

Dark energy dominance and cosmic acceleration in first-order formalismGianluca Allemandi,^{1,*} Andrzej Borowiec,^{2,†} Mauro Francaviglia,^{1,‡} and Sergei D. Odintsov^{3,§}¹*Dipartimento di Matematica, Università di Torino, Via C. Alberto 10, 10123 Torino, Italy*²*Institute of Theoretical Physics, University of Wrocław, Pl. Maksa Borna 9, 50-204 Wrocław, Poland*³*Inst. Catalana de Recerca i Estudis Avancats (ICREA), Barcelona and Inst. d'Estudis Espacials de Catalunya (IEEC)/ICE(CSIC), Campus UAB, Fac. Ciencias, Torre C5-Par-2a E-08193 Bellaterra (Barcelona), Spain*

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The current accelerated universe could be produced by modified gravitational dynamics as it can be seen, in particular, in its Palatini formulation. We analyze here a specific nonlinear gravity-scalar system in the first-order *Palatini* formalism which leads to a Friedmann-Robertson-Walker cosmology different from the purely metric one. It is shown that the emerging Friedmann-Robertson-Walker cosmology may lead either to an effective quintessence phase (cosmic speed-up) or to an effective phantom phase. Moreover, the already known *gravity assisted dark energy dominance* occurs also in the first-order formalism. Finally, it is shown that a dynamical theory able to resolve the cosmological constant problem exists also in this formalism, in close parallel with the standard metric formulation.

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I. INTRODUCTION

The dark energy problem (for astrophysical data indicating dark energy existence and related universe acceleration see [1] and references therein) is considered to be a challenge for modern cosmology. There are various approaches to its construction. In general, one can classify most efforts as (a) modified gravity (for instance, the so-called $f(R)$ theory); or (b) introducing some exotic matter with negative pressure. The *truth*, as usual, may lie in between. Indeed, since the two directions above are completely phenomenological one can go further and try to modify gravity via a nonstandard (nonminimal) coupling with matter. This may in fact provide the effective description for a dark energy universe. The importance of such models already has been pointed out in [2] where a possible origin for dark energy dominance was suggested. In [3,4] a suitable nonlinear matter-gravity theory was used to construct the dynamical approach to the resolution of the cosmological constant problem. (Such coupling may be used for cosmological purposes as in [5]). Here, the gravity-matter coupling part is assumed to be a $f(R)$ -like theory, nonminimally coupled with a scalar-field Lagrangian. It is interesting that in such models the standard conformal transformation to scalar-tensor theory is overcomplicated (if possible at all). This implies that in the purely metric formalism only one possible frame really exists. A first aim of ours is to see how much the picture changes if one allows a first-order à la Palatini approach to the gravitational field.

In the recent study of modified gravity with a nonlinear $1/R$ term [6] (for its one-loop quantization see [7]) in

relation with cosmic acceleration it was further drawn the important lesson that the metric formalism is not equivalent to the Palatini (first-order) formalism. (Actually, this fact was known in general after the early proof given in [8].) Indeed, while modified gravity of only special form may be viable in the metric version [9] (due to a possible instability [10]) its Palatini version may be instead viable (see [11,12] and references therein). Of course, the cosmological dynamics of both versions are in principle different. Notice, however, that modified gravity may be always rewritten as a (well-studied) scalar-tensor theory. It is clear that a generic nonlinear matter-gravity theory may lead also to nonequivalent gravitational physics, depending on the choice of either metric or Palatini description.

In [2] it has been shown how dark energy may become dominant over standard matter at small curvatures in the metric formulation. This is directly caused by the choice of the Lagrangian. Nonlinear gravity-scalar-field systems and modified gravity may nevertheless provide an effective phantom or quintessence dark energy with the possibility of explaining the present speed-up of the universe (and without the need of introducing negative kinetic energy fields) [2].

It has been proven in [3] that a dynamical approach to the cosmological constant based on a suitable nonlinear gravity-scalar-field system with a scalar-field potential having a minimum at a generic but negative value provides a possible solution to the cosmological constant problem. This is based on the presence of a nonstandard kinetic term in the scalar-field Lagrangian, which diverges at zero curvature. In this particular model the cosmological constant freezes at a nearly zero value, while field equations are stable under radiative corrections and they provide the expected asymptotic limit.

The same class of nonlinear matter-gravity theories are here studied in the Palatini formalism (where the purely gravitational part is suitably generalized) and applied to

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alternative gravity. This formalism, considering the metric and the connection as independent fields, produces second order field equations which are not affected by instability problems and seem to be in a fairly acceptable accordance with solar system experiments [12]. The Palatini version leads to a completely new formulation when compared with the standard metric formulation and it provides cosmological models which are able to explain also the present time cosmic speed-up.

It is shown also that the Palatini formalism in the case of nonlinear gravity-scalar systems gives an acceptable realization of the dark energy dominance and the resolution of the cosmological constant problem. Thus, even if the metric and the Palatini formulations apparently lead to different gravitational physics, one can see that the same cosmological phenomena may qualitatively occur in both frameworks, which therefore deserve to be carefully compared.

II. FIRST-ORDER NONLINEAR GRAVITY-MATTER SYSTEMS: THE FIRST-ORDER FORMALISM

We consider a four-dimensional Lorentzian manifold M with a metric $g_{\mu\nu}$ and an independent connection $\Gamma_{\mu\nu}^\alpha$. The starting nonlinear gravity-matter Lagrangian is

$$L = \sqrt{g}(F(R) + f(R)L_d) + \kappa L_{\text{mat}}(\Psi), \quad (1)$$

where $f(R)$ and $F(R)$ are some analytic functions of the scalar-field $R = g^{\alpha\beta}R_{\alpha\beta}(\Gamma)$ and L_d is a scalar-field Lagrangian; here we set $L_d = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi) = -\frac{1}{2}\|\nabla\phi\|^2 + V(\phi)$. In principle, L_d could be any other matter Lagrangian (describing spinors, vectors, etc.). Moreover, one can think of more general cases when $f(R)$ (or even a function of Ricci squared terms [13]) couples to the Riemann (Ricci) tensor [14] contracted with some derivatives of matter and multiplied by a similar matter Lagrangian. It is important to remark that in all cases (even if L_d is chosen to be the Lagrangian of a scalar theory) the general class of theories described by phenomenological Lagrangians of the form (1) cannot be conformally transformed into an equivalent scalar-tensor theory, with the only exception of the theories already treated in [11,12] [i.e. all the theories in which $L_d = 1$, i.e. $f(R)$ theories]. The term L_{mat} represents a matter Lagrangian, functionally depending on arbitrary matter fields Ψ together with their first derivatives, equipped with a gravitational coupling constant $\kappa = 8\pi G$, which can be supposed to be $\kappa = 8\pi G = 1$ in natural units.

In the first-order *Palatini* formalism the Ricci-like scalar is $R \equiv R(g, \Gamma) = g^{\alpha\beta}R_{\alpha\beta}(\Gamma)$ where $R_{\mu\nu}(\Gamma)$ is the Ricci tensor of any independent torsionless connection Γ ; see [8,11] for details. Equations of motion are respectively:

- (1) *Field equations for the metric field g :*

$$F'(R)R_{\mu\nu} - \frac{1}{2}F(R)g_{\mu\nu} + f'(R)L_dR_{\mu\nu} - \frac{1}{2}f(R)L_dg_{\mu\nu} + f(R)T_{\mu\nu} = \tau_{\mu\nu}, \quad (2)$$

where $T_{\mu\nu} = -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi$ is stress-energy tensor of the scalar field, and $\tau_{\mu\nu} \equiv T_{\mu\nu}^{\text{mat}}$ is the matter stress-energy tensor generated by taking the variational derivative of $L_{\text{mat}}(\Psi)$. Taking the trace of Eq. (2) with respect to the metric $g^{\mu\nu}$ we obtain the scalar-valued master equation:

$$F'(R)R - 2F(R) + (f'(R)R - 2f(R))L_d + f(R)T = \tau, \quad (3)$$

where $\tau = g^{\mu\nu}\tau_{\mu\nu}$ and $T = g^{\mu\nu}T_{\mu\nu} = L_d - V(\phi) = -\frac{1}{2}\|\nabla\phi\|^2$. The above equation can be rewritten, more conveniently, as

$$2F(R) - F'(R)R + \tau = (f'(R)R - f(R))L_d - f(R)V(\phi).$$

Hence, the scalar-valued equation (3) controls solutions of Eq. (2). We shall refer to this scalar-valued equation as the *structural equation* of spacetime. For any real solution $R = F(\tau, T)$ of (3) it follows that $f(R) = f(F(\tau, T))$ and $f'(R) = f'(F(\tau, T))$ can be seen as functions of the traces τ and T of the stress-energy tensors of matter and of the scalar-field ϕ . For convenience we shall use the abuse of notation $f(\tau, T) = f(F(\tau, T))$ and $f'(\tau, T) = f'(F(\tau, T))$. Thus, in the same way as in [11,13], the generalized Einstein equations may be introduced:

$$R_{\mu\nu}(\Gamma) = g_{\mu\alpha}P_\nu^\alpha, \quad (4)$$

where one defines the operator $P_\nu^\mu = \frac{c}{b}\delta_\nu^\mu - \frac{f(R)}{b}T_\nu^\mu + \frac{1}{b}\tau_\nu^\mu$. The scalars b and c , depending on R , are here defined as

$$\begin{cases} b = b(R) = F'(R) + f'(R)L_d \\ c = c(R) = \frac{1}{2}(F(R) + f(R)L_d) = \frac{(L - L_{\text{mat}})}{2}\sqrt{g} \end{cases} \quad (5)$$

(see also Sec. III).

- (2) *Field equations for the connection field Γ :* varying the Lagrangian (1) with respect to the connection one gets

$$\nabla_\lambda^\Gamma[\sqrt{g}g^{\mu\nu}(F'(R) + f'(R)L_d)] = 0, \quad (6)$$

which states that Γ can be chosen to be the Levi-Civita connection of the metric $h_{\mu\nu} = (F'(R) + f'(R)L_d)g_{\mu\nu} \equiv bg_{\mu\nu}$, providing the spacetime manifold with the typical bimetric structure ensuing from the first-order Palatini formalism [12]. This

implies that the generalized Einstein equation (4) can be immediately rewritten as

$$R_{\mu\nu}(h) = g_{\mu\alpha} P_{\nu}^{\alpha}, \quad (7)$$

where $R_{\mu\nu}(h)$ is now the Ricci tensor of the new metric h , being $\Gamma = \Gamma_{\text{LC}}(h)$.

- (3) *Field equations for the scalar (dilatonlike) field ϕ* : performing the variation of the Lagrangian (1) with respect to ϕ we obtain:

$$\partial_{\nu}(\sqrt{g}f(R)g^{\mu\nu}\partial_{\mu}\phi) = -\sqrt{g}f(R)V'(\phi). \quad (8)$$

- (4) *Field equations for the matter fields Ψ* : they can be obtained from the Lagrangian (1) by the usual prescription $\frac{\delta L_{\text{mat}}(\Psi)}{\delta \Psi} = 0$, depending on the particular form of $L_{\text{mat}}(\Psi)$.

Before proceeding further let us make two important notes. First, in order to get Eq. (6) one has to additionally assume that L_{mat} is functionally independent of an arbitrary connection Γ ; however it may contain metric covariant derivatives ∇^g of some matter fields denoted here by Ψ . This means that the matter stress-energy tensor $T_{\mu\nu}^{\text{mat}} \equiv \tau_{\mu\nu}$ depends solely on the metric g and fields Ψ together with their partial derivatives. Thus, usually this tensor is covariantly conserved, in a sense of metric covariant derivative, i.e. $\nabla_g^{\mu}\tau_{\mu\nu} = 0$. Nevertheless, we do not assume *a priori* that the (metric) covariant derivative of the left-hand side of Eq. (2) identically vanishes (in fact it does not in general). However this does not provide any inconsistency in our model. When we solve the field equations of motion with a source—here the right-hand side of Eq. (2)—having it a vanishing covariant divergence thus also the left-hand side of Eq. (2) taken on this particular solution (i.e. “on shell”) will have automatically vanishing covariant derivative as well.

Second, we are now close to discussing energy-momentum conservation in modified gravity models, as it is well known that any generally covariant field theory admits the covariant conservation of energy momentum which can be expressed in the form of generalized Bianchi identity. This general fact has been presented in many books and articles and it is widely recognized as (second) Noether theorem (see e.g. a recent book [15] and references quoted therein). More, in particular, for the case of modified (including scalar-tensor) theories of gravity (also within a Palatini formalism) the issue has been nicely discussed in a recent paper [16], where explicit formulas for the Bianchi identity also can be found.

III. FRW COSMOLOGY FROM GENERALIZED EINSTEIN EQUATIONS

Owing to the astonishing cosmological experimental results recently obtained in [1] and to the renewed interest

in alternative theories of gravity for cosmological applications, it is quite interesting to study the cosmological implications of the formalism above. Let g be a Friedmann-Robertson-Walker (FRW) metric:

$$g = -dt^2 + a^2(t) \left[\frac{1}{1 - Kr^2} dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) \right], \quad (9)$$

where $a(t)$ is a scale factor and K is the space curvature ($K = 0, 1, -1$). The stress-energy tensor of a matter (ideal fluid) is

$$T_{\mu\nu}^{\text{mat}} \equiv \tau_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + pg_{\mu\nu},$$

where p is the pressure, ρ is the density of matter, and u^{μ} is a comoving fluid vector, which in a comoving frame [$u^{\mu} = (1, 0, 0, 0)$] becomes simply

$$\tau_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{pa^2(t)}{1-Kr^2} & 0 & 0 \\ 0 & 0 & pa^2(t)r^2 & 0 \\ 0 & 0 & 0 & pa^2(t)r^2\sin^2(\theta) \end{pmatrix}. \quad (10)$$

The standard relations between the pressure p , the matter density ρ , and the expansion factor $a(t)$ are assumed

$$p = w\rho, \quad \rho = \eta a^{-3(1+w)}, \quad (11)$$

where particular values of the parameter $w \in \{-1, 0, \frac{1}{3}\}$ will correspond to the vacuum, dust, or radiation dominated universes. Exotic matter (which is up to now under investigation as a possible model for dark energy) admits instead values of $w < -1$; see e.g. [1]. These expressions (11) follow from the metric covariant conservation law of the energy-momentum $\nabla_g^{\mu}\tau_{\mu\nu} = 0$ (as discussed above) and consequently the continuity equation should hold:

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (12)$$

where $H = \frac{\dot{a}}{a}$ is the *Hubble constant*.

We recall here that the field equations for the metric field g can now be rewritten in terms of the new metric h , as already done in (7). Relying on the methods already developed in [11] it is simple to see that the generalized FRW equations following from (7) are

$$\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{2b} \right)^2 + \frac{K}{a^2} = \frac{1}{2}P_1^1 - \frac{1}{6}P_0^0, \quad (13)$$

where P_1^1 and P_0^0 are the components of $P_{\nu}^{\mu} = \frac{c}{b}\delta_{\nu}^{\mu} - \frac{f(R)}{b}T_{\nu}^{\mu} + \frac{1}{b}\tau_{\nu}^{\mu}$. We recall here that the parameters b and c are defined (5). Owing to the cosmological principle and to the consequent homogeneity and isotropy of spacetime, we argue that $T_{\mu\nu}$, $\tau_{\mu\nu}$, and ϕ depend just on the time parameter t . As already noticed in [11,13] for much simpler cases, we get from the structural equation (see the above

discussion) that both b and c can be expressed just as functions of time.

In our case, we have moreover that $L_d = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, so that

$$\begin{cases} P_1^1 = \frac{c}{b} - \frac{f(R)}{b}T_1^1 + \frac{1}{b}\tau_1^1 = \frac{c}{b} + \frac{p}{b} \\ P_0^0 = \frac{c}{b} - \frac{f(R)}{b}T_0^0 + \frac{1}{b}\tau_0^0 = \frac{c}{b} - \frac{f(R)}{2b}\dot{\phi}^2 - \frac{p}{b}. \end{cases} \quad (14)$$

The modified FRW equation thus can be written as

$$\begin{aligned} \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{2b}\right)^2 + \frac{K}{a^2} &= \frac{1}{6} \frac{f(R)}{b} (L_d - V) + \frac{c}{3b} \\ &+ \frac{3w+1}{6b} \eta a^{-3(1+w)}, \end{aligned} \quad (15)$$

and the nonminimally coupled scalar-field equation reduces to

$$\frac{d}{dt}(\sqrt{g}f(R)\dot{\phi}) = \sqrt{g}f(R)V'(\phi). \quad (16)$$

From now on we assume, for simplicity and following the headlines of [2], that $V(\phi) \equiv 0$. This implies that the field equations can be written as $\sqrt{g}f(R)\dot{\phi} = \text{const}$ or equivalently $gf(R)^2L_d = \beta^2 = \text{const}$. It follows that, on shell with respect to the above field equations for the scalar-field ϕ , the nonminimal coupling term in the Lagrangian takes the form

$$f(R)L_d = \beta^2 f(R)^{-1} a^{-6}, \quad (17)$$

which differs from the results of [2] obtained in the metric formalism in the particular case $F(R) = R$, which is in fact degenerate (see below).

A. A particular case: $F(R) = R$

We start here considering the degenerate case $F(R) = R$. This case is worth considering as it is the simplest case of nonlinear gravity-scalar system reproducing general relativity in the limit $f(R) = 0$; results can moreover be simply compared with results already obtained in [2] in the metric formalism. We stress that to obtain an exactly solvable class of models we will hereafter consider in larger detail the particular cases $f(R) = \alpha R^n$ ($\alpha \neq 0$; $n \neq 1$) for matter-free universe with vanishing spatial curvature ($K = 0$), since these cases are in fact analytically solvable. Nothing would prevent a qualitative study of more general cases. We will specify these conditions step by step in the following, when explicitly necessary.

In the case $F(R) = R$ the master equation gives

$$R + \tau = \beta^2 f(R)^{-2} (f'(R)R - f(R)) a^{-6}, \quad (18)$$

which simply becomes an (analytic) equation for R :

$$R + (3w-1)\eta a^{-3(1+w)} = \beta^2 \frac{f'(R)R - f(R)}{f(R)^2 a^6}. \quad (19)$$

In the case of polynomial (or more generally analytic, if

solvable) functions $f(R)$, solutions R of the above equation are, in general, rational (analytic) functions of the variable a . Consequently, as already remarked before, R can be simply expressed as a function of time. For any solution $R = R(a(t))$ one can further calculate:

$$\begin{cases} b = 1 + f'(R)L_d = 1 + \beta^2 f'(R)f(R)^{-2} a^{-6} \\ c = \frac{1}{2}(R + f(R)L_d) = \frac{1}{2}(R + \beta^2 f(R)^{-1} a^{-6}). \end{cases} \quad (20)$$

The right-hand side of the generalized FRW equation becomes

$$\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{2b}\right)^2 + \frac{K}{a^2} = \frac{\beta^2}{6bf(R)a^6} + \frac{c}{3b} + \frac{(3w+1)\eta}{6ba^{3(1+w)}}, \quad (21)$$

and it is consequently a rational function of a .

Let us now consider the specific example where $f(R) = \alpha R^n$ ($\alpha \neq 0$; $n \neq 1$). It is supposed that there is no matter content in these models. We thus obtain, from the structural equation,

$$R = \alpha R^n (n-1)L_d, \quad (22)$$

so that we have $\alpha R^n L_d \equiv f(R)L_d = \frac{1}{n-1}R$. Apart from the trivial solution $R = 0$ [which implies $b = 1$, $c = 0$ and $(\frac{\dot{a}}{a})^2 + \frac{K}{a^2} = 0$] there exists also a nontrivial solution corresponding to the case $1 = \alpha R^{n-1}(n-1)L_d$. In this nontrivial case the time-dependent parameters are found to be ($n \neq \frac{1}{2}$ to ensure $b \neq 0$):

$$\begin{cases} b = 1 + f'(R)L_d = 1 + \frac{n}{n-1} = \frac{2n-1}{n-1} \\ c = \frac{1}{2}(R + \alpha R^n L_d) = \frac{1}{2}R \left(1 + \frac{1}{n-1}\right) = R \frac{n}{2(n-1)}. \end{cases} \quad (23)$$

The conformal parameter b is constant and the modified Friedmann equations are

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} &= \frac{1}{6} \frac{\alpha R^n L_d}{b} + \frac{c}{3b} = \frac{\alpha R^n L_d}{6b} + \frac{n}{6(2n-1)} R \\ &= \frac{1}{6} R \left(\frac{1}{(n-1)b} + \frac{n}{2n-1} \right), \end{aligned} \quad (24)$$

which in turn can be rewritten in the simplified form

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{6} R \frac{n+1}{2n-1}. \quad (25)$$

As we already stressed before, the coefficient b is now constant; then $R = g^{\mu\nu}R_{\mu\nu}(bg) = R(g)$ is the true Ricci scalar of the FRW metric g , which is known to be equal

$$R(g) = 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 \right] = 6H^2(1-q), \quad (26)$$

where q is the *deceleration parameter*. Hence the two metrics g and h are equivalent, apart from a constant conformal factor. The generalized Friedmann equation (specified now to the exactly solvable case $K = 0$) takes the form:

$$H^2 = H^2 \frac{n+1}{2n-1} (1-q), \quad (27)$$

which immediately gives the value of the deceleration parameter

$$q = \frac{2-n}{1+n} = -1 + \frac{3}{n+1} \quad (28)$$

without having to solve explicitly an ordinary differential equation for the scale factor $a(t)$. It is however possible to obtain the explicit dependence of the scale factor on time. We recall that it has been assumed from the beginning [see (16)] that $V(\phi) = 0$ and consequently the field equation for the scalar field is

$$\frac{d}{dt}(\sqrt{g}R^n\dot{\phi}) = 0, \quad (29)$$

which in turn implies that $\sqrt{g}R^n\dot{\phi} = \text{const}$ or equivalently that on shell $R^{2n}L_d = \beta^2 a^{-6}$. From the structural equation (20) we consequently obtain, in the case under discussion, that $R = [(n-1)\alpha\beta^2]^{1/(n+1)} a^{-[6/(n+1)]}$. Therefore, again under the explicit further assumption $K = 0$, we have

$$\begin{aligned} \dot{a} &= \left[\frac{n+1}{6(2n-1)} \right]^{1/2} [(n-1)\alpha\beta^2]^{1/[2(n+1)]} a^{1-[3/(n+1)]} \\ &= L(n, \alpha, \beta) a^{1-[3/(n+1)]}, \end{aligned} \quad (30)$$

where we have defined for simplicity the constant factor $L(n, \alpha, \beta) = \left[\frac{n+1}{6(2n-1)} \right]^{1/2} [(n-1)\alpha\beta^2]^{1/[2(n+1)]}$. Solving this simple differential equation we get the explicit expression of the scale factor as a function of time:

$$a(t) = \left[\frac{3L(n, \alpha, \beta)}{n+1} \right]^{(n+1)/3} (t_s \pm t)^{(n+1)/3}, \quad (31)$$

where t_s is some integration constant. More precisely, Friedmann equation in the simplest form $H^2 \sim a^{-2\gamma}$ (or equivalently $\dot{a} \sim a^{1-\gamma}$), where $\gamma = \frac{3(w_{\text{eff}}+1)}{2}$, implies that the deceleration parameter is $q = \gamma - 1$. Moreover, in the case $\gamma > 0$, choosing the solution corresponding to an expanding universe one has $a \sim t^{1/\gamma}$, i.e. a forever expanding universe with big bang at the origin $t_s = 0$. Conversely in the case $\gamma < 0$, changing the arrow of time in order to avoid shrinking solutions (see [2]), we obtain that $a \sim (t_s - t)^{1/\gamma}$, i.e. we find a universe with a final big rip singularity. Thus $H \sim (t_s - t)^{-1} \rightarrow \infty$ at finite future time t_s .

Coming back to the case under consideration governed by Eq. (30) it follows that the effective value of w (for $n \neq -1$) is

$$w_{\text{eff}} = \frac{1-n}{n+1} = -1 + \frac{2}{n+1}. \quad (32)$$

The reliability conditions one has to assume are respectively $\frac{n+1}{2n-1} > 0$ and $(n-1)\alpha > 0$, which follow directly from (30). Models for accelerating universe occur conse-

quently for $n > 2$ or $n < -1$. The first case corresponds to an effective quintessence when $-1 < w_{\text{eff}} < -\frac{1}{3}$ and one deals with an initial big bang type singularity at the origin. In contrast, the second case leads to an effective phantom ($w_{\text{eff}} < -1$) (without its explicit introduction) with a big rip type final singularity (for a discussion of it in modified gravity see [17]), with the asymptotic behavior $a(t_s) \sim +\infty$.

The case $n = -1$ should be treated separately. In this case $\dot{a} = 0$; this implies that the trivial solution $R = 0 = \dot{\phi}$ emerges and we obtain a nonexpanding universe model with $a = a_0 = \text{constant}$.

1. The radiating universe

One can also combine the above case $F(R) = R$ with the presence of radiation. Adding radiating matter ($w = \frac{1}{3}$) gives rise to the following modified Friedmann equation:

$$\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{2b} \right)^2 + \frac{K}{a^2} = \frac{\beta^2}{6bf(R)a^6} + \frac{c}{3b} + \frac{\eta}{3ba^4}. \quad (33)$$

To obtain exactly solvable models it is again convenient to specify to the case $f(R) = \alpha R^n$; we thus obtain ($n \neq -1, \frac{1}{2}, 1$):

$$\begin{aligned} H^2 &= \frac{n+1}{6(2n-1)} [(n-1)\alpha\beta^2]^{1/(n+1)} a^{-[6/(n+1)]} \\ &\quad + \frac{(n-1)\eta}{3(2n-1)} a^{-4}, \end{aligned} \quad (34)$$

and we are able to compute the deceleration parameter to be

$$q = \frac{\frac{2-n}{3(2n-1)} [(n-1)\alpha\beta^2]^{1/(n+1)} a^{-[6/(n+1)]} + \frac{2(n-1)}{3(2n-1)} \eta a^{-4}}{2H^2}. \quad (35)$$

A detailed analysis of this last equation is more complicated but the final answer is similar to the previous case: accelerated universe can be obtained only for $n > 2$ or $n < -1$.¹ We stress, moreover, that the asymptotic value of the deceleration parameter for big enough radius $a(t)$ in both cases remains the same, i.e.

$$q_{\text{asympt}} = \frac{2-n}{n+1},$$

as in the radiation-free example (28). Because of this the singularity types should be the same as well. For $n < -1$ (which, by the way, corresponds to the case $\gamma < 0$), besides the final big rip singularity one can encounter the initial big bang singularity caused by radiation. To be more precise this follows from the qualitative behavior of H^2 [see (34)]:

¹Notice that some cases are excluded by the condition $H^2 > 0$ and others give rise to deceleration.

$$H^2 \sim Aa^{-2\gamma} + Ba^{-4} \quad (n < -1),$$

where A and B can be simply deduced from (34). In the case $\gamma < 0$ the first term dominates in late time universe, giving rise to a big rip like singularity. On the contrary, in the same case $\gamma < 0$, we get that small values of a correspond to a behavior of $H^2 \sim Ba^{-4}$, which implies the presence of a big bang singularity at early time universe. In the case $n > 2$ ($\gamma > 0$) one obtains an accelerated and expanding model, such that the initial big bang is still present. Finally we remark that for $n > 2$ there exists a critical value for the radius

$$a_{\text{crit}}^2 = \left[\frac{2(n-1)}{n-2} \eta \right]^{(n+1)/(2n-1)} [(n-1)\alpha\beta^2]^{-[1/(2n-1)]}, \quad (36)$$

which provides a transition from deceleration to acceleration.

To summarize we have thus proven that, also in the Palatini formalism, nonlinear gravity-matter systems based on the Lagrangian (1) with $F(R) = R$ lead to accelerating universe in the same way as it happens in the metric formalism. Big rip singularities appear in the case of effective phantom models in vacuum universe, while quintessence models contain big bang like singularities. We have moreover shown that in the case of radiation models, where a radiationlike fluid is additionally considered, we have both big rip and big bang singularities appearing. A full set of possible FRW cosmological solutions of the above nonlinear gravity-matter system based on the Lagrangians (1) with $F(R) = R$ has been found, under the specific hypotheses $f(R) \sim R^n$, $V(\phi) = 0$, and $K = 0$ both in the case of vacuum and radiating universes. It turns out that the same solutions were found already and studied, for the same class of Lagrangians, in the framework of the purely metric formalism [2] where they, however, represent a proper subset in the set of all possible solutions. We think that a bit of discussion is here appropriate on these arguments.

It is well known that, in the matter-free case $L_d = 0$, $L_{\text{mat}} = 0$, i.e. for pure $F(R)$ gravity, solutions of the Palatini formalism represent in general a subset in the set of all possible solutions allowed by the purely metric formalism; see [18]. As a matter of fact in that case the conformal factor $b = F'(R)$ relating the two metrics $g_{\mu\nu}$ and $h_{\mu\nu}$ becomes constant by virtue of the master equation; this implies that the scalar R is equal to the Ricci scalar of

the metric g . As far as the field equations in the metric formalism are concerned, it implies also that the second order differential operator $\nabla_\mu \nabla_\nu - g_{\mu\nu} \square$ acting on $F'(R)$ vanish identically; see e.g. [18]. This finally implies in turn that the field equations in the metric formalism, expressed in the metric $g_{\mu\nu}$, almost reproduce the field equations of the Palatini formalism.

It is easy to check that exactly the same mechanism acts in our case, provided that $f(R) \sim R^n$. As a consequence we obtain that, in the particular example we have here analyzed [i.e. $F(R) = R$], the Palatini formalism provides at the same time solutions to the metric one.

B. First-order scalar $F(R)$ - $f(R)$ nonminimal coupling

We generalize hereafter the analysis to the more general family of models already introduced in (1) for nonlinear gravity-scalar system. To obtain exactly solvable models, which are preferable as they allow us to discuss relevant physical consequences, we will specify step by step the necessary conditions [$f(R) = \alpha R^n$ ($\alpha \neq 0$; $n \neq 1$), $\tau = 0$, and $K = 0$]. We thus consider the more general class of Lagrangians:

$$L = \sqrt{g}(F(R) + f(R)L_d) + L_{\text{mat}}, \quad (37)$$

where $L_d = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)$ and L_{mat} represents a matter Lagrangian but now $F(R) \neq R$. We get now that the right-hand side of the generalized Friedmann equation (17) becomes:

$$\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{2b}\right)^2 + \frac{K}{a^2} = \frac{\beta^2}{6bf(R)a^6} + \frac{c}{3b} + \frac{(3w+1)\eta}{6ba^{3(1+w)}}. \quad (38)$$

We proceed by specifying the model to the concrete particular case ($n \neq 1$, $m \neq 2$)

$$\begin{cases} f(R) = \alpha R^n \\ F(R) = \omega R^m \end{cases} \quad (39)$$

without matter (i.e. $\tau = 0$ and $\tau_{\mu\nu}^{\text{mat}} = 0$) which leads again to an exactly solvable class of models. Combining the master equation with the on-shell value $f(R)^2 L_d = \beta^2 a^{-6}$ [see (19)] it is simply possible to obtain the Ricci scalar as a function of the radius a :

$$R = \xi a^{-[6/(m+n)]}, \quad (40)$$

where the coefficient ξ is given by $\xi = [\frac{\beta^2(n-1)}{\omega(2-m)}]^{1/(n+m)}$. In this case it follows that

$$\begin{cases} b = (m\omega\xi^{m-1} + n\alpha^{-1}\beta^2\xi^{-(n+1)})a^{-[6(m-1)/(m+n)]} = b_0 a^{-[6(m-1)/(m+n)]} \\ c = \frac{1}{2}(\omega\xi^m + \alpha\beta\xi^{-n})a^{-[(6m)/(m+n)]} = c_0 a^{-[(6m)/(m+n)]}. \end{cases} \quad (41)$$

Since now $b = b_0 a^{-\lambda}$ is proportional to a power of a , it follows that the generalized Hubble parameter $\dot{H} = \frac{\dot{a}}{a} + \frac{\dot{b}}{2b}$ is, up to the constant factor $1 - \frac{3(m-1)}{m+n}$, proportional to $\dot{H}$. Therefore one has (again in the solvable case $K = 0$):

$$H^2 \sim \left[\frac{\beta^2}{6b_0\alpha\xi^n} + \frac{c_0}{3b_0} \right] a^{-[6/(m+n)]} \sim a^{-[6/(m+n)]}. \quad (42)$$

This further implies that $\dot{a} \sim a^{1-[3/(m+n)]}$ (which resembles

the case $m = 1$, already treated in Sec. III). Consequently,

$$a(t) \sim (t_s \pm t)^{(m+n)/3}.$$

The deceleration parameter turns out to be $q = -1 + \frac{3}{n+m}$. As for the very particular case, already considered in Sec. III, we obtain by a similar analysis an expanding universe in the case $m + n > 3$, presenting a big bang like singularity; conversely, for $m + n < 0$ the expanding solution leads to a big rip finite time singularity.

We stress that now the Palatini formalism and the metric one are completely different both from a qualitative and from a quantitative viewpoint. This is related to the fact that in this case the conformal factor b as given by (5) is no longer constant. Moreover we have seen that in the Palatini formalism field equations are easily solvable also in this more general case, while in the metric formalism this is by no means true.

1. The case $F(R) = R + \mu G(R)$

We can consider the interesting case when $F(R) = R + \mu G(R)$, which reproduces the case already treated in Sec. III A in the case $\mu = 0$ [or in the case $G(R) = 0$]. The corresponding Lagrangian is then

$$L = \sqrt{g}(R + \mu G(R) + f(R)L_d) + L_{\text{mat}}(\Psi), \quad (43)$$

and the master equation is

$$\begin{aligned} R + 2\mu G(R) - \mu G'(R)R + \tau \\ = (f'(R)R - f(R))L_d - f(R)V(\phi). \end{aligned} \quad (44)$$

We specialize moreover to the case $f(R) = \alpha R^n$, $G(R) = R^{2n}$, $V(\phi) = 0$, and for simplicity $L_{\text{mat}} = 0$ (the case of vacuum universe). Equation (5) gives in this case:

$$\begin{cases} b = 1 + f'(R)L_d + \mu G'(R) = 1 + \frac{n}{n-1} = \frac{2n-1}{n-1} \\ c = \frac{1}{2}(R + f(R)L_d + \mu G(R)) = R \frac{n}{2(n-1)} - \frac{1}{2}\mu G(R), \end{cases} \quad (45)$$

where the value of the conformal factor b is again constant, as much as $f'(R)L_d + \mu G'(R) = b - 1$ is constant too. This implies that the solutions of the Palatini formalism are at the same time solutions of the metric formalism, as much as in the case $F(R) = R$ previously examined [see our discussion after formula (36)]. It is however clear that in this case the Palatini formalism is much easier to handle and more effective than the purely metric formalism (even if it provides the same solutions). Substituting the actual value of L_d along solutions into the master equation, as given by (17), we obtain now

$$R^{n+1} + 2\mu(1-n)R^{3n} = \alpha(n-1)\beta^2 a^{-6}. \quad (46)$$

Since R is positive, in order to get an approximate solution of the polynomial equation above, one needs to distinguish two cases:

- (1) Small values of R ($R \ll 1$) and $n \gg 0$ or $R \gg 1$ and $n \ll 0$ lead to

$$R \sim a^{-[6/(n+1)]}.$$

- (2) Big values of $R \gg 1$ and $n \gg 0$ or $R \ll 1$ and $n \ll 0$ lead instead to

$$R \sim a^{-(2/n)}. \quad (47)$$

In any case we obtain a power law $R \sim a^{-\lambda}$, where $\lambda = \frac{6}{n+1}$ in the first case and $\lambda = \frac{2}{n}$ in the second one.

Owing to the fact that b is constant it is now easy to obtain the modified Hubble equations from (38) (in that case g is obviously chosen to be a FRW metric for cosmological applications):

$$\begin{aligned} H^2 \sim -Ka^{-2} + \frac{\beta^2(n-1)}{6(2n-1)}R^{-n}a^{-6} + \frac{n}{6(2n-1)}R \\ - \frac{\mu(n-1)}{6(2n-1)}R^{2n}, \end{aligned} \quad (48)$$

which further rewrites as

$$H^2 \sim -Ka^{-2} + Aa^{-(6-n\lambda)} + Ba^{-\lambda} + Ca^{-2n\lambda} \quad (49)$$

with suitable chosen constants A, B, C, D . In the first case, recalling that $\lambda = \frac{6}{n+1}$, one has

$$H^2 \sim -Ka^{-2} + (A+B)a^{-[6/(n+1)]} + Ca^{-[12n/(n+1)]}.$$

For late universe ($a \gg 1$) the term $a^{-[6/(n+1)]}$ dominates and the deceleration parameter is $q = \frac{2-n}{n+1}$. This gives the same qualitative behavior already found in the case $F(R) = R$ previously discussed. For early universe ($a \ll 1$), instead, the term $a^{-[12n/(n+1)]}$ dominates, providing a big bang initial singularity.

Correspondingly, for $R \sim a^{-(2/n)}$ the Friedmann equation reduces to

$$H^2 \sim -Ka^{-2} + Ba^{-(2/n)} + (A+C)a^{-4},$$

and consequently the term $a^{-(2/n)}$ dominates in late universe. However this contradicts to our initial assumptions $R \gg 1$ and $n \gg 0$ or $R \ll 1$ and $n \ll 0$ for that case; see Eq. (47). For early universe, conversely, the term a^{-4} dominates and we obtain a universe fulfilled by radiation with a big bang initial singularity.

IV. THE COSMOLOGICAL CONSTANT PROBLEM

One of the alternative proposals to explain the present acceleration of the universe is related to dynamical cosmological constant models. This approach, however, has a lot of unsolved issues related to the fact that there is no known mechanism, up to now, that guarantees zero or nearly zero energy in a stable energy configuration for the universe. Using cosmological constantlike models,

we have to face with some problems concerning: (i) the small amount of energy of the vacuum, which is much smaller than we estimate it to be (the so-called *cosmological constant problem*); (ii) the nature of the dark energy which seems to dominate the universe and; (iii) the *coincidence problem* between the actual density of dark energy in the universe and the actual matter density [6].

It has been recently proposed in [3] that the apparent value of the cosmological constant can be alternatively determined by dynamical considerations in the framework of alternative theories of gravity. The true value of the vacuum energy is not zero, but settles down to a nearly zero value when the curvature of the universe reaches nearly zero values. This mechanism is produced by means of a feedback process in which the scalar equation depends on curvature, in such a way that the divergent coefficient of the kinetic term of the scalar field forbids the lowest energy state to be reached. The potential of the scalar field is chosen to have a minimum at a generic negative value for the vacuum energy. However, the singular kinetic term of the scalar field makes the dynamics of the scalar field to stop at zero, where the field itself gets frozen. (Of course, some fine-tuning of dynamical Lagrangian is assumed). This theory has been proven to have some very important properties, i.e. it is stable under radiative corrections and it leads to stable dynamics. No fine-tuning of the potential is necessary in this case, neither the introduction of an anthropic principle. The phenomenological theory introduced in [3] slightly differs from the one we studied in (1). The Lagrangian density is

$$L = \sqrt{g} \left(\frac{R}{2\kappa^2} + \alpha R^2 + f(R)L_{\text{kin}} - V(\phi) \right), \quad (50)$$

where $L_{\text{kin}} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is the kinetic term of a scalar-field Lagrangian, and $f(R)$ is a function of the Ricci scalar, which is assumed to be divergent at $R = 0$. It should be noticed also that (50) coincides with (1) for $\alpha = 0$, $F(R) = R$, and $V(\phi) = 0$.

We want here to examine if the same mechanism, generating a dynamical cosmological constant with suitable properties, works in the first-order formalism. Let g be a flat FRW metric (9). First of all, even in the Palatini formalism the same field equations hold for the scalar, i.e. they have the same form as ones in the metric formalism [3]:

$$\frac{d}{dt}[\dot{\phi}f(R)] + 3H\dot{\phi}f(R) + V'(\phi) = 0. \quad (51)$$

As far as field equations for the metric field g are concerned, we obtain in analogy with (2) that

$$\left(\frac{\sqrt{g}}{2\kappa^2} + \sqrt{g}\alpha R \right) \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \right) + f'(R)L_{\text{kin}}R_{\mu\nu} - \frac{1}{2}f(R)L_{\text{kin}}g_{\mu\nu} - \frac{1}{2}V(\phi)g_{\mu\nu} = 0, \quad (52)$$

and the scalar-valued structural field equation, controlling the solutions of (52), can be simply obtained by taking the trace of (52) itself. This implies that, considering ϕ to evolve slowly, we have that $V(\phi)$ can be approximated as a linear function, see [3]:

$$V(\phi) \simeq c\kappa^{-3}(\phi - \phi_0), \quad (53)$$

and the asymptotic behavior of ϕ is consequently

$$\kappa^2\dot{\phi}f(R) \sim -c\kappa^{-1}H^{-1}. \quad (54)$$

This expression follows from the consideration that the term $\frac{\dot{H}}{H^2}$ is nearly zero at small energies. From field Eqs. (52), in the approximation that the kinetic term is small compared to the potential term and assuming $\frac{\dot{H}}{H^2} \simeq 0$, one obtains in analogy with [3]:

$$3H^2 \simeq \kappa^2 V. \quad (55)$$

This is exactly the same result already obtained in [3] in the metric formalism, proving that field equations from the Lagrangian (50) have the same limit at low energies, both in the Palatini and the metric formalism. It follows (see [3] for details), that at low energies ϕ stalls at nearly $V = 0$, where V includes all the contributions to the cosmological constant. This consequently implies that the value of the cosmological constant stalls at a very small but nonzero value, in the limit of small curvatures of the metric g , i.e. at small energies for the gravitational field. It also has been proven in [3] that this behavior is stable under the effect of radiative corrections and leads to stable dynamics, provided that the minimum of the potential always occurs at a negative value. Hence, the same dynamical mechanism to solve the cosmological constant problem works in Palatini formulation.

V. CONCLUSIONS

In summary, the Palatini formulation of nonlinear gravity-matter system has been developed. Using a scalar-field Lagrangian as matter it is shown that such a theory provides the effective quintessence or effective phantom cosmology at late times in the same qualitative way as in the metric version. In addition, the gravity assisted dark energy dominance mechanism also occurs. It is shown that an account of radiation may be done with possible emergence of accelerating cosmology again. It is interesting to notice that in this case the emerging cosmology may contain big bang at early times and big rip at late times (for general review of late time singularities, see Ref. [19] and references therein). Eventually, the account of quantum effects near to singularities is necessary to see if the singularities are realistic or not. It would be really interesting to perform such a study within the Palatini approach.

Any Palatini model of gravity possess its purely metric counterpart which leads to fourth order field equations. An

advantage of Palatini formulation rely on second order field equations which turns out to be more easy to solve. In some very specific situations both approaches give rise to the same solutions but in general it is not true that they admit the same solutions. The same occurred here in the case of cosmological application. Although, one can expect that metric formulation may also produce accelerated solutions thus, however, it is much more difficult (if possible at all) to find them out explicitly. In this sense the Palatini formalism is more easy to handle and simpler to analyze than the corresponding metric formalism.

It is obvious that any reasonable model of gravity should satisfy the standard solar system tests. In the context of modified gravity this problem has been studied recently by several authors [20] (both in the metric and Palatini approaches). It has been shown that, in principle, Palatini formalism provides a good Newtonian approximation. Particularly, the Schwarzschild limit of the theory can be recovered for positive powers n of the scalar R , greater than one $n \geq 1$. In the case of negative powers one should expect de Sitter or anti-de Sitter black holes instead [21].

It also has been explicitly demonstrated that the dynamical mechanism to resolve the cosmological constant problem suggested in metric formulation works also in Palatini formulation (within the same choice of scalar-tensor Lagrangian). Again, it is very interesting to understand the role of quantum effects (as this is exactly the region where they are expected to be essential) in the above mechanism. However, to do so one needs to evaluate quantum effects in the Palatini formulation, something that is rather far from being a trivial task.

Finally, in this paper we have limited ourselves to the consideration of nonlinear scalar-gravity theories in the Palatini formulation. However, it is clear that nonlinear spinor-gravity system (where the spinor part can be constructed from Nambu-Jona-Lasinio-like model coupled to gravity [22]) may be more interesting from a cosmological point of view. Indeed, unlike the covariant derivative of a scalar which is nothing but a partial derivative, the covariant derivative of a spinor includes instead the connection, what definitely leads to new features in the spinor analog of Eq. (7). This will be discussed elsewhere.

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APPENDIX: SOME USEFUL RELATIONS IN THE PALATINI FORMALISM

We include hereafter some useful relation in the Palatini formalism for the benefit of the general reader. The Palatini formalism can be considered as a metric-affine theory for relativistic theories. Metric and connection are first considered as independent objects. We remark however that, without needing any reference to a metric, a connection can be introduced from which one can construct covariant derivatives. In order to produce covariant derivatives which transform as true tensors, under a coordinate transformation the connection itself needs to transform as

$$\Gamma_{\mu\nu}^{\lambda} \rightarrow \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma_{\tau\sigma}^{\rho} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}}.$$

Then for such a connection the quantity

$$R_{\mu\nu\kappa}^{\lambda} = \frac{\partial \Gamma_{\mu\nu}^{\lambda}}{\partial x^{\kappa}} - \frac{\partial \Gamma_{\mu\kappa}^{\lambda}}{\partial x^{\nu}} + \Gamma_{\mu\nu}^{\eta} \Gamma_{\kappa\eta}^{\lambda} - \Gamma_{\mu\kappa}^{\eta} \Gamma_{\nu\eta}^{\lambda}$$

will then be a true tensor, regardless of whether the connection is or is not related to the metric in the standard metric based Christoffel symbol way

$$\Gamma_{\mu\nu}^{\lambda}(g) = \frac{1}{2} g^{\mu\sigma} [\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu}].$$

However, noting that the metric based $\Gamma_{\mu\nu}^{\lambda}(g)$ also transforms as

$$\Gamma_{\mu\nu}^{\lambda}(g) \rightarrow \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma_{\tau\sigma}^{\rho}(g) + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}},$$

we see that the difference between the general connection and the Christoffel symbol connection transforms as

$$\Gamma_{\mu\nu}^{\lambda} - \Gamma_{\mu\nu}^{\lambda}(g) \rightarrow \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} [\Gamma_{\tau\sigma}^{\rho} - \Gamma_{\tau\sigma}^{\rho}(g)],$$

with the difference thus being a true tensor. Then, with covariant derivatives being linear in the connection, the difference between the quantity $T_{;\mu}^{\mu\nu}$ (where $T^{\mu\nu}$ is the stress-energy tensor of matter) as calculated with $\Gamma_{\tau\sigma}^{\rho}$, and $T_{;\mu}^{\mu\nu}$ as calculated with $\Gamma_{\tau\sigma}^{\rho}(g)$, is itself a true tensor; and since $T_{;\mu}^{\mu\nu}$ as calculated with $\Gamma_{\tau\sigma}^{\rho}$ is necessarily a true tensor, it follows that $T_{;\mu}^{\mu\nu}$ as calculated with $\Gamma_{\tau\sigma}^{\rho}(g)$ is one as well. It is consequently meaningful to take the covariant derivative of equation of motion (2) with respect to the Christoffel symbol based connection: it transforms like a true tensor.

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