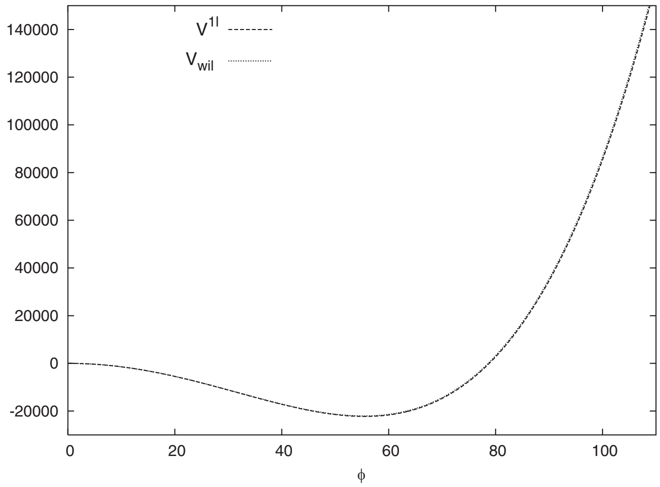


FIG. 6. The Wilsonian, $V_{\rm wil} = V_{k=0}$, effective potential. The boundary values of the parameters are as in Fig. 4. Only the region external to the minimum has to be considered. For comparison, we have also plotted the one-loop effective potential of Fig. 4. We see that, as explained in the text, $V_{\rm wil}$ and V^{1l} are very close one to the other.

easily see, $V_{\rm wil}$ and V^{1l} are very close one to the other. This result could have been guessed. As we have already said, in fact, in the external region the path integral that defines the effective potential is dominated by a single saddle point. As a consequence, we expect that the loop expansion, and, in particular, the one-loop potential, provide a good approximation for $V_{\rm eff}$. The close coincidence between V^{1l} (perturbative) and $V_{\rm wil}$ (nonperturbative) supports this expectation.

By its own construction, the Wilsonian method does not contain any *ad hoc* subtraction of terms. This is why we have compared the effective potential found with Eqs. (46) with the original one-loop result, the potential of Fig. 4, where the quadratically divergent terms were kept.

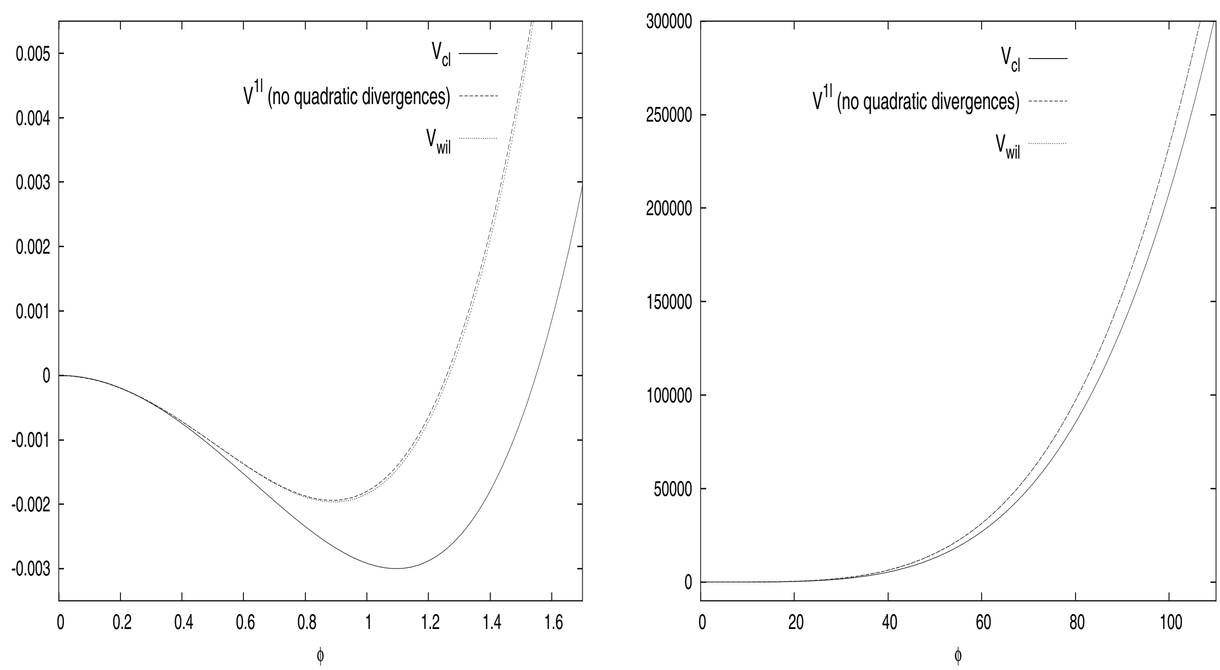
If we want to make contact with the perturbative V^{1l} where the quadratic divergences are subtracted (Fig. 5), we need to implement this operation in the flow equations.

Performing a polynomial expansion of $V_k(\phi)$ and $G_k(\phi)$, we easily see that the subtraction of the quadratic divergences in our flow equations amounts to adding the term

$$\left(-\frac{\lambda_k}{32\pi^2} + \frac{g_k{}^2}{4\pi^2}\right)k\phi^2 \tag{48}$$

to the first of Eqs. (46). In Eq. (48), λ_k is the coefficient of ϕ^4 in the expansion of $V_k(\phi)$, while g_k is the coefficient of ϕ in the expansion of $G_k(\phi)$. Moreover, at each step of the RG iteration, λ_k and g_k are determined via a polynomial fit of V_k and G_k, respectively. Their boundary values, of course, are λ_Λ and g_Λ.

Taking for m_Λ^2, λ_Λ, g_Λ, and Λ the same values considered above, we now run the modified system of RG equa-



(a) (b)

FIG. 7. The bare together with the Wilsonian potential after subtraction of the quadratic divergences. The boundary values of the parameters are as in Fig. 4. For $V_{\rm wil}$, only the external region has to be considered. For comparison, we have also plotted the one-loop effective potential of Fig. 5. We see that, even after subtracting the quadratic divergences, $V_{\rm wil}$ and V^{1l} are quite close.

tions and get for $V_{\rm wil}$ the result plotted in Fig. 7. As before, we note that $V_{\rm wil}$ and V^{1l} are very close.

The results of the present section strongly support our previous findings. Even within the nonperturbative framework considered here, the effective potential does not show any sign of instability.

VII. SUMMARY AND CONCLUSIONS

Starting with the analysis of some popular, but misleading, arguments, we have studied the instability problem of the EW vacuum with the help of a Higgs-Yukawa model.

Combining the Bogoliubov approach to symmetry breaking, namely, the criterion of dynamical instability, with the Wilsonian RG method, we have shown that there is no conflict between the convexity of the effective potential (effective action) and the existence of broken phase vertex functions. This preliminary step was helpful in establishing the incorrectness of the above quoted arguments.

Successively, we have shown that the potential instability is due to an illegal extrapolation of the renormalized effective potential into a region where the results of renormalized perturbation theory do not hold. Moreover, in agreement with what is expected from general theorems, we have found that the effective potential of the cutoff Higgs-Yukawa model is convex all over the region where it is defined.

To establish these results, it was necessary to go beyond the usual application of the perturbation theory conditions. In this respect, we note that the dimensional regularization scheme, by its own construction, directly gives the results of renormalized perturbation theory. As the subject of this paper shows, however, the connection between the UV and the IR sectors of the theory (the relation between bare and renormalized theory) can present aspects which are hidden to a naive application of dimensional regularization.

In our case, the consistency constraint for the theory ($\phi \leq \Lambda$) and Eq. (32) imply that the combination $(\lambda_v/24) + (g^4/16\pi^2)\ln(v^2/\phi^2)$ cannot be negative. When we blindly jump to the perturbation theory results, this information is lost. Actually, Eqs. (27) and (28), typically considered as the only conditions for the renormalized perturbation theory to hold, do not contain the above independent constraint. The effective potential appears to be unstable when this condition is ignored.

We started our analysis within the framework of the momentum cutoff regularization scheme. Successively, with the help of the Wilsonian RG method, our results were established in a more general nonperturbative context.

Moreover, despite the stability of the potential, we have shown that lower bounds on the Higgs mass can still be derived. In fact, for a given renormalized value of λ, the corresponding cutoff can be found by looking for the inflection point of $V_{\rm eff}$ in the external region ($\phi > v$). If the scale of new physics is not too high, a sizable difference between our bounds and the usual ones is obtained. For Λ in the TeV region, we find a value of m_H which is some 10–11 GeV higher than the current determination.

In addition to these phenomenological applications, it is worth noting that there is a deep conceptual difference between our analysis and the usual one. While in our case the stability of the potential, as well as the bounds on m_H, come as a manifestation of the internal consistency of the theory, in the usual approach the bounds are the result of an (apparently) additional constraint to be imposed on the potential, the requirement of stability. The instability is considered as a theoretically legitimate possibility. In fact, the metastability scenario explores the consequences of having a minimum lower than the EW one. Our results exclude this scenario.

In the present work, we have been interested in the instability issue only. However, we believe that our results come as a manifestation of a general problem, the (somehow delicate) connection between the UV and the IR sectors of a theory and that a similar analysis can be applied to other cases where this connection is expected to play an important role. We hope to come to this point in the future.

As already said, we come now to the comparison of our work with Refs. [26,27]. First of all, we note that the instability problem concerns the renormalized effective potential. Therefore, it is important to perform the analysis within a range of ϕ where renormalized perturbation theory is (or is supposed to be) valid. In Refs. [26,27], however, the potential has a minimum at the cutoff, i.e. at $\phi \sim \Lambda = \frac{\pi}{a}$ (see Fig. 2 of Ref. [26] and Fig. 4 of Ref. [27]), and all the relevant scales, namely, the "low energy scale" μ, the cutoff scale Λ, and the minimum v, are of the same order. In our opinion, this hardly helps in understanding the origin of the instability problem.

Moreover, the renormalized potential [see Eq. (2) in Ref. [27]] is obtained from the (subtracted) bare potential [see Eq. (5) in Ref. [27]] after expanding in $\frac{\phi}{\Lambda}$ and neglecting negative powers of Λ. Insisting on the difference between the bare and the renormalized potentials for values of ϕ beyond Λ, as done by the authors, once more does not help in clarifying the problem.

We believe that we have clearly identified the origin of the apparent instability of the effective potential. Contrary to what is stated in Ref. [27], it seems to us that it has nothing to do with triviality.

To understand this point, it is worth stressing once more that we are dealing with an effective theory. Triviality simply means the growing of λ for growing values of the energy scale. Now, as anybody familiar with the problem knows, the instability can be read from the running of $\lambda(\phi)$ and is due to the contribution of the fermion loops which drive λ to the negative.

The role of triviality, in the sense explained above, shows up later on, i.e. for larger values of ϕ. Because of triviality, in fact, soon or later $\lambda(\phi)$ rises up again, thus generating a new minimum of $V_{\rm eff}$. By the way, it is precisely this (''trivial'') behavior of λ which is invoked by the proposers of the metastability scenario.

As is well known, this peculiar flow of λ is due to the presence in the beta function of the $-g^4$ term (which initially drives λ to the negative) and to the usual $+\lambda^2$ term, which takes over later [see the first of our Eqs. (8)].

The conclusion is that triviality has nothing to do with the instability of $V_{\rm eff}$. It would occur even if, hypothetically, λ, after being driven negative by the fermion contributions, would happen to saturate to zero or to any other value for larger values of ϕ (which, by the way, would require a term different than λ^2 in the beta function).

Finally, we note that, in Refs. [26,27], in order to avoid problems with the convexity of $V_{\rm eff}$ (as stated by the authors), the constrained potential is used. On the contrary, insisting on the convexity of $V_{\rm eff}$ as a guiding property, we have found the flaw that artificially makes $V_{\rm eff}$ unstable in the external region.

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APPENDIX A: MAXWELL CONSTRUCTION AND BOGOLIUBOV CRITERION

In the present appendix, we briefly show how the $\Gamma_n^{(v)}$'s are obtained from the convex effective action $\Gamma_{\rm eff}$. For illustrational purposes, it is sufficient to consider the case of a constant background field, i.e. to consider $V_{\rm eff}$ rather than the full effective action. For the sake of simplicity, we also limit ourselves to the case of a single component scalar theory.

General theorems [14,37], together with several analytical and numerical nonperturbative studies [17–24], indicate that $V_{\rm eff}$ is a convex function of ϕ with a flat bottom between $-v$ and v, the minima of the classical potential. At the lowest order, $V_{\rm eff}$ coincides with the well known Maxwell (or double tangent) construction sketched in Fig. 8.

The (zero momentum) $\Gamma_n^{(v)}$'s should be obtained by taking the derivatives of $V_{\rm eff}$ at $\phi = v$. Because of the shape of the potential, however, this operation is ambiguous and has to be defined with a certain care.

The approach that we are going to consider now [28,29], far from being a technical point, has a deep physical meaning. Following Bogoliubov, in fact, we interpret the occurrence of symmetry breaking as a manifestation of the ''dynamical instability'' of the otherwise equivalent vacua of the potential. Adding to the Lagrangian an infinitesimal source term which explicitly breaks the classical symmetry

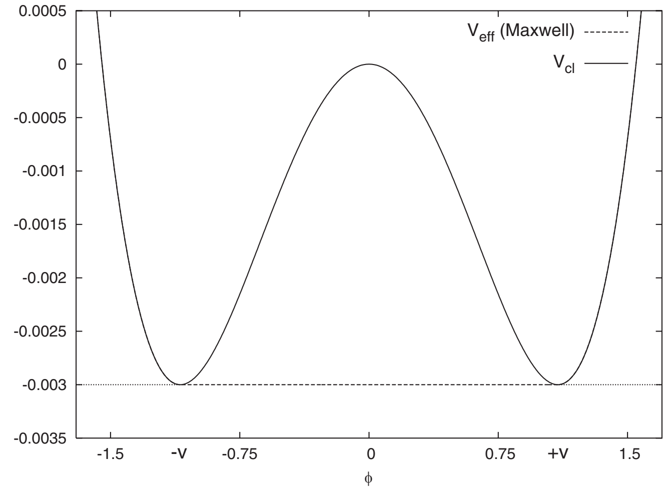

FIG. 8. The Maxwell construction for the classical potential of the single component scalar theory considered in the text. The parameters are chosen as $\lambda = 5 \times 10^{-2}$ and $m^2 = -10^{-2}$.

of the theory $-\varepsilon\phi$, we select one of the two classical vacua (see Fig. 9). More precisely, this additional term creates an absolute minimum v_ε close to the old v.

As for the symmetric case, the lowest order for $V_{\rm eff}$ can be obtained with the help of the double tangent construction (Fig. 9). A simple inspection of Fig. 9 shows that the derivatives at $\phi = v_\varepsilon$ of the resulting modified effective potential $V_{\rm eff}(\phi; \varepsilon)$ can be safely taken. In fact, while in the symmetric case (Fig. 8) the flat region extends from one of the classical minima to the other (the minima coincide with the tangent points), in Fig. 9 the effective potential (as the classical one) has an absolute minimum v_ε, and the flat

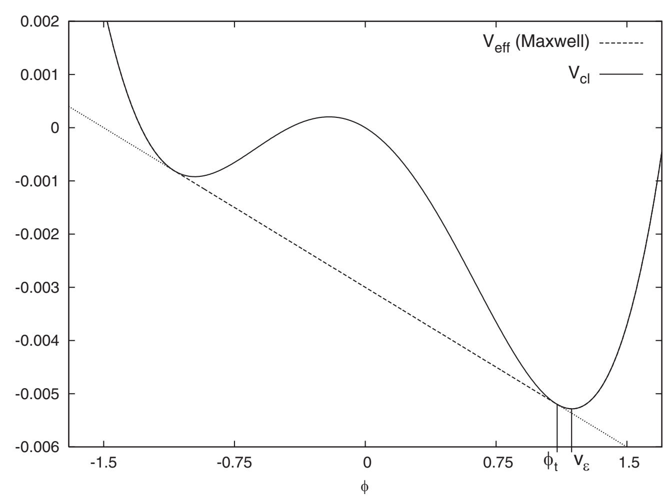

FIG. 9. The Maxwell construction for the classical potential of the single component scalar theory with an explicit symmetry breaking term $-\varepsilon\phi$. The parameters are chosen as $\lambda = 5 \times 10^{-2}$, $m^2 = -10^{-2}$. and $\varepsilon = 2 \times 10^{-3}$.

region starts at $\phi_t < v_\varepsilon$. The corresponding $\Gamma_n^{(v_\varepsilon;\varepsilon)}$'s at this order are then easily obtained. The successive $\varepsilon \to 0$ limit[7] gives the desired $\Gamma_n^{(v)}$'s.

Clearly, the $\Gamma_n^{(v)}$'s that we get this way are nothing but the usual tree level $\Gamma_n^{(v)}$'s. To get higher order approximations, we need to go beyond this lowest order Maxwell construction. As we have shown in the main text (Sec. II), with the help of the Wilsonian RG approach, the above results can be established beyond this lowest order.

APPENDIX B: RENORMALIZATION CONDITIONS

In this appendix, we compute the renormalized potential of Eq. (23), where the renormalization conditions that keep the minimum and the curvature around the minimum fixed at their classical values are implemented. Clearly, these conditions are

$$\left(\frac{dV^{1l}}{d\phi}\right)_{\phi=v} = 0, \tag{B1}$$

$$\left(\frac{d^2V^{1l}}{d\phi^2}\right)_{\phi=v} = \frac{\lambda v^2}{3} = -2m^2, \tag{B2}$$

with $v = \sqrt{-6m^2/\lambda}$. From Eq. (15) we get

$$\frac{dV^{1l}}{d\phi} = \phi\bigg(m^2 + \delta m^2 + (\lambda + \delta\lambda)\frac{\phi^2}{6} + \left(\frac{\lambda}{32\pi^2} - \frac{g^2}{4\pi^2}\right)\Lambda^2 + \frac{\lambda}{32\pi^2}\left(m^2 + \frac{\lambda}{2}\phi^2\right)\ln\frac{m^2 + \frac{\lambda}{2}\phi^2}{\Lambda^2} - \frac{g^4\phi^2}{4\pi^2}\ln\frac{g^2\phi^2}{\Lambda^2}\bigg), \tag{B3}$$

so that the condition (B1) becomes

$$0 = \delta m^2 + \delta\lambda\frac{v^2}{6} + \left(\frac{\lambda}{32\pi^2} - \frac{g^2}{4\pi^2}\right)\Lambda^2 + \frac{\lambda}{32\pi^2}\left(m^2 + \frac{\lambda}{2}v^2\right)\ln\frac{m^2 + \frac{\lambda}{2}v^2}{\Lambda^2} - \frac{g^4v^2}{4\pi^2}\ln\frac{g^2v^2}{\Lambda^2}. \tag{B4}$$

Deriving V^{1l} once more with respect to ϕ, we get

$$\frac{d^2V^{1l}}{d\phi^2} = m^2 + \delta m^2 + \left(\frac{\lambda}{32\pi^2} - \frac{g^2}{4\pi^2}\right)\Lambda^2 + \frac{\lambda}{32\pi^2}\left(m^2 + \frac{3}{2}\lambda\phi^2\right)\ln\frac{m^2 + \frac{\lambda}{2}\phi^2}{\Lambda^2} + \frac{\phi^2}{2}\left(\lambda + \delta\lambda + \frac{\lambda}{16\pi^2} - \frac{3g^4}{2\pi^2}\left(\ln\frac{g^2\phi^2}{\Lambda^2} + \frac{2}{3}\right)\right), \tag{B5}$$

and the condition (B2) reads:

$$0 = \delta m^2 + \delta\lambda\frac{v^2}{2} + \left(\frac{\lambda}{32\pi^2} - \frac{g^2}{4\pi^2}\right)\Lambda^2 + \frac{\lambda}{32\pi^2}\left(m^2 + \frac{3}{2}\lambda v^2\right)\ln\frac{m^2 + \frac{\lambda}{2}v^2}{\Lambda^2} + \frac{v^2}{2}\left(\frac{\lambda^2}{16\pi^2} - \frac{3g^4}{2\pi^2}\left(\ln\frac{g^2v^2}{\Lambda^2} + \frac{2}{3}\right)\right). \tag{B6}$$

From Eqs. (B4) and (B6) we find:

$$\delta\lambda = \frac{3g^4}{2\pi^2}\left(\ln\frac{g^2v^2}{\Lambda^2} + 1\right) - \frac{3\lambda^2}{32\pi^2}\left(\ln\frac{m^2 + \frac{\lambda}{2}v^2}{\Lambda^2} + 1\right), \tag{B7}$$

$$\delta m^2 = \left(\frac{g^2}{4\pi^2} - \frac{\lambda}{32\pi^2}\right)\Lambda^2 - \frac{\lambda m^2}{32\pi^2}\left(\ln\frac{m^2 + \frac{\lambda}{2}v^2}{\Lambda^2} + 3\right) - \frac{g^4v^2}{4\pi^2}. \tag{B8}$$

Inserting Eqs. (B7) and (B8) in V^{1l}, i.e. in Eq. (15), we finally get

$$V^{1l} = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4 + \frac{(m^2 + \frac{\lambda}{2}\phi^2)^2}{64\pi^2}\left(\ln\frac{m^2 + \frac{\lambda}{2}\phi^2}{m^2 + \frac{\lambda}{2}v^2} - \frac{3}{2}\right) - \frac{g^4\phi^4}{16\pi^2}\left(\ln\frac{\phi^2}{v^2} - \frac{3}{2}\right) + \frac{v^2}{2}\phi^2\left(\frac{3\lambda^2}{32\pi^2} - \frac{g^4}{4\pi^2}\right). \tag{B9}$$

Finally, neglecting the bosonic contribution to the quantum fluctuation determinant, we see that Eq. (B9) is nothing but the renormalized one-loop potential of Eq. (23).

APPENDIX C: RG-IMPROVED POTENTIAL FOR THE SM

In the present appendix, we provide some useful relations needed for the computation of the RG-improved one-loop effective potential of the SM (Sec. VI). Following Ref. [11], the matching conditions for the Higgs and the top masses are taken as:

$$M_H^2(t) = m_H^2(t^*)\frac{\xi^2(t^*)}{\xi^2(t)} + \mathrm{Re}(\Pi(p^2 = M_H^2) - \Pi(p^2 = 0)), \tag{C1}$$

[7] Although in this brief presentation we do not aim at complete rigor, it is worth pointing out that, to construct the $\Gamma_n^{(v)}$'s, we begin first with a finite volume system and successively take the infinite volume limit. The latter has to be taken previous to the $\varepsilon \to 0$ limit.

$$M_t = m_t(M_t)\left(1 + \frac{g_s(M_t)^2}{3\pi^2}\right), \tag{C2}$$

where Π is the self-energy of the Higgs boson (for the full explicit expression, see Appendix A of Ref. [11]). Moreover, although the exact effective potential is scale independent, for V^{1l} and $V_{\rm RGI}$ this is true only approximately. The value t^* of the parameter t that appears in Eq. (C1) is chosen as to minimize the dependence of $V_{\rm RGI}$ on the choice of the running scale $\mu(t) = M_Z e^t$. The corresponding $\mu(t^*)$, in our case, is $\mu(t^*) \sim 130$ GeV.

Accordingly, omitting the Higgs and the Goldstone (negligible) contributions, the value of $m_H^2(t^*)$ is secured as [11]:

$$\begin{aligned} m_H^2(t^*) = \xi^2(t^*)v^2\Bigg(&\frac{\lambda(t^*)}{3} + \frac{3}{64\pi^2} \\ &\times \Bigg\{g_1^4(t^*)\left[\log\frac{g_1^2(t^*)\xi^2(t^*)v^2}{4\mu^2(t^*)} + \frac{2}{3}\right] \\ &+ \frac{1}{2}[g_1^2(t^*) + g_2^2(t^*)]^2 \\ &\times \left[\log\frac{[g_1^2(t^*) + g_2^2(t^*)]\xi^2(t^*)v^2}{4\mu^2(t^*)} + \frac{2}{3}\right] \\ &- 8g^4(t^*)\log\frac{g^2(t^*)\xi^2(t^*)v^2}{2\mu^2(t^*)}\Bigg\}\Bigg), \end{aligned} \tag{C3}$$

where, in the first term of the right-hand side, we recognize the tree-level relation for m_H^2, while the other terms come from the loop corrections.

The boundary values for the coupling constants are chosen as [11]:

$$\begin{aligned} &g_1(M_Z) = 0.650, \qquad g_2(M_Z) = 0.355, \\ &g_s(M_Z) = 1.218, \qquad \gamma(M_Z) = 0, \\ &\Omega(M_Z) = 0, \qquad g(M_t) = \frac{\sqrt{2}m_t(M_t)}{\xi(M_t)v} = 0.9635, \end{aligned} \tag{C4}$$

which correspond to $M_W = 80$ GeV, $M_Z = 91.2$ GeV, $\alpha_s = 0.118$, and $M_t = 175$ GeV.

The coupling $\lambda(M_Z)$ is kept as a free parameter. As explained in the text, by considering different values of $\lambda(M_Z)$, we obtain different values for the physical cutoff, thus getting lower bounds for the Higgs mass as a function of the scale of new physics.

Note also that, in order to keep the location of the minimum to its phenomenological value, m^2 has to be fixed by the condition: $\langle\phi(t^*)\rangle/\xi(t^*) = v = 246.22$ GeV, which gives [11]

$$\begin{aligned} m^2(t^*) = -\xi^2(t^*)v^2\Bigg(&\frac{\lambda(t^*)}{6} + \frac{3}{64\pi^2} \\ &\times \Bigg\{\frac{1}{2}g_1^4(t^*)\left[\log\frac{g_1^2(t^*)\xi^2(t^*)v^2}{4\mu^2(t^*)} - \frac{1}{3}\right] \\ &+ \frac{1}{4}[g_1^2(t^*) + g_2^2(t^*)]^2 \\ &\times \left[\log\frac{[g_1^2(t^*) + g_2^2(t^*)]\xi^2(t^*)v^2}{4\mu^2(t^*)} - \frac{1}{3}\right] \\ &- 4g^4(t^*)\left[\log\frac{g^2(t^*)\xi^2(t^*)v^2}{2\mu^2(t^*)} - 1\right]\Bigg\}\Bigg). \end{aligned} \tag{C5}$$

Now, solving numerically the system of RG equations for the running coupling constants, we get Eq. (39) of Sec. VI for $V_{\rm RGI}(\phi)$.

We end this appendix giving the boundary values of the coupling constants corresponding to the updated values of M_Z, M_W, α_S, and M_t reported in Sec. VI: $g_1(M_Z) = 0.653$, $g_2(M_Z) = 0.349$, $g_s(M_Z) = 1.223$, and $g(M_t) = 0.980$.

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