

Green's function of a free massive scalar field on the lattice

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We propose a method to calculate the Green's function of a free massive scalar field on the lattice numerically to very high precision. For masses $m < 2$ (in lattice units) the massive Green's function can be expressed recursively in terms of the massless Green's function and just two additional mass-independent constants.

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Lattice regularization offers a convenient tool to study nonperturbative features of field theories. The Lagrangian is formulated on a discrete space-time lattice and the corresponding finite path integral is evaluated numerically. In most instances, one is ultimately interested in the continuum limit by decreasing the lattice spacing to zero, but for very small lattice spacings the lattice path integral is plagued with ultraviolet divergences rendering numerical lattice simulations impossible.

Perturbation theory plays an important role in providing the missing link between numerical simulations and the desired physical continuum limit. Perturbative calculations on the lattice are performed, e.g., to obtain renormalization constants of lattice operators or nonuniversal coefficients of β functions. If the physical situation contains separate scales, they are also necessary in order to match the perturbative short distance physics with the nonperturbative long distance physics.

However, perturbative calculations on the lattice tend to be difficult, since the Feynman integrals are rather complicated functions of the involved momenta. In this respect, coordinate-space methods have been proven useful in the evaluation of Feynman diagrams and allow a very precise determination of the continuum limit of two- and even higher-loop integrals [1]. In [1] this technique was applied to massless propagators. The massive case was addressed in [2], but for space-time dimension less than four, and explicit results for the Green's functions were obtained only in one and two dimensions. In the present investigation, we extend the work of Lüscher and Weisz [1] to massive scalar fields by presenting an efficient method to calculate the associated Green's function to very high precision.

We will first generalize the recursion relation for the Green's function presented in [1] to massive propagators. The free propagator for a massive scalar field in position space reads in lattice units

$$G(n) = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{e^{ik \cdot n}}{\hat{k}^2 + m^2}, \quad (1)$$

where $n = (n_1, n_2, n_3, n_4)$ and

$$\hat{k}^2 = \sum_{\mu=1}^4 \hat{k}_{\mu}^2 = 4 \sum_{\mu=1}^4 \sin^2(k_{\mu}/2). \quad (2)$$

We define the forward and backward lattice derivatives

$$\begin{aligned} \nabla_{\mu} G(n) &= G(n + \mu) - G(n), \\ \nabla_{\mu}^* G(n) &= G(n) - G(n - \mu) \end{aligned} \quad (3)$$

which yield the lattice Laplacian

$$\Delta = \sum_{\mu=1}^4 \nabla_{\mu}^* \nabla_{\mu} \quad (4)$$

with

$$\Delta G(n) = \sum_{\mu=1}^4 (G(n + \mu) + G(n - \mu) - 2G(n)). \quad (5)$$

The Green's function satisfies the equation

$$(-\Delta + m^2)G(n) = \delta_{n0} \quad (6)$$

from which one obtains after making use of hypercubic symmetry the identity

$$(8 + m^2)G(0) - 8G(\mu) = 1 \quad (7)$$

for any μ . The central identity utilized in [1] is given by

$$(\nabla_{\mu}^* + \nabla_{\mu})G(n) = n_{\mu}H(n), \quad (8)$$

where

$$H(n) = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} e^{ik \cdot n} \ln(\hat{k}^2 + m^2) \quad (9)$$

is independent of μ . By eliminating $H(n)$ via

$$H(n) = \frac{2}{\rho} \sum_{\mu=1}^4 \left(\left[1 + \frac{m^2}{8} \right] G(n) - G(n - \mu) \right) \quad (10)$$

for $\rho = \sum_{\mu=1}^4 n_{\mu} \neq 0$ one obtains a recursion relation for the Green's function at different lattice sites

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$$G(n + \mu) = G(n - \mu) + \frac{2n_\mu}{\rho} \sum_{\nu=1}^4 \left(\left[1 + \frac{m^2}{8} \right] G(n) - G(n - \nu) \right). \quad (11)$$

This recursion relation is at the heart of this method, as it allows to express the Green's function at any lattice site in terms of a few values close to the origin. Utilizing hypercubic symmetry of the lattice $G(n)$ can be written as a linear combination of the values of the propagator at the corners of the unit hypercube

$$\begin{aligned} G(0, 0, 0, 0), & \quad G(1, 0, 0, 0), & \quad G(1, 1, 0, 0), \\ G(1, 1, 1, 0), & \quad G(1, 1, 1, 1). \end{aligned} \quad (12)$$

The second value $G(1, 0, 0, 0)$ can be eliminated by making use of the identity (7). In the massless case, two more restrictions on the remaining four values of the Green's functions can be derived by reducing the relation in Eq. (11) to a one-dimensional recursion along the lattice axes and constructing two linear combinations of the Green's function values along the lattice axes which are independent of the lattice site n [1]. One can then perform the limit $n \rightarrow \infty$ and compare it with the asymptotic form of the massless Green's function G_0 yielding two more relations for the initial propagator values in (12). Therefore, out of the five initial values three can be eliminated and the entire massless Green's function can be expressed in terms of just two values, say $G_0(0, 0, 0, 0)$ and $G_0(1, 1, 0, 0)$, which have been calculated by Lüscher and Weisz to very high precision.

For general nonvanishing mass, on the other hand, linear combinations of the propagator values along the axes which are independent of the lattice site cannot be constructed. (An exception to this are the imaginary masses $m^2 = -4, -8, -12$. For a similar observation in two dimensions cf. [2].) Therefore, with the method described above the initial values of the Green's function close to the origin cannot be reduced further by employing its asymptotic form.

Nonetheless, there exists an additional constraint for the massive Green's function which allows us to express its values at any lattice site in terms of the massless Green's function G_0 and just two additional constants which are independent of the mass. As we will see, these additional two coefficients can be calculated to very high precision yielding a precise determination of the Green's function for $m < 2$. This constraint is based on the identity for the modified Bessel functions I_n [3]

$$\lambda [I_{n-1}(\lambda) - I_{n+1}(\lambda)] = 2nI_n(\lambda). \quad (13)$$

The Green's function can be written as an integral over the Bessel functions I_n

$$G(n_1, n_2, n_3, n_4) = \frac{1}{2} \int_0^\infty d\lambda e^{-m^2\lambda/2-4\lambda} I_{n_1}(\lambda) I_{n_2}(\lambda) I_{n_3}(\lambda) \times I_{n_4}(\lambda), \quad (14)$$

and utilizing the identity (13) we obtain the relation

$$\begin{aligned} G(n_1 + 1, n_2, n_3, n_4) &= G(n_1 - 1, n_2, n_3, n_4) \\ &+ G_0(n_1 + 1, n_2, n_3, n_4) \\ &- G_0(n_1 - 1, n_2, n_3, n_4) \\ &+ n_1 \int_0^{m^2} d\mu^2 G(n_1, n_2, n_3, n_4; \mu^2), \end{aligned} \quad (15)$$

where μ is the mass of the Green's function in the integral and G_0 is the massless Green's function. Therefore, knowledge of the functional dependence of G on the mass at lower lattice sites yields the mass-dependence at higher lattice sites. This constraint in combination with the recursion relation (11) allows us to calculate $G(1, 1, 0, 0)$, $G(1, 1, 1, 0)$, $G(1, 1, 1, 1)$ starting from the Green's function at the origin, the tadpole $G(0, 0, 0, 0)$, if we make use of its functional dependence on m . As a matter of fact, it is well known that the tadpole can be expressed by the following small mass expansion [4,5]

$$G(0, 0, 0, 0) = \sum_{i=0}^{\infty} a_i m^{2i} + m^2 \ln m^2 \sum_{i=0}^{\infty} b_i m^{2i} \quad (16)$$

with expansion coefficients a_i and b_i . We will show below that this expansion converges absolutely for masses $m^2 < 4$. Employing this expression, one uses then the relation

$$G(2, 0, 0, 0) = G(0, 0, 0, 0) + G_0(2, 0, 0, 0) - G_0(0, 0, 0, 0) + \int_0^{m^2} d\mu^2 G(1, 0, 0, 0; \mu^2), \quad (17)$$

where $G(1, 0, 0, 0)$ can be directly derived from $G(0, 0, 0, 0)$ via Eq. (7) and is easily integrated utilizing

$$\begin{aligned} \int_0^{m^2} \mu^i d\mu &= \frac{m^{2(i+1)}}{i+1} \quad \text{and} \\ \int_0^{m^2} \mu^i \ln \mu d\mu &= \frac{m^{2(i+1)}}{(i+1)^2} [(1+i) \ln m^2 - 1]. \end{aligned} \quad (18)$$

Next, the recursion relation (11) can be employed to obtain $G(1, 1, 0, 0)$

$$\begin{aligned} G(1, 1, 0, 0) &= \frac{1}{6} ([8 + m^2] G(1, 0, 0, 0) - G(2, 0, 0, 0) \\ &- G(0, 0, 0, 0)). \end{aligned} \quad (19)$$

Similar steps can be taken to obtain $G(2, 1, 0, 0)$

$$G(2, 1, 0, 0) = G(1, 0, 0, 0) + G_0(2, 1, 0, 0) - G_0(1, 0, 0, 0) + \int_0^{m^2} d\mu^2 G(1, 1, 0, 0; \mu^2), \quad (20)$$

which in turn leads to

$$G(1, 1, 1, 0) = \frac{1}{2} \left(\left[4 + \frac{m^2}{2} \right] G(1, 1, 0, 0) - G(2, 1, 0, 0) - G(1, 0, 0, 0) \right). \quad (21)$$

Finally, one has

$$G(2, 1, 1, 0) = G(1, 1, 0, 0) + G_0(2, 1, 1, 0) - G_0(1, 1, 0, 0) + \int_0^{m^2} d\mu^2 G(1, 1, 1, 0; \mu^2) \quad (22)$$

and

$$G(1, 1, 1, 1) = \frac{1}{2} ([8 + m^2]G(1, 1, 1, 0) - 3G(2, 1, 1, 0) - 3G(1, 1, 0, 0)). \quad (23)$$

We have thus calculated the Green's function at four lattice

sites of the unit hypercube $G(1, 0, 0, 0)$, $G(1, 1, 0, 0)$, $G(1, 1, 1, 0)$, $G(1, 1, 1, 1)$ recursively by making use of Eqs. (7), (11), and (15) which is sufficient to calculate $G(n_1, n_2, n_3, n_4)$ at all other lattice sites. As input we merely need the massless Green's function which is already known to high precision and the Green's function at the origin, i.e. the tadpole $G(0, 0, 0, 0)$.

For the calculation of the massive Green's function it is therefore sufficient to determine the coefficients a_i and b_i in Eq. (16) to very high precision. Moreover, one can express all expansion coefficients a_i and b_i just in terms of the leading coefficients a_0, a_1, a_2 and b_0 . Since a_0 is the massless tadpole $G_0(0, 0, 0, 0)$ and b_0 can be calculated exactly, one is then left with the task to determine the constants a_1, a_2 very precisely. To this end, consider the differential equation for the modified Bessel function $I_n(\lambda)$

$$\lambda^2 \frac{d^2}{d\lambda^2} I_n(\lambda) + \lambda \frac{d}{d\lambda} I_n(\lambda) - (\lambda^2 + n^2) I_n(\lambda) = 0. \quad (24)$$

It follows that the product $I_0(\lambda)^4$ of Bessel functions satisfies the differential equation

$$\left(\lambda^4 \frac{d^5}{d\lambda^5} + 10\lambda^3 \frac{d^4}{d\lambda^4} - 5\lambda^2(4\lambda^2 - 5) \frac{d^3}{d\lambda^3} - 15\lambda(8\lambda^2 - 1) \frac{d^2}{d\lambda^2} + (64\lambda^4 - 152\lambda^2 + 1) \frac{d}{d\lambda} + 32\lambda(4\lambda^2 - 1) \right) I_0(\lambda)^4 = 0. \quad (25)$$

With the representation

$$G(0, 0, 0, 0) = \frac{1}{2} \int_0^\infty d\lambda e^{-m^2\lambda/2 - 4\lambda} I_0(\lambda)^4, \quad (26)$$

this translates into a differential equation for the tadpole [6]

$$\left((8t^5 - 160t^3 + 512t) \frac{d^4}{d(m^2)^4} + (40t^4 - 480t^2 + 512) \frac{d^3}{d(m^2)^3} + (50t^3 - 304t) \frac{d^2}{d(m^2)^2} + (15t^2 - 32) \frac{d}{d(m^2)} + \frac{t}{2} \right) G(0, 0, 0, 0) = 0, \quad (27)$$

where $t = 4 + m^2/2$. Inserting the ansatz (16) into Eq. (27) each power m^{2i} and $m^{2i} \ln m^2$ must vanish separately. This leads to recursive relations between the expansion coefficients a_i, b_i which are then expressible as linear functions of the leading coefficients a_0, a_1, a_2 and b_0 . The relations between the coefficients $b_{0,1,2,3}$ are given by

$$384b_0 + 3072b_1 = 0, \quad 208b_0 + 3968b_1 + 18432b_2 = 0, \quad 62b_0 + 2512b_1 + 31104b_2 + 110592b_3 = 0, \quad (28)$$

while for the higher coefficients they read

$$i^4 b_{i-2} + (8 + 40i + 80i^2 + 80i^3 + 40i^4) b_{i-1} + (832 + 2784i + 3632i^2 + 2240i^3 + 560i^4) b_i + (15872 + 43008i + 43136i^2 + 19200i^3 + 3200i^4) b_{i+1} + (73728 + 172032i + 141312i^2 + 49152i^3 + 6144i^4) b_{i+2} = 0, \quad (29)$$

with $i \geq 2$. The relations between the coefficients a_i and b_i are

$$2a_0 + 208a_1 + 3968a_2 + 18432a_3 + 696b_0 + 10752b_1 + 43008b_2 = 0, \quad (30)$$

$$a_0 + 248a_1 + 10048a_2 + 124416a_3 + 442368a_4 + 600b_0 + 19008b_1 + 199680b_2 + 626688b_3 = 0, \quad (31)$$

and for $i \geq 2$

$$i^4 a_{i-1} + (8 + 40i + 80i^2 + 80i^3 + 40i^4) a_i + (832 + 2784i + 3632i^2 + 2240i^3 + 560i^4) a_{i+1} + (15872 + 43008i + 43136i^2 + 19200i^3 + 3200i^4) a_{i+2} + (73728 + 172032i + 141312i^2 + 49152i^3 + 6144i^4) a_{i+3} + 4i^3 b_{i-2} + (40 + 160i + 240i^2 + 160i^3) b_{i-1} + (2784 + 7264i + 6720i^2 + 2240i^3) b_i + (43008 + 86272i + 57600i^2 + 12800i^3) b_{i+1} + (172032 + 282624i + 147456i^2 + 24576i^3) b_{i+2} = 0. \quad (32)$$

The constant a_0 has already been given in [7] with a precision of 10^{-396}

$$a_0 = 0.154933390231060214084837208107375088769161133645219\dots, \quad (33)$$

and we are left with the precise determination of a_1 and a_2 . This can be accomplished by splitting the tadpole integral $G(0, 0, 0, 0)$ into two parts [6]

$$G(0, 0, 0, 0) = \frac{1}{2} \int_0^\Lambda d\lambda e^{-m^2\lambda/2-4\lambda} I_0(\lambda)^4 + \frac{1}{2} \int_\Lambda^\infty d\lambda e^{-m^2\lambda/2-4\lambda} I_0(\lambda)^4, \quad (34)$$

where we leave the parameter Λ unspecified for the time being. The first integral is analytic in the mass and can be expanded in m^2 , while the logarithmic pieces proportional to $\ln m^2$ are contained in the second integral. The integration limit Λ is now chosen large enough, say $\Lambda = 200 \sim 300$, such that the integrand in the second integral can be replaced by the asymptotic form for the Bessel functions at large arguments. Hence we obtain

$$G(0, 0, 0, 0) = \frac{1}{2} \int_0^\Lambda d\lambda e^{-m^2\lambda/2-4\lambda} I_0(\lambda)^4 + \frac{1}{2} \sum_{n=2}^\infty c_n \int_\Lambda^\infty d\lambda e^{-m^2\lambda/2} \lambda^{-n} \quad (35)$$

with well-known expansion coefficients c_n . The second integration can be performed analytically, while the first integral can be evaluated numerically to very high precision, e.g., by employing a Gauss-Legendre algorithm.¹ Although the two single integrals depend on Λ , the dependence cancels out in the sum of both terms. Equation (35) is therefore suited to determine the constants a_0, a_1, a_2 and b_0 . As a check we first calculate a_0

$$a_0 = G_0(0, 0, 0, 0) = \frac{1}{2} \int_0^\Lambda d\lambda e^{-4\lambda} I_0(\lambda)^4 + \frac{1}{2} \sum_{n=2}^\infty \frac{c_n}{(n-1)\Lambda^{n-1}} \quad (36)$$

and obtain agreement with the value given in Eq. (33) up to a precision of 10^{-150} . This precision is sufficient for our purposes, although it could be easily improved further by using slightly more computer time. The direct integration proposed here is thus an alternative method to the one advocated in [1] for obtaining the massless Green's function at the origin to very high precision. For the coefficients a_1 and a_2 this procedure yields²

$$a_1 = -\frac{1}{4} \int_0^\Lambda d\lambda \lambda e^{-4\lambda} I_0(\lambda)^4 + \frac{1}{2} \sum_{n=2}^\infty \frac{c_n}{\Lambda^{n-1}} T_1 \left\{ E_n \left(\frac{\Lambda m^2}{2} \right) \right\} = -0.030345755097111005404345333329331191871789937096381870985508127022928862366546240927128051477698201507815845611452036369186167716864948113458657909428\dots, \quad (37)$$

$$a_2 = \frac{1}{16} \int_0^\Lambda d\lambda \lambda^2 e^{-4\lambda} I_0(\lambda)^4 + \frac{1}{2} \sum_{n=2}^\infty \frac{c_n}{\Lambda^{n-1}} T_2 \left\{ E_n \left(\frac{\Lambda m^2}{2} \right) \right\} = 0.00277592745728397958697102673077239223991230261503630812094831799582013729662936744646239160211547072551443823584893971316374294223003262761367623512\dots, \quad (38)$$

where E_n is the exponential integral function defined by

$$E_n(z) = \int_1^\infty \frac{\exp(-zt)}{t^n} dt, \quad (39)$$

and T_i denotes the i th coefficient in the asymptotic expansion of E_n in m^2 . Utilizing the series representation of the exponential integral function

¹Since the integrand $e^{-m^2\lambda/2-4\lambda} I_0(\lambda)^4$ is an analytic function in λ , the interpolation procedure converges.

²The first nine digits of a_1 can be found, e.g., in [8].

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!} [-\ln z + \psi(n)] - \sum_{m=0, m \neq n-1}^{\infty} \frac{(-z)^m}{(m-n+1)m!}, \quad (40)$$

where $\psi(n)$ is digamma function defined by

$$\psi(1) = -\gamma, \quad \psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m}, \quad (41)$$

one can immediately read off the coefficients T_i

$$\frac{1}{2} \sum_{n=2}^{\infty} \frac{c_n}{\Lambda^{n-1}} T_1 \left\{ E_n \left(\frac{\Lambda m^2}{2} \right) \right\} = -\frac{1}{4} \left[c_2 \left\{ \psi(2) - \ln \left(\frac{\Lambda}{2} \right) \right\} - \sum_{n=3}^{\infty} c_n \frac{\Lambda^{2-n}}{2-n} \right], \quad (42)$$

$$\frac{1}{2} \sum_{n=2}^{\infty} \frac{c_n}{\Lambda^{n-1}} T_2 \left\{ E_n \left(\frac{\Lambda m^2}{2} \right) \right\} = \frac{1}{16} \left[c_3 \left\{ \psi(3) - \ln \left(\frac{\Lambda}{2} \right) \right\} - c_2 \Lambda - \sum_{n=4}^{\infty} c_n \frac{\Lambda^{3-n}}{3-n} \right]. \quad (43)$$

We have varied Λ within the range $\Lambda = 200 \dots 400$ and confirmed that the results are indeed independent of Λ up to a precision of 10^{-150} . The parameter b_0 , on the other hand, originates solely from the second integral and can be calculated exactly: $b_0 = 1/16\pi^2$. From Eqs. (28) and (29) it follows then that all coefficients b_i can be calculated analytically. Having determined the constants a_0, a_1 and a_2 to very high precision, we use the recursion relations for the coefficients a_i and b_i which follow from Eq. (27) to calculate the last remaining value of the Green's function on the unit hypercube, $G(0, 0, 0, 0)$. As an additional check, we have calculated the Green's function values $G(0, 0, 0, 0), G(1, 1, 0, 0), G(1, 1, 1, 0), G(1, 1, 1, 1)$ by splitting the integral representation of Eq. (14) into two ranges $[0, \Lambda]$ and $[\Lambda, \infty)$ analogous to Eq. (34) and evaluating the integrals directly. The results obtained in this way agree with the ones achieved by applying the recursion relation up to 10^{-135} . With the method outlined above we thus confirm the form of the tadpole in Eq. (16).

Employing the recursion relations Eqs. (29) and (32) we now illustrate that the small mass expansion of the tadpole (16) converges absolutely for $m^2 < 4$. For large enough i the recursion relation for the b_i simplifies to

$$\frac{b_{i+1}}{b_i} = -\frac{25}{48} - \frac{b_{i-3}}{6144b_i} - \frac{5b_{i-2}}{768b_i} - \frac{35b_{i-1}}{384b_i}. \quad (44)$$

If we assume that for large i the ratio b_{i+1}/b_i converges asymptotically towards a constant value, then Eq. (44) has the solutions $b_{i+1}/b_i = -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{12}, -\frac{1}{16}$. We have confirmed numerically that the ratio tends indeed towards

$-\frac{1}{4}$. This implies that for large i the parameters b_i scale as $|b_i| \propto 4^{-i}$. The logarithmic part in Eq. (16) converges thus absolutely for $m^2 < 4$.

Assuming also a constant ratio a_{i+1}/a_i for large enough i and the boundedness of b_i/a_i we arrive at the same simplified recursion relation for the a_i

$$\frac{a_{i+1}}{a_i} = -\frac{25}{48} - \frac{a_{i-3}}{6144a_i} - \frac{5a_{i-2}}{768a_i} - \frac{35a_{i-1}}{384a_i}. \quad (45)$$

Again, we have checked numerically that both the ratio a_{i+1}/a_i tends towards $-\frac{1}{4}$ and b_i/a_i is bounded. This implies that the polynomial part of the tadpole in (16) and hence the entire small mass expansion of the tadpole converges absolutely for $m^2 < 4$.

Next, we would like to confirm the results for the asymptotic behavior of the lattice Green's function for large $|n| = \sqrt{n_1^2 + n_2^2 + n_3^2 + n_4^2}$. In [9] Paladini and Sexton have derived an asymptotic form of $G(n)$ for large $|n|$, $G_{as}(n)$, in terms of modified Bessel functions of the second kind, $K_l(m|n|)$. In the left panel of Fig. 1 we have plotted our results $\ln[G(n, 0, 0, 0)]$ for four different masses. We observe that with increasing n $\ln[G(n, 0, 0, 0)]$ becomes an asymptotically linear function in n . As for large n the Bessel functions $K_l(mn)$ decrease exponentially as $K_l(mn) \sim \sqrt{\pi/2mn} \exp(-mn)$, the different slopes in the plot reflect the different masses. In the right panel, the ratio $G_{as}(n, 0, 0, 0)/G(n, 0, 0, 0)$ is plotted for $m^2 = 1/2$, demonstrating clearly the fast convergence of the asymptotic expansion towards the exact result.

So far, we have discussed the case $m < 2$. The case $m \geq 1$, on the other hand, can be calculated in a straightforward manner. Starting from Eq. (14), we replace the Bessel functions I_ν by their Taylor series to obtain

$$G(n_1, n_2, n_3, n_4) = \frac{1}{2} \int_0^\infty d\lambda e^{-m^2 \lambda/2 - 4\lambda} \times \sum_{i=0}^{\infty} d_i(n_1, n_2, n_3, n_4) \lambda^i, \quad (46)$$

where the expansion coefficients d_i are derived from the product of the Taylor series of the I_ν . Since the integrand in Eq. (46) is absolutely convergent, summation and integration can be interchanged leading to

$$G(n_1, n_2, n_3, n_4) = \frac{1}{2} \sum_{i=0}^{\infty} d_i(n_1, n_2, n_3, n_4) \times \frac{i!}{4^{(i+1)}} \left(1 + \frac{m^2}{8}\right)^{-(i+1)}. \quad (47)$$

We have checked numerically that this series converges sufficiently fast for masses $m \geq 1$. E.g., for $m = 1$ we obtain a precision of 10^{-160} by summing up the first $n = 7000$ terms in (47), while for $m^2 = 8$ the precision is 10^{-191} when taking the first 700 terms into account. Note, that the convergence behavior of the sum in

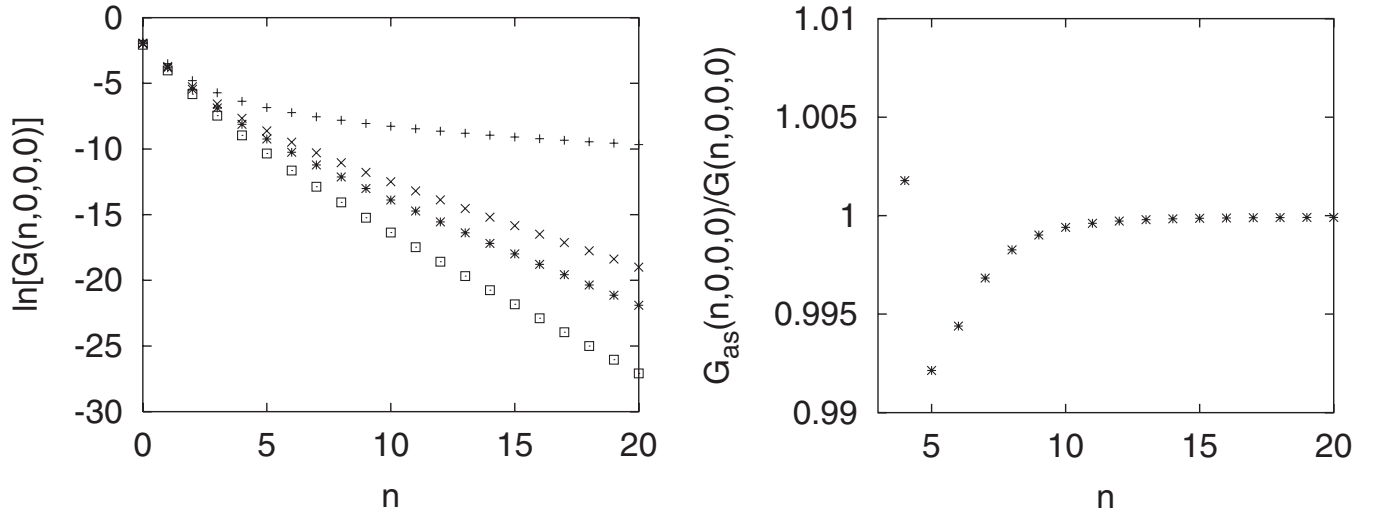


FIG. 1. Left panel: Shown are the logarithms of the Green's functions $G(n, 0, 0, 0)$ for masses $m^2 = 0, \frac{3}{10}, \frac{1}{2}, 1$ (from top to bottom). Right panel: Ratio of $G_{as}(n, 0, 0, 0)/G(n, 0, 0, 0)$ for $m^2 = \frac{1}{2}$.

Eq. (47) worsens with decreasing m and is extremely inefficient for $m \ll 1$. It is therefore not suited to perform the limit $m \rightarrow 0$.

To conclude, we have presented the precise determination of the Green's function of a massive scalar field on the lattice which is based on a recursion relation for the values of the Green's function values at different lattice sites similar to the one obtained in the massless case [1]. We have shown that the Green's function for masses $m < 2$ is fixed by the knowledge of the massless Green's function and only two additional mass-independent constants a_1 and a_2 , which can be determined to very high precision. We have also confirmed the usefulness of the large $|n|$ asymptotic expansion of $G(n)$ given by Paladini and Sexton [9] and demonstrated its fast convergence towards

the exact result. For $m \gtrsim 1$ the Green's function can be calculated directly by writing the Green's function as an integral over the Bessel functions and replacing these by the corresponding Taylor series.

With a minimal amount of computer time we achieve precisions of 10^{-150} for the Green's function values at the origin. Such precise knowledge of the Green's function is expected to be sufficient for carrying out two- and higher-loop calculations. The applicability to higher-loop integrals is currently under investigation and could prove useful, e.g., in the recently proposed framework of lattice regularized chiral perturbation theory [10].

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