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We study the Abelian projected $SU(2)$ lattice gauge theory after gauge fixing to the maximally Abelian gauge (MAG). In order to check the universality of the Abelian dominance we employ the tadpole improved (TI) tree level action. We show that the density of monopoles in the largest cluster (the IR component) is finite in the continuum limit which is approximated already at relatively large lattice spacing. The value itself is smaller than in the case of Wilson action. We present results for the ratio of the Abelian to non-Abelian string tension for both Wilson and TI actions for a number of lattice spacings in the range $0.06 \text{ fm} < a < 0.35 \text{ fm}$. These results show that the ratio is between 0.90 and 0.95 for all considered values of lattice couplings and both actions. We compare the properties of the monopole clusters in two gauges—in MAG and in the Laplacian Abelian gauge (LAG). Whereas in MAG the infrared component of the monopole density shows a good convergence to the continuum limit, we find that in LAG it is even not clear whether a finite limit exists.

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I. INTRODUCTION

The dual superconductor scenario of confinement has received support from many observations made as well in gluodynamics [1,2] as in full lattice QCD [3]. The most intensively investigated case was $SU(2)$ gluodynamics. The scaling properties of many gauge dependent observables such as the Abelian string tension, the effective monopole action, the monopole density, etc. have been checked. It has been shown that some properties of the monopoles in maximally Abelian gauge (MAG) can be explained by percolation theory or by free particle field theory [4]. Despite this progress there is lack of universality checks, i.e. the independence of the choice of action has not been confirmed. Apart from papers [5,6], always the Wilson action for the gauge field has been employed. In comparison with Ref. [5], where the same tadpole improved (TI) action was used, we have much better statistics and better gauge fixing, i.e. lower effects of gauge fixing ambiguities. In [6], where a different improved action was considered, the study was made for one value of lattice spacing only and thus no scaling studies were attempted. In the present paper we are aiming to make a contribution to closing this gap. The other, perhaps even more important problem is the dependence on the gauge condition used for Abelian projection. There are various opinions on this problem. Some authors believe that the occurrence of monopole condensation itself and, correspondingly, the dual superconductor properties of the vacuum have to exist and do exist in any Abelian gauge [7]. On the other hand recent results [8] obtained with the Fröhlich-Marchetti monopole creation operator show that the monopole condensate depends on the choice of the Abelian gauge. In [9] it was argued that in the Abelian gauge defined by diagonalization of the Polyakov loop operator the condensation

of monopoles does not necessarily lead to formation of the Abelian flux tube between static quarks. We share the opinion, that any Abelian projection is made with the goal of separating degrees of freedom responsible for infrared physics, which thus should carry all low momenta of the original non-Abelian gauge field, from ultraviolet degrees of freedom which are responsible for short distance physics (if they have a sensible continuum limit) or might even be mere lattice artefacts. Such separation does not need to be accomplished in any conceivable Abelian gauge. Rather we expect that there might exist a class of gauges which indeed have this property. The MAG is a very likely candidate to belong to this class, and the Laplacian Abelian gauge (LAG) [10] is widely considered as another good candidate. Since the analytical study of these gauges in the nonperturbative regime is very difficult and has not been accomplished so far despite many recent attempts (for MAG studies see, e.g. [11] and references therein), the numerical study is the only way to approach this problem in practice. Therefore, in this study we also compare some of the properties of these two gauges. From the above point of view, the issue of universality, it turns out that MAG is really unique to allow the separation of scales attempted by Abelian projection.

The paper is organized as follows. In Section II we specify the technical tools, in particular, the improved action used here in contrast to the Wilson action and the method of gauge fixing. Then, in Section III, we briefly report on the evaluation of the string tension for the purpose of calibrating the lattice scale corresponding to the improved action. Section IVA contains our observations concerning the scaling properties of the monopole densities and their IR and UV components for both actions in the maximally Abelian gauges. In section IV B we show that similarly to MAG the monopole clusters obtained in

the Laplacian Abelian gauge might be splitted into IR and UV components but their scaling properties are quite different from those observed in MAG. Section V is devoted to the Abelian dominance study. Our results indicate universality of the Abelian dominance in the continuum limit. Finally we summarize our findings in Section VI.

II. SIMULATION DETAILS

To address again the question of universality, we employ here the tree level improved action of the form [12]

$$S = \beta_{\text{imp}} \sum_{pl} S_{pl} - \frac{\beta_{\text{imp}}}{20u_0^2} \sum_{rt} S_{rt}, \quad (1)$$

where S_{pl} and S_{rt} denote plaquette and 1×2 rectangular loop terms in the action

$$S_{pl,rt} = \frac{1}{2} \text{Tr}(1 - U_{pl,rt}), \quad (2)$$

the parameter u_0 is the *input* tadpole improvement factor taken here equal to the fourth root of the average plaquette $P = \langle \frac{1}{2} \text{tr} U_{pl} \rangle$.

In our simulations we have not included one-loop corrections to the coefficients, for the sake of simplicity and also to be able to compare with the results of Ref. [5] after making a few improvements in comparison with this work in other directions. First, we improved substantially the gauge fixing as will be explained later. Second, we have now better statistics and worked on larger physical volumes. This has allowed to determine more reliably various Abelian observables and their infrared part. Third, we have used a new smearing technique which enabled us to make more precise measurements of the non-Abelian string tension. This was necessary to assess the nonperturbative scaling of various monopole densities.

We also make a comparison of these Abelian observables obtained in MAG and LAG, respectively. The MAG is fixed by the maximization of the lattice functional

$$F(U) = \frac{1}{8V} \sum_{n,\mu} \text{Tr}(\sigma_3 U_{n,\mu} \sigma_3 U_{n,\mu}^\dagger), \quad (3)$$

with respect to local gauge transformations

$$U_{n,\mu} \rightarrow U_{n,\mu}^g = g_n U_{n,\mu} g_{n+\mu}^\dagger. \quad (4)$$

For MAG we applied the simulated annealing algorithm. The details of the gauge fixing procedure can be found in [13]. We have applied the algorithm to 10 randomly replicated gauge copies of each Monte Carlo configuration in the hope to find among the 10 local maxima one closest to the global maximum. This procedure proved to be the best so far to fix MAG as well as to fix center gauges [14]. Although there is no proof we hope that our results for gauge noninvariant observables are numerically close to those we would obtain evaluating it at the gauge equivalent

representant carrying the global maximum of (3) for every configuration of the gauge field.

Fixing the LAG amounts to finding the eigenvector with the lowest eigenvalue of the covariant Laplacian operator in the adjoint representation,

$$-\square_{nm}^{ab} = \sum_{\mu} (2\delta_{nm} \delta^{ab} - R_{n,\mu}^{ab} \delta_{m,n+\hat{\mu}} - R_{n-\hat{\mu},\mu}^{ba} \delta_{m,n-\hat{\mu}}) \quad (5)$$

with the adjoint link variable

$$R_{n,\mu}^{ab} = \text{Tr}(\sigma_a U_{n,\mu} \sigma_b U_{n,\mu}^\dagger). \quad (6)$$

The gauge transformation g_n is then determined by the requirement to rotate this eigenvector ϕ_n^a to the 3rd color axis at every site n :

$$\rho_n \sigma_3 = \sum_{a=1}^3 \phi_n^a g_n \sigma_a g_n^\dagger, \quad \rho_n = \sqrt{\phi_n^2}. \quad (7)$$

The simulations with the action (1) have been performed with parameters given in Table I. The parameter u_0 has been iterated over a series of Monte Carlo runs in order to match the fourth root of the average plaquette P . The corresponding entries give an impression of the accuracy of matching. For two values of β_{imp} we simulated lattices of two sizes. The value of the parameter u_0 fixed on smaller lattice was used as an input for larger lattice simulations. The string tensions obtained on these lattices are in agreement within error bars. The smaller (larger) lattices were used to study LAG (MAG).

III. THE NON-ABELIAN STRING TENSION

In order to fix the physical lattice scale we need to compute one physical dimensionful observable the value of which is known. For this purpose we choose the string tension σ . The string tension for action (1) was computed long ago in [5] but we will improve this measurement according to present standards. We use the hypercubic blocking (HYP) invented by the authors of Ref. [15] to reduce the statistical errors. This method has been successfully applied to static potential calculations in $SU(3)$ gluo-

TABLE I. Details of the simulations with improved action.

β_{imp}	L	N_{conf}	u_0	$\langle P \rangle^{1/4}$	$\sqrt{\sigma a^2}$
2.7	12	60	0.87164	0.87165(2)	0.60(5)
3.0	12	200	0.89485	0.89478(2)	0.366(8)
3.1	12	200	0.90069	0.90069(1)	0.309(6)
3.2	16	200	0.90578	0.905765(3)	0.258(5)
3.3	16	100	0.91015	0.910152(4)	0.219(3)
3.3	20	50	0.91015	0.910153(3)	0.215(3)
3.4	20	100	0.91402	0.914020(2)	0.180(3)
3.5	20	100	0.91747	0.917481(1)	0.151(3)
3.5	24	50	0.917475	0.917484(2)	0.152(2)

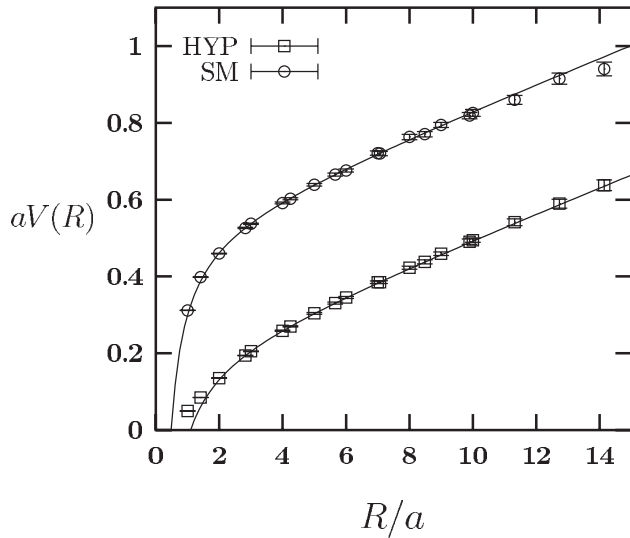


FIG. 1. Non-Abelian potentials for the TI action obtained without (circles, SM only) and after (squares, with HYP) hyper-cubic blocking (both with spatial smearing) vs R/a for $T/a = 5$ at $\beta_{\text{imp}} = 3.4$.

dynamics [15–17] and in full lattice QCD at finite temperature [18]. After one step of HYP, about 20 sweeps of APE smearing [19] were applied to the spacelike links. The spatial smearing (SM) is made, as usually, in order to variationally improve the overlap with a mesonic flux tube state. In Fig. 1 we compare potentials obtained with and without HYP procedure. As was observed in the cited above papers the HYP potential differs essentially by a constant shift corresponding to reducing the static source self-energy. One can see from the figure that HYP decreases both statistical errors and effects of rotational invariance breaking. Since HYP changes the potential at small distances we included only distances $R/a > 2$ into our fits of the static potential. The resulting values for $\sqrt{\sigma a^2}$ are also included in Table I. The string tension was also calculated with Wilson action. In this case APE smearing for space links and the additional trick of link integration [20] for time links were used in the evaluation of Wilson loops. The results for $\sqrt{\sigma a^2}$ and details of simulations with Wilson action are presented in Table II.

TABLE II. Details of the simulations with Wilson action.

β	L	N_{conf}	$\sqrt{\sigma a^2}$
2.40	32	35	0.264(7)
2.45	24	100	0.226(3)
2.50	24	100	0.185(2)
2.55	28	100	0.159(2)
2.60	28	100	0.1319(15)
2.65	32	40	0.114(2)

IV. THE MONOPOLE DENSITY

A. MAG

After fixing the Abelian gauge the Abelian projection can be made:

$$U_{n,\mu} = C_{n,\mu} u_{n,\mu} \quad (8)$$

where the Abelian field is contained in $u_{n,\mu} = \text{diag}(e^{i\theta_{n,\mu}}, e^{-i\theta_{n,\mu}})$, $\theta_{n,\mu} \in (-\pi, \pi]$, and $C_{n,\mu}$ is the coset field describing charged gluons. The Abelian plaquette angle

$$\theta_{n,\mu\nu} = \partial_\mu \theta_{n,\nu} - \partial_\nu \theta_{n,\mu} \quad (9)$$

is decomposed into regular and singular parts:

$$\theta_{n,\mu\nu} = \bar{\theta}_{n,\mu\nu} + 2\pi m_{n,\mu\nu}, \quad \bar{\theta}_{n,\mu\nu} \in (-\pi, \pi]. \quad (10)$$

$\bar{\theta}_{n,\mu\nu}$ is a physical Abelian flux through the lattice plaquette $\{n, \mu\nu\}$, and $m_{n,\mu\nu}$ counts the number of Dirac strings through this plaquette. The magnetic currents are then defined as follows:

$$k_{n,\mu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \partial_\nu \bar{\theta}_{n,\alpha\beta} = -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \partial_\nu m_{n,\alpha\beta}. \quad (11)$$

We will measure the monopole density in lattice units ρ^{lat} defined as

$$\rho^{\text{lat}} = \sum_{n,\mu} \frac{|k_{n,\mu}|}{4L^4}. \quad (12)$$

Since monopoles are three-dimensional objects their physical density is related to the lattice density by $a^3 \rho = \rho^{\text{lat}}$. With Wilson action, the first measurement of the monopole density in MAG gauge has been made in [21] with participation of the authors of the present paper. That result was interpreted in the sense of asymptotic scaling. In fact, the observation of asymptotic scaling at β values in the range from 2.4 to 2.6 would seem rather strange today. It is known that the string tension does not follow the two-loop renormalization group formula over this range in β , such that the result obtained in [21] actually implies the divergence of the monopole density with $\beta \rightarrow \infty$. The situation has been partially clarified by Hart and Teper [22,23]. These authors found that on large enough lattices the network of magnetic currents in each configuration consists of one large cluster and many other clusters with much smaller size. The spectrum of cluster sizes falls into two very distinctive parts, disconnected by a gap. The density of currents forming the largest (percolating) cluster ρ_{IR} has been measured in units of the string tension and a first indication of scaling of the ratio $\rho_{\text{IR}}/\sigma^{3/2}$ has been found. More accurate measurements [24] have corroborated this kind of scaling behavior. The continuum limit for this ratio was determined as $\rho_{\text{IR}}/\sigma^{3/2} = 0.65(2)$.

Another important result obtained in Refs. [22,23] was the observation that the largest cluster alone produces al-

most the full monopole string tension. This fact has allowed the authors of Ref. [25] to call the monopoles belonging to this cluster “infrared monopoles” (IR) while the monopoles from the remaining clusters were called ultraviolet (UV) monopoles, implying that these monopoles are not relevant for IR physics. We should mention that, despite the fact that they are not relevant for the *confining* properties of the vacuum as supported by numerical observation, their relevance for the topology and therefore for the chiral properties of the vacuum has not yet been explored. We will keep (and have already used) the above notation, quoting IR and UV monopoles in the following. In Ref. [24] it has been demonstrated that the density of UV monopoles, hence the total density, diverges in the continuum limit.

It should be noticed that in [22,23] a single, supposedly percolating, cluster with a size much larger than all other clusters in the given configuration was only found on large enough lattices. For decreasing lattice size L and fixed lattice spacing this gap disappears, i.e. the largest and the second-largest cluster are of similar size. Consequently, the important property that the largest cluster alone produces almost the full monopole-related string tension, is lost. This implies “splitting” of the largest cluster when the lattice volume decreases. It was also found in [22,23] that the critical value of the lattice size L_{crit} , below which the largest cluster splits, is in the range of lattice sizes, where physical quantities do not show large finite volume effects, and, moreover, L_{crit} , measured in physical units, increases with decreasing lattice spacing. Such behavior implies that in the continuum limit the gap in the spectrum of cluster sizes might disappear and a clear separation of IR and UV monopoles may become impossible.

The solution of this problem was suggested in [24]. It was found empirically that the splitting of the largest cluster leads to formation of clusters (rare for large enough lattices) with nonzero winding

$$w_{\mu} = \frac{1}{L_{\mu}} \sum_{k_{x,\mu} \in \text{cluster}} k_{x,\mu}. \quad (13)$$

Such clusters might be very large on given configurations or be of moderate size. In both cases they extend through the whole lattice at least in one direction and thus should be considered as relevant for the infrared physics. Furthermore, two or more of such clusters (forming together a combination of clusters with zero total winding w_{μ}) might form boundaries of the same Dirac sheet, which is closed in one or more directions due to periodic boundary conditions. Let us note that in case of two wrapping clusters present in one configuration they unambiguously form the boundary of the same Dirac sheet. When three wrapping clusters are present they also form the boundary of one Dirac sheet or boundaries of two Dirac sheets. In the latter case one of the clusters forms part of the boundaries of both Dirac sheets while two others form part of the

boundary of one of those Dirac sheets. It is natural to consider such clusters as one cluster when it concerns the determination of clusters relevant for IR physics. It is also clear that the splitting phenomena can be, at least partially, ascribed to the annihilation of parts of the percolating cluster through boundary conditions leading to formation of two disconnected clusters which still form a boundary of one Dirac sheet.

Based on these observations, it was suggested in [24] to define, for each configuration, a single IR cluster as the union of the largest cluster (which might have trivial or nontrivial winding) and all clusters with nonzero winding. Numerical evidence was further presented in [24] that under such definition the size of the largest cluster changes smoothly with the lattice size for physically large lattices. There are preliminary results of ours to be published elsewhere, showing that with this definition the IR cluster alone reproduces almost the full monopole-related string tension. We will use this definition in what follows.

As for lattice fields generated with TI action (1), the monopole density has been measured by Poulis [5]. He concluded that the total density has correct scaling in the continuum limit. In the light of the discussion above this would mean that the TI action would take an exclusive role. However, as we show below, this conclusion was wrong.

In Fig. 2 we show the distributions of the monopole clusters length for Wilson action at $\beta = 2.55$ and for TI action at $\beta_{\text{imp}} = 3.5$. Note, that a single IR cluster per configuration has entered the histograms which has been defined for the configuration at hand as described above. One can see that these distributions are qualitatively similar, i.e. for the TI action we also observe a clear splitting of the clusters into IR and UV clusters.

Our results for the densities of IR and UV monopoles and the total density, taken in lattice units, are presented in Table III. In physical units of $\sigma^{3/2}$, the results are shown as a function of lattice spacing (in units of $\sqrt{\sigma a^2}$) in Fig. 3. One can see that the IR density converges to a finite value in the limit $a \rightarrow 0$. In contrast to this, the density of UV monopoles, and thus the total density, behaves divergent in the continuum limit. We thus find that the TI action leads qualitatively to the same picture as was observed before with the Wilson action.

To make a quantitative comparison we plotted in Fig. 4 ρ_{IR} (left) and ρ_{UV} (right) for two actions. From these figures one can see quantitative differences. The continuum value of $\rho_{\text{IR}}/\sigma^{3/2}$, obtained with a quadratic fit, is 0.50(1) for TI action and 0.71(2) for Wilson action, i.e. they differ by factor 1.4. We should make a comment on the different procedures of calculation of $\rho_{\text{IR}}/\sigma^{3/2}$ for Wilson action in the present paper and in Refs. [24,26]. One difference is that in these papers only subsets of the full ensembles of gauge field configurations used in the present paper were employed. Another, more important difference

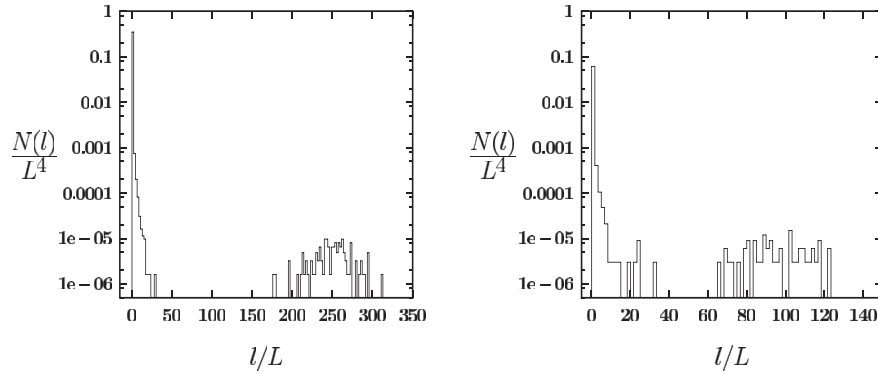


FIG. 2. The monopole cluster length distribution $N(l)$ at $\beta = 2.55$ for the Wilson action (left) and at $\beta_{\text{imp}} = 3.5$ for the TI action (right).

is that, in these earlier papers values of σ from literature were used, while here we are using values of σ calculated, as described in the previous section, i.e. on the same set of configurations on which the monopole density was calculated.

The observed difference in ρ_{IR} measured for TI and Wilson actions, though not being drastically large, means that the present definition of IR density is not universal. This makes it difficult to ascribe to it a meaning as a physical, gauge invariant density. It is evident that the source of the discrepancy in the values of ρ_{IR} might be the appending of UV monopoles, i.e. small loops, to IR monopole clusters. Since the TI action suppresses UV degrees of freedom stronger than Wilson action, it is natural to expect that this additional length assigned to the IR cluster is smaller for TI action. Whether this is the only reason deserves further investigation.

The density of UV monopoles is reduced much more substantially, roughly by a factor 2.5, as can be seen from Fig. 4 (right). We can say that the TI action indeed suppresses (part of) the UV degrees of freedom. As generally expected for an improved action, one can also see earlier and faster convergence to the continuum limit.

As it has been mentioned above, the UV monopole density diverges. It is natural to ask to which power of a^{-1} this divergence is compatible. It was first found in [24] that for the Wilson action $\rho_{\text{UV}} \sim 1/a$. In Ref. [26] this was confirmed with higher confidence. Figure 4 (right) shows

$a\rho_{\text{UV}}/\sigma$ for both actions. For TI action this ratio seems to rapidly converge to a finite value in the continuum limit as soon as $\sqrt{\sigma a^2} < 0.25$. This implies that $\rho_{\text{UV}} \sim 1/a$ also for TI action. Note, that the convergence for the TI action is faster than for the Wilson action, as can be seen from Fig. 4 (right). If the existence of a reasonable continuum limit in the latter case would be confirmed, then only for $\sqrt{\sigma a^2} \ll 0.1$, and the limit value would be markedly larger than for the improved action. In any case, the data clearly show that the UV monopole density and therefore the total density of monopoles is not universal.

The TI improved action has corrections of order $O(\alpha_s a^2)$ and $O(a^4)$ while the Wilson action has corrections of order $O(a^2)$. Thus it is natural to expect that some contribution of lattice artifacts to the monopole density is suppressed in case of the improved action. Now we are able to conclude that a considerable part of the UV monopoles

TABLE III. Density of monopoles.

β_{imp}	$\rho_{\text{tot}}^{\text{MA}}$	$\rho_{\text{IR}}^{\text{MA}}$	$\rho_{\text{tot}}^{\text{LA}}$	$\rho_{\text{IR}}^{\text{LA}}$
2.7	0.08845(25)	0.08014(30)	0.1103(5)	0.1032(5)
3.0	0.03537(16)	0.02560(23)		
3.1	0.02397(15)	0.01529(20)	0.0387(3)	0.0282(3)
3.2	0.01534(11)	0.00869(14)		
3.3	0.00982(6)	0.00509(8)	0.0200(2)	0.0118(2)
3.4	0.00618(4)	0.00297(5)		
3.5	0.00384(3)	0.00170(4)	0.0103(2)	0.0049(2)

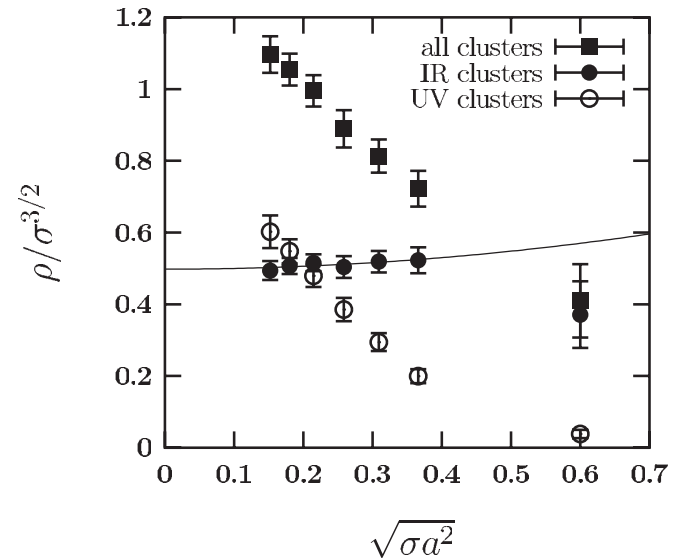


FIG. 3. The monopole densities in MAG for the case of the TI action. The dashed line shows a quadratic fit.

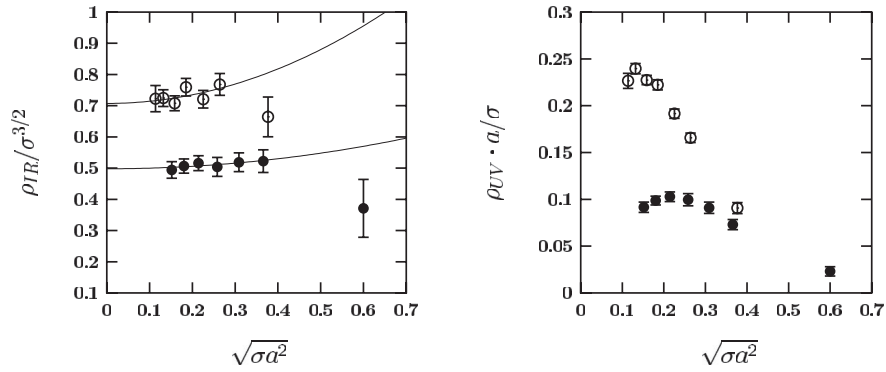


FIG. 4. Comparison of the monopole densities obtained with TI (full symbols) and Wilson actions (open symbols) in MAG: left—IR monopole density; right—UV monopole density.

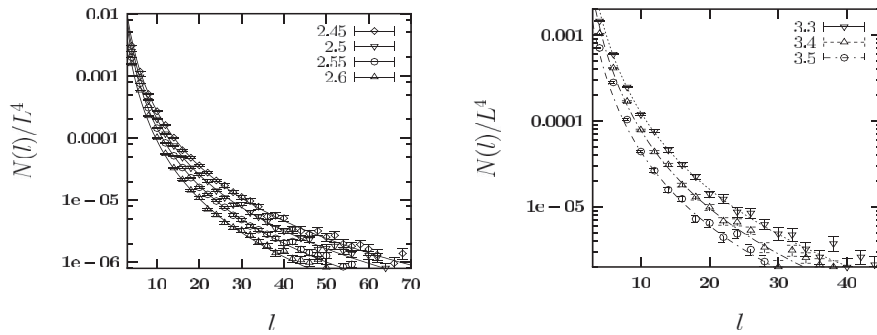


FIG. 5. The small (UV) clusters' length distribution in MAG for the Wilson action at various β (left) and for the TI action at various β_{imp} (right). Curves are fits to Eq. (14).

in the Wilson action case (more than 50%) are lattice artifacts. In contrast to this, we notice that IR monopoles are not much affected by lattice artifacts.¹ Whether the value of ρ_{IR} obtained with the improved action is already the final universal one is an open question. This should be checked in simulations with other improved actions.

The interesting, yet unanswered question is the physical role of UV monopoles. It was found in [22,23] and then confirmed with higher precision in [26] that the number $N(l)$ of small clusters of length l falls like

$$\frac{N(l)}{L^4} = c(\beta)/l^\gamma, \quad (14)$$

where $\gamma \approx 3$. The value $\gamma = 3$ was shown to be in agreement with percolation theory and also to be derivable within the polymer approach to the field theory of free or Coulomb-like interacting scalar particles [4]. Our data for both actions are also in agreement with relation (14), with values of parameter γ close to 3, as can be seen from Fig. 5 and Table IV.

¹The IR density is definitely affected by the roughness of monopole currents inside the largest cluster, and hence not completely free of discretization artifacts.

Let us come back to the observation that $\rho_{\text{UV}} \sim \frac{1}{a}$. This implies that magnetic currents from small clusters have a finite density per unit of 2D volume rather than per unit of 3D volume which is actually the case for the IR magnetic currents. It has been recently verified that the density of P -vortices in the indirect $Z(2)$ center gauge is finite in the continuum limit [27]. On the other hand, it is known that in this gauge P -vortices and monopoles are highly correlated [28,29]. It is then natural to assume that magnetic currents belonging to the small clusters “populate” the surfaces formed by P -vortices with some constant density. The strong reduction of the density of UV monopoles in the case of TI action in comparison with Wilson action sug-

TABLE IV. Parameters of the fits to Eq. (14) of the small clusters' length distribution in MAG for both actions. Respective fit ranges ($l_{\text{min}}, l_{\text{max}}$) are also shown.

β	γ	$c(\beta)$	l_{min}	l_{max}	β_{imp}	γ	$c(\beta)$	l_{min}	l_{max}
2.45	2.93(2)	0.23(2)	10	70	3.3	2.97(4)	0.12(1)	8	40
2.50	2.99(2)	0.20(1)	12	50	3.4	3.01(7)	0.08(2)	10	30
2.55	3.08(2)	0.17(1)	10	50	3.5	2.87(9)	0.03(1)	10	30
2.60	3.11(6)	0.12(2)	14	40					

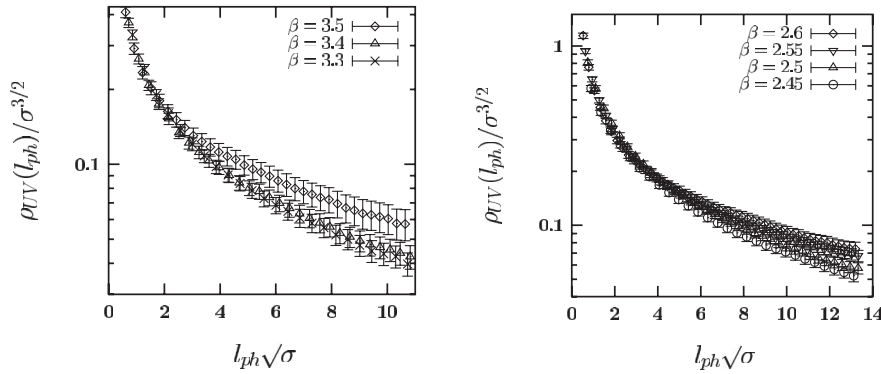


FIG. 6. The constrained UV monopole density derived from the small clusters' length distribution Eq. (15) for TI (left) and Wilson (right) actions.

gests that the density of P -vortices will be suppressed, too. This should be checked in a future calculation.

We now introduce a regularized UV monopole density by summation of the average number of small clusters of length l per lattice volume, i.e. $N(l)$, multiplied by the length l . In this definition we shall exclude clusters below a certain (minimal) length scale l_{ph} specified in natural units:

$$\rho_{UV}(l_{ph}) = \frac{1}{4L^4 a^3} \sum_{l \geq l_{ph}/a} N(l)l. \quad (15)$$

We call this UV monopole density “constrained density.” This definition counts all monopole currents in small clusters with a length above or equal l_{ph} , and the emerging density depends on it as a parameter. Thus, very small loops sensitive to the ultraviolet cutoff are excluded. In Fig. 6 we show $\rho_{UV}(l_{ph})/\sigma^{3/2}$ as a function of l_{ph} for both actions. One can see that scaling is very good as long as $l_{ph}\sqrt{\sigma} < 2$ in the case of the TI action and $l_{ph}\sqrt{\sigma} < 4$ in the case of Wilson action. In general, scaling becomes worse at larger values of $l_{ph}\sqrt{\sigma}$. This might be the con-

sequence of the IR cluster splitting discussed above which underlies the splitting into UV and IR monopoles: some large clusters which actually should belong (are akin) to IR monopole clusters were counted as small ones because of trivial winding. Although such clusters are relatively seldom their number increases with increasing β . To exclude the effect of these ambiguously identified “UV” clusters we subtract the contribution of clusters with $l > \tilde{l}_{ph} = c/\sqrt{\sigma}$ and plot in Fig. 7 the difference

$$\rho_{UV}(l_{ph}) - \rho_{UV}(\tilde{l}_{ph}) \quad (16)$$

with coefficients $c \approx 13.6$ and $c \approx 10.65$ for Wilson action and TI action, respectively. One can see that now scaling is uniformly well satisfied for all lower cutoffs $l_{ph} < \tilde{l}_{ph}$ except very small ones. Thus we come to the conclusion that the UV monopole density derived from the small cluster density $N(l)$ as defined in Eq. (15), i.e. when clusters close to the cutoff scale are excluded, shows good scaling, i.e. is independent of a similar to the IR monopoles density (derived from the IR clusters).

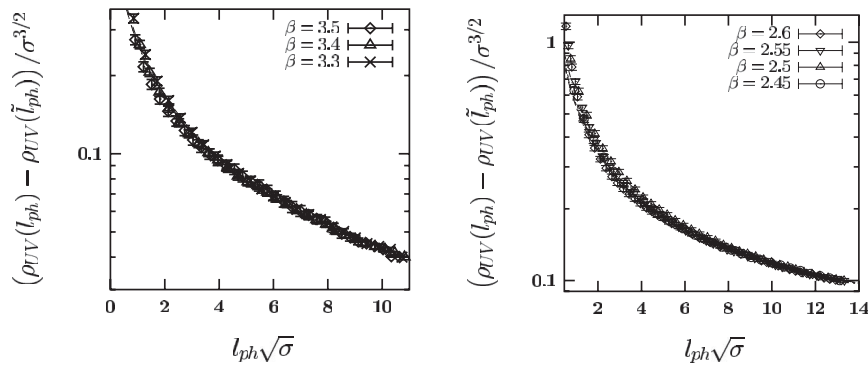


FIG. 7. The constrained UV monopole density derived from the small clusters' length distribution and corrected for ambiguous clusters according to Eq. (16), for TI (left) and Wilson (right) actions. The curves show fits by Eq. (18). The data were lifted by a constant for better readability of the figures.

Using Eq. (14) with $\gamma = 3$ we get

$$\rho_{UV}(l_{ph}) = \frac{c(\beta)}{4a^3} \psi'(l_{ph}/a), \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \quad (17)$$

We fitted the data in Fig. 7 by the function

$$c_1 \psi'(l_{ph}/a) + c_2 \quad (18)$$

and found that this function describes well our data, especially for the TI action case, with constants c_i only weakly dependent on β . This implies that for small a (large l_{ph}/a) the density behaves as $\rho_{UV}(l_{ph}) \sim \sigma/l_{ph}$ since $\psi'(z) \sim 1/z$ for large $|z|$. This fact is in agreement with the $1/a$ behavior of ρ_{UV} we argued for, since it can be unambiguously regulated by the assignment $\rho_{UV}(l_{ph} = 4a) \rightarrow \rho_{UV}$.

B. The Laplacian Abelian gauge (LAG)

To calculate the lowest eigenvector of the covariant adjoint Laplacian operator (5) we used the Arnoldi algorithm [30]. This algorithm, as well as others used to solve this problem, requires large memory increasing fast with lattice size. Therefore, the measurements in LAG for Wilson action have been made on smaller lattices than shown in Table II for $\beta = 2.45, 2.55, 2.60$, but with large

enough physical size, which was never smaller than 1.4 fm. In case of the TI action we made measurements in LAG only for four values of the coupling constant, because this proved to be enough for our purposes of comparison with MAG.

We first present the cluster length distribution for the two actions in Fig. 8. One can see that in LAG the separation into IR and UV clusters works very well. We note that there are only rare cases of clusters with nontrivial winding even for most fine lattices. Our results for various densities are presented in Fig. 9. We found that the total monopole density in LAG is substantially higher than in MAG in agreement with earlier observations [10] made for Wilson action. We further looked at IR and UV densities separately and found that increasing of the density is true for both of them. For our finest lattice these densities in LAG are 2–3 times higher than in MAG. It is not clear from our data whether the IR density in LAG converges to a finite value as it is the case in MAG. This is most probably due to a substantial (more fractal) admixture of lattice artifacts to the IR clusters. We also observed that the UV monopole density ρ_{UV} is much more strongly diverging in LAG than in MAG and is compatible with $\rho_{UV} \sim 1/a^2$ rather than with $\rho_{UV} \sim 1/a$, as observed in MAG. Accordingly, the

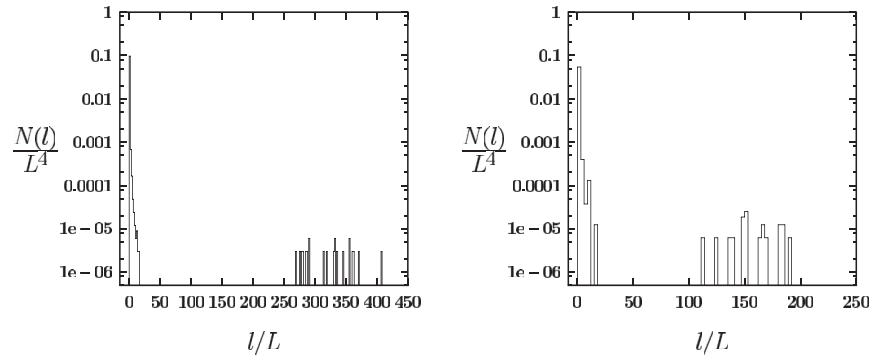


FIG. 8. The monopole cluster length distribution $N(l)$ in LAG at $\beta = 2.6$ for the Wilson action (left) and at $\beta_{\text{imp}} = 3.5$ for the TI action (right).

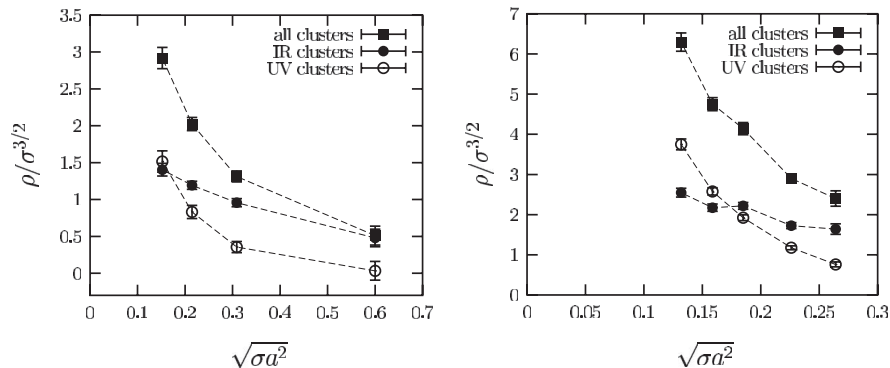


FIG. 9. Various monopole densities in LAG for TI action (left) and for Wilson action (right). Lines are drawn to guide the eye.

constrained density $\rho_{UV}(l_{ph})$ was found to be divergent as $\frac{\sqrt{\sigma}}{al_{ph}}$. The small clusters' length distribution was fitted to Eq. (14) with a resulting parameter γ in the range 2.7–2.8, i.e. significantly smaller than for MAG.

V. THE ABELIAN STRING TENSION IN MAG

To estimate the Abelian string tension we calculate the Abelian potential $V_{ab}(R)$ using (spatially) smeared Abelian Wilson loops $W_{ab}(R, T)$. As usually $V_{ab}(R)$ is defined as a limit

$$V_{ab}(R) = \lim_{T \rightarrow \infty} V_{ab}(R, T), \quad (19)$$

where the potential estimator $V_{ab}(R, T)$ is

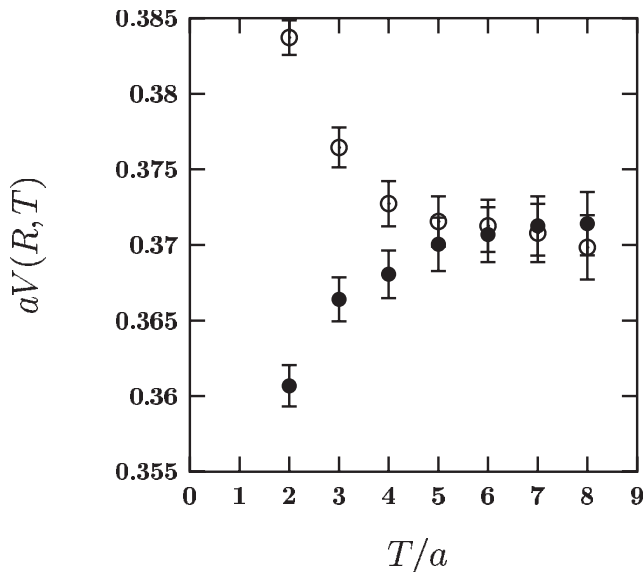


FIG. 10. The Abelian potential $V_{ab}(R, T)$ in MAG at $\beta_{imp} = 3.3$ vs T/a for $R/a = 4$ for 3 (open circles) and 100 (full circles) smearing sweeps.

$$aV_{ab}(R, T) = -\log \frac{W_{ab}(R, T+a)}{W_{ab}(R, T)}. \quad (20)$$

It is important to check that $V_{ab}(R)$ is unique, i.e. independent of the operators used to create the Abelian flux tube. One can get different such operators varying the number of smearing sweeps N_{sm} . In Fig. 10 we show $V_{ab}(R, T)$ for $N_{sm} = 3$ and 100. For large number of sweeps the behavior of $V_{ab}(R, T)$ clearly shows absence of the positivity. For small number of sweeps the behavior is similar to the case when positivity is fulfilled. This is presumably due to higher excitations: in this case they are not suppressed for small T/a . Lack of positivity for gauge dependent correlators in covariant gauges was discussed recently in [31]. The most important observation for us to be read off from Fig. 10 is that at large enough T results agree with each other. This implies that $V_{ab}(R)$ is indeed defined uniquely.

We found that the behavior of the ratio σ_{ab}/σ for TI action is qualitatively similar to that for the Wilson action, see Fig. 11. For both actions the ratio is between 0.90 and 0.95 for all considered values of lattice spacing. We thus definitely confirm the universality of Abelian dominance in the continuum limit. Our results, due to large statistical errors, coming mainly from the determination of the non-Abelian string tension, do not allow to determine precisely the continuum limit of the ratio σ_{ab}/σ .

VI. SUMMARY

In this paper two important questions on the properties of the Abelian projection were addressed: universality and gauge dependence. Comparing results obtained with Wilson and TI actions on lattices with varying lattice spacing we confirmed that the Abelian dominance passes the universality check in MAG. Moreover, this universality holds in the continuum limit. We found that in the continuum limit the ratio σ_{ab}/σ seems to be in the range from 0.90 to 0.95. No convergence to 1 was observed contrary to our earlier results [24] seen with smaller statistics. We have

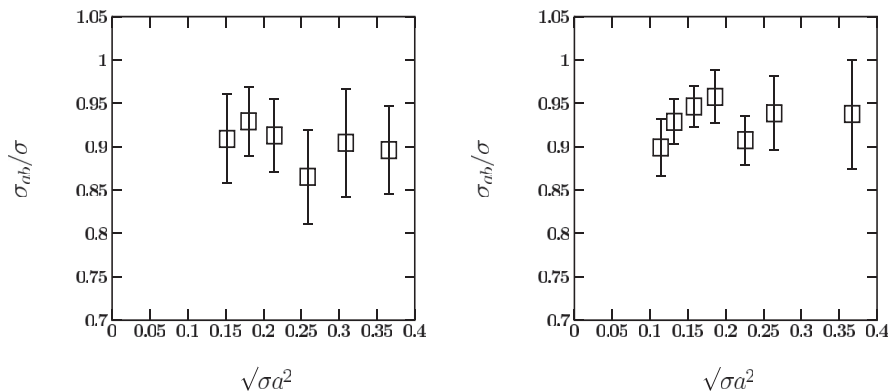


FIG. 11. Ratio between the Abelian string tension (in MAG) and the non-Abelian string tension vs lattice spacing for TI action (left) and for Wilson action (right).

not yet accomplished the check of the monopole dominance universality for the string tension although our preliminary results confirm it. They will be published elsewhere.

For the monopole density we found qualitative similarities: for both actions ρ_{IR} is finite in the continuum limit, while ρ_{UV} is divergent as $1/a$. On the quantitative level we found a violation of universality for the densities. This implies that an UV contribution might be substantial in the measured ρ_{IR} , or in other words, ρ_{IR} needs to be properly renormalized.

We further introduced a constrained UV density $\rho_{\text{UV}}(l_{ph})$, Eq. (15) which is determined by counting only monopole loops longer than some physical length l_{ph} . We found that this density scales properly in the continuum limit:

$$\rho_{\text{UV}}(l_{ph}) = \frac{c\sigma}{l_{ph}}, \quad (21)$$

where the coefficient c is independent of a but has a different value for Wilson and TI actions, i.e. is nonuniversal.

A comparison of the monopole densities in MAG and LAG, made for both actions, revealed that both ρ_{IR} and

ρ_{UV} are 2–3 times higher in LAG. It is not clear from our data whether ρ_{IR} in LAG is finite in the continuum limit. The UV component ρ_{UV} behaves like $\sim 1/a^2$ in LAG contrary to the divergence $\sim 1/a$ found in MAG. This adds to other doubts expressed in the literature [32] about the usefulness of the LAG. Thus, the maximally Abelian gauge turns out to be particularly suited to the separation of IR and UV degrees of freedom.

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