Supersymmetry in gauge theories with extra dimensions

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(Received 1 February 2005; published 14 September 2005)

We show that a quantum-mechanical $N = 2$ supersymmetry is hidden in 4d mass spectrum of any gauge invariant theories with extra dimensions. The $N = 2$ supercharges are explicitly constructed in terms of differential forms. The analysis can be extended to extra dimensions with boundaries, and for a single extra dimension we clarify a possible set of boundary conditions consistent with 5d gauge invariance, although some of the boundary conditions break 4d gauge symmetries.

DOI: 10.1103/PhysRevD.72.064006 PACS numbers: 11.10.Kk, 11.15.−q, 11.30.Pb, 12.60.Jv

I. INTRODUCTION

Much attention has been paid recently to gauge theories with extra dimensions to explore new possibilities for gauge symmetry breaking and solving the hierarchy problem without introducing additional Higgs fields [1–10]. For instance, in the gauge-Higgs unification scenario, extra components of gauge fields play a role of Higgs fields [2,3]. Attractive models of grand unified theories have been constructed on orbifolds, in which gauge symmetry breaking is caused by orbifolding [4]. With extra dimensions with boundaries, Higgsless gauge symmetry breaking can be realized via boundary conditions [5–7,9]. The interesting scenario of dimensional deconstruction [8] can be regarded as a gauge theory with latticized extra dimension.

In those models, the notorious quadratic divergence problem of scalar fields is absent. In a higher-dimensional point of view, this is easily understood because any divergences of mass corrections to gauge fields are protected by a higher-dimensional gauge invariance. In a 4-dimensional point of view, however, the cancellation of the divergences does not seem to be manifest because a remnant of higher-dimensional gauge invariance is not apparent in 4d effective theories and the cancellation can occur only after *all* massive Kaluza-Klein (KK) modes are taken into account. (If we truncate massive KK modes at some energy, the cancellation becomes incomplete.) Furthermore, the cancellation still occurs even when 4d gauge symmetries are broken via orbifolding, the Hosotani mechanism, or boundary conditions. Thus, in constructing phenomenological models, it will be important to understand higher-dimensional gauge invariance from a 4d effective theory point of view.

Another appealing and well-known scenario to solve the problem of the quadratic divergence is to invoke supersymmetry. Then, it may be natural to ask a question whether these two kinds of theories, gauge theories with extra dimensions and supersymmetric theories, ever have some relation. The immediate answer is negative, since supersymmetry necessarily needs fermionic degree of freedom, while the cancellation mechanism of the quadratic divergence in the higher-dimensional gauge theory does not necessitate it. We will, however, see that actually these two are related: (quantum-mechanical) supersymmetry is hidden in the higher-dimensional gauge theories. The main purpose of this paper is to show it. We note that quantum-mechanical supersymmetry, being $0 + 1$-dimensional field theory, can be described without using any spinors.

There exists some evidences for the existence of some kind of supersymmetry in 4d mass spectrum already at the truncated low-energy theory, as the remnant of higher-dimensional gauge symmetry: If a 4d gauge symmetry is not broken, a massless 4d gauge field appears because 4d gauge invariance guarantees the gauge field to be massless. This may be explained from a supersymmetry point of view because supersymmetry ensures that the ground state has zero energy and the zero energy state is interpreted as the massless gauge field. The second evidence is that in a 5d gauge theory with a single extra dimension, every massive mode of $A_{5,n}$ (a massive KK mode of the gauge field in the direction of the extra dimension) can be absorbed into the longitudinal mode of $A_{\mu,n}$ (a massive KK mode of the 4d gauge field) by gauge transformations. This fact implies that there should exist a one-to-one correspondence between $A_{5,n}$ and $A_{\mu,n}$. The correspondence may be interpreted as supersymmetry between a "bosonic" state and a "fermionic" one, although both modes are bosonic. The last evidence is that a massless mode $A_{5,0}$ (if exists) cannot be gauged away and it appears as a physical state, in contrast to the massive modes $A_{5,n}$, which can be gauged away and hence are unphysical modes. This observation is again consistent with supersymmetry because zero energy states do not form any supermultiplets between bosonic and fermionic states.

In this paper, we show that the above observations are true. To see this, we expand the gauge fields into infinite towers of KK modes by use of eigenfunctions of differen-

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tial equations with respect to extra-space coordinates to determine the 4d mass-squared. There is a one-to-one correspondence between a KK mode and a mass eigenfunction (see Eqs. (3.14) and (3.19)). In a four-dimensional point of view, the KK modes of the extra-dimensional components of the gauge fields are regarded as scalars and contain would-be Nambu-Goldstone bosons that can be absorbed into the longitudinal modes of the four-dimensional components of the gauge fields. One of the main purposes of this paper is to show that for each KK mode, the mass eigenfunction of the four-dimensional gauge field forms a supermultiplet with that of the would-be Nambu-Goldstone boson, except for massless modes and that the mass eigenfunctions really obey the algebra of $N = 2$ supersymmetric quantum mechanics given by Witten [11]. We emphasize that this statement is true for any compact Riemannian manifold without a boundary as extra dimensions. The statement still holds even if we add a warped factor to the metric [12] and/or also a weight function to the action (see Section III for details).

The question, however, of whether the whole action possesses the quantum-mechanical supersymmetry cannot be addressed in the present paper. In particular, though we deal with only free Lagrangians in this paper, once we intend to include interactions of gauge fields with other fields, we have to define the supersymmetry transformation of matter fields, for instance. The interaction terms would be inevitable for the investigation of the cancellation mechanism of the quadratic divergence. These issues are out of scope of the present paper.

An important implication of the above observation is that the 4d mass spectrum of the 4d vector bosons is governed by an $N = 2$ supersymmetric quantum mechanics, and it suggests a mechanism to solve the hierarchy problem at tree level [5,13]. The supersymmetric structure for the mass eigenfunctions turns out to be useful to determine allowed boundary conditions when the extra dimensions have boundaries. The choice of boundary conditions is very crucial to construct Higgsless models of gauge symmetry breaking [5–7,9]. In a four-dimensional gauge invariant theory, we will see that the boundary conditions compatible with the supersymmetric structure are very restrictive and that the requirement of the compatibility is powerful enough to obtain all possible sets of boundary conditions on the gauge fields.

This paper is organized as follows. We construct an $N = 2$ superalgebra in terms of differential forms in Section II. In Section III, applying the technology developed in Section II, we show that the $N = 2$ supersymmetric structure appears in 4d mass eigenfunctions (mass spectrum) of gauge theories with extra dimensions. In Section IV, extra dimensions with boundaries are discussed. For a 4 + 1-dimensional gauge invariant theory on an interval, a consistent set of boundary conditions with 5d gauge invariance is successfully obtained from a supersymmetric point of view. The Section V is devoted to summary and discussions. A simple proof of the Hodge decomposition theorem is given in an Appendix.

II. $N = 2$ SUPERSYMMETRY ALGEBRA AND DIFFERENTIAL FORMS

In this section, we construct an $N = 2$ supersymmetry algebra in terms of differential forms. We will see later that the $N = 2$ supersymmetry is realized in 4d mass spectrum of any gauge invariant theories with extra dimensions.

Let K be a D-dimensional compact Riemannian manifold with a metric $g_{ij}(i, j = 1, 2, \cdots, D)$. A k form $\omega^{(k)}$ on K is given by

$$\omega^{(k)} = \frac{1}{k!}\omega_{i_1 i_2\cdots i_k} dy^{i_1} \wedge dy^{i_2} \wedge \cdots \wedge dy^{i_k}, \tag{2.1}$$

where $y^i (i = 1, 2, \cdots, D)$ are coordinates on K and $\wedge$ denotes the wedge product. The coefficient $\omega_{i_1\cdots i_k}$ is totally antisymmetric in all k indices. The Hodge star (Poincaré dual) operator on the k form $\omega^{(k)}$ is defined by

$$*\omega^{(k)} = \frac{\sqrt{g}}{k!(D-k)!}\omega_{i_1\cdots i_k} g^{i_1 j_1}\cdots g^{i_k j_k}\epsilon_{j_1\cdots j_k j_{k+1}\cdots j_D} dy^{j_{k+1}} \wedge \cdots \wedge dy^{j_D}, \tag{2.2}$$

where $g = \det g_{ij}$ and $\epsilon_{i_1 i_2\cdots i_D}$ is a totally antisymmetric tensor with $\epsilon_{12\cdots D} = 1$. Repeated applications of $*$ on any k form give

$$* * \omega^{(k)} = (-1)^{k(D-k)}\omega^{(k)}. \tag{2.3}$$

The inner product of any two k forms $\omega^{(k)}$ and $\eta^{(k)}$ is defined by

$$\begin{aligned}(\eta^{(k)}, \omega^{(k)})_\Delta &\equiv \frac{1}{k!}\int_K d^D y\sqrt{g}\Delta\eta_{i_1\cdots i_k}\omega_{j_1\cdots j_k} g^{i_1 j_1}\cdots g^{i_k j_k} \\ &= \int_K \Delta\eta^{(k)} \wedge *\omega^{(k)}.\end{aligned} \tag{2.4}$$

Here, we have introduced a weight function $\Delta(y)$ for later convenience with a property

$$\Delta(y) > 0. \tag{2.5}$$

Then, it follows that

$$(\eta^{(k)}, \omega^{(k)})_\Delta = (\omega^{(k)}, \eta^{(k)})_\Delta, \tag{2.6}$$

$$(\omega^{(k)}, \omega^{(k)})_\Delta \geq 0, \tag{2.7}$$

$$(\eta^{(k)}, *\omega^{(D-k)})_\Delta = (-1)^{k(D-k)}(*\eta^{(k)}, \omega^{(D-k)})_\Delta. \tag{2.8}$$

The adjoint of the exterior derivative d is defined, with respect to the inner product (2.4), by

$$(\eta^{(k-1)}, d^\dagger \omega^{(k)})_\Delta \equiv (d\eta^{(k-1)}, \omega^{(k)})_\Delta. \tag{2.9}$$

For compact manifold without a boundary,[1] the action of $d^\dagger$ on a k form $\omega^{(k)}$ turns out to be of the form

$$d^\dagger \omega^{(k)} = -(-1)^{(k-1)D} \Delta^{-1} * d\Delta * \omega^{(k)}. \tag{2.10}$$

For $k = 0$ and 1, $d^\dagger \omega^{(k)}$ are explicitly written as

$$d^\dagger \omega^{(0)} = 0, \tag{2.11}$$

$$d^\dagger \omega^{(1)} = -\frac{1}{\Delta\sqrt{g}} \partial_j (\Delta \sqrt{g} g^{ij} \omega_i), \tag{2.12}$$

where $\omega^{(1)} = \omega_i dy^i$. The first equation comes from the fact that $d^\dagger$ maps k forms into $k - 1$ forms and there is no -1 form. We notice that the nilpotency of $d^\dagger$ still holds irrespective of $\Delta(y)$, i.e.

$$(d^\dagger)^2 = 0. \tag{2.13}$$

We can now construct an $N = 2$ supersymmetry algebra. To this end, we introduce a 2-component vector

$$|\Omega^{(k)}\rangle \equiv \begin{pmatrix} \omega^{(k)} \\ \phi^{(k+1)} \end{pmatrix}, \tag{2.14}$$

where the upper (lower) component consists of a k ($k + 1$) form. The inner product of two 2-component vectors $|\Omega_1^{(k)}\rangle$ and $|\Omega_2^{(k)}\rangle$ is defined by

$$\langle \Omega_2^{(k)} | \Omega_1^{(k)} \rangle \equiv (\omega_2^{(k)}, \omega_1^{(k)})_\Delta + (\phi_2^{(k+1)}, \phi_1^{(k+1)})_\Delta. \tag{2.15}$$

Then, the $N = 2$ supercharges Q_a ($a = 1, 2$) are given by[2]

$$Q_1 = \begin{pmatrix} 0 & d^\dagger \\ d & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & -id^\dagger \\ id & 0 \end{pmatrix}. \tag{2.16}$$

We note that the action of Q_a on $|\Omega^{(k)}\rangle$ is well defined. It is easy to show that they form the following $N = 2$ supersymmetry algebra:

$$\{Q_a, Q_b\} = 2\delta_{ab} H, \tag{2.17}$$

$$[Q_a, H] = 0, \tag{2.18}$$

$$[(-1)^F, H] = 0, \tag{2.19}$$

$$\{(-1)^F, Q_a\} = 0, \quad \text{for } a, b = 1, 2, \tag{2.20}$$

where the Hamiltonian H and the operator $(-1)^F$ with F being the "fermion" number operator are defined by

$$H = \begin{pmatrix} d^\dagger d & 0 \\ 0 & dd^\dagger \end{pmatrix}, \tag{2.21}$$

$$(-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.22}$$

We may call states with $(-1)^F = +1(-1)$ "bosonic" ("fermionic") ones. All the operators Q_a, H, and $(-1)^F$ are Hermitian with respect to the inner product (2.15).

To examine the structure of the $N = 2$ supersymmetry algebra, let us consider the following Schrödinger-type equations:

$$H|\Omega_n^{(k)}\rangle = (m_n^{(k)})^2 |\Omega_n^{(k)}\rangle. \tag{2.23}$$

Since $H = (Q_1)^2$ and $Q_1^\dagger = Q_1$, the eigenvalues $(m_n^{(k)})^2$ are positive semidefinite, i.e.

$$(m_n^{(k)})^2 \geq 0. \tag{2.24}$$

Since $(-1)^F$ commutes with H, we can have simultaneous eigenfunctions of H and $(-1)^F$ such that

$$H|m_n^{(k)}, \pm\rangle = (m_n^{(k)})^2 |m_n^{(k)}, \pm\rangle, \tag{2.25}$$

$$(-1)^F |m_n^{(k)}, \pm\rangle = \pm |m_n^{(k)}, \pm\rangle. \tag{2.26}$$

Since Q_1 commutes (anticommutes) with $H((-1)^F)$, $Q_1|m_n^{(k)}, \pm\rangle$ have the same (opposite) eigenvalues of H ($(-1)^F$) as $|m_n^{(k)}, \pm\rangle$, i.e.

$$H(Q_1|m_n^{(k)}, \pm\rangle) = (m_n^{(k)})^2 (Q_1|m_n^{(k)}, \pm\rangle), \tag{2.27}$$

$$(-1)^F (Q_1|m_n^{(k)}, \pm\rangle) = \mp (Q_1|m_n^{(k)}, \pm\rangle). \tag{2.28}$$

Actually, the states $|m_n^{(k)}, \pm\rangle$ and $Q_1|m_n^{(k)}, \mp\rangle$ are mutually related, with an appropriate phase convention, as

$$Q_1|m_n^{(k)}, \pm\rangle = m_n^{(k)} |m_n^{(k)}, \mp\rangle. \tag{2.29}$$

Since $|m_n^{(k)}, \pm\rangle$ have the form

$$|m_n^{(k)}, +\rangle = \begin{pmatrix} \omega_n^{(k)} \\ 0 \end{pmatrix}, \tag{2.30}$$

[1] An extension with boundaries will be discussed in Section IV.

[2] Here, the inner product (2.4) should be extended for complex forms as

$$(\eta^{(k)}, \omega^{(k)})_\Delta \equiv \frac{1}{k!} \int_K d^D y \sqrt{g} \Delta (\eta_{i_1 \cdots i_k})^* \omega_{j_1 \cdots j_k} g^{i_1 j_1} \cdots g^{i_k j_k},$$

and the relation (2.6) is then replaced by $(\eta^{(k)}, \omega^{(k)})_\Delta = ((\omega^{(k)}, \eta^{(k)})_\Delta)^*$.

$$|m_n^{(k)}, -\rangle = \begin{pmatrix} 0 \\ \phi_n^{(k+1)} \end{pmatrix}, \tag{2.31}$$

the relations (2.29) are rewritten as

$$d\omega_n^{(k)} = m_n^{(k)} \phi_n^{(k+1)}, \tag{2.32}$$

$$d^\dagger \phi_n^{(k+1)} = m_n^{(k)} \omega_n^{(k)}. \tag{2.33}$$

Therefore, there is a one-to-one correspondence between the eigenstates of $d^\dagger d$ and $dd^\dagger$ for k and $k+1$ forms, respectively, and the eigenvalues are, in general, doubly degenerate except for $m_n^{(k)} = 0$.

In the appendix, we have shown that any k form $A^{(k)}$ can be expanded as

$$A^{(k)} = \sum_{p=1}^{b_k} c_p \eta_p^{(k)} + \sum_{n_k}{}' a_{n_k} \omega_{n_k}^{(k)} + \sum_{n_{k-1}}{}' b_{n_{k-1}} \phi_{n_{k-1}}^{(k)}, \tag{2.34}$$

where $\{\eta_p^{(k)}, p = 1, 2, \cdots, b_k\}$ is a complete set of the harmonic k forms and $\omega_{n_k}^{(k)}$ and $\phi_{n_{k-1}}^{(k)}$ are eigenfunctions of the equations

$$d^\dagger d\omega_{n_k}^{(k)} = (m_{n_k}^{(k)})^2 \omega_{n_k}^{(k)} \quad \text{for } m_{n_k}^{(k)} \neq 0, \tag{2.35}$$

$$dd^\dagger \phi_{n_{k-1}}^{(k)} = (m_{n_{k-1}}^{(k)})^2 \phi_{n(k-1)}^{(k)} \quad \text{for } m_{n_{k-1}}^{(k)} \neq 0. \tag{2.36}$$

The summations in Eq. (2.34) should be taken over the eigenfunctions with nonzero eigenvalues. Thus, we have found the following structure among a sequence of differential forms:

$$\begin{tikzcd}
\vdots & & \ddots & & \\
k+2 \text{ form} & & \phi^{(k+2)} \arrow[dd, shift left, "d^\dagger"] & & \\
& 0 & & 0 & \\
k+1 \text{ form} & (\ \eta^{(k+1)}, \arrow[u, "d"] \arrow[d, "d^\dagger"] & \omega^{(k+1)}, \arrow[uu, shift left, "d"] \arrow[d, "d^\dagger"] & \phi^{(k+1)}\ ) \arrow[u, "d"] \arrow[dd, shift left, "d^\dagger"] & \\
& 0 & 0 & & 0 \\
k \text{ form} & & (\ \eta^{(k)}, \arrow[u, "d"] \arrow[d, "d^\dagger"] & \omega^{(k)}, \arrow[uu, shift left, "d"] \arrow[d, "d^\dagger"] & \phi^{(k)}\ ) \arrow[u, "d"] \arrow[dd, shift left, "d^\dagger"] \\
& & 0 & 0 & \\
k-1 \text{ form} & & & & \omega^{(k-1)} \arrow[uu, shift left, "d"] \\
\vdots & & & & \ddots
\end{tikzcd}$$

III. SUPERSYMMETRY IN GAUGE THEORIES WITH EXTRA DIMENSIONS

In this section, with the help of the previous analysis, we show that an $N = 2$ supersymmetry is hidden in 4d mass spectrum of any gauge invariant theories with compact extra dimensions without a boundary. We should notice, to avoid confusion, that the $N = 2$ supersymmetric structure can be seen for mass eigenfunctions appearing in the mode expansion of gauge fields, rather than a symmetry of the action. We can introduce $N = 2$ supercharges, that make a mass eigenfunction associated with a 4d vector boson transform into a mass eigenfunction associated with a 4d would-be Nambu-Goldstone boson, and vice versa, which can be absorbed into the longitudinal mode of the 4d vector boson.

To this end, we consider a $(4 + D)$-dimensional Abelian gauge theory with a weight function. The $(4 + D)$-dimensional metric is assumed to be of the form

$$d\tilde{s}^2 = e^{-(4/D)W(y)}(\eta_{\mu\nu}dx^\mu dx^\nu + g_{ij}(y)dy^i dy^j). \tag{3.1}$$

The $(4 + D)$-dimensional coordinates are denoted by $x^M = (x^\mu, y^i)$, where $x^\mu (\mu = 0, 1, 2, 3)$ are the 4-dimensional coordinates and $y^i (i = 1, 2, \cdots, D)$ are the extra D-dimensional coordinates. The $\eta_{\mu\nu}$ is a 4d Minkowski metric with $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, and $W(y)$ and $g_{ij}(y)$ are assumed to depend only on the coordinates y^i. We note that for $D = 1$ and $W(y) = \frac{1}{2}k|y|$ with $g_{55}(y) = e^{4W(y)}$, the metric reduces to the warped metric discussed by Randall and Sundrum [12]. The action we consider is

$$S = \int d^4x \mathcal{L}_K = \int d^4x \int d^Dy \sqrt{-G}\tilde{\Delta}\Big\{-\frac{1}{4}G^{MM'}G^{NN'}F_{MN}F_{M'N'}\Big\}, \tag{3.2}$$

where $\tilde{\Delta}(y)$ is a weight function depending on y^i and

$$F_{MN}(x, y) = \partial_M A_N(x, y) - \partial_N A_M(x, y), \tag{3.3}$$

$$G_{MN}(y) = \begin{pmatrix} e^{-(4/D)W(y)}\eta_{\mu\nu} & 0 \\ 0 & e^{-(4/D)W(y)}g_{ij}(y) \end{pmatrix}, \tag{3.4}$$

$$G(y) = \det G_{MN}(y). \tag{3.5}$$

For our purpose, it is convenient to rewrite $\mathcal{L}_K$ into the form

$$\mathcal{L}_K = \int d^Dy \sqrt{g}\Delta\Big\{-\frac{1}{4}g^{MM'}g^{NN'}F_{MN}F_{M'N'}\Big\}, \tag{3.6}$$

where

$$g_{MN}(y) = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{ij}(y) \end{pmatrix}, \tag{3.7}$$

$$g(y) = \det g_{ij}(y), \tag{3.8}$$

$$\Delta(y) = \tilde{\Delta}(y)e^{-2W(y)}. \tag{3.9}$$

Thus, the system becomes identical to that of the gauge theory with the metric

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + g_{ij}(y)dy^i dy^j \tag{3.10}$$

and the weight function $\Delta(y)$.

In a viewpoint of the extra D-dimensions, the 4-dimensional gauge fields $A_\mu(x, y)$ are regarded as 0 forms, while the extra D-dimensional components $A_i(x, y)$ are regarded as a 1 form, so that we may write

$$A^{(1)}(x, y) \equiv A_i(x, y)dy^i. \tag{3.11}$$

Here, the exterior derivative d is defined by $d = dy^i\partial_i$ with respect to the coordinates on the extra dimensions. As was done in the previous section, we introduce the inner product for k forms as

$$(\eta^{(k)}, \omega^{(k)})_\Delta = \frac{1}{k!}\int d^D y\sqrt{g(y)}\Delta(y)\eta_{i_1\cdots i_k}(y)\omega_{j_1\cdots j_k}(y) \times g^{i_1j_1}(y)\cdots g^{i_kj_k}(y). \tag{3.12}$$

It turns out that $\mathcal{L}_K$ can be written into the form

$$\begin{aligned}\mathcal{L}_K = &-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu, \partial^\mu A^\nu - \partial^\nu A^\mu)_\Delta \\ &-\frac{1}{2}(dA_\mu - \partial_\mu A^{(1)}, dA^\mu - \partial^\mu A^{(1)})_\Delta \\ &-\frac{1}{2}(dA^{(1)}, dA^{(1)})_\Delta.\end{aligned} \tag{3.13}$$

As proved in the appendix, 0 forms $A_\mu(x, y)$ can be expanded as[3]

$$\begin{aligned}A_\mu(x, y) &= A_{\mu,0}(x)\eta^{(0)} + \sum_{n_0}{}' A_{\mu,n_0}(x)\omega^{(0)}_{n_0}(y) \\ &= A_{\mu,0}(x)\eta^{(0)} + \sum_{n_0}{}' \frac{A_{\mu,n_0}(x)}{m^{(0)}_{n_0}} d^\dagger\phi^{(1)}_{n_0}(y),\end{aligned} \tag{3.14}$$

where

$$\eta^{(0)} = \left(\int d^D y\sqrt{g}\Delta\right)^{-1/2}, \tag{3.15}$$

$$d^\dagger d\omega^{(0)}_{n_0} = (m^{(0)}_{n_0})^2\omega^{(0)}_{n_0}, \tag{3.16}$$

$$dd^\dagger\phi^{(1)}_{n_0} = (m^{(0)}_{n_0})^2\phi^{(1)}_{n_0}, \quad \text{for } m^{(0)}_{n_0} \neq 0, \tag{3.17}$$

and

$$\omega^{(0)}_{n_0} = \frac{1}{m^{(0)}_{n_0}}d^\dagger\phi^{(1)}_{n_0} \quad \text{or} \quad \phi^{(1)}_{n_0} = \frac{1}{m^{(0)}_{n_0}}d\omega^{(0)}_{n_0}. \tag{3.18}$$

Since $A^{(1)}(x, y) = A_i(x, y)dy^i$ is a 1 form, it can be expanded as

$$\begin{aligned}A^{(1)}(x, y) &= \sum_{p=1}^{b_1}\varphi_p(x)\eta^{(1)}_p(y) + \sum_{n_1}{}'\Phi_{n_1}(x)\omega^{(1)}_{n_1}(y) \\ &\quad + \sum_{n_0}{}' h_{n_0}(x)\phi^{(1)}_{n_0}(y) \\ &= \sum_{p=1}^{b_1}\varphi_p(x)\eta^{(1)}_p(y) + \sum_{n_1}{}'\frac{\Phi_{n_1}(x)}{m^{(1)}_{n_1}}d^\dagger\phi^{(2)}_{n_1}(y) \\ &\quad + \sum_{n_0}{}'\frac{h_{n_0}(x)}{m^{(0)}_{n_0}}d\omega^{(0)}_{n_0}(y),\end{aligned} \tag{3.19}$$

where b_1 is the 1st Betti number and

$$d\eta^{(1)}_p = 0 = d^\dagger\eta^{(1)}_p, \qquad p = 1, 2, \cdots, b_1, \tag{3.20}$$

$$d^\dagger d\omega^{(1)}_{n_1} = (m^{(1)}_{n_1})^2\omega^{(1)}_{n_1}, \tag{3.21}$$

$$dd^\dagger\phi^{(2)}_{n_1} = (m^{(1)}_{n_1})^2\phi^{(2)}_{n_1}, \quad \text{for } m^{(1)}_{n_1} \neq 0, \tag{3.22}$$

and

$$\omega^{(1)}_{n_1} = \frac{1}{m^{(1)}_{n_1}}d^\dagger\phi^{(2)}_{n_1} \quad \text{for } \phi^{(2)}_{n_1} = \frac{1}{m^{(1)}_{n_1}}d\omega^{(1)}_{n_1}. \tag{3.23}$$

It is now clear that the $N = 2$ supersymmetric structure discussed in the previous section can be seen in the mode expansion of the gauge fields. The mass eigenfunction $\omega^{(0)}_{n_0}$ and $\phi^{(1)}_{n_0}$ associated with the 4d vector $A_{\mu,n_0}(x)$ and the (would-be Nambu-Goldstone) boson $h_{n_0}(x)$, respectively, obey the $N = 2$ supersymmetric quantum mechanics. They transform into each other by the action of the $N = 2$ supercharges and hence form a supermultiplet, i.e.

$$H\begin{pmatrix}\omega^{(0)}_{n_0}\\ \phi^{(1)}_{n_0}\end{pmatrix} = (m^{(0)}_{n_0})^2\begin{pmatrix}\omega^{(0)}_{n_0}\\ \phi^{(1)}_{n_0}\end{pmatrix}, \tag{3.24}$$

$$Q_1\begin{pmatrix}\omega^{(0)}_{n_0}\\ 0\end{pmatrix} = m^{(0)}_{n_0}\begin{pmatrix}0\\ \phi^{(1)}_{n_0}\end{pmatrix}, \qquad Q_1\begin{pmatrix}0\\ \phi^{(1)}_{n_0}\end{pmatrix} = m^{(0)}_{n_0}\begin{pmatrix}\omega^{(0)}_{n_0}\\ 0\end{pmatrix}, \tag{3.25}$$

[3]We have used the fact that the 0th Betti number b_0 is equal to 1.

$$Q_2\begin{pmatrix}\omega_{n_0}^{(0)}\\ 0\end{pmatrix} = im_{n_0}^{(0)}\begin{pmatrix}0\\ \phi_{n_0}^{(1)}\end{pmatrix},\qquad Q_2\begin{pmatrix}0\\ \phi_{n_0}^{(1)}\end{pmatrix} = -im_{n_0}^{(0)}\begin{pmatrix}\omega_{n_0}^{(0)}\\ 0\end{pmatrix}, \tag{3.26}$$

where

$$H = (Q_1)^2 = (Q_2)^2 = \begin{pmatrix} d^\dagger d & 0\\ 0 & dd^\dagger\end{pmatrix}, \tag{3.27}$$

$$Q_1 = \begin{pmatrix}0 & d^\dagger\\ d & 0\end{pmatrix}, \qquad Q_2 = \begin{pmatrix}0 & -id^\dagger\\ id & 0\end{pmatrix}. \tag{3.28}$$

We note that the mass eigenfunction $\omega_{n_0}^{(0)}$ corresponds to 4d vector boson $A_{\mu,n_0}(x)$ and hence the mass spectrum of the 4d vector bosons is governed by the above $N = 2$ supersymmetric quantum mechanics.

Inserting the mode expansions (3.14) and (3.19) into the Lagrangian $\mathcal{L}_K$ and using the orthogonal relations of the eigenfunctions, we have

$$\begin{aligned}\mathcal{L}_K = &-\frac{1}{4}(F_{\mu\nu,0}(x))^2 + \sum_{n_0}{}'\bigg[-\frac{1}{4}(F_{\mu\nu,n_0}(x))^2\\ &-\frac{(m_{n_0}^{(0)})^2}{2}\bigg(A_{\mu,n_0}(x) - \frac{1}{m_{n_0}^{(0)}}\partial_\mu h_{n_0}(x)\bigg)^2\bigg]\\ &-\frac{1}{2}\sum_{p=1}^{b_1}(\partial_\mu\varphi_p(x))^2 + \sum_{n_1}{}'\bigg[-\frac{1}{2}(\partial_\mu\Phi_{n_1}(x))^2\\ &-\frac{(m_{n_1}^{(1)})^2}{2}(\Phi_{n_1}(x))^2\bigg],\end{aligned} \tag{3.29}$$

where

$$F_{\mu\nu,n}(x) = \partial_\mu A_{\nu,n}(x) - \partial_\nu A_{\mu,n}(x). \tag{3.30}$$

Therefore, we conclude that in a 4-dimensional point of view, the field contents of the model are given as follows: $A_{\mu,0}$ is a massless gauge field. A_{μ,n_0} are massive vector bosons with mass $m_{n_0}^{(0)}$. h_{n_0} are would-be Nambu-Goldstone bosons and can be absorbed into the longitudinal modes of A_{μ,n_0}. $\varphi_p (p = 1, 2, \cdots, b_1)$ are massless scalars and cannot be gauged away. They could play a role of Higgs fields for non-Abelian gauge theories [2]. Φ_{n_1} are massive scalars with mass $m_{n_1}^{(1)}$. The origin of the scalar fields φ_p and Φ_{n_1} are the extra-dimensional components of the gauge fields.

IV. EXTRA DIMENSIONS WITH BOUNDARIES

In this section, we extend the previous analysis to extra dimensions with boundaries. In this case, we have to impose boundary conditions at the boundaries. The criteria of obtaining a possible set of boundary conditions are, however, less obvious. For instance, the Dirichlet boundary conditions are used in Higgsless gauge symmetry breaking scenario [5–7], but the boundary conditions break 4d gauge symmetries explicitly. Thus, it is not clear whether such boundary conditions lead to consistent gauge theories. Recently, a criterion to select a possible set of boundary conditions has been proposed in Ref. [6]. The authors require the boundary conditions to obey the least action principle. Since the requirement does not, however, rely on gauge invariance directly, it is still unclear that such boundary conditions lead to consistent gauge theories. Since gauge symmetry breaking can occur via boundary conditions, it is important to clarify a class of boundary conditions compatible with higher-dimensional gauge invariance.

In the following, we discuss how to obtain a possible set of boundary conditions compatible with gauge invariance from a supersymmetry point of view. To this end, let us consider a 4 + 1-dimensional Abelian gauge theory on an interval

$$S = \int d^4x \int_0^L dy\sqrt{-g(y)}\bigg\{-\frac{1}{4}F_{MN}(x, y)F^{MN}(x, y)\bigg\} \tag{4.1}$$

with a nonfactorizable metric

$$ds^2 = e^{-4W(y)}\eta_{\mu\nu}dx^\mu dx^\nu + g_{55}(y)dy^2. \tag{4.2}$$

The metric reduces to the warped metric discussed by Randall and Sundrum [12] when $g_{55}(y) = 1$ and $W(y) = \frac{1}{2}k|y|$. Another choice of $g_{55}(y) = e^{-4W(y)}$ leads to the model discussed in Ref. [5], in which a hierarchical mass spectrum has been observed.

In order to expand the 5d gauge fields $A_\mu(x, y)$ and $A_5(x, y)$ into mass eigenstates, we follow the discussions in Section III and consider the supersymmetric Hamiltonian,[4]

$$H = Q^2 = \begin{pmatrix} -\frac{1}{\sqrt{g_{55}}}\partial_y\frac{e^{-4W}}{\sqrt{g_{55}}}\partial_y & 0\\ 0 & -\partial_y\frac{1}{\sqrt{g_{55}}}\partial_y\frac{e^{-4W}}{\sqrt{g_{55}}}\end{pmatrix}, \tag{4.3}$$

$$Q = \begin{pmatrix} 0 & -\frac{1}{\sqrt{g_{55}}}\partial_y\frac{e^{-4W}}{\sqrt{g_{55}}}\\ \partial_y & 0\end{pmatrix} \tag{4.4}$$

which act on two-component vectors

$$|\Psi\rangle = \begin{pmatrix} f(y)\\ g(y)\end{pmatrix}. \tag{4.5}$$

The inner product of two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ is defined by

[4]Here, we have represented H and Q in terms of the differential operator ∂_y, instead of d and $d^\dagger$, so that they act on functions rather than forms.

$$\langle\Psi_2|\Psi_1\rangle = \int_0^L dy\sqrt{g_{55}(y)}\bigg\{f_2(y)f_1(y) + \frac{e^{-4W(y)}}{g_{55}(y)}g_2(y)g_1(y)\bigg\}. \tag{4.6}$$

To obtain consistent boundary conditions for the functions $f(y)$ and $g(y)$ in $|\Psi\rangle$, we first require that the supercharge Q is Hermitian with respect to the inner product (4.6), i.e.

$$\langle\Psi_2|Q\Psi_1\rangle = \langle Q\Psi_2|\Psi_1\rangle. \tag{4.7}$$

It turns out that the functions $f(y)$ and $g(y)$ have to obey one of the following four types of boundary conditions[5]:

$$\text{(i) } g(0) = g(L) = 0, \tag{4.8}$$

$$\text{(ii) } f(0) = f(L) = 0, \tag{4.9}$$

$$\text{(iii) } g(0) = f(L) = 0, \tag{4.10}$$

$$\text{(iv) } f(0) = g(L) = 0. \tag{4.11}$$

We further require that the state $Q|\Psi\rangle$ obeys the same boundary conditions as $|\Psi\rangle$, otherwise Q is not a well-defined operator and "bosonic" and "fermionic" states would not form supermultiplets. The requirement leads to

$$\partial_y f(0) = \partial_y f(L) = 0 \quad \text{for (i)}, \tag{4.12}$$

$$\partial_y\bigg(\frac{e^{-4W}}{\sqrt{g_{55}}}g\bigg)(0) = \partial_y\bigg(\frac{e^{-4W}}{\sqrt{g_{55}}}g\bigg)(L) = 0 \quad \text{for (ii)}, \tag{4.13}$$

$$\partial_y f(0) = \partial_y\bigg(\frac{e^{-4W}}{\sqrt{g_{55}}}g\bigg)(L) = 0 \quad \text{for (iii)}, \tag{4.14}$$

$$\partial_y\bigg(\frac{e^{-4W}}{\sqrt{g_{55}}}g\bigg)(0) = \partial_y f(L) = 0 \quad \text{for (iv)}. \tag{4.15}$$

Combining the conditions (4.8), (4.9), (4.10), and (4.11) together with (4.12), (4.13), (4.14), and (4.15), we have found the four types of boundary conditions compatible with supersymmetry,

$$\text{Type (N, N)}: \begin{cases} \partial_y f(0) = \partial_y f(L) = 0, \\ g(0) = g(L) = 0, \end{cases} \tag{4.16}$$

$$\text{Type (D, D)}: \begin{cases} f(0) = f(L) = 0, \\ \partial_y(\frac{e^{-4W}}{\sqrt{g_{55}}}g)(0) = \partial_y(\frac{e^{-4W}}{\sqrt{g_{55}}}g)(L) = 0, \end{cases} \tag{4.17}$$

$$\text{Type (N, D)}: \begin{cases} \partial_y f(0) = f(L) = 0, \\ g(0) = \partial_y(\frac{e^{-4W}}{\sqrt{g_{55}}}g)(L) = 0, \end{cases} \tag{4.18}$$

$$\text{Type (D, N)}: \begin{cases} f(0) = \partial_y f(L) = 0, \\ \partial_y(\frac{e^{-4W}}{\sqrt{g_{55}}}g)(0) = g(L) = 0. \end{cases} \tag{4.19}$$

It follows that the above boundary conditions ensure the Hermiticity of the Hamiltonian, i.e.

$$\langle\Psi_2|H\Psi_1\rangle = \langle H\Psi_2|\Psi_1\rangle. \tag{4.20}$$

Therefore, we have succeeded to obtain the consistent set of boundary conditions that ensure the Hermiticity of the supercharges and the Hamiltonian and also that the action of the supercharge on $|\Psi\rangle$ is well defined. Since the supersymmetry is a direct consequence of higher-dimensional gauge invariance, our requirements on boundary conditions should be, at least, necessary conditions to preserve it. It turns out that the boundary conditions obtained above are consistent with those in Ref. [6], although it is less obvious how the requirement of the least action principle proposed in Ref. [6] is connected to gauge invariance. We should emphasize that the supercharge Q is well defined for all the boundary conditions (4.16), (4.17), (4.18), and (4.19) and hence that the supersymmetric structure always appears in the spectrum, even though the boundary conditions other than the type (N,N) break 4d gauge symmetries, as we will see below.

From the above analysis, the 5d gauge fields $A_\mu(x, y)$ and $A_5(x, y)$ are expanded in the mass eigenstates as follows:

$$A_\mu(x, y) = \sum_n A_{\mu,n}(x)f_n(y), \tag{4.21}$$

$$A_5(x, y) = \sum_n h_n(x)g_n(y), \tag{4.22}$$

where $f_n(y)$ and $g_n(y)$ are the eigenstates of the Schrödinger-like equations

$$-\frac{1}{\sqrt{g_{55}}}\partial_y\frac{e^{-4W}}{\sqrt{g_{55}}}\partial_y f_n(y) = m_n^2 f_n(y), \tag{4.23}$$

$$-\partial_y\frac{1}{\sqrt{g_{55}}}\partial_y\frac{e^{-4W}}{\sqrt{g_{55}}}g_n(y) = m_n^2 g_n(y) \tag{4.24}$$

with one of the four types of the boundary conditions (4.16), (4.17), (4.18), and (4.19). Since the massless states are especially important in phenomenology, let us investigate the massless states of the Eqs. (4.23) and (4.24). Thanks to supersymmetry, the massless modes would be the solutions to the first order differential equation $Q|\Psi_0\rangle = 0$, i.e.

$$\partial_y f_0(y) = 0, \tag{4.25}$$

[5]If we allow $f(L)$ ($g(L)$) to be connected with $f(0)$ ($g(0)$), we have a one parameter family of the boundary conditions [14]:

$$\sin\theta f(0) - \cos\theta f(L) = \cos\theta\bigg(\frac{e^{-4W}}{\sqrt{g_{55}}}g\bigg)(0) - \sin\theta\bigg(\frac{e^{-4W}}{\sqrt{g_{55}}}g\bigg)(L) = 0.$$

$$\partial_y\left(\frac{e^{-4W}}{\sqrt{g_{55}}}g_0(y)\right) = 0. \tag{4.26}$$

The solutions are easily found to be

$$f_0(y) = c, \tag{4.27}$$

$$g_0(y) = c' e^{4W(y)}\sqrt{g_{55}(y)}, \tag{4.28}$$

where c and c' are some constants. We should emphasize that the above solutions do not necessarily imply physical massless states of $A_{\mu,0}(x)$ and $h_0(x)$ in the spectrum. This is because the boundary conditions exclude some or all of them from the physical spectrum. Indeed, $f_0(y)$ (($g_0(y)$)) satisfies only the boundary conditions of the type (N,N) (type (D,D)). Thus, a massless vector $A_{\mu,0}(x)$ (a massless scalar $h_0(x)$) appears only for the type (N,N) (type (D,D)) boundary conditions. This implies that the 4d gauge symmetry is broken except for the type (N,N) boundary conditions.

Let us next discuss geometrical meanings of the boundary conditions. To this end, it is convenient to rewrite the Eqs. (4.23) and (4.24) into a familiar form of the $N = 2$ supersymmetric quantum mechanics. Reparametrizing the coordinate y such that

$$ds^2 = e^{-4W(\tilde{y})}(\eta_{\mu\nu}dx^\mu dx^\nu + d\tilde{y}^2), \tag{4.29}$$

where $W(\tilde{y}) = W(y(\tilde{y}))$, we can rewrite the Eqs. (4.23) and (4.24) into the form

$$-\bar{\mathcal{D}}_{\tilde{y}}\mathcal{D}_{\tilde{y}}\tilde{f}_n(\tilde{y}) = m_n^2\tilde{f}_n(\tilde{y}), \tag{4.30}$$

$$-\mathcal{D}_{\tilde{y}}\bar{\mathcal{D}}_{\tilde{y}}\tilde{g}_n(\tilde{y}) = m_n^2\tilde{g}_n(\tilde{y}), \tag{4.31}$$

where

$$e^{W(\tilde{y})}\tilde{f}_n(\tilde{y}) = f_n(y), \tag{4.32}$$

$$e^{3W(\tilde{y})}\sqrt{g_{55}(\tilde{y})}\tilde{g}_n(\tilde{y}) = g_n(y), \tag{4.33}$$

$$\mathcal{D}_{\tilde{y}} = \partial_{\tilde{y}} + W'(\tilde{y}), \tag{4.34}$$

$$\bar{\mathcal{D}}_{\tilde{y}} = \partial_{\tilde{y}} - W'(\tilde{y}). \tag{4.35}$$

Here, the prime denotes the derivative with respect to $\tilde{y}$. The Hamiltonian and the supercharge can be written, in this basis, as

$$\tilde{H} = \tilde{Q}^2 = \begin{pmatrix} -\bar{\mathcal{D}}_{\tilde{y}}\mathcal{D}_{\tilde{y}} & 0 \\ 0 & -\mathcal{D}_{\tilde{y}}\bar{\mathcal{D}}_{\tilde{y}} \end{pmatrix} = \begin{pmatrix} -\partial_{\tilde{y}}^2 - W''(\tilde{y}) + (W'(\tilde{y}))^2 & 0 \\ 0 & -\partial_{\tilde{y}}^2 + W''(\tilde{y}) + (W'(\tilde{y}))^2 \end{pmatrix}, \tag{4.36}$$

$$\tilde{Q} = \begin{pmatrix} 0 & -\bar{\mathcal{D}}_{\tilde{y}} \\ \mathcal{D}_{\tilde{y}} & 0 \end{pmatrix}. \tag{4.37}$$

These expressions are nothing but the $N = 2$ supersymmetric quantum mechanics given by Witten [11], and $W(\tilde{y})$ is called a superpotential. In this basis, the boundary conditions (4.16), (4.17), (4.18), and (4.19) become

$$\text{Type (N, N)}: \begin{cases} \tilde{f}'(\tilde{y}_0) + W'(\tilde{y}_0)\tilde{f}(\tilde{y}_0) = \tilde{f}'(\tilde{y}_L) + W'(\tilde{y}_L)\tilde{f}(\tilde{y}_L) = 0, \\ \tilde{g}(\tilde{y}_0) = \tilde{g}(\tilde{y}_L) = 0, \end{cases} \tag{4.38}$$

$$\text{Type (D, D)}: \begin{cases} \tilde{f}(\tilde{y}_0) = \tilde{f}(\tilde{y}_L) = 0, \\ \tilde{g}'(\tilde{y}_0) - W'(\tilde{y}_0)\tilde{g}(\tilde{y}_0) = \tilde{g}'(\tilde{y}_L) - W'(\tilde{y}_L)\tilde{g}(\tilde{y}_L) = 0, \end{cases} \tag{4.39}$$

$$\text{Type (N, D)}: \begin{cases} \tilde{f}'(\tilde{y}_0) + W'(\tilde{y}_0)\tilde{f}(\tilde{y}_0) = \tilde{f}(\tilde{y}_L) = 0, \\ \tilde{g}(\tilde{y}_0) = \tilde{g}'(\tilde{y}_L) - W'(\tilde{y}_L)\tilde{g}(\tilde{y}_L) = 0, \end{cases} \tag{4.40}$$

$$\text{Type (D, N)}: \begin{cases} \tilde{f}(\tilde{y}_0) = \tilde{f}'(\tilde{y}_L) + W'(\tilde{y}_L)\tilde{f}(\tilde{y}_L) = 0, \\ \tilde{g}'(\tilde{y}_0) - W'(\tilde{y}_0)\tilde{g}(\tilde{y}_0) = \tilde{g}(\tilde{y}_L) = 0, \end{cases} \tag{4.41}$$

where $\tilde{y}_0 = \tilde{y}(y = 0)$ and $\tilde{y}_L = \tilde{y}(y = L)$. For the Dirichlet boundary conditions of $\tilde{f}(\tilde{y}) = 0$ and $\tilde{g}(\tilde{y}) = 0$ at $\tilde{y} = \tilde{y}_0, \tilde{y}_L$, we can interpret them as the existence of rigid walls at the boundaries. For the other boundary conditions of $\tilde{f}'(\tilde{y}) + W'(\tilde{y})\tilde{f}(\tilde{y}) = 0$ and $\tilde{g}'(\tilde{y}) - W'(\tilde{y})\tilde{g}(\tilde{y}) = 0$ at $\tilde{y} = \tilde{y}_0, \tilde{y}_L$, we can also interpret them as the existence of delta function potentials at the boundaries. Since for a delta function potential a localized (bound) state can appear, the low-energy spectrum will have interesting properties for a nontrivial function $W(\tilde{y})$ [5,12].

Before closing this section, it is instructive to investigate 5d gauge invariance in non-Abelian gauge theories. Let G and H be a non-Abelian gauge group and its subgroup, respectively. We consider the situation that the 4d gauge symmetry G is broken to H via boundary conditions. We denote the generators of G, H, and G/H by $\{T^I\}$, $\{T^a\}$, and

$\{T^{\hat{a}}\}$, respectively. The 4d gauge symmetry breaking of $G \to H$ may be realized by imposing the type (N,N) boundary conditions on the gauge fields $A_M^a(x, y)$, which correspond to the unbroken generators of H, and one of the other boundary conditions on $A_M^{\hat{a}}(x, y)$, which correspond to the broken generators of G/H.

Infinitesimal gauge transformations will be given by

$$\delta A_M^I(x, y) = \partial_M \varepsilon^I(x, y) + g f^{IJK} A_M^J(x, y) \varepsilon^K(x, y), \quad (4.42)$$

where g is the 5d gauge coupling constant and f^{IJK} are the structure constants of G. The boundary conditions for the gauge parameters $\varepsilon^I(x, y)$ should be taken to be the same as $A_\mu^I(x, y)$. This requirement comes from the consistency with the inhomogeneous terms in Eq. (4.42). In order for the 5d gauge invariance under the transformations (4.42) to preserve, the homogeneous terms on the right-hand side of Eq. (4.42) have to obey the same boundary conditions as $A_M^I(x, y)$. Then, it turns out that this is the case provided that [10]

$$f^{\hat{a}\hat{b}\hat{c}} = 0. \quad (4.43)$$

Although the above conditions are not, in general, satisfied for arbitrary choice of G and H, they can be realized, for instance, if the Z_2 parity for the generators of G are assigned as

$$\mathcal{P}(T^a) = +T^a, \quad (4.44)$$

$$\mathcal{P}(T^{\hat{a}}) = -T^{\hat{a}}. \quad (4.45)$$

It is interesting to note that this happens for gauge symmetry breaking via Z_2 orbifolding [4]. Thus, we have found that the 5d gauge invariance is preserved under the infinitesimal gauge transformations (4.42) with the conditions (4.43), even though the 4d gauge symmetry G is broken to H in the 4d effective theory.

V. SUMMARY AND DISCUSSIONS

We have investigated gauge invariant theories with extra dimensions and observed the quantum-mechanical $N = 2$ supersymmetric structure between 4d and extra-space components of gauge fields. The supersymmetry manifests itself in their 4d mass spectrum and massless 4d modes are found to be the solutions to the first order differential equation

$$Q|\Psi_0\rangle = 0. \quad (5.1)$$

It is then clear that the massless modes possess distinct analytic properties from other massive modes, which obey the 2nd order differential equations.

We have also discussed boundary conditions in gauge theories on extra dimensions with boundaries. In a gauge symmetry point of view, it is less obvious to obtain a possible set of boundary conditions consistent with gauge invariance because some of the boundary conditions explicitly break 4d gauge symmetries. On the other hand, in a supersymmetry point of view, the requirement of 5d gauge invariance is replaced by the conditions that the supercharges are Hermitian and also that the action of the supercharges are well defined on a functional space with definite boundary conditions. We should emphasize that the supercharges are well defined in quantum mechanics even if there is no zero energy state, and hence that the degeneracy between "bosonic" and "fermionic" states still holds. In this sense, the boundary conditions we obtained are consistent with 5d gauge invariance, even if some of 4d gauge symmetries are broken via boundary conditions.

Since the origin of the supersymmetry is the gauge invariance in higher dimensions, we expect that any higher-dimensional theories with gaugelike symmetries possess supersymmetry. Such an example is a gauge theory with an antisymmetric field, which often appears in string theory. Since the action of the antisymmetric gauge field can be written in terms of differential forms, it will be straightforward to show the $N = 2$ supersymmetric structure of the theory. It turns out that the $N = 2$ supersymmetry is actually enhanced, in particular, dimensions and that the $N = 2$ supersymmetry algebra given in Section II can be extended to an $N = 4$ supersymmetry algebra by adding a duality operator. The results will be reported elsewhere [15].

We finally make comments on some previous works related to this paper. Howe *et al.* [16] discussed an $N = 2$ worldline supersymmetry for a relativistic spin $N/2$ particle and succeeded to present field equations for massless and massive antisymmetric tensors in arbitrary spacetime dimensions. The $N = 2$ worldline supersymmetry seems to have some connections to the $N = 2$ supersymmetry found in this paper, but a direct relation between them is not clear. Shaposhnikov and Tinyakov [13] considered a gauge invariant theory with a noncompact extra dimension with a weight function depending on the extra-dimensional coordinate and observed that mass eigenfunctions for $A_\mu(x, y)$ satisfy a similar equation to Eq. (4.30). They, however, missed the full supersymmetric structure, since they worked on the gauge $A_5(x, y) = 0$. Although we found an $N = 2$ quantum-mechanical supersymmetry in the spectrum of gauge fields, a similar $N = 2$ supersymmetric structure has already been found in the spectrum of spinor fields. Jackiw and Rebbi [17] showed that a dynamical localization of a chiral fermion can occur in a nontrivial soliton background and that mode functions for left-handed and right-handed spinors are governed by an $N = 2$ supersymmetric quantum mechanics, though they did not use the word "supersymmetry." Arkani-Hamed and Schmaltz [18] considered a five-dimensional fermion coupled to a background scalar that has the dependence on the extra-dimensional coordinate and explicitly constructed a Hamiltonian and $N = 2$ supercharges that form an $N = 2$ supersymmetry algebra. The connection between the de Rham cohomology and the quantum-mechanical superalgebra was argued by R. P. Malik in the context of noncommutative geometry [19].

ACKNOWLEDGMENTS

C. S. L., H. S., and M. S. are supported in part by the Grant-in-Aid for Scientific Research (No. 15340078, No. 14340077, and No. 15540277) by the Japanese Ministry of Education, Science, Sports, and Culture.

APPENDIX: HODGE DECOMPOSITION THEOREM

In this appendix, we give a simple proof of the Hodge decomposition theorem by use of the eigenfunctions of the differential operators $d^\dagger d$ and $dd^\dagger$.

Since $d^\dagger d$ is a Hermitian operator, the eigenfunctions will form a complete set. Thus, any k form $A^{(k)}$ can be expanded as

$$A^{(k)} = \omega_0^{(k)} + \sum_{n_k}{}' a_{n_k}\omega_{n_k}^{(k)}, \tag{A1}$$

where

$$d\omega_0^{(k)} = 0, \tag{A2}$$

$$d^\dagger d\omega_0^{(k)} = (m_{n_k}^{(k)})^2\omega_{n_k}^{(k)} \quad \text{for } m_{n_k}^{(k)} \neq 0. \tag{A3}$$

Here, $\sum_{n_k}'$ denotes the summation over all eigenstates with $m_{n_k}^{(k)} \neq 0$. Since $dd^\dagger$ is also a Hermitian operator, $\omega_0^{(k)}$ can be expanded as

$$\omega_0^{(k)} = \eta_0^{(k)} + \sum_{n_{k-1}}{}' b_{n_{k-1}}\phi_{n_{k-1}}^{(k)}, \tag{A4}$$

where

$$d^\dagger\eta_0^{(k)} = d\eta_0^{(k)} = 0, \tag{A5}$$

$$dd^\dagger\phi_{n_{k-1}}^{(k)} = (m_{n_{k-1}}^{(k-1)})^2\phi_{n_{k-1}}^{(k)} \quad \text{for } m_{n_{k-1}}^{(k-1)} \neq 0. \tag{A6}$$

It is convenient to further introduce the eigenfunctions of $dd^\dagger$ for $k+1$ forms and $d^\dagger d$ for $k-1$ forms as

$$dd^\dagger\phi_{n_k}^{(k+1)} = (m_{n_k}^{(k)})^2\phi_{n_k}^{(k+1)}, \tag{A7}$$

$$d^\dagger d\omega_{n_{k-1}}^{(k-1)} = (m_{n_{k-1}}^{(k-1)})^2\omega_{n_{k-1}}^{(k-1)}. \tag{A8}$$

As shown in Section II, $\omega_{n_k}^{(k)}$ and $\phi_{n_{k-1}}^{(k)}$ are related to $\phi_{n_k}^{(k+1)}$ and $\omega_{n_{k-1}}^{(k-1)}$ as

$$m_{n_k}^{(k)}\omega_{n_k}^{(k)} = d^\dagger\phi_{n_k}^{(k+1)} \quad \text{or} \quad m_{n_k}^{(k)}\phi_{n_k}^{(k+1)} = d\omega_{n_k}^{(k)}, \tag{A9}$$

$$m_{n_{k-1}}^{(k-1)}\omega_{n_{k-1}}^{(k-1)} = d^\dagger\phi_{n_{k-1}}^{(k)} \quad \text{or} \quad m_{n_{k-1}}^{(k-1)}\phi_{n_{k-1}}^{(k)} = d\omega_{n_{k-1}}^{(k-1)}. \tag{A10}$$

A k form satisfying Eqs. (A5) is called a harmonic k form which can be expanded, in terms of a complete set of the harmonic k forms $\{\eta_p^{(k)}, p = 1, 2, \cdots, b_k\}$, as

$$\eta_0^{(k)} = \sum_{p=1}^{b_k} c_p\eta_p^{(k)}. \tag{A11}$$

The integer b_k, which is the number of the independent harmonic k forms, is called the kth Betti number and is known as a topological number of the manifold.

We have thus found that any k form $A^{(k)}$ can be expanded as

$$\begin{aligned} A^{(k)} &= \sum_{p=1}^{b_k} c_p\eta_p^{(k)} + \sum_{n_k}{}' a_{n_k}\omega_{n_k}^{(k)} + \sum_{n_{k-1}}{}' b_{n_{k-1}}\phi_{n_{k-1}}^{(k)} \\ &= \sum_{p=1}^{b_k} c_p\eta_p^{(k)} + \sum_{n_k}{}' \frac{a_{n_k}}{m_{n_k}^{(k)}} d^\dagger\phi_{n_k}^{(k+1)} \\ &\quad + \sum_{n_{k-1}}{}' \frac{b_{n_{k-1}}}{m_{n_{k-1}}^{(k-1)}} d\omega_{n_{k-1}}^{(k-1)}. \end{aligned} \tag{A12}$$

This implies that any k form has the decomposition of the form

$$A^{(k)} = \eta_0^{(k)} + d^\dagger\alpha^{(k+1)} + d\beta^{(k-1)}. \tag{A13}$$

This completes a proof of the Hodge decomposition theorem.

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