Fit to the Bjorken, Ellis-Jaffe and Gross-Llewellyn-Smith sum rules in a renormalon based approach

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We study the large order behavior in perturbation theory of the Bjorken, Ellis-Jaffe and Gross-Llewellyn-Smith sum rules. In particular, we consider their first infrared renormalons, for which we obtain their analytic structure with logarithmic accuracy and also an approximate determination of their normalization constant. Estimates of higher order terms of the perturbative series are given. The renormalon subtracted scheme is worked out for these observables and compared with experimental data. Overall, good agreement with experiment is found. This allows us to obtain $\hat{a}_0$ and some higher-twist nonperturbative constants from experiment: $\hat{a}_0 = 0.141 \pm 0.089$; $f_{3,\mathrm{RS}}(1\ \mathrm{GeV}) = -0.124^{+0.137}_{-0.142}\ \mathrm{GeV}^2$.

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I. INTRODUCTION

Deep inelastic scattering (DIS) is one of the few places where one can test, on a very solid theoretical ground, asymptotic freedom and the operator product expansion against experiment. This is so because, for the moments of the different structure functions, the transfer momentum lies in the Euclidean region and far away of the physical cuts (for Q^2 large). Therefore, the theoretical predictions do not rely on any kind of quark-hadron duality and may provide with solid and very clean determinations of α_s and some nonperturbative matrix elements. Equally interesting is the study of the interplay between the perturbative and nonperturbative regime. This can only be done if a full control on the perturbative series is achieved.

Within DIS, special consideration deserves the sum rules for which the matrix elements are related to symmetry generators (in this case they can be computed in absolute value within perturbation theory), like the Gross-Llewellyn-Smith (GLS) sum rule [1], or to some low energy constants (which can be directly measured from experiment), like the Bjorken sum rule [2]. In this paper, we will concentrate on the Bjorken, Ellis-Jaffe [3] and GLS sum rules. Their leading-twist term has been computed with next-to-next-to-leading order (NNLO) accuracy and they have been measured with increasingly good accuracy over the years.

By the operator product expansion, the short- and long-distance contributions are separated and a sum rule M can be expressed in the following way:

$$M = C\langle J\rangle + B\frac{\langle R\rangle}{Q^2} + \ldots, \tag{1}$$

with short-distance Wilson coefficients $C, B, \ldots$ and long-distance matrix elements $\langle J\rangle, \langle R\rangle, \ldots$. The perturbative series of C are expected to be asymptotic and, therefore, to diverge for a high enough order in perturbation theory. Moreover, in schemes without strict separation of large and small momenta, such as $\overline{\mathrm{MS}}$, they are believed to be non-Borel summable. This is because when calculating the matching coefficients $C, \ldots$, the integrals run over all loop momenta, including small ones. Therefore, they also contain, in addition to the main short-distance contributions, contributions from large distances, where perturbation theory is ill-defined. These contributions produce infrared renormalon singularities [4], factorially growing contributions to coefficients of the perturbative series, which lead to ambiguities $\sim(\Lambda_{\mathrm{QCD}}/Q)^{2n}$ in the matching coefficients $C, \ldots$. Similarly, matrix elements of higher-dimensional operators $\langle R\rangle, \ldots$ also contain, in addition to the main large-distance contributions, contributions from short distances, which produce ultraviolet-renormalon singularities. They lead to ambiguities of the order $\Lambda^{2n}_{\mathrm{QCD}}$ times lower-dimensional matrix elements (e.g., $\langle J\rangle$). These two kinds of renormalon ambiguities should cancel in physical observables [5–10], in this case M.

The intrinsic (minimal) error associated to the perturbative series is of the order of the higher-twist correction. Thus, one can not unambiguously determine the higher-twist terms, unless a prescription to deal with the perturbative series that has powerlike accuracy is given. In this paper we will adapt to this case the prescription used for heavy quark physics in Refs. [11–13]. The idea is that the leading divergent behavior of the perturbative series is related to the closest singularities in the Borel plane of its Borel transform. In heavy quark physics, they lie on the positive axis (infrared renormalons). In the case of the sum rules considered in this paper, the closest singularities lie on the positive and negative axis at equal distance to the origin. We will assume that the one in the positive axis will dominate the asymptotics of the perturbative series. Since these singularities cancel against the ultraviolet renormalons of the low energy dynamics of the twist-4 operators, the proposal will be to shift the singularities from the perturbative series to the twist-4 operators. We will refer to this prescription as the renormalon subtracted (RS) scheme and apply it to the Bjorken, Ellis-Jaffe and GLS

sum rules. We will obtain the contribution of the leading infrared renormalon, subtract it from the perturbative series, and add it to the low energy matrix elements, in our case, to the twist-4 operators. In this way one enlarges the range of convergence of the Borel transform of the perturbative series, which can be defined with powerlike accuracy. This procedure has proven to be extremely successful in heavy quark physics, where it has been shown that perturbation theory works very well and good determinations of nonperturbative subleading corrections have been obtained using either lattice or experimental data. One should be aware, however, that in heavy quark physics one was in an optimal situation, since the singularity in the Borel plane was quite close to the origin ($u = 1/2$). We are now going to be in a less optimal situation, since the closest singularities lie at $u = \pm 1$. The physical situation is also completely different since now we are talking of a system made with light fermions. Therefore, it is interesting to investigate if a similar improvement is obtained in this case. We will do so in this paper.

The paper is organized as follows. In the next section we will introduce the relevant sum rules. In Sec. III the Borel transform of the first infrared renormalon of the leading-twist Wilson coefficient will be calculated with leading log accuracy, as well as the normalization constant and estimates of the higher order terms of the perturbative series. The RS scheme will be worked out in Sec. IV. In Sec. V the comparison with the experimental data will be done allowing us the extraction of some nonperturbative matrix elements. Finally, the conclusions are presented in Sec. VI.

II. SUM RULES

The Bjorken and Ellis-Jaffe sum rules are related to polarized deep inelastic electron-nucleon scattering, which is described by the hadronic tensor

$$\begin{aligned} W_{\mu\nu} &= \frac{1}{4\pi}\int d^4z e^{iqz}\langle p,s|J_\mu(z)J_\nu(0)|p,s\rangle \\ &= \left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)F_1(x,Q^2)+\left(p_\mu-\frac{p\cdot q}{q^2}q_\mu\right) \\ &\quad\times\left(p_\nu-\frac{p\cdot q}{q^2}q_\nu\right)\frac{1}{p\cdot q}F_2(x,Q^2) \\ &\quad+i\epsilon_{\mu\nu\rho\sigma}q_\rho\left(\frac{s_\sigma}{p\cdot q}g_1(x,Q^2)\right. \\ &\quad\left.+\frac{s_\sigma p\cdot q-p_\sigma q\cdot s}{(p\cdot q)^2}g_2(x,Q^2)\right). \end{aligned} \tag{2}$$

Here $J_\mu=\sum_{i=1}^{n_f}e_i\overline{\psi_i}\gamma_\mu\psi_i$ is the electromagnetic quark current where $e_i = 2/3, -1/3, -1/3, \cdots$ is the electromagnetic charge of a quark with the corresponding flavour u, d, s. $x = Q^2/(2p\cdot q)$ is the Bjorken scaling variable and $Q^2 = -q^2$ is the square of the transferred momentum. $|p,s\rangle$ is the nucleon state that is normalized as $\langle p,s|p',s'\rangle = 2p^0(2\pi)^3\delta^{(3)}(p-p')\delta_{ss'}$. The polarization vector of the nucleon is expressed as $s_\sigma = \overline{U}(p,s)\gamma_\sigma\gamma_5 U(p,s)$ where $U(p,s)$ is the nucleon spinor $\overline{U}(p,s)U(p,s) = 2m_N$.

The Ellis-Jaffe sum rule then reads

$$\begin{aligned} M_1^{p/n}(Q^2) &= \left(\pm\frac{1}{12}g_A+\frac{1}{36}a_8\right)C_B(\alpha_s)+\frac{1}{9}\hat{a}_0C_{EJS}(\alpha_s) \\ &\quad-\frac{8}{9Q^2}\left[\left(\pm\frac{1}{12}f_3(Q_0)+\frac{1}{36}f_8(Q_0)\right)\right. \\ &\quad\times\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-(\gamma^0_{NS})/2\beta_0}(1+\mathcal{O}(\alpha_s)) \\ &\quad+\frac{1}{9}f_0(Q_0)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-(1/2\beta_0)(\gamma^0_{NS}+(4/3)N_f)} \\ &\quad\left.\times(1+\mathcal{O}(\alpha_s))\right]+\mathcal{O}\left(\frac{1}{Q^4}\right), \end{aligned} \tag{3}$$

and the Bjorken sum rule is the difference between the proton and neutron sum rule:

$$\begin{aligned} M_1^B &\equiv M_1^p(Q^2)-M_1^n(Q^2) \\ &= \frac{g_A}{6}C_B(\alpha_s)-\frac{4}{27}\frac{1}{Q^2}f_3(Q_0)\left[\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right]^{(-\gamma^0_{NS})/2\beta_0} \\ &\quad\times(1+\mathcal{O}(\alpha_s))+\mathcal{O}\left(\frac{1}{Q^4}\right), \end{aligned} \tag{4}$$

where the definitions are the following:

$$\gamma^0_{NS}=\frac{16}{3}C_F;\qquad \gamma^0_S=\gamma^0_{NS}+\frac{4}{3}n_f; \tag{5}$$

$$C_X(Q)=1+\sum_{s=0}^{\infty}C_X^{(s)}\alpha_s^{s+1}(\nu), \tag{6}$$

and $X = \{B, EJ, GLS\}$, the latter to be defined below. For the Bjorken sum rule, the α_s correction [14], the α_s^2 correction [15], and the α_s^3 correction [16] have been calculated in the leading-twist approximation. Higher-twist corrections have also been calculated [17]. The Ellis-Jaffe sum rule for the proton and neutron was calculated to order α_s [18], to order α_s^2 [19], and to order α_s^3 [20] in the leading-twist approximation. Power corrections were calculated in [21]. The LO renormalization group running of the twist-4 operators have been computed in Ref. [22]. The nonperturbative matrix elements are defined in the following way:

$$\begin{aligned} |g_A|s_\sigma &= 2\langle p,s|J^{5,3}_\sigma|p,s\rangle = (\Delta u-\Delta d)s_\sigma, \\ a_8 s_\sigma &= 2\sqrt{3}\langle p,s|J^{5,8}_\sigma|p,s\rangle = (\Delta u+\Delta d-2\Delta s)s_\sigma, \\ a_0(\mu^2)s_\sigma &= \langle p,s|J^5_\sigma|p,s\rangle = (\Delta u+\Delta d+\Delta s)s_\sigma, \end{aligned} \tag{7}$$

where $J^{5,a}_\sigma(x)=\overline{\psi}\gamma_\sigma\gamma_5 t^a\psi(x)$ is the nonsinglet axial current, where t^a is a generator of the flavor group, and $J^5_\sigma(x)=\sum_{i=1}^{n_f}\overline{\psi_i}\,\gamma_\sigma\gamma_5\psi_i(x)$ is the singlet axial current. $|g_A|$ is the absolute value of the constant of the neutron

beta-decay, $|g_A/g_V| = F + D = 1.2695 \pm 0.0029$ [23]. $a_8 = 3F - D = 0.572 \pm 0.019$ is the hyperon decay constant.[1] The matrix element of the singlet axial current $a_0(\mu^2)$ will be redefined in a proper invariant way as a constant $\hat{a}_0$:

$$\hat{a}_0 = \exp\left(-2\int^{\alpha_s(\nu)} d\alpha_s' \frac{\gamma^s(\alpha_s')}{\beta(\alpha_s')}\right) a_0(\nu). \tag{8}$$

We use the notation $\Delta q(\mu^2) s_\sigma = \langle p, s|\overline{q}\gamma_\sigma\gamma_5 q|p, s\rangle$, $q = u, d, s$, for the polarized quark distributions. f_0, f_3, and f_8 are the twist-4 counter parts of a_0, a_3, and a_8. f_i's are scale dependent and here they are defined at Q_0^2, i.e. f_i is the reduced matrix element of $R^i_{2\sigma}$, renormalized at Q_0^2, which is defined for the general flavor indices, with t^i being the flavor matrices, as

$$\begin{aligned} R^i_{2\sigma} &= g\overline{\psi}\tilde{G}_{\sigma\nu}\gamma^\nu t^i\psi, \\ \langle p, s|R^i_{2\sigma}|p, s\rangle &= f_i s_\sigma \quad (i = 0, \cdots, 8), \end{aligned} \tag{9}$$

and $\tilde{G}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}$ is the dual field strength.

In the left-hand side of Eqs. (3) and (4) target-mass effects have been included using the Nachtmann variable [25]. They read

$$\begin{aligned} M_1^N(Q^2) &\equiv \int_0^1 dx \frac{\xi^2}{x^2}\left\{g_1^N(x, Q^2)\left[\frac{x}{\xi} - \frac{1}{9}\frac{m_N^2x^2}{Q^2}\frac{\xi}{x}\right]\right. \\ &\quad \left. - g_2^N(x, Q^2)\frac{m_N^2x^2}{Q^2}\frac{4}{3}\right\} \\ &= \int_0^1 g_1^N(x, Q^2)dx + \frac{\mu_4^N}{Q^2} + \mathcal{O}\left(\frac{1}{Q^4}\right) \\ &= \Gamma_1^N(Q^2) + \frac{\mu_4^N}{Q^2} + \mathcal{O}\left(\frac{1}{Q^4}\right), \end{aligned} \tag{10}$$

where $\xi = 2x/(1 + \sqrt{1 + 4m_N^2x^2/Q^2})$ is the Nachtmann scaling variable, m_N is the nucleon mass. The quantity M_1 is the first Nachtmann moment of g_1 that absorbs all the target-mass corrections, $\sim(m_N^2/Q^2)^n$, and

$$\mu_4^N = -\frac{m_N^2}{9}(a_2^N + 4d_2^N), \tag{11}$$

where a_2^N is the target-mass correction given by the x^2-weighted moment of the leading-twist g_1 structure function, and d_2^N is a twist-3 matrix element given by

$$d_2^N = \int_0^1 dx x^2(2g_1^N + 3g_2^N). \tag{12}$$

We will consider the Ellis-Jaffe proton sum rule and the Bjorken sum rule. For the former we will use the data points given in Ref. [26], which are already given in terms of M_1^p, and for the latter we will use the data points given in Ref. [27], which used the values[2]:

$$d_2^{p-n} = -0.0029, \qquad a_2^{p-n} = 0.0279. \tag{13}$$

If one also considers DIS of neutrinos with nucleons, the GLS sum rule appears, for which the leading twist has been computed with NNLO accuracy (see for instance [28]):

$$\begin{aligned} M_3^{GLS} &\equiv \frac{1}{2}(M_3^p(Q^2) + M_3^n(Q^2)) \\ &= 3C_{GLS}(\alpha_s) - \frac{8}{9}\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-(\gamma_{NS}^0)/2\beta_0} \\ &\quad \times \frac{\langle\langle O_5^{S(p+n)}\rangle\rangle(Q_0)}{Q^2}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} M_3^N(Q^2) &= \frac{1}{3}\int_0^1 dx F_3^{\nu N}(x; Q^2)\frac{\xi^2}{x^2}\left[1 + 2\left(1 + \frac{4m_N^2x^2}{Q^2}\right)^{1/2}\right] \\ &= \int_0^1 dx F_3^{\nu N}(x; Q^2)\left[1 - \frac{2}{3}\frac{m_N^2x^2}{Q^2} + \mathcal{O}\left(\frac{m_N^4}{Q^4}\right)\right], \end{aligned} \tag{15}$$

$$\langle P|O_{5\mu}^S|P\rangle_{\text{spin averaged}} \equiv 2P_\mu\langle\langle O_5^S\rangle\rangle \equiv 2P_\mu f_5^S, \tag{16}$$

and

$$O_{5\mu}^S = \bar{u}\tilde{G}_{\mu\nu}\gamma^\nu\gamma_5 u + \bar{d}\tilde{G}_{\mu\nu}\gamma^\nu\gamma_5 d. \tag{17}$$

For this sum rule, we will use the data of the CCFR Collaboration [29].

The difference between C_{GLS} and C_B first starts at $\mathcal{O}(\alpha_s^3)$ and is proportional to the number of light fermions. This new term is of a ''light-by-light'' nature and proportional to a new Casimir. The anomalous dimension of the higher twist is equal to the Bjorken case.

The difference between the three C_X is the light flavor dependence. In the limit $n_f \to 0$ they are all equal.

We also consider the correction due to the charm quark (with finite mass) to the perturbative series of C_B and C_{GLS}. They have been computed in Ref. [30]. The $\mathcal{O}(\alpha_s^2)$ correction is equal for both of them and rather small (actually negligible compared with other sources of errors). Note however that the leading-order correction is different in each case (zero for the Bjorken case). This correction depends on the Cabibbo angle. We take the value $\sin\theta = 0.224$ [23]. According to Ref. [30], it is a good approximation to work with 3 light flavors plus one massive flavor up to rather large energies. This is the situation we will consider in this paper.

[1] We will obtain this number from hyperon decays, see [24]. $F/D = 2/3$ in the large N_c.

[2] To take them as constants is an approximation, nevertheless, their effect on the fit is small in comparison with other source of errors.

III. RENORMALONS

The Wilson coefficients C_X can be expressed in terms of S_X, their Borel transform, defined as

$$S_X(u) = \sum_{n=0}^{\infty} \frac{C_X^{(n)}}{n!} \left(\frac{4\pi}{\beta_0} u\right)^n, \tag{18}$$

in the following way:

$$C_X(Q) = 1 + \frac{4\pi}{\beta_0} \int_0^{\infty} S_X(u) e^{(-4\pi)/(\beta_0 \alpha_s(\nu))u} du. \tag{19}$$

However, the perturbative expansions in α_s of the Wilson coefficients C_X are expected to be asymptotic and non-Borel sumable. In other words, we expect to have singularities in the real axis of $S_X(u)$. Those lying in the positive axis are called infrared renormalons, and those lying on the negative axis are called ultraviolet renormalons. The position and strength of the singularities can be obtained by using the renormalization group and consistency with the operator product expansion. In particular, the infrared renormalons of the perturbative series are obtained by demanding their cancellation with the ultraviolet renormalons of the higher-twist terms. Although the renormalon cancellation has only been explicitly shown in some cases in the large-β_0 limit, it is assumed to hold beyond this approximation. Based on this assumption, one may obtain additional information on the structure of the infrared renormalon singularities of the matching coefficients, based on the knowledge of the ultraviolet renormalons in higher-dimensional matrix elements, which are controlled by the renormalization group [5]. This model-independent approach has been applied in heavy quark effective theory (HQET) in [10,31,11,32]. In our case, the singularities closest to the origin are located in the real axis at $u = \pm 1$.

The ultraviolet-renormalon structure of the moments of the DIS structure functions have been computed in Ref. [33]. For those one gets the ultraviolet renormalon for the sum rules we are discussing here, which is the same in all cases up to a constant. For the case of the GLS or Bjorken sum rule, the ultraviolet renormalon formally dominates for $n_f > 2$ for $n \to \infty$. For the Ellis-Jaffe sum rule, since the infrared renormalon is weaker, the ultraviolet-renormalon dominance appears for even a smaller number of flavors. Nevertheless, at low orders in perturbation theory the infrared renormalon appears to be dominant. This can be seen from the fact that the sign of the known terms of the perturbative series is equal whereas if the ultraviolet renormalon were to be dominant we would find a sign alternating series. Nevertheless, we will perform a conformal mapping to avoid the ultraviolet renormalon. The fact that we will obtain a similar number than without conformal mapping will support the view that the normalization constant of the ultraviolet renormalon is small in comparison with the infrared one. Indeed, a similar conclusion was obtained in Ref. [34] using Pade approximants. Therefore, we will neglect ultraviolet-renormalon effects in the leading-twist Wilson coefficients in what follows.

The Borel transform near the closest infrared renormalon singularity has the following structure:

$$S_X(u) = \frac{\nu^2}{Q^2} N_X^{(IR)} \frac{1}{(a-u)^{1+b+b_X}} (1 + d_1^X (a-u) + d_2^X (a-u)^2 + \cdots) + S_{\rm reg}(u), \tag{20}$$

where $S_{\rm reg}(u)$ is an analytic function at $u = a$ and we define $b = \beta_1/\beta_0^2$. The procedure to fix the coefficients of this expansion (except $N_X^{(IR)}$) is by demanding consistency with the operator product expansion. In other words, we demand the ambiguity of the Borel transform to cancel with the ambiguity of the ultraviolet renormalons of the twist-4 matrix elements (see Eqs. (3), (4), and (14)). Therefore,

$$\begin{aligned} {\rm Im}\left[\int_0^{\infty} S_X(u) e^{(-4\pi)/(\beta_0 \alpha_s(\nu))u} du\right] &\propto \frac{\Lambda^2_{\overline{\rm MS}}}{Q^2} \alpha_s(Q)^{-b_X} \\ &= \frac{\nu^2}{Q^2} e^{(-4\pi)/(\beta_0 \alpha_s(\nu))} \alpha_s(\nu)^{-b-b_X} \left(1 - \frac{\beta_0}{4\pi} \alpha_s(\nu) \times \ln\left(\frac{\nu^2}{Q^2}\right)\right)^{b_X}. \end{aligned} \tag{21}$$

This fixes $a = 1$ and b_X [6]:

$$b_{GLS} = b_B = -\frac{\gamma_{NS}^0}{2\beta_0}, \qquad b_{EJ} = -\frac{\gamma_S^0}{2\beta_0}. \tag{22}$$

b_X dictates the strength of the singularity. It is interesting to study its dependence on n_f. In the Bjorken and GLS sum rules, for $n_f \in (0, 6) \Rightarrow 1 + b + b_X \in (1, 2)$ so, formally, one could just keep the first two terms of the series in Eq. (20), since the next term would go to zero for $u \to 1$. This is also the case for Ellis-Jaffe if $n_f < 4$, otherwise one could even stick to the first term only.

If the Wilson coefficients multiplying the higher-twist operators were known exactly, we could also fix the coefficients d_r^X. Unfortunately, we only know their leading log running. Nevertheless, by performing the matching at a generic scale ν, we will be able to resum the terms of the type $(1-u)^n \ln^n(Q^2/\nu^2)$ and obtain the logarithmically dominant contribution to $d_r^X \sim \ln^r(Q^2/\nu^2)$. We obtain

$$S_X(u) = \frac{\nu^2}{Q^2} N_X^{(IR)} \frac{1}{(1-u)^{1+b+b_X}} {}_1F_1(-b_X, -b-b_X, (1-u) \times \ln(Q^2/\nu^2)) + S_{\rm reg}(u=1). \tag{23}$$

The leading asymptotic behavior of the perturbative series due to the first infrared renormalon reads

$$C_X^{(n)} \underset{n\to\infty}{=} N_X \frac{\nu^2}{Q^2}\left(\frac{\beta_0}{4\pi}\right)^n \frac{\Gamma(n+1+b+b_X)}{\Gamma(1+b+b_X)} \times \sum_{s=0}^{\infty} d_s^X \frac{\Gamma(1+b+b_X)}{\Gamma(1+b+b_X-s)} \times \frac{\Gamma(1+b+b_X+n-s)}{\Gamma(1+b+b_X+n)}, \quad (24)$$

where $d_0^X \equiv 1$. The logarithmically enhanced contribution to d_s^X is known. By introducing it to Eq. (24), we obtain

$$C_X^{(n)} \underset{n\to\infty}{=} N_X \frac{\nu^2}{Q^2}\left(\frac{\beta_0}{4\pi}\right)^n \frac{\Gamma(1+b_X+b+n)}{\Gamma(1+b+b_X)} \times {}_1F_1(-b_X, -b-b_X-n, \ln(Q^2/\nu^2)). \quad (25)$$

The above expression contains subleading terms in the $1/n$ expansion. In the strict $1/n$ expansion, it simplifies to:

$$C_X^{(n)} \underset{n\to\infty}{=} N_X \frac{\nu^2}{Q^2}\left(\frac{\beta_0}{4\pi}\right)^n \frac{n! n^{b_X+b}}{\Gamma(1+b+b_X)} \times \left(1+\frac{1}{n}\ln(Q^2/\nu^2)\right)^{b_X}. \quad (26)$$

This result is numerically not very different from the previous expression for large n. On the other hand, it will give more stable results than Eq. (25) when working in the RS scheme for small n.

With the above results we can identify the contribution to C_X that comes from the renormalon. It reads

$$\delta C_X(\nu) = \sum_{n=n^*}^{\infty} N_X \frac{\nu^2}{Q^2}\left(\frac{\beta_0}{4\pi}\right)^n \frac{n! n^{b_X+b}}{\Gamma(1+b+b_X)} \times \left(1+\frac{1}{n}\ln(Q^2/\nu^2)\right)^{b_X} \alpha_s^{n+1}(\nu), \quad (27)$$

where n^* indicates the freedom to add and subtract finite-order contributions in perturbation theory.

At this stage, it is interesting to notice that $\delta C_X(\nu)$ can be written in the following form

$$\delta C_X(\nu) = \left(\frac{\alpha_s(Q)}{\alpha_s(\nu)}\right)^{-b_X} \sum_{n=n^*}^{\infty} N_X \frac{\nu^2}{Q^2}\left(\frac{\beta_0}{4\pi}\right)^n \times \frac{n! n^{b_X+b}}{\Gamma(1+b+b_X)} \alpha_s^{n+1}(\nu), \quad (28)$$

up to subleading terms. This expression will be more convenient for our purposes, since it has the same scale dependence on Q as the higher-twist contribution. Therefore, it can be moved from the leading to the subleading twist term without jeopardizing the Q scale dependence predicted by the factorization of scales.

A. Determination of the normalization constant

In this subsection we will obtain N_X. Let us momentarily neglect the ultraviolet renormalon. If this were the case, we could concentrate on the singularity closest to the origin in the Borel plane located at $u = 1$. Then, we can proceed in analogy with Refs. [11,13,35] and define the new function

$$D_X(u) = (1-u)^{1+b+b_X} S_X(u) = \sum_{n=0}^{\infty} D_X^{(n)} u^n. \quad (29)$$

One would then obtain N_X from the following identity

$$D_X(u=1) = \sum_{n=0}^{\infty} D_X^{(n)} = N_X \frac{\nu^2}{Q^2}, \quad (30)$$

where the first three terms of the expansion are known. Note that, formally, the expansion parameter is $u = 1$, but one does not really know if there are small factors multiplying the powers of u. In practice, the series is quite convergent, and stable numbers for N_X can be obtained from most of the sum rules and values of n_f. Nevertheless, we still have the problem of the ultraviolet renormalon located at $u = -1$. Formally, this renormalon would make the series in Eq. (30) nonconvergent. In order to avoid this problem we will perform the conformal mapping [36],

$$w(u) = \frac{\sqrt{1+u} - \sqrt{1-u/2}}{\sqrt{1+u} + \sqrt{1-u/2}}. \quad (31)$$

This transformation maps the first infrared renormalon to $w = 1/3$ and all other singularities to the unit circle $|w| = 1$. In the conformal mapping the expansion parameter is $w = 1/3$. In practice the effect of doing the conformal mapping is small, which points to the fact that the effect of ultraviolet renormalons is small in comparison with the effect of the infrared renormalon located at $u = 1$. We will only give numbers for the conformal mapping case for theoretical reasons. Nevertheless, as we have already mentioned, they will be quite similar to the computation without conformal mapping. Our best values for N_X can be found in Table I. They have been computed with NNLO accuracy, after conformal mapping, at the scale of minimal sensitivity to the scale variation. The scale dependence of the results as well as the convergence is shown in Fig. 1 for some selected values of n_f. We can see that in most cases the scale dependence becomes smoother as we go to higher orders. The convergence depends on the number of flavors.

TABLE I. Values of the infrared renormalon residue of the Bjorken, Ellis-Jaffe and GLS leading-twist Wilson coefficient C_X after conformal mapping. "*" means that no stable result is obtained.

n_f	N_B	N_{EJ}	N_{GLS}
0	-0.523 ± 0.154	-0.523 ± 0.154	-0.523 ± 0.154
1	-0.487 ± 0.126	-0.423 ± 0.120	-0.479 ± 0.121
2	-0.451 ± 0.101	-0.291 ± 0.085	-0.436 ± 0.094
3	-0.414 ± 0.079	-0.103 ± 0.035	-0.393 ± 0.070
4	-0.378 ± 0.058	*	-0.351 ± 0.059
5	-0.343 ± 0.102	*	-0.311 ± 0.134
6	-0.311 ± 0.194	*	-0.272 ± 0.232

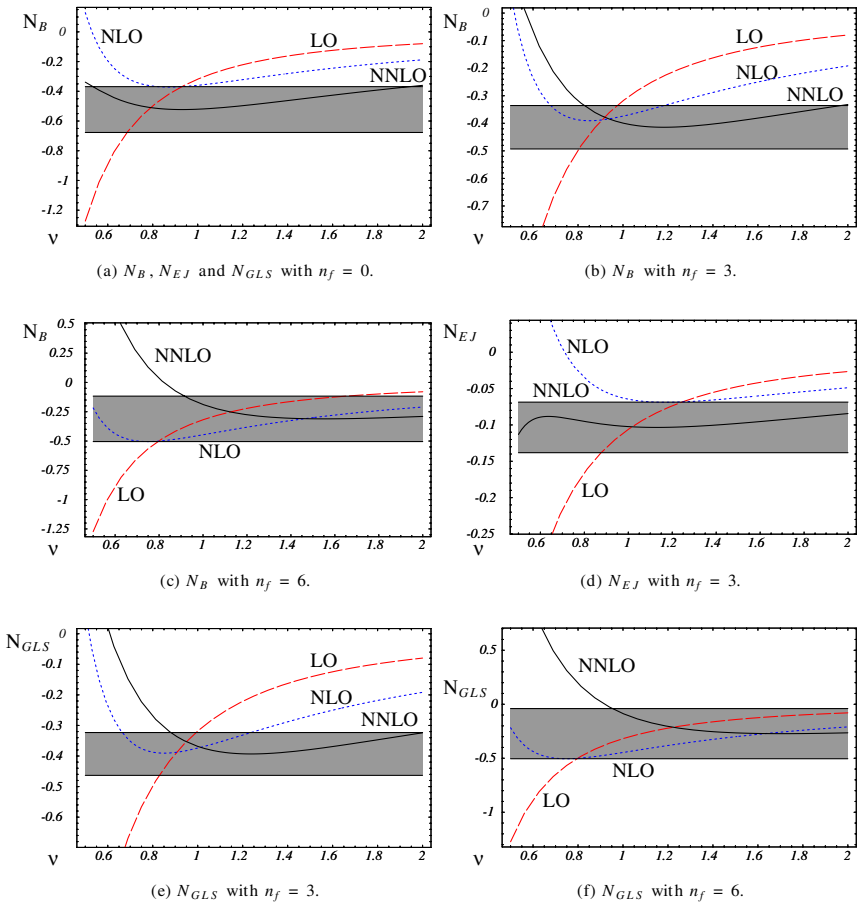


FIG. 1 (color online). Scale dependence of N_X with conformal mapping for $n_f = 0, 3, 6$. The dashed line is the LO result, the dotted line is the NLO result, and the continuous line the NNLO result. The band represents the error.

It is optimal for $n_f = 3$, which actually happens to be the most interesting case from the physical point of view and it deteriorates for large n_f. The error quoted in Table I, and represented in Fig. 1 by the gray band, stands by the maximum between the difference between the NNLO and NLO evaluation at the scale of minimal sensitivity of the NNLO evaluation and the difference between the NNLO and NLO at the scale of minimal sensitivity for each of them.

The values of N_B and N_{GLS} are consistent with each other within errors. This is consistent with the interpretation that the light-by-light term does not contribute to the renormalon as it was done in Ref. [36]. On the other hand our value for N_B appears to be smaller than the number obtained in Ref. [34].

We are then able to give some estimates for the coefficients of the perturbative series. We provide them in Tables II, III, and IV. We should stress that our numbers incorporate the right asymptotic behavior, which is not the case for large-β_0 estimates. For the Bjorken and GLS sum rule have been calculated in [37]. The comparison with the exact result works reasonably well for n_f smaller than 6 for the Bjorken or GLS perturbative series. For large n_f the comparison with the exact result gets worse. Note that for $n_f = 6$, the normalization constant of the renormalon could be almost compatible with zero if the errors are included. This fits with the picture that the renormalon is less important when the number of flavors grows and one can reach to the point where the infrared renormalon disappears. For the Ellis-Jaffe perturbative series the same

TABLE II. Renormalon-based estimates of the perturbative coefficients $C_B^{(s)}$ for $\nu = Q$ and for different number of flavors. We use the expression from Eq. (26) except for $C_B^{(0)}$ for which we use the expression from Eq. (25), otherwise the result is 0.

n_f	0	1	2	3	4	5	6
$-C_B^{(0)}(Q)$	0.523	0.487	0.451	0.414	0.378	0.343	0.311
$-C_B^{(1)}(Q)$	0.516	0.452	0.391	0.334	0.280	0.228	0.175
$-C_B^{(2)}(Q)$	1.295	1.045	0.824	0.630	0.461	0.318	0.198
$-C_B^{(3)}(Q)$	4.199	3.149	2.285	1.588	1.042	0.630	0.334
$-C_B^{(4)}(Q)$	17.07	11.94	8.008	5.097	3.023	1.625	0.750
$-C_B^{(5)}(Q)$	83.91	54.77	34.08	19.93	10.75	5.175	2.100

TABLE III. Renormalon-based estimates of the perturbative coefficients $C_{EJ}^{(s)}$ for $\nu = Q$ and for different number of flavors. We do not display the column with $n_f = 0$ since the numbers are equal to the Bjorken case. We use the expression from Eq. (26) except for $C_{EJ}^{(0)}$ for which we use the expression from Eq. (25), otherwise the result is 0.

n_f	1	2	3
$-C_{EJ}^{(0)}(Q)$	0.423	0.291	0.103
$-C_{EJ}^{(1)}(Q)$	0.392	0.250	0.080
$-C_{EJ}^{(2)}(Q)$	0.868	0.478	0.129
$-C_{EJ}^{(3)}(Q)$	2.547	1.253	0.298
$-C_{EJ}^{(4)}(Q)$	9.476	4.222	0.896
$-C_{EJ}^{(5)}(Q)$	42.87	17.42	3.336

TABLE IV. Renormalon-based estimates of the perturbative coefficients $C_{GLS}^{(s)}$ for $\nu = Q$ and for different number of flavors. We do not display the column with $n_f = 0$ since the numbers are equal to the Bjorken case. We use the expression from Eq. (26) except for $C_{GLS}^{(0)}$ for which we use the expression from Eq. (25), otherwise the result is 0.

n_f	1	2	3	4	5	6
$-C_{GLS}^{(0)}(Q)$	0.479	0.436	0.393	0.351	0.311	0.272
$-C_{GLS}^{(1)}(Q)$	0.449	0.379	0.317	0.260	0.206	0.154
$-C_{GLS}^{(2)}(Q)$	1.029	0.797	0.598	0.428	0.288	0.174
$-C_{GLS}^{(3)}(Q)$	3.100	2.211	1.507	0.967	0.569	0.293
$-C_{GLS}^{(4)}(Q)$	11.75	7.750	4.838	2.807	1.469	0.658
$-C_{GLS}^{(5)}(Q)$	53.92	32.97	18.92	9.980	4.680	1.841

discussion applies with n_f smaller than 4 (for $n_f = 4$ no stable determination of the normalization constant can be obtained). Again this fit with the picture that the infrared renormalon becomes weaker when the number of active flavors increases.

We can also compare with other estimates one may find in the literature for the Bjorken (GLS) sum rules [34,36,38,39]. We find that our predictions somewhat lie on the upper limit of the range of values obtained in these references for $C_{B/GLS}^{(3)}$.

As a final comment it should be mentioned that the introduction of the leading logarithms in the renormalon, which is novel, does not actually improve the agreement with the running on ν of the perturbative coefficients of the perturbative series, when they are known. If this is an artifact of the leading-order analysis and would be solved at higher orders remains to be seen.

IV. RENORMALON SUBTRACTED SCHEME

In the previous section we have obtained the contribution to the perturbative series of the leading-twist Wilson coefficient that produces its leading asymptotic behavior. This behavior limits the accuracy that can be obtained for the observable from the perturbative series alone, which it will always have an error of the order of the higher-twist corrections.[3] Therefore, one cannot obtain these higher-twist corrections unless the perturbative series is defined with powerlike accuracy. The specific value for the higher twist will depend upon the specific prescription used. Here we will adapt the procedure used in Refs. [11–13]. In those references the nonanalytic behavior in the Borel plane that produced the asymptotic behavior of the perturbative series was subtracted from the perturbative series and added to the subleading nonperturbative contributions. Actually the definition of the nonanalytic piece is ambiguous and analytic terms can always be added. The specific quantity we will substract to the perturbative series will be Eq. (28) with $n^* = 1$.[4] This quantity fulfils the requirement that its imaginary part cancels the imaginary part of the perturbative series and that its dependence on Q complies with the structure of the higher-twist contribution. For illustration, the Bjorken sum rule would read

$$M_1^p(Q^2) - M_1^n(Q^2) = \frac{g_A}{6} C_{B,\mathrm{RS}}(Q;\nu_f) - \frac{4}{27}\frac{1}{Q^2} f_{3,\mathrm{RS}}(\nu_f) \times \left[\frac{\alpha_s(\nu_f^2)}{\alpha_s(Q^2)}\right]^{(-\gamma_{\mathrm{NS}}^0)/2\beta_0} (1 + \mathcal{O}(\alpha_s)) + \mathcal{O}\left(\frac{1}{Q^4}\right), \quad (32)$$

where

$$C_{B,\mathrm{RS}}(Q;\nu_f) = C_B(Q) - \delta C_{B,\mathrm{RS}}(\nu_f) = 1 + \sum_{s=0}^{\infty} C_{B,\mathrm{RS}}^{(s)}(Q/\nu;\nu_f/\nu)\alpha_s^{s+1}(\nu), \quad (33)$$

and

$$f_{3,\mathrm{RS}}(\nu_f) = f_3(\nu_f) - \frac{9}{8} Q^2 g_A \left[\frac{\alpha_s(Q)}{\alpha_s(\nu_f)}\right]^{(-\gamma_{\mathrm{NS}}^0)/2\beta_0} \delta C_{B,\mathrm{RS}}(Q;\nu_f). \quad (34)$$

Similar changes would apply to the Ellis-Jaffe and GLS sum rules. In Eq. (33) and Eq. (34), we have subtracted and

[3] Actually, this is so for the order in α_s for which the difference between the exact and the finite-order result is minimal. If one goes to higher order in perturbation theory the series will deteriorate and the error will increase.

[4] This is equivalent to what was called the RS' scheme in Ref. [11], whereas the case $n^* = 0$ was named the RS scheme. In our case here both schemes coincide, since the $n = 0$ contribution from Eq. (28) vanishes because we expand the Γ's in $1/n$. Obviously, we could also set n^* different from 1 (as far as it is not too large). This would be equivalent to a change of scheme. The values of the nonperturbative matrix elements would change accordingly.

added the contributions coming from the first infrared renormalon at the scale ν_f, respectively. Thus, in this scheme, the Wilson coefficient $C_{B,\mathrm{RS}}(Q;\nu_f)$ is free of the first infrared renormalon and the associated $n!$ behavior. Therefore, the series is expected to converge better. On the other hand, the higher twist, $f_{3,RS}(\nu_f)$, is free of the first ultraviolet renormalon.

In principle, the above series could be improved by incorporating the running in ν_f to any order in α_s, which is renormalon free and therefore it could be obtained with good accuracy. In this situation it is legitimate to use the principal value (PV) prescription (or any other), since the renormalon ambiguity will cancel in the ratio. In this situation $\delta C_{X,\mathrm{RS}}$ reads

$$\delta C^{(PV)}_{X,\mathrm{RS}}(Q;\nu_f) = \left(\frac{\alpha_s(Q)}{\alpha_s(\nu_f)}\right)^{-b_X} \frac{\nu_f^2}{Q^2} N_X \alpha_s(\nu_f) \times \left[D_{b+b_X}\left(-\frac{4\pi}{\beta_0 \alpha_s(\nu_f)}\right) - 1 \right], \quad (35)$$

where

$$D_b(-x) = -xe^{-x}\{x^b \cos(\pi b)\Gamma(-b) - (-x)^b[\Gamma(-b) - \Gamma(-b,-x)]\}, \quad (36)$$

and

$$\Gamma(b,x) = \int_x^\infty dt\, t^{b-1} e^{-t}, \quad (37)$$

denotes the incomplete Γ function. One would then work with the following quantity

$$C_{B,\mathrm{RS}}(Q;\nu_f) = C_{B,\mathrm{RS}}(Q;\nu) + [\delta C^{(PV)}_{X,\mathrm{RS}}(Q;\nu) - \delta C^{(PV)}_{X,\mathrm{RS}}(Q;\nu_f)], \quad (38)$$

where one would set $\nu \sim Q$ and work order by order in perturbation theory in the first term in the right-hand side (where there are no large logs), whereas for the other terms the PV prescription is used. This allows to partially resum the dependence on ν_f. Let us also note that the first term in Eq. (36) corresponds to $\Lambda_{\overline{\mathrm{MS}}}$ (up to the anomalous dimension). Therefore it cancels in the difference, $\delta C^{(PV)}_{X,\mathrm{RS}}(Q;\nu) - \delta C^{(PV)}_{X,\mathrm{RS}}(Q;\nu_f)$, and can be neglected. Working with the principal value prescription has produced very good results in heavy quark physics, where it was possible to check the running of ν_f with ''experiment'' (lattice), see [13]. However, in that situation, the running in ν_f was known with a very high accuracy, since there was no anomalous dimension and the running on ν_f could be deduced simply from the running of α_s. In our case we only know the leading order and we expect a less accurate result.

Finally, we would like to mention that our knowledge of the running in ν of the higher-twist terms is much more limited than in heavy quark physics. Here, we only know the leading log (LL) running. This has consequences in Eq. (33) since, once we expand in $\alpha_s(\nu)$, there are some subleading logs which are unknown.

V. COMPARISON WITH EXPERIMENT

In this section we will compare our theoretical predictions with the experimental data. Our aim is to perform a combined global fit for the three sum rules. We will use the experimental data for the Bjorken sum rule from [27,40–43] analyzed according to Ref. [27]. Note that the elastic contribution has to be included in the experimental numbers in order for the sum rule to be fully inclusive, i.e.

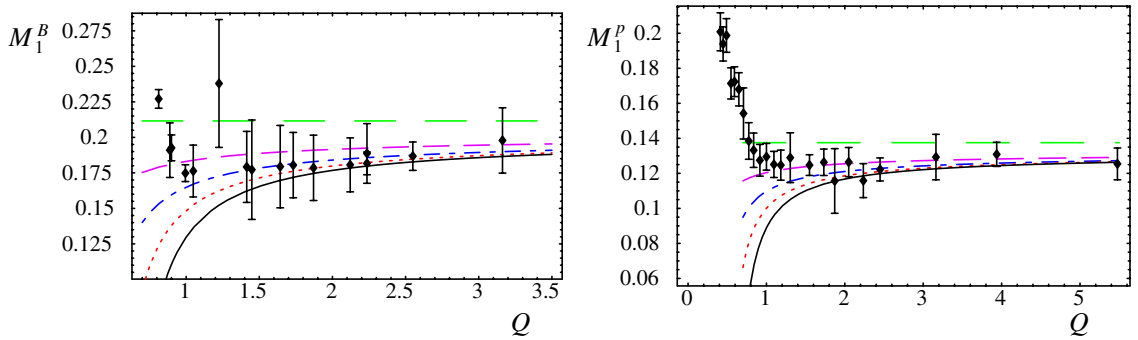
$$\Gamma(Q^2) = \frac{1}{2} F_1(Q^2)(F_1(Q^2) + F_2(Q^2)) + \Gamma_{\mathrm{inel.}}(Q^2). \quad (39)$$

The empirical parametrization of the elastic form factors was taken from [44]. For large momentum this contribution is completely negligible.

We take the experimental data points for $M_1^p(Q^2)$ from Ref. [26], which have used the experimental results for the structure functions from [45–52].

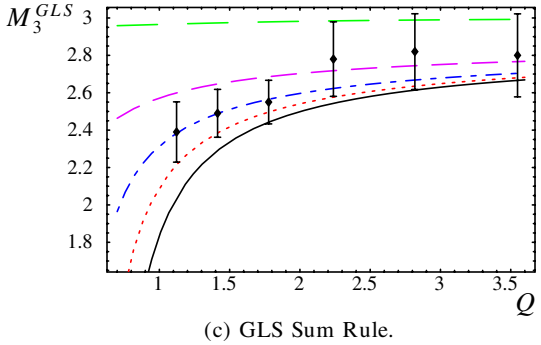
For the GLS sum rule we will use the experimental data from the CCFR Collaboration [29].

We first illustrate the problem of convergence of the perturbative series by drawing the perturbative series in the $\overline{\mathrm{MS}}$ scheme at different orders in α_s in Fig. 2. The perturbative series has a relative good convergence. However, this convergence deteriorates when we approach to low energies. We also see how the perturbative theoretical result diverges from the experimental numbers at low energies. As we have already stated, the solution to this problem comes from using the RS scheme. We plot again the perturbative series in the RS scheme in Fig. 3 for two values of ν_f: $\nu_f = 0.8$ and 1 GeV. We can see that in both cases the convergence of the perturbative series improves. This is especially so around the 1 GeV region. For $\nu_f = 1$ GeV we can also see a qualitative change in the figure with a much better agreement with experiment around the 1 GeV region. This alone does not mean much, since it only reflects the ν_f scale dependence of the pure perturbative piece. Nevertheless, this scale dependence is known and can be predicted by perturbation theory. This scale dependence cancels with the scale dependence of the higher-twist terms. Therefore, the complete result, including the higher-twist terms, should be independent of ν_f. Thus, consistency demands that if we perform the fit to experiment including the higher twist $f_i(\nu_f)$ for different values of ν_f, in particular, for 0.8 and 1 GeV, the results obtained for the $f_i(\nu_f)$ should be consistent with the result obtained by performing the perturbative running in ν_f with respect to these two values. We have checked that this is so within the errors of our evaluation. This reassures the reliability of our fit. This also tells us that it is reasonable to use the operator product expansion formulas for Q larger



(a) Bjorken Sum Rule.

(b) Ellis-Jaffe Sum Rule.



(c) GLS Sum Rule.

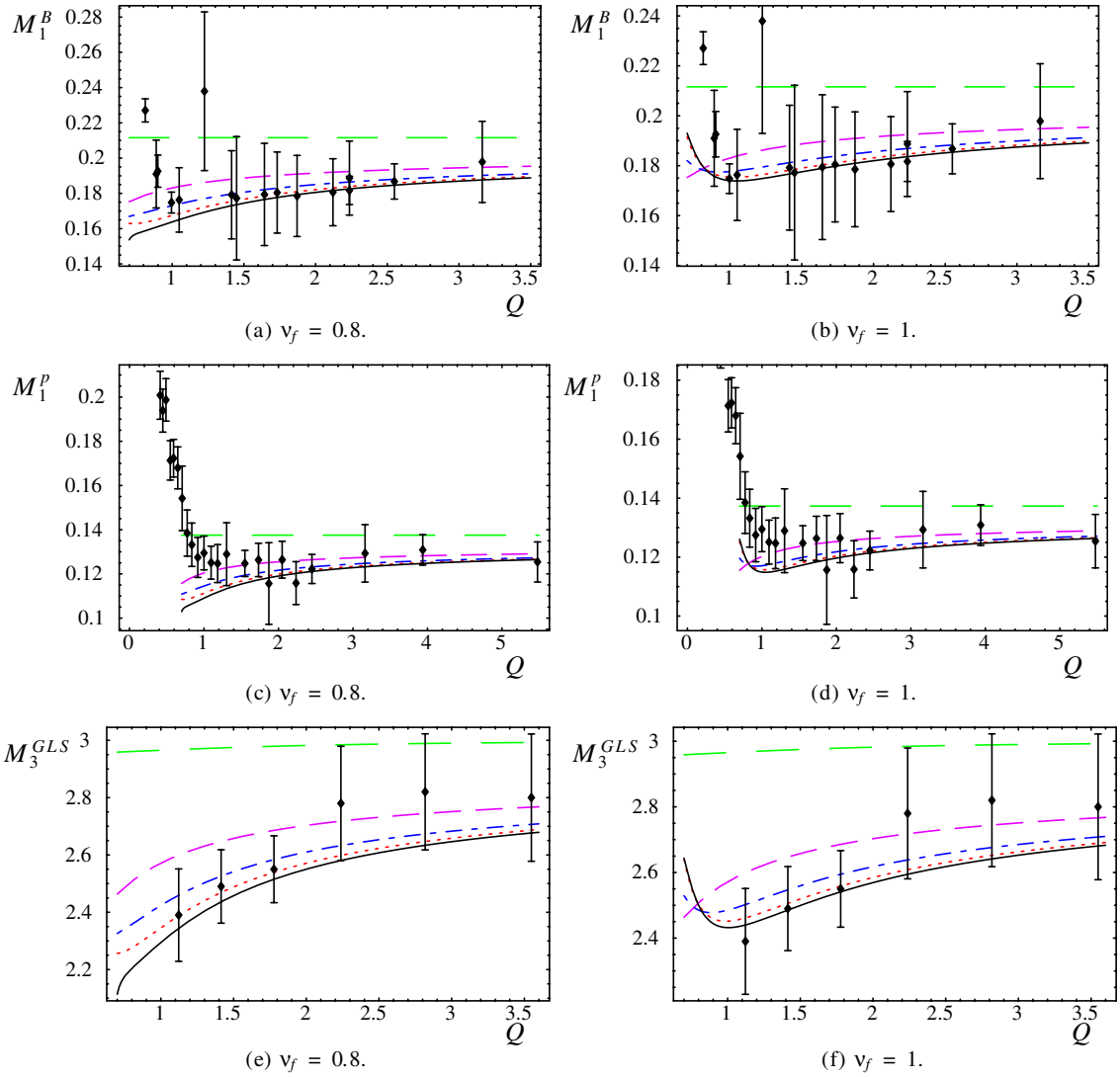
FIG. 2 (color online). Leading-twist contribution to the sum rules at different orders in perturbation theory in the $\overline{\text{MS}}$ scheme with $\nu = Q$ compared with the experimental data. The long-dashed line is the LO result. The dashed line is the NLO result. The dot-dashed line is the NNLO result. The dotted line is the NNNLO result and the continuous black line is the (estimate) N^{4}LO*. At this order we have used a renormalon-based estimate for $C_X^{(3)}(Q)$ from Tables II, III, and IV. $\hat{a}_0 = 0.141$.

than 0.8 GeV. This is the attitude we will take in this paper where we will include (unless otherwise indicated) the experimental points for $Q > 0.8$ GeV. Therefore, the perturbative plots in Fig. 3 with $\nu_f = 1$ GeV can be reinterpreted as that a piece of the higher-twist correction has been included in the pure perturbative term. This piece corresponds to the running of ν_f from 0.8 to 1 GeV of the higher-twist term and can be obtained from the renormalization group. Therefore, in this sense, the change of slope observed in Fig. 3 can be interpreted as having its origin in perturbation theory. Note that the change of slope, from the experimental point of view, comes from the elastic term.

In Fig. 3 higher-twist effects have not been included. Our next aim is to perform a fit of $\hat{a}_0$ and the subleading twist matrix elements from the available experimental data. We refrain from trying to fit α_s, since the experimental errors appear to be too large. Actually, they will be one of the major sources of uncertainty of our analysis. We perform a global fit of all the available data from the different sum rules at the same time. The size of the experimental errors is the largest for the GLS sum rule, whereas the most accurate data come from the $M_1^p(Q^2)$ experimental points. In any case, f_5^S will be obtained from the GLS sum rule alone, since this fit is independent of the other two sum rules. We perform the global fit to different orders in the expansion in α_s of the leading-twist perturbative Wilson coefficient. We work with the running α_s consistent with the accuracy one is working at each order.[5] We show the results of the fit at different orders in perturbation theory in Table V. We can see how the series shows convergence. We can then obtain relatively good estimates for $\hat{a}_0$ and f_3, for the other nonperturbative parameters the situation is less conclusive. The values have been obtained with $\nu_f = 1$ GeV.

In order to estimate the errors, we allow for a variation of $\alpha_s(M_z) = 0.118 \pm 0.003$, of N_X (according to the error given in Table I), of the allowed set of experimental points (we consider two situations: a) all the data points for $q > 0.8$ GeV and b) all the data points for $q > 1$ GeV; our central values will be the ones obtained with option a), and also consider the experimental errors. We also consider the difference between the N^{3}LO* and NNLO result as an

[5]We have also performed the analysis using the four-loop running α_s at any order. The final results are very similar. The use of the four-loop running α_s somewhat accelerates/improves the convergence of the series. Nevertheless, we prefer to keep ourselves consistent and only resum the logs associated to each order in perturbation theory. The fit using the principal value prescription is consistent with the finite-order computation. Nevertheless, it is less precise because of the reasons mentioned in Sec. IV. Therefore, we will not consider it further in this analysis.



(a) $\nu_f = 0.8$. (b) $\nu_f = 1$.

(c) $\nu_f = 0.8$. (d) $\nu_f = 1$.

(e) $\nu_f = 0.8$. (f) $\nu_f = 1$.

FIG. 3 (color online). Leading-twist contribution to the sum rules at different orders in perturbation theory in the RS scheme with $\nu = Q$ for two different values of the subtraction point ν_f compared with the experimental data. The long-dashed line is the LO result. The dashed line is the NLO result. The dot-dashed line is the NNLO result. The dotted line is the NNNLO result and the continuous black line is the (estimate) N^{4}LO*. At this order we have used a renormalon-based estimate for $C_X^{(3)}(Q)$ from Tables II, III, and IV. $\hat{a}_0 = 0.141$.

estimate of the error due to the convergence of the series. This error happens to be very small in comparison with the other source of errors. Summarizing, we obtain (with the f_i in units of GeV^2)

$$\hat{a}_0 = 0.141^{+0.006}_{-0.004}(\delta\alpha_s)^{+0.002}_{-0.002}(\delta N_X)^{+0.088}_{-0.088}(\exp)^{+0.001}_{-0.001} \times (\text{pert})^{+0.010}_{-0.010}(q > 1), \quad (40)$$

$$f_{0,\text{RS}}(1\ \text{GeV}) = 0.790^{+0.241}_{-0.399}(\delta\alpha_s)^{-0.489}_{+0.489}(\delta N_X)^{+0.159}_{-0.159} \times (\exp)^{+0.238}_{-0.238}(\text{pert})^{+1.060}_{-1.060}(q > 1), \quad (41)$$

$$f_{3,\text{RS}}(1\ \text{GeV}) = -0.124^{-0.050}_{+0.032}(\delta\alpha_s)^{-0.049}_{+0.049}(\delta N_X)^{-0.121}_{+0.121} \times (\exp)^{+0.007}_{-0.007}(\text{pert})^{+0.026}_{-0.026}(q > 1), \quad (42)$$

$$f^S_{5,\text{RS}}(1\ \text{GeV}) = -0.029^{-0.162}_{+0.124}(\delta\alpha_s)^{-0.065}_{+0.065}(\delta N_X)^{-0.344}_{+0.344} \times (\exp)^{+0.042}_{-0.042}(\text{pert}) \pm 0(q > 1), \quad (43)$$

$$f_{8,\text{RS}}(1\ \text{GeV}) = -3.30^{-0.98}_{+1.61}(\delta\alpha_s)^{+1.95}_{-1.95}(\delta N_X)^{-0.27}_{+0.27} \times (\exp)^{+0.96}_{-0.96}(\text{pert})^{+4.27}_{-4.27}(q > 1). \quad (44)$$

If we combine all the errors in quadrature we obtain

$$\hat{a}_0 = 0.141 \pm 0.089, \quad (45)$$

$$f_{0,\text{RS}}(1\ \text{GeV}) = 0.790^{+1.225}_{-1.266}\ \text{GeV}^2, \quad (46)$$

$$f_{3,\text{RS}}(1\ \text{GeV}) = -0.124^{+0.137}_{-0.142}\ \text{GeV}^2, \quad (47)$$

TABLE V. Determination of $\hat{a}_0$ and the higher-twist nonperturbative parameters from the global fit to the sum rules at different orders in perturbation theory.

	$\hat{a}_0$	$f_{0,\mathrm{RS}}$ (1 GeV)	$f_{3,\mathrm{RS}}$ (1 GeV)	$f^S_{5,\mathrm{RS}}$ (1 GeV)	$f_{8,\mathrm{RS}}$ (1 GeV)
LO	0.043	−1.571	0.119	1.030	5.90
NLO	0.120	−0.817	−0.075	0.304	3.11
NNLO	0.136	0.014	−0.106	0.097	−0.18
NNNLO	0.139	0.552	−0.117	0.014	−2.34
N^{4}LO*	0.141	0.790	−0.124	−0.029	−3.30

$$f^S_{5,\mathrm{RS}}(1\ \mathrm{GeV}) = -0.029^{+0.374}_{-0.388}\ \mathrm{GeV}^2, \quad (48)$$

$$f_{8,\mathrm{RS}}(1\ \mathrm{GeV}) = -3.30^{+5.06}_{-4.90}\ \mathrm{GeV}^2. \quad (49)$$

We have also performed the fit with $\nu_f = 0.8$. We obtain in this case

$$\hat{a}_0 = 0.143, \quad (50)$$

$$f_{0,\mathrm{RS}}(0.8\ \mathrm{GeV}) = 0.365, \quad (51)$$

$$f_{3,\mathrm{RS}}(0.8\ \mathrm{GeV}) = -0.210, \quad (52)$$

$$f^S_{5,\mathrm{RS}}(0.8\ \mathrm{GeV}) = -0.167, \quad (53)$$

$$f_{8,\mathrm{RS}}(0.8\ \mathrm{GeV}) = -1.58, \quad (54)$$

whereas the magnitude of the errors is similar to the fit with $\nu_f = 1$ GeV. As expected, the value of $\hat{a}_0$ almost remains independent of ν_f. The higher-twist parameters do depend on ν_f in a way predicted by the renormalization group. For instance, for f_3 we would have the following expression[6]

$$f_{3,\mathrm{RS}}(\nu_f) = f_{3,\mathrm{RS}}(\nu'_f)\left[\frac{\alpha_s(\nu'_f)}{\alpha_s(\nu_f)}\right]^{(-\gamma^0_{\mathrm{NS}})/2\beta_0} + \frac{9}{8}\nu_f^2 g_A[\delta C_{B,\mathrm{RS}}(Q=\nu_f;\nu'_f) - \delta C_{B,\mathrm{RS}}(Q=\nu_f;\nu_f)] \quad (55)$$

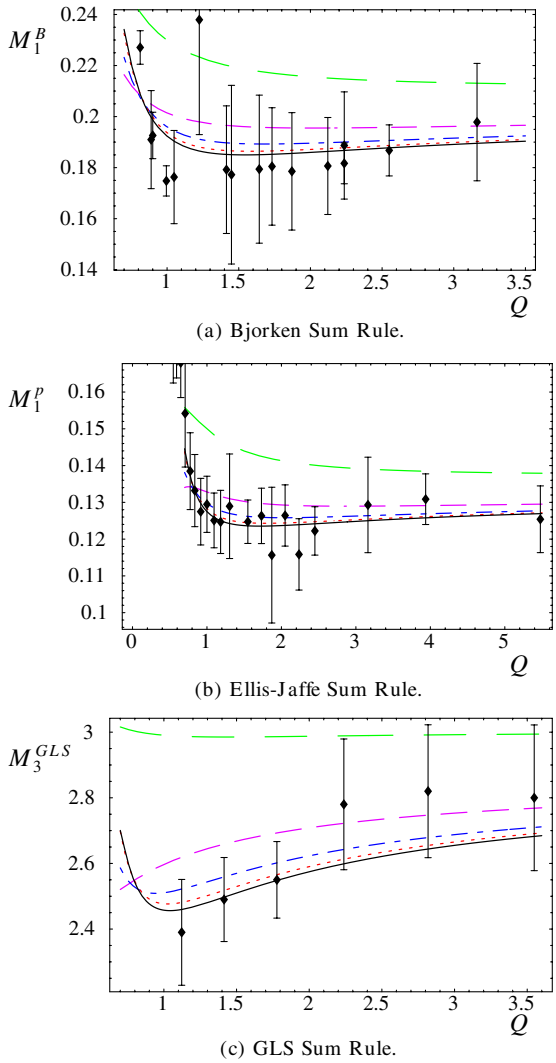
and analogously for the other higher-twist terms. Therefore, we should recover the values in Eqs. (45)–(49) after performing the renormalization group running of Eqs. (51)–(54). If we perform such running we obtain

$$f_{0,\mathrm{RS}}(1\ \mathrm{GeV}) = 0.335\ \mathrm{GeV}^2, \quad (56)$$

$$f_{3,\mathrm{RS}}(1\ \mathrm{GeV}) = -0.130\ \mathrm{GeV}^2, \quad (57)$$

$$f^S_{5,\mathrm{RS}}(1\ \mathrm{GeV}) = -0.004\ \mathrm{GeV}^2, \quad (58)$$

[6] $\delta C_{B,\mathrm{RS}}$ has a renormalon ambiguity, which cancels in the difference we find in the second term in the right-hand side of the equation. We would like to remind the reader that in order to enforce this renormalon cancellation order by order in α_s both terms have to be expanded with α_s taken at the very same scale.



(a) Bjorken Sum Rule.

(b) Ellis-Jaffe Sum Rule.

(c) GLS Sum Rule.

FIG. 4 (color online). Global fit to the Sum Rules in the RS scheme with $\nu = Q$ at different orders in perturbation theory including the leading and next-to-leading twist term. The long-dashed line is the LO result. The dashed line is the NLO result. The dot-dashed line is the NNLO result. The dotted line is the NNNLO result and the continuous black line is the (estimate) N^{4}LO*. At this order we have used a renormalon-based estimate for $C^{(3)}_X(Q)$ from Tables II, III, and IV. The values of $\hat{a}_0$, f_i, and f^S_5 are taken from Eqs. (45)–(49).

$$f_{8,\mathrm{RS}}(1\ \mathrm{GeV}) = -1.46\ \mathrm{GeV}^2. \quad (59)$$

We see that the running goes in the right direction for f_3 and f_5^S, whereas for f_0 and f_8 the values remain constant. Either way, the numbers agree with those obtained in Eqs. (46)–(49) within errors.

Finally, we have also considered the inclusion of $1/Q^4$ corrections. The values of f_0 and f_8 are not stable under the inclusion of these effects, although there is a correlation on the values of f_0 and f_8: a large positive value of f_0 is only possible if we also have a large negative value of f_8. The point is that, even if formally it should be possible to distinguish f_0 and f_8 due to the different anomalous dimension, in practice we do not have enough accuracy. Therefore, the values obtained above for f_0 and f_8 (and its errors) should be taken with caution. This warning also applies to the determination of f_5^S, for which the inclusion of $1/Q^4$ corrections significantly changes its value. For the other coefficients, $\hat{a}_0$ and f_3 the variations are smaller than the errors of our fit.

The error appears to be dominated by the experimental one in $\hat{a}_0$ and f_3, the objects we can compute with better accuracy.

We would also like to note that, in some cases, the values of the higher-twist nonperturbative parameters are compatible with zero within errors.

To illustrate the quality of the fit we plot our final results, the sum rules including the leading and subleading twist, at different orders in perturbation theory with our best fit, Eqs. (45)–(49), compared with the experimental data in Fig. 4.

The analysis of the GLS sum rule taking into account renormalon effects has been considered in Ref. [36]. In this reference only two data points were used and a principal valuelike Borel resummation prescription was used. The authors also included some pure nonperturbative effects arguing that they could be inherited from the renormalon computation. This is still an open question and, actually, it has been criticized by one of the authors in Ref. [53].

VI. CONCLUSIONS

We have studied the large order behavior in perturbation theory of the Bjorken, Ellis-Jaffe and GLS sum rules. In particular, we have considered their first infrared renormalons, for which we have obtained their analytic structure with logarithmic accuracy and also an approximate determination of their normalization constant. Estimates of higher order terms of the perturbative series are given. The RS scheme has been worked out for these observables and compared with experimental data. The convergence of the perturbative series greatly improves in this scheme, especially around the 1 GeV region. In particular, for $\nu_f = 1$ GeV, the agreement between the pure perturbative contribution and experiment is quite good. The fact that we have a convergent series in perturbation theory allows to give meaningful values for the higher-twist condensates with well-defined errors. We have performed a detailed analysis, being able to give predictions for $\hat{a}_0$ and some higher-twist condensates, including error bars. Our best fits for the sum rules can be found in Fig. 4. The values for the nonperturbative matrix elements read

$$\hat{a}_0 = 0.141 \pm 0.089, \quad (60)$$

$$f_{3,\mathrm{RS}}(1\ \mathrm{GeV}) = -0.124^{+0.137}_{-0.142}\ \mathrm{GeV}^2. \quad (61)$$

The experimental situation is not very good for the GLS sum rule, for which we cannot give a precise number for the higher twist. The experimental precision is not good enough to check the assumption of [54] that f_3 and f_5^S are equal. We also do not display here the values of f_0 and f_8, since they have large errors and their values in the fit are somewhat correlated. A large value of one of them could be obtained in the fit to the price of having the other being large with opposite sign.

One of the most important sources of error of the present analysis is the experimental one. Any improvement in this respect will immediately lead to a reduction of the errors of the numbers obtained in this paper.

The quality of the analysis is worse than the one obtained for heavy quark physics analysis. The sensitivity to the renormalon is smaller and the determination of the normalization constant is less accurate than in that case. This was to be expected since the singularities in the Borel plane are more far away here than in heavy quark physics. In any case, the inclusion of the renormalon cancellation introduces a qualitative change on the perturbative behavior around the 1 GeV region making it much closer to the experimental figures.

Another issue we would like to mention is that the resummation of renormalon-related logarithms does not appear to improve the convergence of the series. On the other hand, we have only performed the leading log resummation in this paper. It would be interesting to see what happens at higher orders.

Finally, the possibility to merge with the chiral limit seems closer now but the gap still exists. In particular one should find a systematic way to incorporate higher-twist effects.

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