Approximated seventh order calculation of the vacuum wave function of 2 + 1 dimensional SU(2) lattice gauge theory

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Using the coupled cluster expansion with the random phase approximation, we calculate the long wavelength vacuum wave function and the vacuum state energy of 2 + 1 dimensional Hamiltonian SU(2) lattice gauge theory up to the seventh order. The coefficients μ_0, μ_2 of the vacuum wave function show good scaling behavior and convergence in high order calculations.

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I. INTRODUCTION

Lattice gauge theory was developed into a promising first principles to approach to the nonperturbative aspects of gauge field systems. Most of our present knowledge about lattice gauge theory (LGT) has been obtained from numerical simulations. However, in order to gain more physical insight into the theory, it is desirable to develop more analytical methods. Based on Greensite's proposal [1], we developed a scheme of coupled cluster expansions for the Hamiltonian LGT [2–4]. Although the preliminary researches were inspiring, the scheme suffers from rapid proliferation and nonindependence of clusters in the high order expansive calculations.

In Ref. [2], we used the Cayley-Hamilton relations

$$\mathrm{Tr}\, U = \mathrm{Tr}U^+, \qquad U^2 = U\,\mathrm{Tr}U - 1 \quad \text{for SU(2)}, \tag{1}$$

$$2\,\mathrm{Tr}U^+ = (\mathrm{Tr}U)^2 - \mathrm{Tr}U^2,$$
$$U^3 = U^2\,\mathrm{Tr}U - U\,\mathrm{Tr}U^+ + 1 \quad \text{for SU(3)}, \tag{2}$$

to eliminate redundancies and find the independent cluster bases, where U is any group element. But those relations are too complicated to be used in the high order expansions. We have tried to use the improved Hamiltonian [5] with the tadpole improvement [6], so that the physical results could be obtained in a relatively low order expansion. The research results for 2 + 1 dimensional U(1) LGT show that the improvement of convergence is immaterial compared with the complexity of calculation brought about by the improved Hamiltonian [7]. Recently, we introduced the random phase approximation (RPA) into the coupled cluster expansions to circumvent the above problems. The preliminary results of the investigation were encouraging [8]. In this paper, we use this new method to calculate the vacuum wave function and vacuum energy of 2 + 1 dimensional SU(2) LGT.

The paper is organized as follows. In Sec. II, we briefly review the scheme of the coupled cluster expansion with RPA. Section III is devoted to the calculation of the vacuum wave function. In Sec. IV, the conclusions and discussions are presented.

II. FORMULATION AND APPROXIMATION

The Kogut-Susskind Hamiltonian [9] is

$$H = \frac{g^2}{2a}\left[\sum_l E_l^2 - \frac{4}{g^4}\sum_p \mathrm{Tr}(U_p)\right], \tag{3}$$

where the index l denotes the links between sites, a is the lattice spacing, and U_p is the plaquette. The vacuum wave function can be written as [1]

$$\psi_0(U) = e^{R(U)}, \tag{4}$$

where $R(U)$ consists of various Wilson loops or linked clusters with appropriate symmetries for the state. In the continuum limit ($a \to 0$, or equivalently $\beta = 4/g^2 \to \infty$), the long wavelength behavior of the vacuum state can be approximated by [2]

$$\psi(A) = \exp\left(-\mu_0 \int d^2x\, \mathrm{tr}F^2(x) - \mu_2 \int d^2x\, \mathrm{tr}[D_i F^2(x)] + \cdots\right), \tag{5}$$

with F being the field strength tensor and D_i the covariant derivative. The superrenormalizability of the theory in 2 + 1 dimensions implies that $\mu_0 \to \text{const.}/e^2$ and $\mu_2 \to \text{const.}/e^6$, where e is the invariant charge which is related to the dimensionless coupling constant g by $g^2 = e^2 a$.

The eigenvalue equation for the vacuum state of H is

$$\sum_l \{[E_l^a, [E_l^a, R(U)]] + [E_l^a, R(U)][E_l^a, R(U)]\} - \frac{4}{g^4}\sum_p \mathrm{tr}U_p = \frac{2a}{g^2}\epsilon_0, \quad (6)$$

where ϵ_0 is the vacuum energy. In coupled cluster expansion, R is expanded in terms of a set of linearly independent clusters $G_{n,i}$ with suitable symmetries

$$R(U) = \sum_{n=1}^{M} R_n = \sum_{n,i} c_{n,i} G_{n,i}, \quad (7)$$

with $G_{n,i}$ denoting the ith cluster of order n, $c_{n,i}$ being a coefficient to be determined by Eq. (6), and M the highest order number in the coupled cluster expansion. Substituting Eq. (7) into Eq. (6) and adopting the truncation scheme introduced in Refs. [2–4], we obtain the truncated eigenvalue equation

$$\sum_l \left\{ \left[E_l, \left[E_l, \sum_{n=1}^{M} R_n(U) \right] \right] + \sum_{n+n' \le M} [E_l, R_n(U)][E_l, R_{n'}(U)] \right\} - \frac{4}{g^4}\sum_p \mathrm{tr}U_p = \frac{2a}{g^2}\epsilon_0. \quad (8)$$

According to the symmetry of the vacuum state, the lowest order term R_1 is chosen to be composed of just one cluster, which is an elementary plaquette

$$R_1 = c_{1,1} G_{1,1} = c_{1,1} \sum_p \mathrm{tr}U_p. \quad (9)$$

The term $[E_l^a, R_n][E_l^a, R_{n'}]$ in Eq. (6) will produce the $(n + n')$th order clusters. For example, $[E_l^a, R_1][E_l^a, R_1]$ generates the second order clusters involving two plaquettes. Obviously, clusters with order $(n + n')$ may contain at most $(n + n')$ Wilson loops. But not all clusters produced in such a way are independent. They may be related to each other by Eq. (1). We have to use this relation to identify and get the independent clusters. When clusters have many Wilson loops, the formulas (1) become so complicated that it is impossible to find a set of independent clusters, as pointed out in Sec. I. To avoid this difficulty, the RPA is applied in the expansions [8]. In RPA, one set of operators in a product of two sets of operators is replaced by its average value [10]. Here we replace one Wilson loop by its vacuum average value when a cluster produced by $[E_l^a, R_n][E_l^a, R_{n'}]$ consists of two Wilson loops. It is easy to prove that any new cluster produced in such a way contains only one Wilson loop. Suppose R_n and $R_{n'}$ are linear combinations of clusters which contain only one Wilson loop; then the new clusters produced by $[E_l, R_n] \times [E_l, R_{n'}]$ will contain at most two loops, and one loop will be replaced by its vacuum average after applying the RPA. On the other hand, according to Eq. (9) the first order cluster $G_{1,1}$ has only one Wilson loop. Therefore, all new clusters produced in the calculation consist of only one Wilson loop after using the RPA.

Since the clusters consist of only one Wilson loop, the independent cluster bases for the expansions can be obtained directly. In addition, the number of independent bases of high order expansions is much smaller than that without using the random phase approximation; for example, the number of independent third order clusters is nine in the coupled cluster expansion [2], while it is two after using the RPA (see Sec. III). Therefore, the calculations are simplified considerably.

The vacuum average value of a Wilson loop can be determined by the Feynman-Hellman theorem. Let G be some Wilson loop and $\langle G \rangle$ be its vacuum average. Defining $W = H2a/g^2$, we make a transformation as [11]:

$$W \to W^G = W + \xi_G G, \quad (10)$$

where ξ_G is a variable and will take zero at last. From $W^G|\psi_0\rangle = w_0^G|\psi_0\rangle$, we get

$$\langle G \rangle = \frac{\partial w_0^G}{\partial \xi_G}\bigg|_{\xi_G=0}. \quad (11)$$

III. CALCULATION OF APPROXIMATION

We now present the calculation of expansion. From

$$[E_l^a, G_{1,1}][E_l^a, G_{1,1}] = -4 - 2G_{2,1} + G_1' + G_2', \quad (12)$$

we get three new clusters $G_{2,1}$, G_1', and G_2' in the second order calculation. The corresponding graphs are given in Fig. 1. Two of them, which consist of two Wilson loops, turn to cluster $G_{1,1}$ times $\langle G_{1,1} \rangle$ by RPA. Therefore, there is only one cluster with order two after applying the RPA, that is, $G_{2,1}$, and

$$R_2 = c_{2,1} G_{2,1}. \quad (13)$$

Substituting R_1 and R_2 into Eq. (8), we obtain a set of equations about $c_{1,1}$, $c_{2,1}$, and w_0 with a parameter $\langle G_{1,1} \rangle$ which can be determined by Eq. (11). Solving those equations, we get the second order approximation of vacuum wave function $\psi_0(U) \approx e^{R_1(U)+R_2(U)}$. The long wavelength coefficients up to the second order are

$$\mu_0 = \left[\frac{c_{1,1}}{2} + 2c_{2,1} \right] g^4, \quad (14)$$

$$\mu_2 = -\frac{c_{2,1}}{4} g^8. \quad (15)$$

The third order clusters are produced by term $[E_l^a, R_1] \times [E_l^a, R_2]$, i.e. $c_{1,1}c_{2,1}[E_l^a, G_{1,1}][E_l^a, G_{2,1}]$. Since

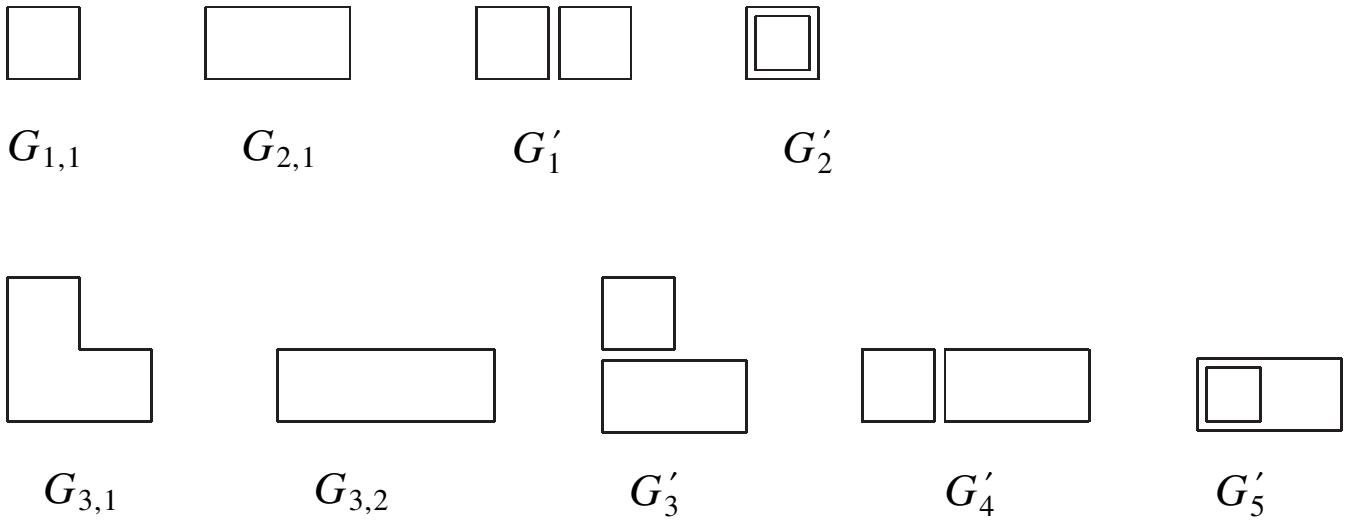

FIG. 1. The linked clusters used in the expansions up to order 3.

$$[E_l^a, G_{1,1}][E_l^a, G_{2,1}] = -2G_{3,1} - G_{3,2} + G'_3 + \tfrac{1}{2}G'_4 + \tfrac{3}{2}G'_5 - 3G_{1,1} \approx -2G_{3,1} - G_{3,2} + 3\langle G_{1,1}\rangle G_{2,1} - 3G_{1,1}, \quad (16)$$

we get

$$R_3 = c_{3,1}G_{3,1} + c_{3,2}G_{3,2}. \quad (17)$$

The number of independent clusters is two at third order, while it is nine in the expansion without RPA [2]. The calculation procedure is simplified considerably. In Eq. (16), when applying RPA to cluster G'_3, G'_4, and G'_5, we replace the smaller Wilson loop with its vacuum average and let the larger one remain unchanged as in Ref. [8]. Thus, only one vacuum average of cluster emerges in the third order calculation. This simplifies the calculation pro-

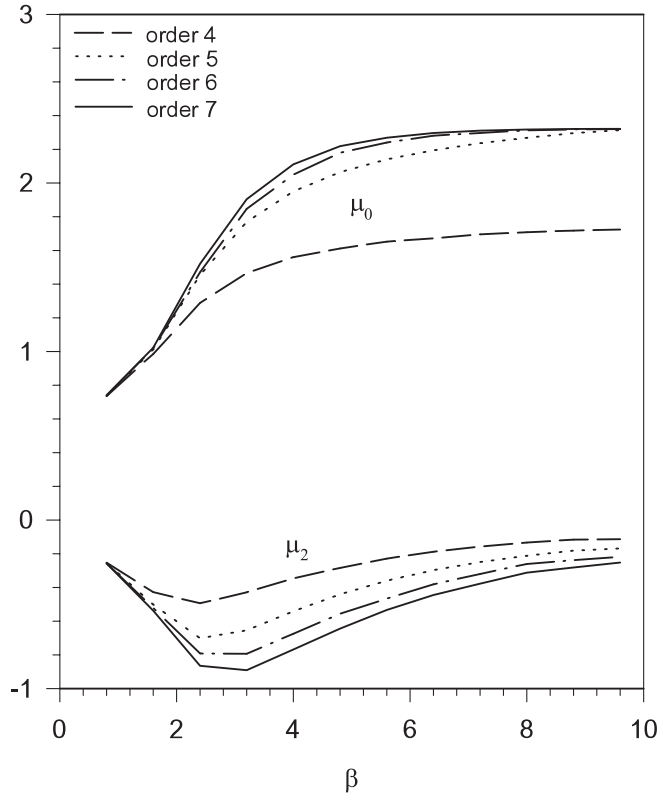

FIG. 2. μ_0 and μ_2 as a function of $\beta = 4/g^2$. The four curves represent the results from the fourth order to the seventh order expansion with RPA, respectively.

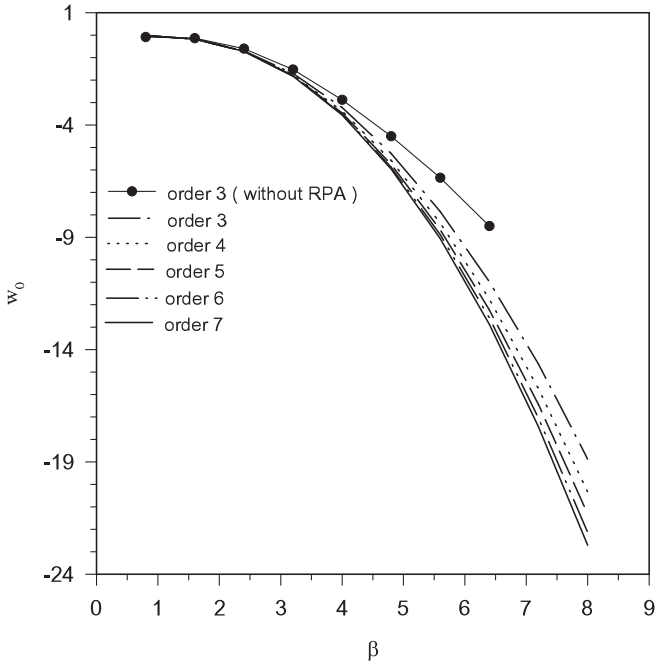

FIG. 3. The vacuum energy vs β. We also give the third order vacuum energy calculated without using RPA.

cedure further. On the other hand, the vacuum or exciting states possess definite correlation lengths. Only when the space occupied by glueball is covered with the Wilson loops is the calculation efficient. To take this into account, we replace the smaller loop with its vacuum average and preserve the larger one. Submitting R_1, R_2, and R_3 into Eq. (8), and solving the equation, we obtain the third order approximation of the vacuum wave function.

Higher order calculation can be carried out similarly. We have done the calculation up to the seventh order. The number of independent clusters is 1, 1, 2, 6, 14, 44, and 109 at order 1, 2, 3, 4, 5, 6, and 7, respectively. The results for μ_0 and μ_2 from the fourth order to the seventh order are presented in Fig. 2. In Fig. 3, the vacuum energy w_0 against β is plotted.

IV. RESULTS AND DISCUSSIONS

From Fig. 2, we see that the curves of μ_0 (or μ_2) show good scaling behavior and a convergent trend in weak coupling region $\beta = 5.5$–9.5. The sixth and seventh order values of μ_0 are coincident in the scaling region, which shows that the approach is rapidly convergent. From the seventh order results, we obtain

$$\mu_0 = 2.3, \tag{18}$$

$$\mu_2 = -0.3. \tag{19}$$

A Monte Carlo measurement was given by Arisue [12]

$$\mu_0 = (0.91 \pm 0.02), \qquad \mu_2 = -(0.19 \pm 0.05). \tag{20}$$

The value of μ_0 in Eq. (18) is larger than Arisue's. At the moment, we do not know the reasons. It is perhaps because there is some systematic error introduced by the random phase approximation in our procedure or due to other causes. What is really the cause needs further study.

In Fig. 3, the curves of the sixth and seventh order results of vacuum energy are almost coincident, which proves the expansion is able to converge rapidly again. We also give the third order result of the vacuum energy calculated without RPA for comparison. The third order values of the vacuum energy with RPA are lower than that without RPA. Such a case was also true in the case of SU(3) LGT [8].

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