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Derivation of the Fradkin-Shenker result from duality: Links to spin systems in external magnetic fields and percolation crossovers

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In this article, we illustrate how the qualitative phase diagram of a gauge theory coupled to matter can be directly proved and how rigorous numerical bounds may be established. Our work reaffirms the seminal result of Fradkin and Shenker from another vista. Our main ingredient is the combined use of the self-duality of the three-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory and an extended Lee-Yang theorem. We comment on extensions of these ideas and firmly establish the existence of a sharp percolation crossover line in the two-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory.

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I. INTRODUCTION

Lattice gauge theories [1] witnessed an accelerated revival in condensed matter physics during the last decade. Their applications are widespread. Amongst others, these include novel theories of liquid crystals [2], the incorporation of Berry phase effects in quantum spin systems [3,4], and stimulating suggestions for long-distance physics of lightly doped Mott-Hubbard insulators [5]. Further research relating to fundamental questions in gauge theories followed, e.g. [6,7]. Central to many of these investigations is the behavior of matter fields minimally coupled to gauge fields. Several key results in these theories were noted long ago by Fradkin and Shenker [8] (complemented by treatments in [9]). Perhaps the best known result of [8] is the demonstration that (when matter fields carry the fundamental unit of charge) the Higgs and confinement phases of gauge theories are smoothly connected to each other and are as different as a liquid is from a gas. This result remains one of the cornerstones of our understanding of the phases of gauge theories. Although derived long ago, the physical origin of this effect does not seem to be universally agreed upon.

In the current article, we revisit this old result and rederive it for the original $\mathbb{Z}_2/\mathbb{Z}_2$ theory investigated in [8]. Our proof relies merely on duality and the Lee-Yang theorem. We further illustrate why similar results are anticipated for other gauge theories. Our derivation highlights the origin of this phenomenon as akin to the absence of phase transitions in spin systems in a magnetic field. Notwithstanding the absence of true nonanalyticities, some such spin models display a percolation crossover line [10] at which the surface tension of an oppositely oriented spin cluster vanishes. In this article, we firmly establish the existence of precisely such *a sharp percolation crossover* line for one of the most trivial $\mathbb{Z}_2/\mathbb{Z}_2$ theories (the $d = 1 + 1$-dimensional theory).

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II. $\mathbb{Z}_2$ MATTER COUPLED TO $\mathbb{Z}_2$ GAUGE FIELDS

In matter coupled gauge theories, matter fields ($\{\sigma_i\}$) reside as lattice sites i while gauge fields U_{ij} reside on the links connecting sites i and j. The $\mathbb{Z}_2$ matter coupled to $\mathbb{Z}_2$ gauge field theory ($\mathbb{Z}_2/\mathbb{Z}_2$ in common notation) is the simplest incarnation of a matter coupled gauge theory. Its action reads

$$S = -\beta \sum_{\langle ij \rangle} \sigma_i U_{ij} \sigma_j - K \sum_{\square} UUUU \tag{1}$$

on a hyper-cubic lattice. Here, the first sum is over all nearest neighbor links $\langle ij \rangle$ in the lattice while the second is the product of the four gauge fields $U_{ij}U_{jk}U_{kl}U_{ki}$ over each minimal plaquette (square) of the lattice. Both matter (σ_i) and gauge (U_{ij}) fields are Ising variables within the $\mathbb{Z}_2/\mathbb{Z}_2$ theory: $\sigma_i = \pm 1$, $U_{ij} = \pm 1$. A trivial yet fundamental observation is that the quantity $z_{ij} \equiv \sigma_i U_{ij} \sigma_j$, where i and j denote two nearest neighboring lattice sites, is invariant under local $\mathbb{Z}_2$ gauge transformations

$$\sigma_i \to \eta_i \sigma_i, \qquad U_{ij} \to \eta_i U_{ij} \eta_j, \tag{2}$$

with the arbitrary on-site $\eta_i = \pm 1$ [11]. The action of Eq. (1) may be trivially written in terms of these gauge invariant bond variables $\{z_{ij}\}$ as

$$S = -\beta \sum_{\text{links}} z_{ij} - K \sum_{\square} zzzz. \tag{3}$$

The matter coupling β acts as a magnetic field on the spin variable z. On a new lattice whose sites reside on the centers of all bonds, this is none other than a model having 4-spin interactions augmented by a $\mathbb{Z}_2$ symmetry breaking (for finite β) magnetic field. For $\beta > 0$, the link expectation value $\langle z_{ij} \rangle \equiv \langle \sigma_i U_{ij} \sigma_j \rangle \neq 0$. As shown by Wegner [12], three- (or 2 + 1) dimensional variants of the $\mathbb{Z}_2/\mathbb{Z}_2$ model with couplings (β, K) are equivalent to the same model at couplings (β^*, K^*) related via the self-duality relations

$$\exp(-2\beta^*) = \tanh K, \qquad \exp(-2K^*) = \tanh \beta. \tag{4}$$

III. DUALITY AND THE LEE-YANG THEOREM

As illustrated above, the matter coupled gauge theory can be reinterpreted as a pure gauge theory with an additional magnetic field applied. Such an analogy immediately triggers a certain intuition regarding the exclusion of phase transitions in certain systems. In standard spin models governed by the classical action

$$S = -\frac{1}{2}\sum_{ij} J_{ij} s_i s_j - \sum_i h s_i, \tag{5}$$

with H the Hamiltonian no phase transition can occur when a symmetry breaking magnetic field ($h \neq 0$) is applied. It is clear that the local magnetization $\langle s \rangle \neq 0$ and this goes hand in hand with an analytic free energy.

Lee and Yang [13] proved that, in the thermodynamic limit, the partition function cannot have zeros. This can be shown to imply an analytic free energy for magnetic fields for which $|\mathrm{Im}\{h\}| < |\mathrm{Re}\{h\}|$ (with Im and Re the imaginary and real components, respectively). This may be extended to many systems. Its generalization to a pure $\mathbb{Z}_2$ lattice gauge action with a magnetic field applied on each gauge link [Eq. (3)] on a general hyper-cubic lattice of dimension d has been done [14]. However, as Eq. (3) is equivalent to the general matter coupled gauge theory of Eq. (1), this implies that the free energy ($-\ln Z$) is analytic for all sufficiently large matter couplings β. More precisely [14], if $\theta_\Box \equiv \tanh K$, and

$$\theta_{\mathrm{link}} \equiv \tanh\left[\frac{\mathrm{Re}\{\beta\} - |\mathrm{Im}\{\beta\}|}{2(d-1)}\right], \tag{6}$$

then the partition function is nonvanishing and the free energy analytic in the region

$$\theta_{\mathrm{link}}^4 \geq \theta_\Box + \theta_\Box^{-1} + 3 - \sqrt{(\theta_\Box + \theta_\Box^{-1} + 3)^2 - 1}. \tag{7}$$

Next we consider the (2 + 1) dimensional case and then briefly remark on the (1 + 1) dimensional theory. For the three (or 2 + 1) dimensional case, Eq. (7) explicitly reads

$$\tanh^4\frac{\beta}{4} \geq \tanh K + \coth K + 3 - \sqrt{(\tanh K + \coth K + 3)^2 - 1}. \tag{8}$$

Let us now insert the self-duality relations Eqs. (4) to obtain

$$\tanh K = \exp[-2\beta^*], \qquad \tanh\frac{\beta}{4} = \left(\frac{\sqrt{1+\lambda} - \sqrt{2\lambda}}{\sqrt{1+\lambda} + \sqrt{2\lambda}}\right)^{1/2}, \tag{9}$$

with

$$\lambda \equiv \sqrt{1 - \exp[-4K^*]}. \tag{10}$$

Inserting Eq. (9) into Eq. (8) gives a domain of analyticity of the free energy in (β^*, K^*). The union of both domains is a region free of nonanalyticities. In particular, we find that for all plaquette couplings

$$K < -\frac{1}{2}\ln \tanh[4\tanh^{-1}(5 - \sqrt{24})^{1/4}] \tag{11}$$

with arbitrary matter coupling β, the partition function of the three-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory has no zeros in the thermodynamic limit and the free energy is analytic. The Lee-Yang line [14] and its dual are displayed in Fig. 1. The entire region bounded by the smaller of these two lines is free of nonanalyticities. Along the $\beta = 0$ line of Eq. (1) (the pure gauge only theory), the value $K = K_c$ at which a confining transition occurs may be inferred from the critical temperature of the three-dimensional Ising model. Within the confining transition of the pure gauge theory [the action of Eq. (1) in the absence of matter coupling—$\beta = 0$], the Wilson loop $W_C = \langle \prod_{ij \in C} U_{ij}\rangle$ for a large loop C changes from an asymptotic perimeter law behavior ($W_c \sim e^{-c_1 l}$ with l the perimeter of C and c_1 a constant) for large plaquette couplings ($K > K_c$) to a much more rapidly decaying area law ($W_c \sim e^{-c_2 A}$ with A the area of the minimal surface bounded by C and c_2 a constant) for weak couplings $K < K_c$ [1]. At $K = K_c$, the free energy is nonanalytic. By duality [Eqs. (4)], the location of this nonanalyticity in K along the $\beta = 0$ axis maps onto the location of nonanalyticity associated with the transition in the 3D Ising model ($S_{\mathrm{3D\ Ising}} = -\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j$)—its critical point $\beta = \beta_c$. Following Eqs. (4), the relation between the two is $\tanh \beta_c^{\mathrm{3D\ Ising\ model}} = \exp(-2K_c^{\mathrm{3D\ Ising\ gauge}})$. Numerically, in the 3D Ising model $\beta_c^{\mathrm{3D\ Ising\ model}} \simeq 0.22165$. This implies that the critical value of K within the pure [$\beta = 0$ in Eq. (1)] 3D Ising gauge theory is $K_c^{\mathrm{3D\ Ising\ gauge}} \simeq 0.761423$, e.g. [15]. The partition function is nonzero and the free energy is analytic within the region given by Eq. (11) which lies within the confining phase of the three-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ model for small β. Thus, as pointed out in a seminal paper by Fradkin and Shenker [8], the Higgs (large β, K) and the confining phases (small β, K) are analytically connected (see Fig. 2). No phase transition need be encountered in going from one phase to the other. Here we explicitly prove this for the three-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ model with explicit rigorous numerical bounds as in Eq. (11).

The bound on a finite region of the phase diagram free of partition function zeros complements the classic works of Marra and Miracle-Sole [16] that show that the small β, K expansion of the free energy corresponding to Eq. (1) converges if K is sufficiently small irrespective of β, or if both K and β are sufficiently large. It is noteworthy that although duality allowed us to generate stringent bounds, the Lee-Yang theorem itself linked points deep within the confining phase $(\beta, K) \to (0, 0)$ to those in the Higgs phase ($\beta \gg 1, K \gg 1$). The extension of the Lee-Yang theorem to gauge theories other than $\mathbb{Z}_2/\mathbb{Z}_2$ is straightforward albeit technically more involved.

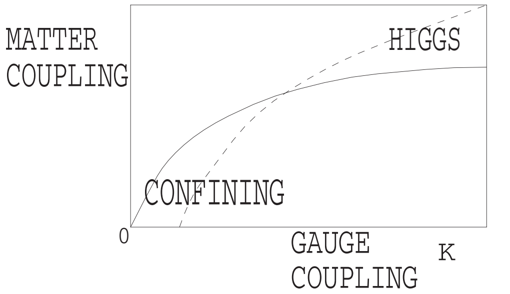

FIG. 1. The region in the phase diagram of the three-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ for which we prove that the partition function is free of zeros and consequently the free energy is analytic. The horizontal axis is K—the strength of the gauge field and the vertical axis depicts β—the strength of the matter coupling. Both axes span the region from 0 to ∞. The solid line is the bound attained from the Lee-Yang theorem. The region above this curve is free of nonanalyticities. By duality, the region above the dashed line is also free of nonanalyticities. Thus the union of both regions is analytic. This connects the Higgs phase (high β, K) to the confining phase (low β and K). The intercept of the dashed line with the $\beta = 0$ axis is found to be $K = -\frac{1}{2} \times \ln \tanh[4 \tanh^{-1}(5 - \sqrt{24})^{1/4}]$, which is within the confining phase for small β as expected (it cannot span the deconfining phase as then a nonanalyticity in the free energy or zero of the partition function would be encountered at $K = K_c^{\text{3D Ising gauge}}$).

We now examine the much more trivial two-dimensional incarnation of the $\mathbb{Z}_2/\mathbb{Z}_2$ theory to illustrate that it displays a single phase. By a duality mapping (see an explicit derivation in the appendix), it is readily seen that the partition function of the two-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ model at matter coupling β and gauge coupling K is equal (up to constants) to the partition function of the two-dimensional Ising model (of unit lattice spacing) given by Eq. (5) of nearest neighbor exchange constant

$$J_{ij} = \left(\frac{1}{2} \ln \coth\beta\right)\delta_{|i-j|,1} \tag{12}$$

and uniform external magnetic field

$$h = \frac{1}{2} \ln \coth K. \tag{13}$$

As the two-dimensional Ising model in a magnetic field displays (via the Lee-Yang theorem) no phase transitions, the two-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory exhibits only a single phase regardless of the strength of the couplings. Notwithstanding, as we report towards the end of this paper, the existence of a line of weak singularities may be firmly established.

IV. GENERAL CONSIDERATIONS FOR A SINGLE HIGGS-CONFINING PHASE

Next, we avoid the use of rigorous Lee-Yang bounds and ask ourselves what statements can be made regarding the phase diagram on general principle alone both in the presence and absence of dualities. First, we illustrate that a phase diagram such as that shown in Fig. 3 is impossible for the $\mathbb{Z}_2/\mathbb{Z}_2$ theory. The phase diagram depicted in Fig. 3 was proposed for the very different theory of $O(3)$ matter fields coupled to $\mathbb{Z}_2$ by [2] in their beautiful theory of liquid crystals.

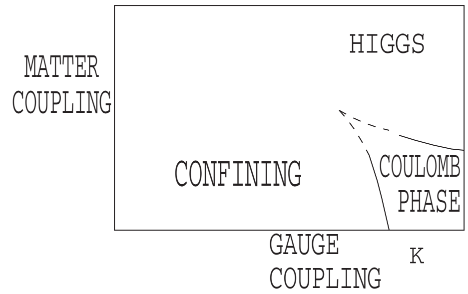

FIG. 2. A schematic representation of the phase diagram of the $\mathbb{Z}_2/\mathbb{Z}_2$ theory in $d = 3$ space dimensions. This phase diagram was proposed by [8]. The boundaries drawn in Fig. 1 are only bounds. The confining transition extend deep beyond the line implied by the Lee-Yang theorem.

The proof of the impossibility of such a phase diagram for a $\mathbb{Z}_2/\mathbb{Z}_2$ theory and the necessity of having a single Higgs-confinement phase is quite straightforward. As the $\mathbb{Z}_2/\mathbb{Z}_2$ theory is self-dual [see Eqs. (4)], the phase diagram must look the same under the duality transformation. The phase boundaries where the partition function vanishes, $Z(\beta, K) = 0$, must be the same as those where $Z(\beta^*, K^*) = 0$. A phase diagram such as Fig. 3 does not satisfy self-duality. A critical line emanating from $(\beta = \beta_1, K = 0)$ immediately implies a line of singularities emanating from $(\beta^* = \infty, K^* = -\frac{1}{2} \ln \tanh\beta_1)$. If $Z(\beta^*, K^*) = 0$ along this line then, as the functional form for $Z(\beta^*, K^*)$ is equivalent to that of $Z(\beta, K)$ with merely

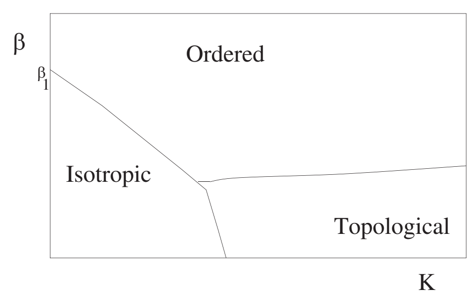

FIG. 3. The phase diagram above was found by [2] for $O(3)$ matter fields coupled to $\mathbb{Z}_2$ gauge links in the context of liquid crystals. Here, the confining, Higgs, and Coulomb phases of the $\mathbb{Z}_2/\mathbb{Z}_2$ theory becomes three different sharp phases (whose siblings are, respectively, denoted in the above as "Isotropic," "Ordered," and "Topological.") We prove, by employing self-duality of the $\mathbb{Z}_2/\mathbb{Z}_2$ theory that a phase diagram having three phases such as that of $O(3)$ matter coupled to $\mathbb{Z}_2$ gauges shown above is impossible. In the $\mathbb{Z}_2/\mathbb{Z}_2$ theory, phase boundaries may only terminate on the $\beta = 0$ or $K = \infty$ axis—no phase boundary can separate the Higgs and confining phases.

the coupling constant tuned to different values, $Z(\beta, K)$ must also have a line of zeros emanating from ($\beta = \infty, K = -\frac{1}{2}\ln\tanh\beta_1$) and the phase diagram must possess, at least, another line of singularities. The same would apply to a line of singularities starting from ($\beta = \infty, K = K_1$) which is easily excluded.

Next, we look at the physics of the models in their limiting incarnations. At $K = 0$ the partition function of the $\mathbb{Z}_2/\mathbb{Z}_2$ theory is trivially $Z = (2\cosh\beta)^{Nd}$ with N the number of lattice sites. Here, the system is simply that of free bonds in a magnetic field and no singularities can occur at any value of $\beta = \beta_1$ with $K = 0$. Self-duality then implies that no singularities can occur in the self-dual $\mathbb{Z}_2/\mathbb{Z}_2$ theory at $\beta = \infty$ and any finite value of K.

Putting all of the pieces together, by employing self-duality, and the absence of singularities at $K = 0$, within the $\mathbb{Z}_2/\mathbb{Z}_2$ theory, lines of singularities in the phase diagram can only originate from ($\beta = 0, K = K_c$) or from ($\beta = \beta_c, K = 0$) [with possibly more than one value of K_c ($\{K_{c,i}\}$) and/or β_c] or form closed loops or lines of transitions terminating in the bulk. States with $\beta = 0$ and $K < \min_i\{K_{c,i}\}$ and those with $K = \infty$ and $\beta < \max_j\{\beta_{c,j}\}$ must be analytically connected to each other. In the standard spin ($K = 0, \beta = \beta_c$) and gauge models ($\beta = 0, K = K_c$) only a single critical value appears. The Higgs and confining phases must, asymptotically, be one and the same. A singularity anywhere along the line $K = 0^+$ is excluded in the self-dual theory as that limit corresponds to $\beta \to \infty$ which is completely ordered ($z_{ij} = 1$) and no transitions occur. This proves the celebrated result of [8].

We now examine the situation in general non-self-dual theories in which the matter fields (σ_i) are in a subgroup of the gauge group (the group G such that all links $U_{ij} \in G$). (This situation does not encompass theories such as those described by [2,5].) In such instances, the bond variables $z_{ij} = \sigma_i^* U_{ij} \sigma_j$ are elements of G. Similar to Eq. (3), we may parameterize the action in terms of the gauge invariant link variables $\{z_{ij}\}$. In what follows, we focus for concreteness on $U(1)$ [or $O(n = 2)$] theories. First, we note that along the $K = 0$ axis, the pure noninteracting links in the effective magnetic field β [leading to $Z = (I_{n/2-1}(\beta))^{Nd}$ [with $I_{n/2-1}$ a Bessel function of order $(n/2 - 1)$] for $O(n)$ fields] display no singularities in the free energy. Along the $\beta = \infty$ axis, irrespective of the value of K, all the gauge invariant bonds z_{ij} in the $U(1)$ theory are pinned to 1. No transitions occur as K is varied along the $\beta = \infty$ line as all bond variables are already frozen at their maximally magnetized unit value. In fact, increasing K for finite β can only make this magnetization stronger. The partition function has no dependence on K along this line. Thus, we see that in general no phase boundaries can traverse the $\beta = \infty$ or the $K = 0$ line even in the absence of self-duality and Lee-Yang results which allow us to make matters more elegant and provide rigorous numerical bounds. Thus, the Higgs and confining phases are one and the same for all of these theories. We must nevertheless mention that in non-self-dual theories, relying only on the above we cannot immediately exclude a transition boundary ending in the bulk at $K = 0^+$. To exclude this for different individual theories, we need to examine the radius of convergence in K.

V. ESTABLISHING NEW PERCOLATION CROSSOVERS BY DUALITY

With all stated thus far, it would appear that the single Higgs-confining phase is one bulk phase and no transitions occur within it. We now illustrate that this is not the case—at least not within the simplest of all matter coupled lattice gauge theories—the two-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory which we now show to possess a richer phase diagram than anticipated (a single phase). With no matter, as is well known e.g. [1,17], the pure two-dimensional $\mathbb{Z}_2$ gauge theory given by the plaquette term of Eq. (1) is equivalent (by a trivial gauge fix, e.g. $U_{ij} = 1$ on all horizontal bonds in the plane) to a stack of decoupled one-dimensional Ising chains (all of which are horizontal Ising chains formed by the vertical bonds in the gauge alluded to here). As Ising chains are disordered at any finite coupling, the two-dimensional $\mathbb{Z}_2$ gauge theory is trivially always confining. As along the $K = 0$ line no transitions can occur (as discussed in Sec. IV), the system must be in this confining phase for all finite β and K. As we will shortly illustrate, although no transitions occur within this single phase, new sharp percolation crossovers may be established.

Now let us introduce matter coupling [a finite β in Eq. (1)] and consider the following thought experiment: we color every appearance of $z_{ij} = \pm 1$ in the two-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory by one of two colors and ask ourselves whether the bonds of a uniform sign (the $z_{ij} = 1$ bonds for $\beta > 0$) percolate, upon a trivial mapping, across the sample and if so whether a transition between percolative and nonpercolative clusters can exist within the single Higgs-confining phase. Although this question is very general, we can make easy progress and establish rigorous results by relying on the exact duality of Eqs. (12) and (13), to the well-studied model of a two-dimensional Ising magnet in a magnetic field. Some time ago, Kertesz argued [10] that although there might not be (via the Lee-Yang theorem) any thermodynamic singularities in various spin models when subjected to an external magnetic field, sharp crossovers related to vanishing surface tension (of droplets of oppositely oriented spins) and a change of character of the high field series (for quantities such as the free energy or magnetization) occur. As is well-established by now, this crossover may be discerned by the percolation of clusters constructed via the Fortuin-Kastelyn representation [18]. We would like to suggest that the massive character of the photons in the confining/Higgs regimes may reflect such a difference. Here, the spins of [10] are replaced by a functional of the gauge invariant meson variables $z_{ij} \equiv \sigma_i^* U_{ij} \sigma_j$. For the $1+1$-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory, we now readily establish this result: By Eqs. (12) and (13), the $d = 1 + 1$-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory may be directly mapped to a two-dimensional Ising model in a magnetic field given by Eq. (5). However, as established by [19] the two-dimensional Ising model in a magnetic field displays a Kertesz line. In general dimension d, with n_V the number of drops of volume V whose spins are oppositely oriented [those opposing the external magnetic field h in Eq. (5)], we have, for large V, the leading order relation

$$\ln n_V \sim -2hV - \Gamma V^{(d-1)/d}. \tag{14}$$

Here, the surface tension Γ vanishes in one phase (phase B of Fig. 4) while it is finite in the other (phase A in Fig. 4) [10]. Equivalently, this crossover may be ascertained via the examination of the radii of expansion [19] in $\mu \equiv e^{-2h}$ [see Eq. (5)] for the magnetization

$$\langle s \rangle = 1 - 2\sum_V V L_V(u) \mu^V, \tag{15}$$

where $u \equiv e^{-2J}$, and L_V is a polynomial in u. Although for any finite h, the radius of convergence in μ is finite (as indeed no transitions occur by the Lee-Yang theorem)—the radius of convergence increases across the percolation line (appearing as jumps in [19]). For couplings J larger than the percolation threshold $J > J_p$, the radius of convergence in μ is up to $\mu = 1$ [i.e. it is convergent for all $h \geq 0$ in Eq. (5)] and to a larger value $\mu > 1$ for $J < J_p$ (the surface tension free regime)—up to finite negative values of h [19]. Upon dualizing [Eqs. (12) and (13)], this implies an identical crossover in the single plaquette expectation value of $\langle z_{ij} z_{jk} z_{kl} z_{li} \rangle$ (which is none other than the minimal Wilson loop $\langle U_{ij} U_{jk} U_{kl} U_{li} \rangle$) when expanded in powers of $\tilde{\mu} \equiv \tanh K$ for fixed $\tilde{u} \equiv \tanh \beta$. The transition is discerned by the convergence of the single plaquette expectation value up to negative K values.

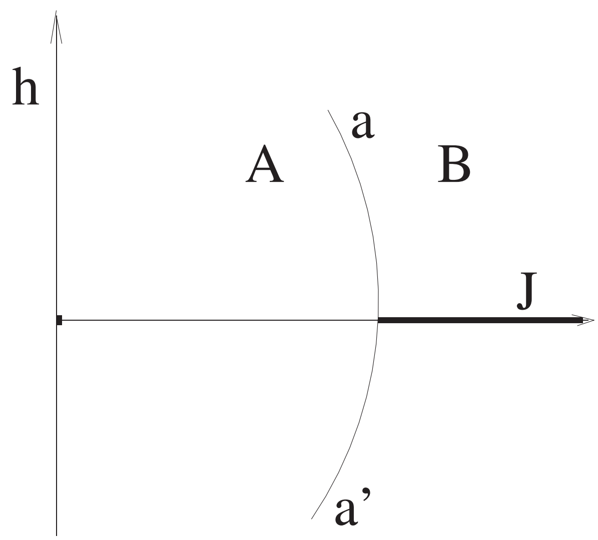

FIG. 4. A caricature of the original [10] phase diagram proposed by Kertesz and later verified in detail by [19]. The J and h axes parameterize the classical two-dimensional Ising Hamiltonian $S = -J\sum_{\langle ij \rangle} s_i s_j - h\sum_i s_i$. The low temperature solid line along the $h = 0$ axis denotes the usual first order transition, while the fainter lines denote Kertesz transitions. Here, in phase A there is an exponentially rare number of droplets whose spin points opposite to the applied field h with a finite surface tension. In phase B, the surface tension of oppositely oriented spin droplets vanishes. The lines a and a' denote the droplet cluster transition across which the surface tension [Γ of Eq. (14)] drops to zero.

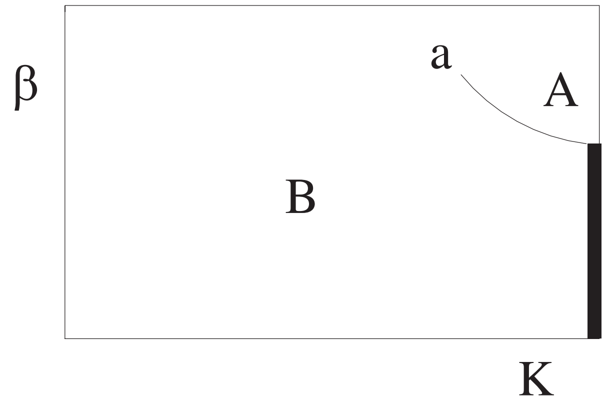

FIG. 5. The phase diagram of the two-dimensional matter coupled Ising gauge theory [Eq. (1) in $d = 2$ dimensions] as derived from Fig. 4 following the duality transformations of Eqs. (12) and (13). As we establish here by an application of the duality relations of Eqs. (12) and (13) linking this gauge theory to the two-dimensional Ising model in a magnetic field, a percolation transition separates phase B (spanning the confining regime) and phase A (which overlaps with the Higgs phase). The solid line along the $K = \infty$ axis signifies the disordered phase of the simple ferromagnet. [At $K = \infty$, Eq. (1) reduces to $S = -\beta\sum_{\langle ij \rangle} \sigma_i \sigma_j$, the action of a ferromagnet.]

Taken together, the duality relations of Eqs. (12) and (13), and the firm results of [19] prove, for the first time, that the two-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory must also exhibit a Kertesz line. A sketch of the original phase diagram of Kertesz [10] and its new gauge theory dual are depicted in Figs. 4 and 5. Percolation transitions established here for the two-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory and others speculated elsewhere might be linked to infinite Wilson loop-like observables [20,21].

VI. CONCLUSIONS

In conclusion, we illustrate that a phase diagram of a gauge theory coupled to matter can be proved directly and stringent numerical bounds provided. Our methods reaffirm the seminal result of Fradkin and Shenker [8]. We further remarked on extensions of this result. Our results suggest that the existence of a single Higgs-confining phase in some theories (as mandated via a generalized Lee-Yang theorem in the $\mathbb{Z}_2/\mathbb{Z}_2$ theory and strongly hinted by general considerations in other general instances) often can be viewed as the analogue of the absence of phase transitions in spin systems subjected to an external magnetic field. Similar to such spin systems, we speculate that the locus of gauge and matter couplings (K, β) at which a correlated percolation of clusters (given by an effective spin state related to gauge invariant bonds variables $z_{ij} \equiv \sigma_i^* U_{ij} \sigma_j$) occurs may constitute an analogue of the Kertesz line known in such spin systems [10]. We establish the validity of this anticipation and *the existence of a Kertesz line* within a simple gauge theory harboring a single confining phase—the $1+1$-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory. Possible manifestations of this effect for more physically pertinent higher group gauge fields [e.g. $SU(3)$] in $d=4$ remain a speculation.

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APPENDIX: DERIVATION OF THE DUALITY OF THE $\mathbb{Z}_2/\mathbb{Z}_2$ THEORY

The $\mathbb{Z}_2/\mathbb{Z}_2$ theory of Eq. (3) in $d=2$ dimensions is dual [via Eqs. (12) and (13)] to the two-dimensional Ising in an external magnetic field of Eq. (5). This duality allowed us to prove the existence of a sharp percolation crossover (a Kertesz [10] line) within the Higgs-confining phase of the simplest of all matter coupled gauge theories—the two-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory. The existence of a duality between the two-dimensional Ising model in a magnetic field and the two-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory was noted in [8]. As this duality is pivotal in proving our new percolation crossovers (Sec. V) even in this simplest of all matter coupled gauge theories, we explicitly illustrate its derivation below.

In what follows, we employ series expansions (a standard approach for deriving many dualities) in the high and low coupling limits to show that the high and low coupling regimes of the two disparate models [the two-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ theory of Eq. (3) and the two-dimensional Ising model in a magnetic field of Eq. (5)] become one and the same upon a change of variables (the duality transformation). Hereafter, we set in Eq. (5) the exchange constant $J_{ij} = J\delta_{|i-j|,1}$. We start by expanding the partition function

$$Z = \sum_{\{z_{ij}\}} \exp[-S], \tag{A1}$$

in a small β, K ("high temperature") series. In Eq. (A1), the action S is given by Eq. (3) and the summation in Eq. (A1) spans all gauge invariant bond variables ($z_{ij} = \pm 1$ on all nearest neighbor links $\langle ij \rangle$). To attain the low coupling expansion of the partition function Z of Eq. (A1), we employ the identities

$$\begin{aligned} \exp[\beta z] &= \cosh\beta(1 + z\tanh\beta), \\ \exp[Kzzzz] &= \cosh K(1 + zzzz\tanh K), \end{aligned} \tag{A2}$$

to obtain a polynomial expansion in

$$x = \tanh K, \qquad y = \tanh\beta. \tag{A3}$$

With these elements in tow, the partition function of Eq. (A1) becomes a sum of diagrams. These diagrams (Fig. 6) correspond to drawing closed contours in the plane and counting the number of dual lattice sites (the centers of plaquettes surrounded by four gauge links—corresponding to the plaquette terms $zzzz$ stemming from the exponentiation of the second term in the action of Eq. (3)—which are labeled by the solid rectangles) and the number of bonds (z), obtained from exponentiation of the first term of Eq. (3), labeled by the crosses residing on the contour boundaries. The sum over all values of $z_{ij} = \pm 1$ allows only diagrams containing closed loops in which each bond (z_{ij}) appears to an even power (all other diagrams necessarily have at least one bond which appears an odd number of times and therefore leads to zero once the sum over $z_{ij} = \pm 1$ is performed). For even n_{ij} ($n_{ij} = 0, 2$), the sum over each bond leads to $\sum_{z_{ij}=\pm 1} z_{ij}^{n_{ij}} = 2$. All in all, the series for the partition function becomes

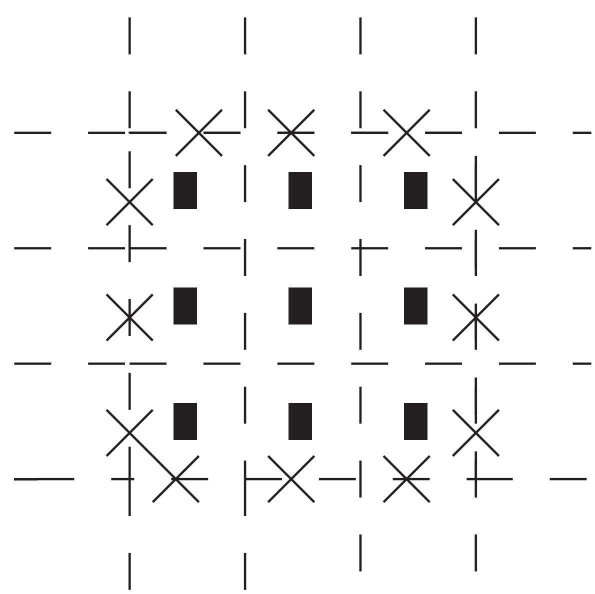

FIG. 6. A contribution to the low coupling series of the two-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ action. The centers of plaquettes are labeled by the solid rectangles. The crosses (x) denote energetic bonds (gauge invariant bond variables z of text) residing on the perimeter of the contour.

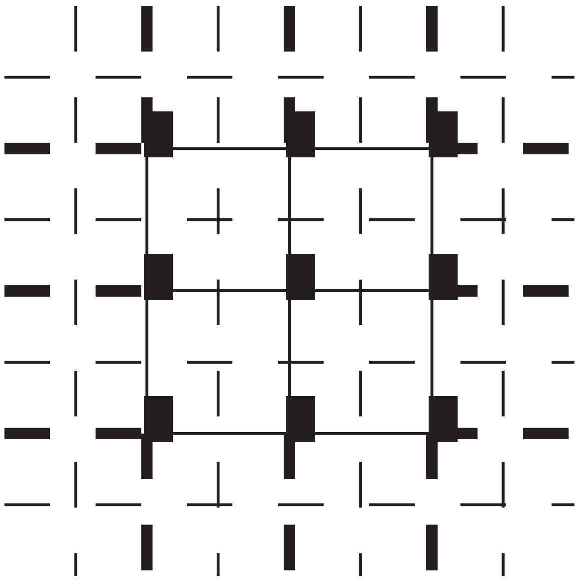

FIG. 7. The corresponding contribution to the strong coupling ("low temperature") series of the two-dimensional Ising model in a magnetic field. The flipped spins are marked by black rectangles with the bad energetic bonds that the flipped spins generate along their perimeter marked by a thick dashed line. The bonds of the dual lattice are marked by a thin dashed line.

$$Z = 4^N (\cosh K)^N (\cosh\beta)^{2N} \sum_{\text{closed loops}} x^A y^{|C|}, \qquad \text{(A4)}$$

where A denotes the net area enclosed by the set of loops C and, $|C|$ marks the net perimeter of all closed loops making up the joint contour C.

If we expand the partition function corresponding to the action of Eq. (5) about $J \to \infty$ [corresponding to flipping the spins s_i from their infinite coupling ("zero temperature") ground state value of one ($h > 0$)] then we will obtain a polynomial expansion in

$$\tilde{x} = \exp[-2h], \qquad \tilde{y} = \exp[-2J]. \qquad \text{(A5)}$$

Explicitly, the partition function reads

$$\tilde{Z} = \tilde{Z}_0 \Bigg[\sum_{\text{clusters of flipped spins}} (\tilde{x})^A (\tilde{y})^{|C|} \Bigg], \qquad \text{(A6)}$$

where $\tilde{Z}_0$ is the zero temperature (infinite h and J) partition function, A is the net area of all clusters of flipped spins ($s_i = -1$), and $|C|$ is the perimeter of all clusters of flipped spins. In Fig. 7, a simple cluster of flipped spins is shown. The flipped spins are marked by black rectangles (each flipped spin incurs a Boltzmann energy penalty of $\tilde{x}$), and the bad energetic bonds that the flipped spins generate along their perimeter (perpendicular to the domain wall) are marked by a thick dashed line. In Fig. 7, the bonds of the dual lattice are marked by a thin dashed line. Note the obvious one-to-one relation between the cluster of flipped spins of Fig. 7 to the high temperature limit in Fig. 6. The polynomials for the partition functions Z and $\tilde{Z}$ in (x, y) and $(\tilde{x}, \tilde{y})$, respectively, are identical. The partition function of the two-dimensional $\mathbb{Z}_2/\mathbb{Z}_2$ model at matter coupling β and gauge coupling K is equal (up to constants) to the partition function of the two-dimensional Ising model of exchange constant and magnetic field given by Eqs. (12) and (13). As the two-dimensional Ising model in a magnetic field of Eq. (5) exhibits no phase transition and thus no nonanalyticities for any nonzero h [corresponding to any finite K in the action of Eq. (3) on the square lattice], the radii of expansion of the series derived above are infinite and the duality transformations of Eqs. (12) and (13) hold throughout.

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[11] These meson-like variables ($\{z_{ij}\}$) have clear physical meaning. For instance, within higher group incarnations of the action of Eq. (3) wherein $z_{ij} = \sigma_i^* U_{ij} \sigma_j$, e.g. the theory of $U(1)$ matter coupled to $U(1)$ gauge [the $U(1)/U(1)$ theory], z_{ij} is directly linked to the gauge invariant current between lattice sites i and j. Within the $U(1)/U(1)$ theory describing a hyper-cubic array of coupled superconducting grains [whose superconducting order parameter is given by a phase ($\sigma_i = e^{i\theta_i}$)] in an external magnetic field [governed by the $U(1)$ gauge field $\vec{A}$ or, on the hyper-cubic lattice with $U_{ij} = \exp[i \int_i^j \vec{A} \cdot d\vec{r}]$], these gauge invariant fields $z_{ij} = \exp[i\phi_{ij}]$, with the Josephson current between sites i and j given by $J_{ij} = \sin\phi_{ij}$. By considering the general relation giving the

current as the variational derivative of the action with respect to the gauge potential, $J = \frac{\delta S}{\delta A}$, it is readily seen that this identification is far more general.
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