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Gravitational dynamics in $s+1+1$ dimensions

László Á. Gergely[1] and Zoltán Kovács[2]

[1]*Departments of Theoretical and Experimental Physics, University of Szeged, Szeged 6720, Dóm tér 9, Hungary*

[2]*Max-Planck-Institut für Astronomie, Königstuhl 17, D-69117 Heidelberg, Germany*

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We present the concomitant decomposition of an $(s+2)$-dimensional space-time both with respect to a timelike and a spacelike direction. The formalism we develop is suited for the study of the initial value problem and for canonical gravitational dynamics in braneworld scenarios. The bulk metric is replaced by two sets of variables. The first set consists of one tensorial (the induced metric g_{ij}), one vectorial (M^i) and one scalar (M) dynamical quantity, all defined on the s space. Their time evolutions are related to the second fundamental form (the extrinsic curvature K_{ij}), the normal fundamental form ($\mathcal{K}^i$) and normal fundamental scalar ($\mathcal{K}$), respectively. The nondynamical set of variables is given by the lapse function and the shift vector, which however has one component less. The missing component is due to the externally imposed constraint, which states that physical trajectories are confined to the $(s+1)$-dimensional brane. The pair of dynamical variables (g_{ij}, K_{ij}), well known from the Arnowitt-Deser-Misner decomposition is supplemented by the pairs $(M^i, \mathcal{K}^i)$ and $(M, \mathcal{K})$ due to the bulk curvature. We give all projections of the junction condition across the brane and prove that for a perfect fluid brane neither of the dynamical variables has jump across the brane. Finally we complete the set of equations needed for gravitational dynamics by deriving the evolution equations of K_{ij}, $\mathcal{K}^i$ and $\mathcal{K}$ on a brane with arbitrary matter.

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I. INTRODUCTION

Recent models motivated by string theory/M theory [1,2] indicate that gravity may act in more than four noncompact dimensions (the bulk). Our every day experiences are preserved by postulating that ordinary standard-model matter remains confined to the brane, which is a single spatial time-evolving 3-surface (a temporal 4-surface). Remarkably, gravity is also shown to be localized on the brane in these models [2]. As energy momentum is the source for gravity, the expectation is that some sort of localization of gravity on the brane is quite generic. More exactly, the Ricci component of the curvature should be localized through the bulk Einstein equations. However the Weyl component of the curvature induced by black holes in the bulk can give rise to the nonlocalization of gravity. It has been known for a while that gravity is not localized on the Friedmann-Lemaître-Robertson-Walker brane embedded in Schwarzschild–anti-de Sitter bulk, whenever the former has a negative cosmological constant [3]. A recent example in this sense was found in [4], where it is explicitly shown how gravity is delocalized on a vacuum Einstein brane by the 5-dimensional (5D) horizon of a bulk black hole. The way to see whether gravity is localized on the brane is a perturbative one, emerging from the perturbative analysis of the massive Kaluza-Klein modes of the bulk graviton by perturbations around a Minkowski brane in anti-de Sitter bulk [2] (or Einstein static vacuum brane [5] in Schwarzshild–anti-de Sitter bulk [4]).

In order to monitor gravitational dynamics from the brane observer's viewpoint and to follow the evolution of matter fields on the brane, the decomposition of bulk quantities and their dynamics with respect to the brane is necessary. This was done in [6], leading to an effective Einstein equation on the brane, which contains additional new source terms: a quadratic expression of the energy-momentum tensor and the ''electric'' part of the bulk Weyl tensor. Brane matter is related to the discontinuity (across the brane) in the second fundamental form of the brane through the Lanczos-Sen-Darmois-Israel matching conditions [7–10]. The more generic situation, allowing for a brane embedded asymmetrically and for matter (nonstandard-model fields) in the bulk was presented in [11]. There it was proven that the 5D Einstein equations are equivalent on the brane with the set of the effective Einstein equations (with even more new source terms, arising from asymmetric embedding and bulk matter), the Codazzi equation and the twice-contracted Gauss equation. For a recent review of other issues related to braneworlds, see [12].

No canonical description of the bulk, similar to the standard Arnowitt-Deser-Misner (ADM) treatment of the 4D gravity has been given until now. That would be straightforward if one simply aims to increase the dimension of space by one. Such a procedure however would not know about the preferred hypersurface which is the brane. Occurrence of effects like the localization of gravity on the brane would be difficult to follow. Therefore we propose to develop a formalism which singles out both the time and the off-brane dimension. Although we primarily have in mind the $3+1+1$ braneworld model, we would like to keep a more generic setup for other possible applications, like the $2+1+1$ decomposition of space-time in general relativity. Therefore we develop the formalism in $s + 1 + 1$ dimensions.

We suppose that the full noncompact space-time $\mathcal{B}$ can be foliated by a family of $(s + 1)$-dimensional spacelike leaves S_t, characterized by the unit normal vector field n. The projection of this foliation onto the observable $(s + 1)$-dimensional space-time $\mathcal{M}$ (the brane) gives the usual $s + 1$ decomposition of $\mathcal{M}$ into the foliation Σ_t, whenever $n(x \in \mathcal{M}) \in T\mathcal{M}$. Let us denote the unit normal vector field to $\mathcal{M}$ by l. We also suppose that $\mathcal{B}$ can be foliated by a family of $(s + 1)$-surfaces $\mathcal{M}_\chi$ (with $\mathcal{M}_0 = \mathcal{M}$), or at least we are able to extend the field l in a suitable manner to some neighborhood of $\mathcal{M}$. We denote by m the unit normal to n in the tangent plane spanned by n and l. The intersection of the leaves S_t and $\mathcal{M}_\chi$ represent spatial s-surfaces $\Sigma_{t\chi}$, from among which $\Sigma_t = \Sigma_{t0}$ (Fig. 1).

In Sec. II we define temporal and off-brane evolution vectors and we decompose them in an orthonormal basis adapted to the foliation S_t. The $(s + 2)$-dimensional metric is replaced by a convenient set of dynamical variables (g_{ij}, M^i, M), together with the lapse and shifts of n. We show that one shift component, $\mathcal{N}$, obeys a constraint due to the Frobenius theorem. This constraint can be traced back to the fact that the trajectories of standard-model particles are confined to a hypersurface. In other words, it is a consequence of the very existence of the brane. The simplest way to fulfill this constraint is to choose a vanishing $\mathcal{N}$. This in turn is equivalent with choosing the two foliations perpendicularly, $m^a = l^a$. Then we introduce the second fundamental forms of the S_t, $\mathcal{M}_\chi$ and $\Sigma_{t\chi}$ hypersurfaces and establish their interconnections. This is done via the normals n and l, their "accelerations," the normal fundamental forms and normal fundamental scalars, all to be defined in Sec. II. Some of the more technical details needed for the results of Sec. II are derived in Appendixes A, B, and C.

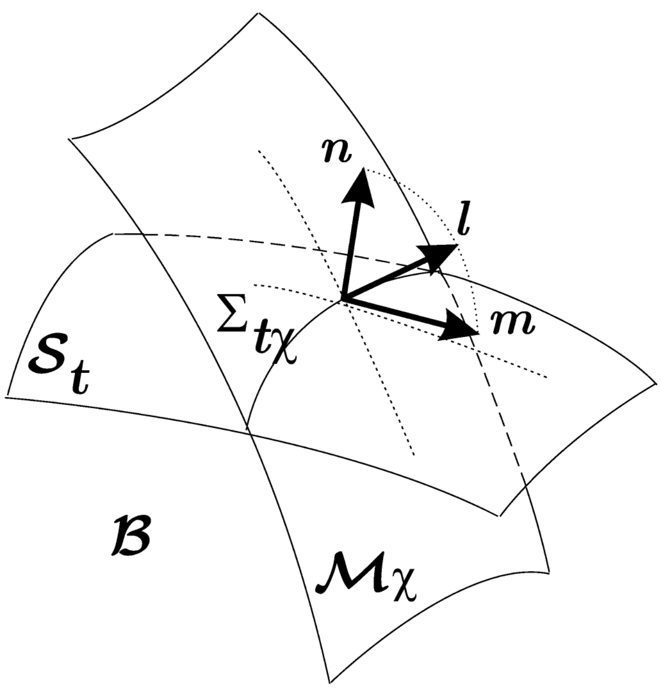

FIG. 1. The two foliations with unit normal vector fields n and l. The unit vector field m belongs to the tangent space spanned by n and l and it is perpendicular to n.

We establish in Sec. III the relation between time derivatives of the dynamical data and various projections of extrinsic curvatures $(K_{ij}, \mathcal{K}^i, \mathcal{K})$. In Sec. IV we write the junction conditions, in particular, the Lanczos equation across a brane containing arbitrary matter in terms of these extrinsic curvature projections.

Section V and Appendix D contain the decompositions with respect to S_t, $\mathcal{M}_\chi$ and $\Sigma_{t\chi}$ of the intrinsic curvatures. We give the decomposition of the connections, Riemann, Ricci and Einstein tensors. Among these we find the Raychaudhuri, Codazzi and Gauss equations. The evolution equations for the set $(K_{ij}, \mathcal{K}^i, \mathcal{K})$ are then readily deduced in Sec. VI. We give them explicitly for a bulk containing nothing but a negative cosmological constant, but for a generic brane. With this we complete the task of giving all gravitational evolution equations in terms of one tensorial, one vectorial and one scalar pair of dynamical quantities.

Section VII contains the concluding remarks. Here we compare our formalism specified for $s = 2$ with previous decompositions of space-time in general relativity.

Notation.—A tilde and a hat distinguish the quantities defined on $\mathcal{B}$ and S_t, respectively. Quantities belonging to $\mathcal{M}_\chi$ possess a distinctive dimension-carrying index while those defined on $\Sigma_{t\chi}$ have no special distinctive mark. For example, the metric 2-forms on $\mathcal{B}$, S_t, $\mathcal{M}_\chi$ and $\Sigma_{t\chi}$ are denoted $\tilde{g}$, $\hat{g}$, ${}^{(s+1)}g$ and g, respectively, while the corresponding metric-compatible connections are $\tilde{\nabla}$, $\hat{D}$, ${}^{(s+1)}D$ and D. Then

$$\begin{aligned} \hat{g}^{a_1\cdots a_r d_1\cdots d_s}_{c_1\cdots c_r b_1\cdots b_s} &= \hat{g}^{a_1}_{c_1}\cdots\hat{g}^{a_r}_{c_r}\hat{g}^{d_1}_{b_1}\cdots\hat{g}^{d_s}_{b_s}, \\ {}^{(s+1)}g^{a_1\cdots a_r d_1\cdots d_s}_{c_1\cdots c_r b_1\cdots b_s} &= {}^{(s+1)}g^{a_1}_{c_1}\ldots{}^{(s+1)}g^{a_r}_{c_r}{}^{(s+1)}g^{d_1}_{b_1}\ldots{}^{(s+1)}g^{d_s}_{b_s}, \\ g^{a_1\cdots a_r d_1\cdots d_s}_{c_1\cdots c_r b_1\cdots b_s} &= g^{a_1}_{c_1}\cdots g^{a_r}_{c_r}g^{d_1}_{b_1}\cdots g^{d_s}_{b_s}, \end{aligned} \tag{1}$$

project any tensor $\tilde{T}^{a_1\cdots a_r}_{b_1\cdots b_s}$ on $\mathcal{B}$ to S_t, $\mathcal{M}_\chi$ and $\Sigma_{t\chi}$, respectively.

Latin indices represent abstract indices running from 0 to $(s + 1)$. Greek and bold Latin indices, running from 0 to $(s + 1)$ and from 1 to s, respectively, either count some specific basis vectors or they denote tensorial components in these bases. Vector fields in Lie derivatives are represented by boldface characters. For example $\tilde{\mathcal{L}}_{\mathbf{V}}T$ denotes the Lie derivative on $\mathcal{B}$ along the integral lines of the vector field V^a.

II. FUNDAMENTAL FORMS

A. First fundamental forms of $\Sigma_{t\chi}$, S_t and $\mathcal{M}_\chi$

By a careful study of the two foliations in Appendix A we arrive to the conclusion that temporal and off-brane evolutions happen along vector fields given as

$$\left(\frac{\partial}{\partial t}\right)^a = Nn^a + N^a + \mathcal{N}m^a, \tag{2}$$

$$\left(\frac{\partial}{\partial \chi}\right)^a = M^a + Mm^a. \tag{3}$$

In the above formulas N^a and N have the well-known interpretation from the decomposition of the 3 + 1-dimensional space-time as shift vector and lapse function. The quantity $\mathcal{N}$ is the component of the shift in the off-brane direction (Fig. 2). Finally the vector M^a and the scalar M are quantities representing the off-brane sector of gravity, all defined on $\Sigma_{t\chi}$ (Fig. 3).

The $(s + 2)$-dimensional metric is

$$\tilde{g}_{ab} = g_{ab} + m_a m_b - n_a n_b, \tag{4}$$

g_{ab} being the induced metric of $\Sigma_{t\chi}$. In Appendix B we prove a simple relationship between the corresponding determinants:

$$\sqrt{-\tilde{g}} = NM\sqrt{g}. \tag{5}$$

The induced metrics on S_t and on $\mathcal{M}_\chi$ are respectively:

$$\hat{g}_{ab} = \tilde{g}_{ab} + n_a n_b = g_{ab} + m_a m_b, \tag{6}$$

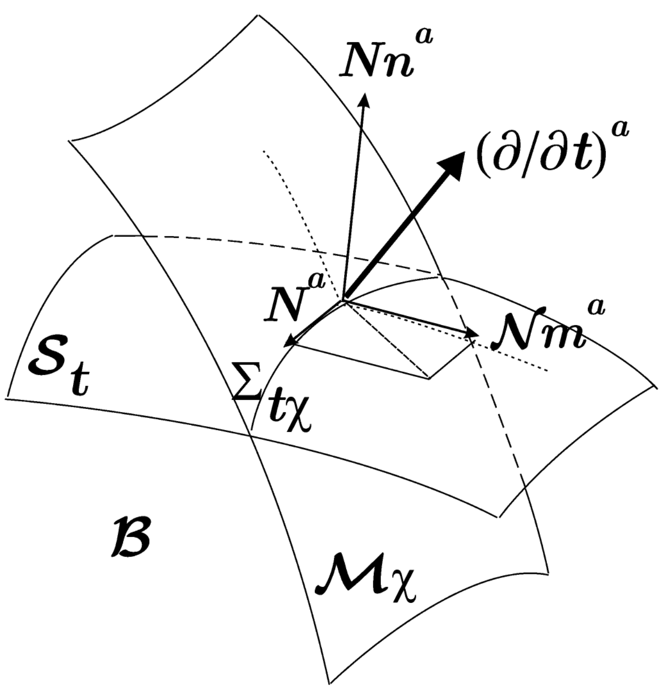

FIG. 2. Decomposition of the time-evolution vector $\partial/\partial t$, when the two foliations are not perpendicular (Fig. 3).

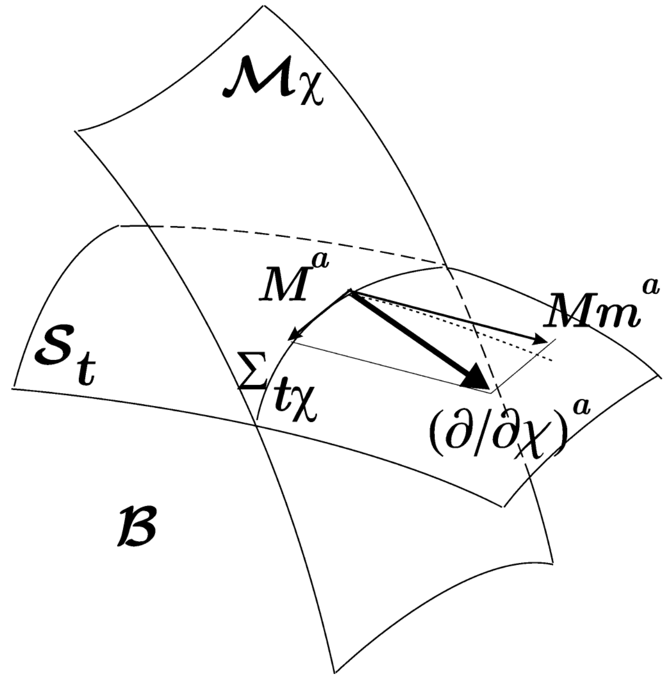

FIG. 3. Decomposition of the off-brane evolution vector $\partial/\partial\chi$ for nonperpendicular foliations.

$${}^{(s+1)}g_{ab} = \tilde{g}_{ab} - m_a m_b = g_{ab} - n_a n_b. \tag{7}$$

They obey $\hat{g}_{ab} n^b = {}^{(s+1)}g_{ab} l^b = 0$.

A simple counting shows that the $(s + 2)(s + 3)/2$ components of the $(s + 2)$-metric $\tilde{g}_{\alpha\beta}$ can be replaced by the equivalent set $\{g_{ab}, M^a, M, N^a, \mathcal{N}, N\}$, for which the restrictions $g_{ab}n^a = g_{ab}m^a = M^a n_a = M^a m_a = N^a n_a = N^a m_a = 0$ apply. In Appendix C we however prove from the Frobenius theorem the constraint (C2) on $\mathcal{N}$ and M. This seems to reduce the number of variables; still the information is not lost. Although the number of gravitational variables is reduced by one, we gain the information that the evolution of standard-model fields is constrained to a hypersurface. The easiest way to satisfy this constraint is to choose

$$\mathcal{N} = 0, \tag{8}$$

which means perpendicular foliations (Fig. 4). We will follow this choice, which leads to the identification

$$l^a = m^a \tag{9}$$

in the rest of the paper [excepting Appendixes A, B, and C, the results of which are indispensable in deriving the constraint Eq. (C2)].

B. Second fundamental forms of $\Sigma_{t\chi}$, S_t and $\mathcal{M}_\chi$

We introduce the $\Sigma_{t\chi}$, S_t and $\mathcal{M}_\chi$ projections of the covariant derivative of an arbitrary tensor $\tilde{T}^{a_1\cdots a_r}_{b_1\cdots b_s}$ defined on $\mathcal{B}$ as

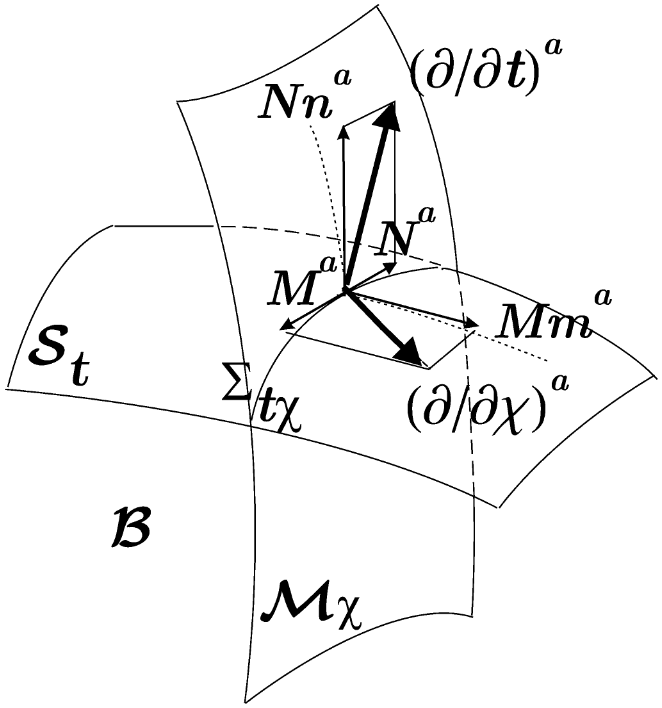

FIG. 4. Decomposition of the temporal and off-brane evolution vectors $\partial/\partial t$ and $\partial/\partial\chi$ for perpendicular foliations.

$$\begin{aligned} D_a\tilde{T}^{a_1\cdots a_r}_{b_1\cdots b_s} &= g^{ca_1\cdots a_r d_1\cdots d_s}_{ac_1\cdots c_r b_1\cdots b_s}\tilde{\nabla}_c\tilde{T}^{c_1\cdots c_r}_{d_1\cdots d_s}, \\ \hat{D}_a\tilde{T}^{a_1\cdots a_r}_{b_1\cdots b_s} &= \hat{g}^{ca_1\cdots a_r d_1\cdots d_s}_{ac_1\cdots c_r b_1\cdots b_s}\tilde{\nabla}_c\tilde{T}^{c_1\cdots c_r}_{d_1\cdots d_s}, \\ {}^{(s+1)}D_a\tilde{T}^{a_1\cdots a_r}_{b_1\cdots b_s} &= {}^{(s+1)}g^{ca_1\cdots a_r d_1\cdots d_s}_{ac_1\cdots c_r b_1\cdots b_s}\tilde{\nabla}_c\tilde{T}^{c_1\cdots c_r}_{d_1\cdots d_s}. \end{aligned} \tag{10}$$

When $\tilde{T}^{a_1\cdots a_r}_{b_1\cdots b_s}$ coincides with its projection to $\Sigma_{t\chi}$, S_t or $\mathcal{M}_\chi$, the expressions (10) are the covariant derivatives in $\Sigma_{t\chi}$, S_t or $\mathcal{M}_\chi$, respectively (they annihilate the corresponding metrics). Despite the notation, projections of derivatives of tensors not lying on $\Sigma_{t\chi}$, S_t and $\mathcal{M}_\chi$ are not derivatives, as they fail to obey the Leibniz rule. For example: $D_a(Ml^b) = MD_a l^b$.

As there are two normals to $\Sigma_{t\chi}$, we can define two kinds of extrinsic curvatures:

$$K_{ab} = D_a n_b = g^{cd}_{ab}\tilde{\nabla}_c n_d, \qquad L_{ab} = D_a l_b = g^{cd}_{ab}\tilde{\nabla}_c l_d. \tag{11}$$

We denote their traces by K and L. It is immediate to see the symmetry of these extrinsic curvatures by noting that $n_b = -N\tilde{\nabla}_b t$ and $l_b = M\tilde{\nabla}_b\chi$ [see Eqs. (A6)].

It is also useful to introduce the extrinsic curvatures of S_t:

$$\hat{K}_{ab} = \hat{D}_a n_b = \hat{g}^{cd}_{ab}\tilde{\nabla}_c n_d = \tilde{\nabla}_a n_b + n_a n^c\tilde{\nabla}_c n_b. \tag{12}$$

As n^a and l^a are hypersurface orthogonal, all extrinsic curvatures defined above are symmetric.

An $(s+1)+1$ decomposition would be the generalization to the arbitrary dimension of the ADM decomposition of gravity, in which the extrinsic curvature $\hat{K}^{ab}$ takes the role of the canonical momenta associated with $\hat{g}_{ab}$. Instead we would like a formalism which singles out both directions n and l. By identifying $\hat{g}_{ab} \equiv \{g_{ab}, M^a, M, \mathcal{N}, N^a\}$ [see Eqs. (6), (A1), and (A6)], it is immediate to choose the set $\{g_{ab}, M^a, M\}$ as canonical coordinates on S_t and to search for the canonical momenta among the various projections of $\hat{K}_{ab}$:

$$\hat{K}_{ab} = K_{ab} + 2l_{(a}\mathcal{K}_{b)} + l_a l_b\mathcal{K}. \tag{13}$$

The quantities

$$\mathcal{K}_a = {}^{(s+1)}g^c_a l^d\hat{K}_{cd}, \qquad \mathcal{K} = l^c l^d\hat{K}_{cd} \tag{14}$$

represent off-brane projections of $\hat{K}_{ab}$. $\mathcal{K}^i$ is the normal fundamental form, introduced in [13] and we call $\mathcal{K}$ the normal fundamental scalar. The extrinsic curvature K_{ab} defined in (11) can be shown to be the projection of $\hat{K}_{ab}$ to $\Sigma_{t\chi}$:

$$K_{ab} = {}^{(s+1)}g^{cd}_{ab}\hat{K}_{cd} \tag{15}$$

from the $g^c_a = {}^{(s+1)}g^c_d\hat{g}^d_a$ property of the projectors. We remark that $l^a K_{ab} = 0$ and $l^a\mathcal{K}_a = 0$ hold, thus K_{ab} and $\mathcal{K}_a$ are tensors defined on $\Sigma_{t\chi}$.

In terms of the $(s+2)$-connection $\tilde{\nabla}$, the projections $\mathcal{K}_a$ and $\mathcal{K}$ are expressed as

$$\begin{aligned} \mathcal{K}_a &= g^b_a l^c\tilde{\nabla}_b n_c = g^b_a l^c\tilde{\nabla}_c n_b, \\ \mathcal{K} &= l^a l^b\tilde{\nabla}_a n_b = l^a\tilde{\mathcal{L}}_{\mathbf{n}} l_a. \end{aligned} \tag{16}$$

The second expression for $\mathcal{K}_a$ follows from the symmetry of $\hat{K}_{ab}$ in Eq. (12).

Alternatively, keeping in mind the bulk-brane scenario, where a decomposition with respect to the normal l^a is frequently desirable [6], we introduce the extrinsic curvature of the space-time leave $\mathcal{M}_\chi$:

$${}^{(s+1)}L_{ab} = {}^{(s+1)}D_a l_b = {}^{(s+1)}g^{cd}_{ab}\tilde{\nabla}_c l_d = \tilde{\nabla}_a l_b - l_a l^c\tilde{\nabla}_c l_b, \tag{17}$$

and we decompose it with respect to the $\Sigma_{t\chi}$ foliation as

$${}^{(s+1)}L_{ab} = L_{ab} + 2n_{(a}\mathcal{L}_{b)} + n_a n_b\mathcal{L}. \tag{18}$$

Here the quantities

$$\mathcal{L}_a = -\hat{g}^b_a n^{c(s+1)}L_{bc}, \qquad \mathcal{L} = n^a n^{b(s+1)}L_{ab} \tag{19}$$

represent timelike projections of ${}^{(s+1)}L_{ab}$ and the previously introduced extrinsic curvature L_{ab} is nothing but the projection of ${}^{(s+1)}L_{ab}$ to $\Sigma_{t\chi}$:

$$L_{ab} = \hat{g}^{cd(s+1)}_{ab}L_{cd}. \tag{20}$$

As $n^a L_{ab} = 0$ and $n^a\mathcal{L}_a = 0$ indicate, both L_{ab} and $\mathcal{L}_a$ are tensors defined on $\Sigma_{t\chi}$.

The projections $\mathcal{L}_a$ and $\mathcal{L}$ can be equally expressed as

$$\mathcal{L}_a = -g_a^b n^c \tilde{\nabla}_b l_c = -g_a^b n^c \tilde{\nabla}_c l_b = \mathcal{K}_a, \quad \mathcal{L} = n^a n^b \tilde{\nabla}_a l_b = n^a \tilde{\mathcal{L}}_{\mathbf{l}} n_a. \tag{21}$$

Thus $\mathcal{L}_a$ will be replaced in all forthcoming computations by $\mathcal{K}_a$.

We have just completed the task to characterize both the extrinsic curvatures of S_t and of $\mathcal{M}_\chi$ in terms of quantities defined on $\Sigma_{t\chi}$ alone.

Finally we define the curvatures α^a and λ^a of the congruences n^a and l^a

$$\alpha^b = n^c \tilde{\nabla}_c n^b = g_d^b n^c \tilde{\nabla}_c n^d - \mathcal{L} l^b, \tag{22}$$

$$\lambda^b = l^c \tilde{\nabla}_c l^b = g_d^b l^c \tilde{\nabla}_c l^d + \mathcal{K} n^b. \tag{23}$$

The second pair of expressions for each of the above curvatures are nothing but their $s+1+1$ decomposition. Physically, the curvature α^a is the nongravitational acceleration of observers with velocity n^a. Detailed expressions for these curvatures are deduced in Appendix C.

Let us note that the sets of Eqs. (12) and (13) and (17) and (18) allow for the following decompositions of the covariant derivatives of the normals:

$$\tilde{\nabla}_a n_b = K_{ab} + 2\mathcal{K}_{(a} l_{b)} + l_a l_b \mathcal{K} - n_a \alpha_b, \quad \tilde{\nabla}_a l_b = L_{ab} + 2\mathcal{K}_{(a} n_{b)} + n_a n_b \mathcal{L} + l_a \lambda_b. \tag{24}$$

From here we find simple expressions for their covariant divergences:

$$\tilde{\nabla}_a n^a = \hat{K} = K + \mathcal{K}, \qquad \tilde{\nabla}_a l^a = \hat{L} = L - \mathcal{L}. \tag{25}$$

We also derive the useful relation

$$\tilde{\nabla}_a n^b \tilde{\nabla}_b n^a = \hat{K}_{ab}\hat{K}^{ab} = K_{ab}K^{ab} + 2\mathcal{K}_a\mathcal{K}^a + \mathcal{K}^2. \tag{26}$$

III. TIME DERIVATIVES OF g_{ab}, M^a AND M

In this section we establish the relations among the time derivatives of the dynamical data $\{g_{ab}, M^a, M\}$ and extrinsic curvatures $\{K_{ab}, \mathcal{K}_a, \mathcal{K}\}$ and $\{L_{ab}, \mathcal{L}_a, \mathcal{L}\}$.

We proceed as follows. First we define suitable derivatives of tensors from S_t and $\mathcal{M}_\chi$ as the respective projections of the Lie derivative in $\mathcal{B}$ along an arbitrary vector flow V^a:

$$\hat{\mathcal{L}}_{\mathbf{V}} \hat{T}^{a_1\cdots a_r}_{b_1\cdots b_s} = \hat{g}^{a_1\cdots a_r d_1\cdots d_s}_{c_1\cdots c_r b_1\cdots b_s} \tilde{\mathcal{L}}_{\mathbf{V}} \hat{T}^{c_1\cdots c_r}_{d_1\cdots d_s}, \tag{27}$$

$$^{(s+1)}\mathcal{L}_{\mathbf{V}} {}^{(s+1)}T^{a_1\cdots a_r}_{b_1\cdots b_s} = {}^{(s+1)}g^{a_1\cdots a_r d_1\cdots d_s}_{c_1\cdots c_r b_1\cdots b_s} \tilde{\mathcal{L}}_{\mathbf{V}} {}^{(s+1)}T^{c_1\cdots c_r}_{d_1\cdots d_s}. \tag{28}$$

When V^a belongs to S_t ($\mathcal{M}_\chi$), the derivative $\hat{\mathcal{L}}_{\mathbf{V}}$ ($^{(s+1)}\mathcal{L}_{\mathbf{V}}$) is the Lie derivative in S_t ($\mathcal{M}_\chi$).

When $\mathbf{V}$ is $\partial/\partial t$ or $\partial/\partial\chi$, the derivatives decouple as

$$\hat{\mathcal{L}}_{\partial/\partial\mathbf{t}} \hat{T}^{a_1\cdots a_r}_{b_1\cdots b_s} = N\hat{\mathcal{L}}_{\mathbf{n}} \hat{T}^{a_1\cdots a_r}_{b_1\cdots b_s} + \hat{\mathcal{L}}_{\mathbf{N}} \hat{T}^{a_1\cdots a_r}_{b_1\cdots b_s}, \tag{29}$$

$$^{(s+1)}\mathcal{L}_{\partial/\partial\chi} {}^{(s+1)}T^{a_1\cdots a_r}_{b_1\cdots b_s} = M {}^{(s+1)}\mathcal{L}_{\mathbf{l}} {}^{(s+1)}T^{a_1\cdots a_r}_{b_1\cdots b_s} + {}^{(s+1)}\mathcal{L}_{\mathbf{M}} {}^{(s+1)}T^{a_1\cdots a_r}_{b_1\cdots b_s}. \tag{30}$$

In particular, for the metrics induced on S_t and $\mathcal{M}_\chi$ we find

$$\hat{\mathcal{L}}_{\partial/\partial\mathbf{t}} \hat{g}_{ab} = 2N\hat{K}_{ab} + 2\hat{D}_{(a}N_{b)}, \tag{31}$$

$$^{(s+1)}\mathcal{L}_{\partial/\partial\mathbf{t}} {}^{(s+1)}g_{ab} = 2M {}^{(s+1)}L_{ab} + 2 {}^{(s+1)}D_{(a}M_{b)}. \tag{32}$$

We have used that the extrinsic curvatures (12) and (17) are expressible as

$$\hat{K}_{ab} = \frac{1}{2}\hat{\mathcal{L}}_{\mathbf{n}}\hat{g}_{ab} = \frac{1}{2}\tilde{\mathcal{L}}_{\mathbf{n}}\hat{g}_{ab}, \quad {}^{(s+1)}L_{ab} = \frac{1}{2} {}^{(s+1)}\mathcal{L}_{\mathbf{l}} {}^{(s+1)}g_{ab} = \frac{1}{2}\tilde{\mathcal{L}}_{\mathbf{l}} {}^{(s+1)}g_{ab}. \tag{33}$$

Next we define a projected derivative of a tensor taken from $\Sigma_{t\chi}$,[1]

$$\mathcal{L}_{\mathbf{V}} T^{a_1\cdots a_r}_{b_1\cdots b_s} = g^{a_1\cdots a_r d_1\cdots d_s}_{c_1\cdots c_r b_1\cdots b_s} \tilde{\mathcal{L}}_{\mathbf{V}} T^{c_1\cdots c_r}_{d_1\cdots d_s}, \tag{34}$$

in terms of which time and off-brane derivatives can be defined as the projections to $\Sigma_{t,\chi}$ of the Lie derivatives $\tilde{\mathcal{L}}$ taken along $\partial/\partial t$ and $\partial/\partial\chi$ directions, respectively:

$$\frac{\partial}{\partial t} T^{a_1\ldots a_r}_{b_1\ldots b_s} = \mathcal{L}_{\partial/\partial\mathbf{t}} T^{a_1\cdots a_r}_{b_1\cdots b_s} = N\mathcal{L}_{\mathbf{n}} T^{a_1\cdots a_r}_{b_1\cdots b_s} + \mathcal{L}_{\mathbf{N}} T^{a_1\cdots a_r}_{b_1\cdots b_s}, \tag{35}$$

$$\frac{\partial}{\partial\chi} T^{a_1\cdots a_r}_{b_1\cdots b_s} = \mathcal{L}_{\partial/\partial\chi} T^{a_1\cdots a_r}_{b_1\cdots b_s} = M\mathcal{L}_{\mathbf{l}} T^{a_1\cdots a_r}_{b_1\cdots b_s} + \mathcal{L}_{\mathbf{M}} T^{a_1\cdots a_r}_{b_1\cdots b_s}. \tag{36}$$

It is not difficult to show that despite being *projected* Lie derivatives, they become partial derivatives in any adapted coordinate system. Thus for any $T^{a_1\cdots a_r}_{b_1\cdots b_s}$ defined on $\Sigma_{t,\chi}$ the property

$$\frac{\partial}{\partial t}\frac{\partial}{\partial\chi} T^{a_1\cdots a_r}_{b_1\cdots b_s} = \frac{\partial}{\partial\chi}\frac{\partial}{\partial t} T^{a_1\cdots a_r}_{b_1\cdots b_s} \tag{37}$$

holds. For the s metric the formulas (35) and (36) reduce to

$$\frac{\partial}{\partial t} g_{ab} = 2NK_{ab} + 2D_{(a}N_{b)}, \tag{38}$$

[1] The expressions (27), (28), and (34), introduced for tensors projected to S_t, $\mathcal{M}_\chi$ or $\Sigma_{t\chi}$, when generalized to arbitrary tensors on $\mathcal{B}$ could fail to obey the Leibniz rule. For example: $\mathcal{L}_{\mathbf{V}}(Ml^b) = M\mathcal{L}_{\mathbf{V}} l^b$.

$$\frac{\partial}{\partial\chi}g_{ab} = 2ML_{ab} + 2D_{(a}M_{b)}, \tag{39}$$

where we have used that the extrinsic curvatures of $\Sigma_{t\chi}$ can be expressed as

$$K_{ab} = \frac{1}{2}\mathcal{L}_{\mathbf{n}}g_{ab}, \qquad L_{ab} = \frac{1}{2}\mathcal{L}_{l}g_{ab}. \tag{40}$$

Equations (38) and (39), when inverted with respect to the extrinsic curvatures, read

$$K_{ab} = \frac{1}{2N}\left(\frac{\partial}{\partial t}g_{ab} - 2D_{(a}N_{b)}\right), \tag{41}$$

$$L_{ab} = \frac{1}{2M}\left(\frac{\partial}{\partial\chi}g_{ab} - 2D_{(a}M_{b)}\right). \tag{42}$$

The comparison of the right-hand sides from Eqs. (22) and (C3a), also from (23) and (C3b) leads to simple expressions for the curvatures α^b and λ^b:

$$\alpha^b = D^b(\ln N) - \mathcal{L}l^b, \tag{43}$$

$$\lambda^b = -D^b(\ln M) + \mathcal{K}n^b, \tag{44}$$

with $\mathcal{L}$ and $\mathcal{K}$ given by

$$\mathcal{L} = -\frac{1}{MN}\left(\frac{\partial}{\partial\chi}N - M^aD_aN\right), \tag{45}$$

$$\mathcal{K} = \frac{1}{MN}\left(\frac{\partial}{\partial t}M - N^aD_aM\right). \tag{46}$$

Equivalently,

$$\mathcal{K} = \frac{1}{M}\mathcal{L}_{\mathbf{n}}M, \qquad \mathcal{L} = -\frac{1}{M}\mathcal{L}_{l}N. \tag{47}$$

Finally the difference of the contractions of Eqs. (24) with l^a and n^a respectively leads to $2\mathcal{K}^a = ([l,n]^a - \mathcal{K}l^a - \mathcal{L}n^a)$. Inserting the commutator (C1c) and keeping in mind that $\partial M^a/\partial t = (F_{\mathbf{i}})^a\partial M^{\mathbf{i}}/\partial t$ and $\partial N^a/\partial\chi = (F_{\mathbf{i}})^a\partial N^{\mathbf{i}}/\partial\chi$ hold in the chosen coordinate basis $\{e_\beta\}$, we get

$$\mathcal{K}^a = \frac{1}{2MN}\left[\frac{\partial}{\partial t}M^a - \frac{\partial}{\partial\chi}N^a + M^bD_bN^a - N^bD_bM^a\right]. \tag{48}$$

In terms of Lie derivatives this gives

$$\mathcal{K}^a = \frac{1}{2M}\mathcal{L}_{\mathbf{n}}M^a - \frac{1}{2N}\mathcal{L}_{l}N^a - \frac{1}{2NM}(M^cK^a_c - N^cL^a_c). \tag{49}$$

Summarizing, Eqs. (41), (46), and (48) express the projections of the extrinsic curvature $\hat{K}_{ab}$ in terms of the time derivatives $\partial g_{ab}/\partial t$, $\partial M^a/\partial t$, $\partial M/\partial t$ and other quantities from $\Sigma_{t\chi}$. Concerning the extrinsic curvature of $\mathcal{M}_\chi$, Eqs. (42) and (45) convince us that L_{ab} and $\mathcal{L}$ are those projections of ${}^{(s+1)}L_{ab}$, which have nothing to do with time evolution, while the projection $\mathcal{L}^a = \mathcal{K}^a$ is dynamical.

IV. THE LANCZOS EQUATION

Whenever the hypersurface $\mathcal{M}$ has a distributional energy momentum τ_{ab}, the extrinsic curvature ${}^{(s+1)}L_{ab}$ will have a jump

$$\Delta^{(s+1)}L_{ab} = -\tilde{\kappa}^2\left(\tau_{ab} - \frac{1}{s}{}^{(s+1)}g_{ab}\tau\right). \tag{50}$$

This is particularly interesting in the braneworld scenario. Then τ_{ab} is composed of the energy-momentum tensor T_{ab} of standard-model fields and of a λ brane tension term:

$$\tau_{ab} = -\lambda^{(s+1)}g_{ab} + T_{ab}. \tag{51}$$

We allow for a completely generic brane, with energy density ρ, homogeneous pressure p, tensor of anisotropic pressures Π_{ab} and energy transport Q_a:

$$T_{ab} = \rho n_an_b + pg_{ab} + \Pi_{ab} + 2n_{(a}Q_{b)}. \tag{52}$$

We would like to write the Lanczos equation (50) in terms of the new variables introduced in the previous sections. For this we employ the decomposition (18) of the extrinsic curvature of $\mathcal{M}$. We find the following projections of the Lanczos equation:

$$\Delta L_{ab} = -\tilde{\kappa}^2\left[pg_{ab} + \Pi_{ab} + \frac{1}{s}g_{ab}(\lambda - 3p + \rho)\right], \tag{53}$$

$$\Delta\mathcal{K}_a = -\tilde{\kappa}^2Q_a, \tag{54}$$

$$s\Delta\mathcal{L} = -\tilde{\kappa}^2[-\lambda + 3p + (s-1)\rho]. \tag{55}$$

We see that among the gravitational degrees of freedom only $\mathcal{K}_a$ will have a jump across the brane due to the distributional energy-momentum tensor. This occurs only when there is energy transport on the brane. Thus in all cosmological models with perfect fluid on the brane none of the dynamical variables characterizing gravity will have a jump across the brane.

V. INTRINSIC CURVATURES

The Riemann tensor, expression of the intrinsic curvature of the $\Sigma_{t\chi}$ hypersurfaces arises from the noncommutativity of the covariant derivative D:

$$R^a{}_{bcd}v^b = (D_cD_d - D_dD_c)v^a. \tag{56}$$

Here $v^a \in T\Sigma_{t\chi}$ is arbitrary. The Riemann tensor of $\mathcal{B}$ is defined in an analogous manner:

$$\tilde{R}^a{}_{bcd}\tilde{v}_b = (\tilde{\nabla}_c\tilde{\nabla}_d - \tilde{\nabla}_d\tilde{\nabla}_c)\tilde{v}^a, \tag{57}$$

for any $\tilde{v}^a \in T\mathcal{B}$. Straightforward computation leads to the Gauss equation:

$$R^a{}_{bcd} = g^a_i g^j_b g^k_c g^l_d \tilde{R}^i{}_{jkl} - K^a{}_c K_{bd} + K^a{}_d K_{bc} + L^a{}_c L_{bd} - L^a{}_d L_{bc}. \quad (58)$$

This standard result holds for any s-dimensional hypersurface embedded in an $(s+2)$-dimensional space [14].

All other independent projections of $\tilde{R}^a{}_{bcd}$ are enlisted in Appendix C.

By contracting the Gauss equation twice, the relation between the scalar curvatures emerges

$$R = \tilde{R} + 2\tilde{R}_{ab}(n^a n^b - l^a l^b) - 2\tilde{R}_{abcd} n^a l^b n^c l^d - K^2 + K^{ab}K_{ab} + L^2 - L^{ab}L_{ab}. \quad (59)$$

Employing

$$n^a n^b \tilde{R}_{ab} = \tilde{\nabla}_a(\alpha^a - n^a \tilde{\nabla}_b n^b) + K^2 - K_{ab}K^{ab} - 2\mathcal{K}_a\mathcal{K}^a + 2K\mathcal{K}, \quad (60a)$$

$$l^a l^b \tilde{R}_{ab} = \tilde{\nabla}_a(\lambda^a - l^a \tilde{\nabla}_b l^b) + L^2 - L_{ab}L^{ab} + 2\mathcal{K}_a\mathcal{K}^a - 2L\mathcal{L}, \quad (60b)$$

$$n^a l^b n^c l^d \tilde{R}_{abcd} = \mathcal{L}^2 - \mathcal{K}^2 - 3\mathcal{K}_a\mathcal{K}^a - l^a\tilde{\nabla}_a\mathcal{L} - n^a\tilde{\nabla}_a\mathcal{K} - \alpha^b\lambda_b, \quad (60c)$$

we find the relation between the s- and $(s+2)$-dimensional curvature scalars:

$$R = \tilde{R} + K^2 - K_{ab}K^{ab} - 2\mathcal{K}_a\mathcal{K}^a + 2\mathcal{K}K - L^2 + L_{ab}L^{ab} + 2\mathcal{L}L + 2\alpha^b\lambda_b + 2\tilde{\nabla}_a(\alpha^a - \lambda^a - Kn^a + Ll^a). \quad (61)$$

Appendix D contains an independent derivation of Eq. (61). Keeping in mind $\mathcal{K}_a = \mathcal{L}_a$, this equation shows a perfect symmetry between quantities related to l^a and n^a. The expressions of the $(s+1+1)$-decomposed Riemann, Ricci and Einstein tensors in terms of tensors on $\Sigma_{t\chi}$ are also given in Appendix D, as well as the relation between the scalar curvatures in a form without total divergences.

VI. TIME DERIVATIVES OF EXTRINSIC CURVATURES

The energy-momentum tensor $\tilde{T}_{ab}$ of the bulk receives two types of contributions: $\tilde{\Pi}_{ab}$ from the nonstandard-model fields in the bulk and τ_{ab} from standard-model fields on the brane. The latter is a distributional source located on the brane, at $\chi = 0$ [or in terms of generic coordinates $\tilde{x}^a$ given covariantly as $\chi(\{\tilde{x}^a\}) = 0$]. In the special case when the bulk contains only a cosmological constant $\tilde{\Lambda}$, but with a generic source on the brane, from Eqs. (51) and (52) we obtain

$$\tilde{T}_{ab} = -\tilde{\kappa}^2\tilde{\Lambda} l_a l_b + [(\rho+\lambda)\delta(\chi) + \tilde{\kappa}^2\tilde{\Lambda}]n_a n_b + [(p-\lambda)\delta(\chi) - \tilde{\kappa}^2\tilde{\Lambda}]g_{ab} + [\Pi_{ab} + 2n_{(a}Q_{b)}]\delta(\chi), \quad (62)$$

and

$$\tilde{T} = -5\tilde{\kappa}^2\tilde{\Lambda} - (\rho - 3p + 4\lambda)\delta(\chi). \quad (63)$$

By employing the bulk Einstein equation

$$\tilde{R}_{ab} = \tilde{T}_{ab} - \frac{\tilde{T}}{s}\tilde{g}_{ab}, \quad (64)$$

we obtain the following projections of the bulk Ricci tensor:

$$g^c_a g^d_b \tilde{R}_{cd} = \frac{5-s}{s}\tilde{\kappa}^2\tilde{\Lambda} + \left[\frac{\rho + (s-3)p + (4-s)\lambda}{s} g_{ab} + \Pi_{ab}\right]\delta(\chi), \quad (65a)$$

$$g^c_a l^d \tilde{R}_{cd} = 0, \quad (65b)$$

$$l^a l^b \tilde{R}_{ab} = \frac{5-s}{s}\tilde{\kappa}^2\tilde{\Lambda} + \frac{\rho - 3p + 4\lambda}{s}\delta(\chi). \quad (65c)$$

Employing these in Eqs. (D2a), (D2c), and (D2f) of Appendix D and transforming the Lie derivatives $\mathcal{L}_{\mathbf{n}}K_{ab}$, $\mathcal{L}_{\mathbf{n}}\mathcal{K}_a$ and $\mathcal{L}_{\mathbf{n}}\mathcal{K}$ to time derivatives by Eq. (35), the time evolution of K_{ab}, $\mathcal{K}_a$ and $\mathcal{K}$ can be readily deduced:

$$\frac{\partial}{\partial t}K_{ab} = N\left[g^c_a g^d_b \tilde{R}_{cd} - R_{ab} + L_{ab}(L - \mathcal{L}) - 2L_{ac}L^c_b - K_{ab}(K + \mathcal{K}) + 2K_{ac}K^c_b + 2\mathcal{K}_a\mathcal{K}_b + \mathcal{L}_{\mathbf{l}}L_{ab} + \frac{D_b D_a M}{M}\right] + D_b D_a N + \mathcal{L}_{\mathbf{N}}K_{ab}, \quad (66a)$$

$$\frac{\partial}{\partial t}\mathcal{K}_a = N[-D^b L_{ab} + D_a(L - \mathcal{L}) - K\mathcal{K}_a] + \mathcal{L}_{\mathbf{N}}\mathcal{K}_a - (L^b_a + \mathcal{L}\delta^b_a)D_b N, \quad (66b)$$

$$\frac{\partial}{\partial t}\mathcal{K} = N\left[l^a l^b \tilde{R}_{ab} - L_{ab}L^{ab} + \mathcal{L}^2 + \frac{D_a D^a M}{M} - 2\mathcal{K}_a\mathcal{K}^a - \mathcal{K}(K + \mathcal{K}) + \mathcal{L}_{\mathbf{l}}(L - \mathcal{L})\right] + \frac{D^a M}{M}D_a N + \mathcal{L}_{\mathbf{N}}\mathcal{K}. \quad (66c)$$

The corresponding expressions containing χ derivatives [which are similar to those of the time-evolution equations (41), (46), and (48) of the complementary set of dynamical data] are

$$\frac{\partial}{\partial t}K_{ab} = N\bigg[g_a^c g_b^d \tilde{R}_{cd} - R_{ab} + L_{ab}(L - \mathcal{L}) - 2L_{ac}L_b^c - K_{ab}(K + \mathcal{K}) + 2K_{ac}K_b^c + 2\mathcal{K}_a\mathcal{K}_b + \frac{D_b D_a M}{M} + \frac{1}{M}\bigg(\frac{\partial}{\partial\chi}L_{ab} - M^c D_c L_{ab} - 2L_{c(a}D_{b)}M^c\bigg)\bigg] + D_b D_a N + N^c D_c K_{ab} + 2K_{c(a}D_{b)}N^c, \tag{67a}$$

$$\frac{\partial}{\partial t}\mathcal{K}_a = N[-D^b L_{ab} + D_a(L - \mathcal{L}) - K\mathcal{K}_a] + N^b D_b\mathcal{K}^a - (L_a^b + \mathcal{L}\delta_a^b)D_b N + \mathcal{K}_b D_a N^b, \tag{67b}$$

$$\frac{\partial}{\partial t}\mathcal{K} = N\bigg\{l^a l^b \tilde{R}_{ab} - L_{ab}L^{ab} + \mathcal{L}^2 + \frac{D_a D^a M}{M} - 2\mathcal{K}_a\mathcal{K}^a - \mathcal{K}(K + \mathcal{K}) + \frac{1}{M}\bigg[\frac{\partial}{\partial\chi}(L - \mathcal{L}) - M^a D_a(L - \mathcal{L})\bigg]\bigg\} + \frac{D^a M}{M}D_a N + N^a D_a\mathcal{K}. \tag{67c}$$

We have employed Eq. (36) in their derivation. Note that in the above formulas L_{ab} and $\mathcal{L}$ are also given in terms of dynamical data, cf. Eqs. (42) and (45).

VII. CONCLUDING REMARKS

We have developed a decomposition scheme of the $(s+2)$-dimensional space-time based on two perpendicular foliations of constant time and constant χ surfaces. In a braneworld scenario the latter contains the brane at $\chi = 0$. A careful geometrical interpretation has allowed for identifying dynamical quantities with geometrical expressions. From among the various projections to $\Sigma_{t\chi}$ of the extrinsic curvature pertinent to the off-brane normal solely $\mathcal{L}^a$ was found to be dynamical, together with *all* components (K_{ab}, $\mathcal{K}^a = \mathcal{L}^a$ and $\mathcal{K}$) of the extrinsic curvature related to n^a. Their expression was given in terms of time derivatives of the metric g_{ab} induced on $\Sigma_{t\chi}$, shift vector components M^a and lapse M of $\partial/\partial\chi$. Time evolution of the second fundamental form K_{ab}, of the normal fundamental form $\mathcal{K}^i$ and normal fundamental scalar $\mathcal{K}$ were also derived for a generic brane. The Lanczos equation was written in terms of the same set of variables.

Our formalism applied for $s = 2$ is different from previous $2 + 1 + 1$ decompositions in general relativity, developed to deal with stationary and axisymmetric space-times. There the temporal and spatial directions singled out are a stationary and a rotational Killing vector. By contrast, we are interested in *evolutions* along the singled-out time-like and off-brane directions. The formalism developed in [15,16] relies on the use of a factor space with respect to the rotational Killing vector. The induced metric is then defined with this Killing vector and the formalism becomes rather complicated. In a more recent approach [17], Gourgoulhon and Bonazzola introduce the induced metrics by using normal vector fields to the 2-space (like we do), hence they avoid the use of twist-related quantities. However their treatment relies on first decomposing space-time with respect to the temporal direction, next with respect to the spatial direction, and this procedure unfortunately lets no counterpart to the brane extrinsic curvature ${}^{(s+1)}L_{ab}$ in their formalism. Thus their formalism is not suited for a generalization to braneworld scenarios, where the Lanczos equation is given precisely in terms of ${}^{(s+1)}L_{ab}$. For convenience we give a comparative table of the quantities appearing in the approaches of [17] and ours (Table I).

The system of equations giving the evolution of (g_{ab}, M^a, M) and $(K_{ab}, \mathcal{K}^a, \mathcal{K})$ represents the gravitational dynamics in terms of variables adapted to the brane. On top of these there are constraints on their initial values to be satisfied. These are the Hamiltonian and diffeomorphism constraints. Their derivation from a variational principle in

TABLE I. Comparison of our notations and quantities employed with those of Ref. [17]. A triple dot denotes the absence of the respective quantity from the formalism. We note that the correspondences $l^a \leftrightarrow m^\alpha$ and $L_{ab} \leftrightarrow L_{\alpha\beta}$ hold only because we have chosen the two foliations perpendicular to each other.

	Our formalism	Gourgoulhon and Bonazzola
Manifolds	$(\mathcal{B}, \mathcal{M}_\chi, S_t, \Sigma_{t\chi})$	$(\mathcal{E}, \ldots, \Sigma_t, \Sigma_{t\phi})$
Metrics	$(\tilde{g}, {}^{(s+1)}g, \hat{g}, g)$	$(g, \ldots, h, k)$
Metric-compatible covariant derivatives	$(\tilde{\nabla}, {}^{(s+1)}D, \hat{D}, D)$	$(;, \ldots, {}_{\vert}, {}_{\Vert})$
Coordinates	(t, χ)	(t, φ)
Singled-out vectors	$(\frac{\partial}{\partial t}, \frac{\partial}{\partial\chi})$	$(\frac{\partial}{\partial t}, \frac{\partial}{\partial\varphi})$
Normals	(n^a, l^a)	(n^α, m^α)
Accelerations	$(\alpha^a, \hat{g}_b^a\lambda^b)$	(a^α, b^α)
Shifts	$(N^a, \mathcal{N}), M^a$	$-N^\alpha = -(q^\alpha, \omega), -M^\alpha$
Extrinsic curvatures	$({}^{(s+1)}L_{ab}, \hat{K}_{ab})$	$(\ldots, -K_{\alpha\beta})$
Extrinsic curvature projections	$(K_{ab}, L_{ab}, \mathcal{K}_a = \mathcal{L}_a, \mathcal{K}, \mathcal{L})$	$(\ldots, -L_{\alpha\beta}, \ldots, \ldots, \ldots)$

terms of our brane-adapted variables, keeping in mind that some of the dynamics is frozen due to the existence of the brane as a hypersurface, is in progress and will be published in a forthcoming paper.

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APPENDIX A: THE TWO FOLIATIONS

The $t =$ const hypersurfaces S_t are defined by the one-form field $\overline{n} = Ndt$. Similarly, $\mathcal{M}_\chi$ is the hypersurfaces $\chi =$ const, defined by $\overline{l} = M'd\chi$. We introduce the one-form field $\overline{m}$ such that the metric in $\mathcal{B}$ can be written as

$$\tilde{g} = g_{\mathbf{ij}} F^{\mathbf{i}} \otimes F^{\mathbf{j}} + \overline{m} \otimes \overline{m} - \overline{n} \otimes \overline{n}. \tag{A1}$$

Here the co-basis $\{f^\alpha\} = \{\overline{n}, F^{\mathbf{i}}, \overline{m}\}$ has the dual basis $\{f_\beta\} = \{n, F_{\mathbf{j}}, m\}$, where $\{F_{\mathbf{j}}\}$ is some basis in $T\Sigma_{t\chi}$. Thus $n_a = -\overline{n}_a$ and $m_a = \overline{m}_a$. The normal vector l^a to $\mathcal{M}_\chi$ can be conveniently parametrized as

$$l^a = n^a \sinh\gamma + m^a \cosh\gamma. \tag{A2}$$

This parametrization assures that l^a has unit norm with respect to the metric (A1). The dual form to l^a is

$$\overline{l} = \tilde{g}(.,l) = -\overline{n}\sinh\gamma + \overline{m}\cosh\gamma. \tag{A3}$$

We introduce the second basis $\{e_\alpha\} = \{\partial/\partial t, E_{\mathbf{i}} = F_{\mathbf{i}}, \partial/\partial\chi\}$ and its dual co-basis $\{e^\beta\} = \{dt, E^{\mathbf{i}}, d\chi\}$. The two sets of bases are related as

$$\begin{aligned} \overline{n} &= Ndt, \qquad F^{\mathbf{i}} = A^{\mathbf{i}}_\beta e^\beta, \\ \overline{m} &= \frac{M'd\chi + N\sinh\gamma dt}{\cosh\gamma}, \end{aligned} \tag{A4}$$

and

$$\frac{\partial}{\partial t} = N^\beta f_\beta, \qquad \frac{\partial}{\partial\chi} = M^\beta f_\beta, \qquad E_{\mathbf{i}} = F_{\mathbf{i}}. \tag{A5}$$

The unknown coefficients $A^{\mathbf{i}}_\beta$, N^β and M^β are constrained by the duality relations $\langle e^\alpha, e_\beta\rangle = \delta^\alpha_\beta = \langle f^\alpha, f_\beta\rangle$. After some algebra we find the relation between the two co-bases

$$\begin{aligned} \overline{n} &= Ndt, \qquad \overline{m} = \mathcal{N}dt + Md\chi, \\ F^{\mathbf{i}} &= N^{\mathbf{i}}dt + E^{\mathbf{i}} + M^{\mathbf{i}}d\chi, \end{aligned} \tag{A6}$$

and the relation between the bases

$$\frac{\partial}{\partial t} = Nn + N^{\mathbf{i}}F_{\mathbf{i}} + \mathcal{N}m, \tag{A7a}$$

$$\frac{\partial}{\partial\chi} = M^{\mathbf{i}}F_{\mathbf{i}} + Mm, \tag{A7b}$$

$$E_{\mathbf{i}} = F_{\mathbf{i}}. \tag{A7c}$$

We have introduced the shorthand notations

$$\mathcal{N} = N\tanh\gamma, \qquad M = \frac{M'}{\cosh\gamma}. \tag{A8}$$

Equation (A7a) shows that the time-evolution vector $\partial/\partial t$ is decomposed as in the ADM formalism of general relativity: N is the lapse function and $(N^{\mathbf{i}}, \mathcal{N})$ are the 4D shift vector components in the chosen basis. The functions $(M^{\mathbf{i}}, M)$ represent the arbitrary tangential and normal contributions to the off-brane evolution vector $\partial/\partial\chi$ with respect to $\Sigma_{t\chi}$. Equation (A7b) shows that there is no n term in $\partial/\partial\chi$, thus off-brane evolution in the coordinate χ happens in S_t, thus at constant time.

For convenience we also give the inverted relations among the bases:

$$\begin{aligned} dt &= \frac{\overline{n}}{N}, \qquad d\chi = \frac{1}{M}\left(-\frac{\mathcal{N}}{N}\overline{n} + \overline{m}\right), \\ E^{\mathbf{j}} &= \frac{1}{N}\left(\frac{\mathcal{N}}{M}M^{\mathbf{j}} - N^{\mathbf{j}}\right)\overline{n} + F^{\mathbf{j}} - \frac{M^{\mathbf{j}}}{M}\overline{m}, \end{aligned} \tag{A9}$$

and

$$\begin{aligned} n &= \frac{1}{N}\left[\frac{\partial}{\partial t} + \left(\frac{\mathcal{N}}{M}M^{\mathbf{i}} - N^{\mathbf{i}}\right)E_{\mathbf{i}} - \frac{\mathcal{N}}{M}\frac{\partial}{\partial\chi}\right], \\ m &= \frac{1}{M}\left(-M^{\mathbf{i}}E_{\mathbf{i}} + \frac{\partial}{\partial\chi}\right), \qquad F_{\mathbf{i}} = E_{\mathbf{i}}. \end{aligned} \tag{A10}$$

Obviously the two foliations will become perpendicular for $\gamma = 0 = \mathcal{N}$. Then the vectors l and m coincide.

APPENDIX B: DECOMPOSITION OF THE METRIC

By substituting Eq. (A6) into the $(s + 2)$-metric (A1) we obtain

$$\tilde{g}_{\alpha\beta} = \begin{pmatrix} g_{\mathbf{ij}}N^{\mathbf{i}}N^{\mathbf{j}} + \mathcal{N}^2 - N^2 & g_{\mathbf{ij}}N^{\mathbf{j}} & g_{\mathbf{ij}}N^{\mathbf{i}}M^{\mathbf{j}} + \mathcal{N}M \\ g_{\mathbf{ij}}N^{\mathbf{i}} & g_{\mathbf{ij}} & g_{\mathbf{ij}}M^{\mathbf{i}} \\ g_{\mathbf{ij}}N^{\mathbf{i}}M^{\mathbf{j}} + \mathcal{N}M & g_{\mathbf{ij}}M^{\mathbf{j}} & g_{\mathbf{ij}}M^{\mathbf{i}}M^{\mathbf{j}} + M^2 \end{pmatrix}. \tag{B1}$$

To prove in a simple way the relationship (5) between the determinants of the $(s + 2)$ and s metrics, we transform the determinant $\tilde{g}$ by suitably combining the columns and lines as follows:

$$\tilde{g} = \begin{vmatrix} \mathcal{N}^2 - N^2 & g_{\mathbf{ij}}N^{\mathbf{j}} & g_{\mathbf{ij}}N^{\mathbf{i}}M^{\mathbf{j}} + \mathcal{N}M \\ 0 & g_{\mathbf{ij}} & g_{\mathbf{ij}}M^{\mathbf{i}} \\ \mathcal{N}M & g_{\mathbf{ij}}M^{\mathbf{j}} & g_{\mathbf{ij}}M^{\mathbf{i}}M^{\mathbf{j}} + M^2 \end{vmatrix}$$
$$= [\mathcal{N}^2 - N^2]\begin{vmatrix} g_{\mathbf{ij}} & g_{\mathbf{ij}}M^{\mathbf{i}} \\ g_{\mathbf{ij}}M^{\mathbf{j}} & g_{\mathbf{ij}}M^{\mathbf{i}}M^{\mathbf{j}} + M^2 \end{vmatrix} + \mathcal{N}M\begin{vmatrix} g_{\mathbf{ij}}N^{\mathbf{j}} & g_{\mathbf{ij}}N^{\mathbf{i}}M^{\mathbf{j}} + \mathcal{N}M \\ g_{\mathbf{ij}} & g_{\mathbf{ij}}M^{\mathbf{i}} \end{vmatrix}$$
$$= [\mathcal{N}^2 - N^2]\begin{vmatrix} g_{\mathbf{ij}} & g_{\mathbf{ij}}M^{\mathbf{i}} \\ 0 & M^2 \end{vmatrix} + \mathcal{N}M\begin{vmatrix} 0 & \mathcal{N}M \\ g_{\mathbf{ij}} & g_{\mathbf{ij}}M^{\mathbf{i}} \end{vmatrix} = -N^2M^2g. \tag{B2}$$

The inverse of the metric (B1) has a more cumbersome expression:

$$\tilde{g}^{\alpha\beta} = \frac{1}{N^2}\begin{pmatrix} -1 & N^{\mathbf{i}} - \frac{\mathcal{N}}{M}M^{\mathbf{i}} & \frac{\mathcal{N}}{M} \\ N^{\mathbf{j}} - \frac{\mathcal{N}}{M}M^{\mathbf{j}} & N^2g^{\mathbf{ij}} - N^{\mathbf{i}}N^{\mathbf{j}} + \frac{N^2-\mathcal{N}^2}{M^2}M^{\mathbf{i}}M^{\mathbf{j}} + 2\frac{\mathcal{N}}{M}N^{(\mathbf{i}}M^{\mathbf{j})} & \frac{\mathcal{N}^2-N^2}{M^2}M^{\mathbf{j}} - \frac{\mathcal{N}}{M}N^{\mathbf{j}} \\ \frac{\mathcal{N}}{M} & \frac{\mathcal{N}^2-N^2}{M^2}M^{\mathbf{i}} - \frac{\mathcal{N}}{M}N^{\mathbf{i}} & \frac{N^2-\mathcal{N}^2}{M^2} \end{pmatrix}. \tag{B3}$$

It is convenient to free ourselves from the particular bases employed above. For this purpose we define the generic tensorial expressions $g_{ab} = g_{\mathbf{ij}}(F^{\mathbf{i}})_a(F^{\mathbf{j}})_b$, $N^a = N^{\mathbf{i}}(F_{\mathbf{i}})^a$, $M^a = M^{\mathbf{i}}(F_{\mathbf{i}})^a$, in terms of which Eqs. (A1), (A7a), and (A7b) give Eqs. (2)–(4).

APPENDIX C: CURVATURES OF NORMAL CONGRUENCES

We start with the computation of the nontrivial Lie brackets of the basis vectors $\{f_\beta\} = \{n, F_{\mathbf{j}} = \partial/\partial y^{\mathbf{j}}, m\}$, employing first the relations (A10), second that $\{e_\beta\}$ is a coordinate basis and finally rewriting the resulting expressions in the $\{f_\beta\}$ basis:

$$[n, F_{\mathbf{j}}]^a = \partial_{\mathbf{j}}(\ln N)n^a + \frac{1}{N}\left(\partial_{\mathbf{j}}N^{\mathbf{i}} - \frac{\mathcal{N}}{M}\partial_{\mathbf{j}}M^{\mathbf{i}}\right)(F_{\mathbf{i}})^a + \frac{M}{N}\partial_{\mathbf{j}}\left(\frac{\mathcal{N}}{M}\right)m^a, \tag{C1a}$$

$$[m, F_{\mathbf{j}}]^a = \partial_{\mathbf{j}}(\ln M)m^a + \frac{1}{M}\partial_{\mathbf{j}}M^{\mathbf{i}}(F_{\mathbf{i}})^a, \tag{C1b}$$

$$[n, m]^a = \frac{1}{M}\left[\frac{\partial}{\partial\chi}(\ln N) - M^{\mathbf{j}}\partial_{\mathbf{j}}(\ln N)\right]n^a + \frac{1}{MN}\left[-\frac{\partial}{\partial t}M + \frac{\partial}{\partial\chi}\mathcal{N} - M^{\mathbf{j}}\partial_{\mathbf{j}}\mathcal{N} + N^{\mathbf{j}}\partial_{\mathbf{j}}M\right]m^a$$
$$+ \frac{1}{MN}[-\tilde{\mathcal{L}}_{\partial/\partial t}M^{\mathbf{i}} + \tilde{\mathcal{L}}_{\partial/\partial\chi}N^{\mathbf{i}} - M^{\mathbf{j}}\partial_{\mathbf{j}}N^{\mathbf{i}} + N^{\mathbf{j}}\partial_{\mathbf{j}}M^{\mathbf{i}}](F_{\mathbf{i}})^a. \tag{C1c}$$

However the Frobenius theorem states that $[n, F_{\mathbf{j}}]^a$ should have no m^a component. Therefore we get the condition

$$\mathcal{N} = \nu M, \qquad D_a\nu = 0. \tag{C2}$$

Thus $\mathcal{N}$ should be proportional to M, with a proportionality coefficient, which is constant along $\Sigma_{t\chi}$.

Next we announce the following:

Theorem.—If $\tilde{\nabla}$ is the connection compatible with the metric $\tilde{g}$, any set of vectors $\{f_\alpha\}$ obeying $\tilde{g}(f_\alpha, f_\beta) = \text{const}$ also satisfy the relation $\tilde{g}(f_\alpha, \tilde{\nabla}_{\mathbf{f}_\beta}f_\beta) = \tilde{g}([f_\alpha, f_\beta], f_\beta)$.

Proof.—$\tilde{g}(f_\alpha, \tilde{\nabla}_{\mathbf{f}_\beta}f_\beta) = -\tilde{g}(\tilde{\nabla}_{\mathbf{f}_\beta}f_\alpha, f_\beta) = -\tilde{g}(\tilde{\nabla}_{\mathbf{f}_\alpha}f_\beta, f_\beta) - \tilde{g}([f_\beta, f_\alpha], f_\beta) = \tilde{g}([f_\alpha, f_\beta], f_\beta)$ Q.E.D.

Finally we employ the above result for f_β either n or m together with the commutators (C1a)–(C1c) to obtain explicit expressions for various components of the curvatures α^a and λ^a:

$$\alpha_b = \tilde{g}(f_\alpha, \tilde{\nabla}_{\mathbf{n}}n)(f^\alpha)_b = D_b(\ln N) + \frac{1}{M}\left[\frac{\partial}{\partial\chi}(\ln N) - M^aD_a(\ln N)\right]m_b, \tag{C3a}$$

$$\lambda_b = \tilde{g}(f_\alpha, \tilde{\nabla}_{\mathbf{l}}l)(f^\alpha)_b = -D_b(\ln M) + \frac{1}{MN}\left[\frac{\partial}{\partial t}M - \frac{\partial}{\partial\chi}\mathcal{N} + M^aD_a\mathcal{N} - N^aD_aM\right]n_b. \tag{C3b}$$

We have employed the relations $D_bN = (F^{\mathbf{i}})_b\partial_{\mathbf{i}}N$ and $N^a\partial_aM = N^{\mathbf{i}}\partial_{\mathbf{i}}M$, deducible from the coordinate basis character of $\{e_\beta\}$.

APPENDIX D: DECOMPOSITION OF CURVATURES

In this Appendix we enlist the relations among the $(s+2)$- and s-dimensional Riemann, Ricci and Einstein tensors, as well as the relation between the scalar curvatures in a fully decomposed form. The formulas are valid for perpendicular foliations, $m^a = l^a$.

The projections of the Riemann tensor are Eq. (58) and

$$n^i g^j_b g^k_c g^l_d \tilde{R}_{ijkl} = D_d K_{bc} - D_c K_{bd} + \mathcal{K}_c L_{bd} - \mathcal{K}_d L_{bc}, \tag{D1a}$$

$$n^i g^j_b n^k g^l_d \tilde{R}_{ijkl} = -\mathcal{L}_{\mathbf{n}} K_{bd} + K_{bk} K^k_d + \mathcal{K}_b \mathcal{K}_d - \mathcal{L} L_{bd} + N^{-1} D_d D_b N, \tag{D1b}$$

$$l^i g^j_b g^k_c g^l_d \tilde{R}_{ijkl} = D_d L_{bc} - D_c L_{bd} + \mathcal{K}_c K_{bd} - \mathcal{K}_d K_{bc}, \tag{D1c}$$

$$l^i g^j_b l^k g^l_d \tilde{R}_{ijkl} = -\mathcal{L}_{\mathbf{l}} L_{bd} + L_{bk} L^k_d - \mathcal{K}_b \mathcal{K}_d + \mathcal{K} K_{bd} - M^{-1} D_d D_b M, \tag{D1d}$$

$$n^i l^j g^k_c g^l_d \tilde{R}_{ijkl} = D_d \mathcal{K}_c - D_c \mathcal{K}_d + L_{ci} K^i_d - L_{di} K^i_c, \tag{D1e}$$

$$n^i g^j_b g^k_c l^l \tilde{R}_{ijkl} = -D_c \mathcal{K}_b + \mathcal{L}_{\mathbf{l}} K_{bc} - K_{kd} L^k_b - \mathcal{K} L_{bc} - 2M^{-1} \mathcal{K}_{(b} D_{c)} M, \tag{D1f}$$

$$n^i l^j g^k_c l^l \tilde{R}_{ijkl} = \mathcal{L}_{\mathbf{l}} \mathcal{K}_c - D_c \mathcal{K} + \mathcal{K}_k L^k_c + M^{-1} K^i_c D_i M - M^{-1} \mathcal{K} D_c M, \tag{D1g}$$

$$n^i l^j n^k g^l_d \tilde{R}_{ijkl} = -D_d \mathcal{L} - \mathcal{L}_{\mathbf{n}} \mathcal{K}_d - \mathcal{K}_i K^i_d - N^{-1} L^i_d D_i N - N^{-1} \mathcal{L} D_d N, \tag{D1h}$$

$$n^i l^j n^k l^l \tilde{R}_{ijkl} = \mathcal{L}^2 - \mathcal{K}^2 - 3\mathcal{K}_i \mathcal{K}^i - \mathcal{L}_{\mathbf{l}} \mathcal{L} - \mathcal{L}_{\mathbf{n}} \mathcal{K} + (NM)^{-1} D^i N D_i M. \tag{D1i}$$

Contractions and multiplications with the normal vectors of the above formulas give, respectively

$$g^c_a g^d_b \tilde{R}_{cd} = R_{ab} + K_{ab}(K + \mathcal{K}) - 2K_{ac} K^c_b + \mathcal{L}_{\mathbf{n}} K_{ab} - N^{-1} D_b D_a N - 2\mathcal{K}_a \mathcal{K}_b - L_{ab}(L - \mathcal{L}) + 2L_{ac} L^c_b - \mathcal{L}_{\mathbf{l}} L_{ab} - M^{-1} D_b D_a M, \tag{D2a}$$

$$n^a n^b \tilde{R}_{ab} = -\mathcal{L}_{\mathbf{n}}(K + \mathcal{K}) + K_{ab} K^{ab} + N^{-1} D_a D^a N - 2\mathcal{K}_a \mathcal{K}^a - \mathcal{K}^2 - \mathcal{L}_{\mathbf{l}} \mathcal{L} - \mathcal{L}(L - \mathcal{L}) + (NM)^{-1} D_a N D^a M, \tag{D2b}$$

$$l^a l^b \tilde{R}_{ab} = -\mathcal{L}_{\mathbf{l}}(L - \mathcal{L}) + L_{ab} L^{ab} - M^{-1} D_a D^a M + 2\mathcal{K}_a \mathcal{K}^a - \mathcal{L}^2 + \mathcal{L}_{\mathbf{n}} \mathcal{K} + \mathcal{K}(K + \mathcal{K}) - (NM)^{-1} D_a N D^a M, \tag{D2c}$$

$$n^a l^b \tilde{R}_{ab} = D_a \mathcal{K}^a - \mathcal{L}_{\mathbf{l}} K + K_{ab} L^{ab} + \mathcal{K} L + M^{-1} \mathcal{K}^a D_a M, \tag{D2d}$$

$$g^c_a n^d \tilde{R}_{cd} = D_c K^c_a - D_a(K + \mathcal{K}) + \mathcal{K}_a L + \mathcal{L}_{\mathbf{l}} \mathcal{K}_a + M^{-1} K^i_a D_i M - M^{-1} \mathcal{K} D_a M, \tag{D2e}$$

$$g^c_a l^d \tilde{R}_{cd} = D_c L^c_a - D_a(L - \mathcal{L}) + \mathcal{K}_a K + \mathcal{L}_{\mathbf{n}} \mathcal{K}_a + N^{-1} L^i_a D_i N + N^{-1} \mathcal{L} D_a N. \tag{D2f}$$

The last two equations are the Codazzi equations. Note that $\mathcal{L}_{\mathbf{n}} K$ denotes the trace of $\mathcal{L}_{\mathbf{n}} K_{ab}$. The scalar curvatures are related as

$$\tilde{R} = R + K^2 - 3K_{ab} K^{ab} + 2\mathcal{L}_{\mathbf{n}}(K + \mathcal{K}) - 2N^{-1} D_a D^a N + 2\mathcal{K} K + \mathcal{K}_a \mathcal{K}^a - L^2 + 3L_{ab} L^{ab} - 2\mathcal{L}_{\mathbf{l}}(L - \mathcal{L}) - 2M^{-1} D_a D^a M + 2\mathcal{L} L - (NM)^{-1} D_a N D^a M. \tag{D3}$$

By virtue of the (25) decompositions of covariant derivatives, Eqs. (D2b) and (D2c) can be put into the forms (60) containing divergence terms. So can the scalar curvature.

Finally, the Einstein tensors are related as

$$g^c_a g^d_b \tilde{G}_{cd} = G_{ab} + K_{ab}(K + \mathcal{K}) - 2K_{ac} K^c_b + \mathcal{L}_{\mathbf{n}} K_{ab} - N^{-1} D_b D_a N - L_{ab}(L - \mathcal{L}) + 2L_{ac} L^c_b - \mathcal{L}_{\mathbf{l}} L_{ab} - M^{-1} D_b D_a M - \frac{1}{2} g_{ab}[K^2 - 3K_{cd} K^{cd} + \mathcal{L}_{\mathbf{n}}(K + \mathcal{K}) - 2N^{-1} D_c D^c N] - \frac{1}{2} g_{ab}[-L^2 + 3L_{cd} L^{cd} - 2\mathcal{L}_{\mathbf{l}}(L - \mathcal{L}) - 2M^{-1} D_c D^c M] - g_{ab}(\mathcal{K} K + \mathcal{K}_c \mathcal{K}^c - (NM)^{-1} D_c N D^c M), \tag{D4a}$$

$$n^a n^b \tilde{G}_{ab} = \frac{1}{2}(R + K^2 - K_{ab} K^{ab} - L^2 + 3L_{ab} L^{ab}) + \mathcal{K} K - \mathcal{K}_a \mathcal{K}^a - \mathcal{K}^2 - \mathcal{L}_{\mathbf{l}} L + \mathcal{L}^2 - M^{-1} D_a D^a M, \tag{D4b}$$

$$l^a l^b \tilde{G}_{ab} = -\frac{1}{2}(R - L^2 + L_{ab} L^{ab} + K^2 - 3K_{ab} K^{ab}) - \mathcal{L} L + \mathcal{K}_a \mathcal{K}^a - \mathcal{L}^2 - \mathcal{L}_{\mathbf{n}} K + \mathcal{K}^2 + N^{-1} D_a D^a N, \tag{D4c}$$

$$n^a l^b \tilde{G}_{ab} = n^a l^b \tilde{R}_{ab}, \tag{D4d}$$

$$g^c_a n^d \tilde{G}_{cd} = g^c_a n^d \tilde{R}_{cd}, \tag{D4e}$$

$$g^c_a l^d \tilde{G}_{cd} = g^c_a l^d \tilde{R}_{cd}. \tag{D4f}$$

By performing first the traditional $(s+1)+1$ decomposition and then further splitting the $(s+1)$-dimensional spacelike hypersurface into the s-dimensional brane and off-brane direction $\partial/\partial\chi$, we can give an independent proof of the relation between the scalar curvatures. The twice-contracted Gauss equation for the hypersurface S_t of $\mathcal{B}$ is

$$\tilde{R} = \hat{R} - \hat{K}^2 + \hat{K}_{ab}\hat{K}^{ab} - 2\tilde{\nabla}_a(\alpha^a - \hat{K}n^a). \qquad \text{(D5)}$$

The extrinsic curvature $\hat{K}_{ab}$ of S_t can be further decomposed employing Eq. (13), while for $\hat{R}$ there is another twice-contracted Gauss equation, this time for the hypersurface $\Sigma_{t\chi}$ of S_t:

$$\hat{R} = R + L^2 - L_{ab}L^{ab} + 2\hat{D}_a(l^c\hat{D}_c l^a - Ll^a). \qquad \text{(D6)}$$

The latter relation and Eq. (13) inserted in Eq. (5) and employing

$$\hat{D}_a(l^c\hat{D}_c l^a - Ll^a) = \tilde{\nabla}_a(\lambda^a - Ll^a - \mathcal{K}n^a) - L\mathcal{L} - \lambda^a\alpha_a, \qquad \text{(D7)}$$

gives Eq. (61) once again.

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