

Synchrotron radiation in Lorentz-violating electrodynamics: The Myers-Pospelov modelR. Montemayor¹ and L. F. Urrutia²¹*Instituto Balseiro and CAB, Universidad Nacional de Cuyo and CNEA, 8400 Bariloche, Argentina*²*Departamento de Física de Altas Energías, Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, 04510 México D.F.*

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We develop a detailed analysis of synchrotron radiation in the effective Lorentz invariance violating (LIV) model of Myers-Pospelov, considering explicitly both the dynamics of the charge producing the radiation and the dynamics of the electromagnetic field itself. Within the radiation approximation we compute exact expressions in the LIV parameters for the electric and magnetic fields, the angular distribution of the power spectrum, the total emitted power in the m th harmonic and the polarization. We also perform expansions of the exact results in terms of the LIV parameters to identify the dominant effects and study the main features of the high energy limit of the spectrum. A very interesting consequence is the appearance of rather unexpected and large amplifying factors associated with the LIV effects, which go along with the usual contributions of the expansion parameter. This opens up the possibility of looking for astrophysical sources where these amplifying factors are important to further explore the constraints imposed upon the LIV parameters by synchrotron radiation measurements. We briefly sketch some phenomenological applications in the case of supernova remnants and gamma ray bursts.

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I. INTRODUCTION

The isotropy together with the Lorentz covariance of the laws of nature in locally inertial frames are properties which must be ultimately determined on experimental grounds. This is particularly true for boosts which, due to the noncompact character of the Lorentz group, can always be tested at higher energies. The experimental study of these questions began around 1960 with the works of Hughes *et al.* and Drever [1] and has continued since then involving a huge variety of very different testing methods with ever increasing experimental precision [2]. It has been possible to correlate such highly disparate observations in terms of the standard model extension (SME) proposed in Ref. [3]. This model is an effective theory that parameterizes all the possible Lorentz-violating interactions consistent with the standard model, which arise in a spontaneously broken formulation of such symmetry violation motivated by string theory [4]. The SME has been recently generalized to incorporate gravity effects [5].

A possible physical realization for such Lorentz invariance violations has recently been suggested within the context of dynamical modifications induced by quantum gravity in the propagation of particles in flat space, as, for example, in Ref. [6]. Most theories of nonperturbative quantum gravity predict that our notion of space as a continuum manifold has to be abandoned at distances of the order of the Planck length $\ell_P \approx 1/M_P \approx 10^{-33}$ cm, where space should be regarded as a bunch of quantum excitations. A first question related to this idea is whether or not such a drastic change in the description could produce observable residual modifications to our large scale description of particle dynamics, manifested as addi-

tional terms in the corresponding actions that are suppressed by powers of ℓ_P . Some approaches suggest that standard Lorentz covariance is consistent with the granular structure of space predicted by quantum gravity [7,8], expressed as the absence of ℓ_P corrections to large scale dynamics, while others propose that such corrections should arise, but with different interpretations of their manifestations. The most direct interpretation of such corrections is through the existence of a preferred reference frame associated to the space granularity in which the existence of a minimum length provides a maximum physical upper bound for the momenta accessible in such a frame. Nevertheless, in Ref. [9] it was shown that this interpretation leads to severe fine-tuning problems when considering the radiative corrections induced by the standard interactions. Supersymmetry has been proposed as a way to improve such situations [10]. An alternative proposal can be found in Ref. [11]. A second possibility arises when considering the dynamical modifications as signaling a spontaneous breaking of the Lorentz symmetry. In this case the corresponding effective theory also possesses a class of preferred frames that are those where the chosen vacuum expectation values of the respective fields are only a very small perturbation with respect to the Lorentz symmetric case. These are the so-called concordant frames [12], and they are not directly related to the idea of a maximum momentum. The SME together with effective models like those of Refs. [10,13] belong to this category. A third possibility, generically included in the so-called deformed special relativity approach, allows for dynamical corrections without the introduction of preferred reference frames by generalizing the concept of relative inertial frames on the basis of nonlinear representations of the

Poincaré group [14–18]. Finally, alternative ways of incorporating such dynamical modifications can be found in Ref. [19]. For recent reviews of the above topics see, for example, Refs. [20–22].

One of the most studied consequences of the Lorentz-violating corrections to the dynamics are related with multiple phenomena associated to the modified dispersion relations (MDR) for photons and fermions

$$\omega^2(k) = k^2 \pm \xi \frac{k^3}{M}, \quad E^2(p) = p^2 + m^2 + \eta_{R,L} \frac{p^3}{M}, \quad (1)$$

respectively. Here M is a scale that signals the onset of LIV and is usually associated with the Planck mass M_P . In this context the main emphasis has been set on the determination of observational bounds for the LIV parameters ξ and $\eta_{R,L}$ appearing in the MDR, thus opening the door to quantum gravity phenomenology [23]. A partial list of references is given in [24–27].

The possibility of deriving such modified dynamics from a more fundamental theory has been further explored in loop quantum gravity, where constructions of increasing degree of sophistication have managed to produce modified effective actions for photons and fermions encoding the relations (1). Basically, the starting point of these approaches has been the well-defined Hamiltonian operators of the quantum theory, together with a heuristical characterization of the semiclassical ground state. The first derivation of such a type of dispersion relations in the context of loop quantum gravity was developed in Ref. [28] and leads to a consistent extension of Maxwell electrodynamics with correction terms linear in ℓ_P . Subsequently, an alternative approach inspired by Thiemann's regularization [29] reproduced and extended such a result for the photon case [30] and produced the corresponding modified Hamiltonian for the fermion case [31]. For a review see, for example, Ref. [32]. Coherent states [33] also have been constructed, leading to similar types of dynamical modifications. All previous approximations have been kinematical, i.e. without incorporating the dynamical constraint of general relativity. In other words, a physically sound characterization of the semiclassical ground state corresponding to a given asymptotic classical metric that dictates the appropriate symmetries of the effective low energy theory is still lacking. This issue remains one of the most important open problems in loop quantum gravity. String theory has also provided a possible connection between quantum gravity and LIV in the case of photons and fermions [34,35].

Let us make some very general comments regarding some of the bounds already found in the literature for the LIV parameters appearing in MDR of the form given in Eq. (1) for photons, electrons, and protons, arising from dimension five corrections to standard electrodynamics. The main scenarios that we consider here are phenomena

related to observations of electromagnetic radiation arising from astrophysical objects and from the arrival of ultra high energy cosmic rays (UHECR).

The absence of birefringence effects from distant galaxies provides the bound $|\xi| \lesssim 10^{-4}$ [36]. Observations from the Crab nebula, a type of supernova remnant (SNR), give the following bounds on the parameters. On one hand the birefringence limit has been improved up to $|\xi| \lesssim 10^{-6}$ for the parameter region $|\eta_e| < 10^{-4}$ with η_e being at least one of the $\eta_{L,R}$ for electrons or positrons [22]. On the other hand, the intersection of the corresponding synchrotron, synchrotron-Cerenkov, and the inverse Compton constraints produces the two-sided upper bound $|\eta_{L,R}| < 10^{-2}$ on one of the two parameters $\eta_{L,R}$ [22]. Typical upper bounds for the corresponding energies in the Crab nebula are 10^6 GeV for the radiating electrons (positrons), 10^{-1} GeV for the synchrotron photons and 5×10^4 GeV for the inverse Compton photons. These give an idea of the energy scale in which the above bounds are obtained.

The assumption of a dispersion relation of the type (1) with a universal LIV parameter η for all particles in the study of UHECR has produced the stringent bound $|\eta| < 10^{-32}$ GeV $^{-1}$ [37]. This reference also reports fits of the UHECR spectrum starting from 10^{18} eV which include the AGASA data.

The vacuum Cerenkov effect becomes significant for charged particles with high energy. It already has been discussed, both using a kinematical approach [26,38,39] and on the basis of a dynamical analysis in the context of the Maxwell-Chern-Simons action [40]. An analogous analysis could be interesting in the Myers-Pospelov model, but it is out of the scope of the present work. Very stringent boundaries for the LIV parameters from the absence of vacuum Cerenkov radiation have been presented recently using protons in the ultra high energy region $E_{\text{proton}} \simeq 3 \times 10^{11}$ GeV, based on reasonable assumptions about the nature of the high energy cosmic rays. Assuming that the values for the LIV parameter η are comparable for different standard model elementary species, or at least the same within species of a given spin, leads to consider unmodified dispersion relations for a structureless proton, i. e. $\eta_p = 0$. With these assumptions the bound $|\tilde{\xi}| \lesssim 2\alpha_p = (4m_{\text{proton}}^2/E_{\text{proton}}^3) \simeq 10^{-34}$ GeV $^{-1}$ is obtained in Ref. [39]. Allowing for MDR for the proton, $\tilde{\eta}_p \neq 0$, the less stringent bounds of Ref. [26] are recovered: $\tilde{\eta}_p < \alpha_p/4$, $|\tilde{\xi}| < |\tilde{\eta}_p - \alpha_p - \alpha_p \sqrt{1 - 4\tilde{\eta}_p/\alpha}|$. Here $\tilde{\eta}_p = \eta_p/M_P$ and refers to the proton, while $\tilde{\xi} = \xi/M_P$. These constraints apply to both photon helicities, which is the origin for the bound on the absolute value $|\tilde{\xi}|$. Considering the proton polarization we can write $\tilde{\eta} = \lambda_p k_p$, where $\lambda_p = \pm 1$ is the corresponding polarization. Thus, if we have $k_p > \alpha_p/4$ ($k_p < -\alpha_p/4$) the protons with positive (negative) helicity emit vacuum Cerenkov radiation, while the protons with opposite helicity do not radiate. Hence, to

establish boundaries from Cerenkov radiation it is necessary not only to know if there are protons at a given energy, but also what polarizations they have. At energies $E \simeq 10^{11}$ GeV the UHECR are detected by the cascade they produce in the atmosphere, where the information about primary proton polarization is lost. To determine their polarization they must be observed directly as primary cosmic rays. The highest energy for which this could be done is $E \simeq 10^5$ GeV, with instruments on board of satellites [41]. In this case $\alpha_p \simeq 10^{-15}$ GeV $^{-1}$ and thus we would get $|\tilde{\xi}| \lesssim 10^4$, which is 10^{10} times weaker than the boundary obtained from vacuum birefringence. It is interesting to remark that the highest electron energies measured in primary cosmic rays are estimated to be of the order of 10^3 GeV with a flux 0.05 times that of the protons [41]. This gives $\alpha_e = 10^{-15}$ GeV $^{-1}$. In the Crab nebula we have $\alpha_e \simeq 10^{-24}$ GeV $^{-1}$ which produces a much better bound than electron vacuum Cerenkov radiation in cosmic rays.

For a detailed and updated discussion of the LIV parameter constraints see, for example, Refs. [20–22].

Now let us go back to the observation of 100 MeV synchrotron radiation from the Crab nebula. This fact alone leads to the bound $\pm \eta_{L,R} > -7 \times 10^{-8}$ for at least one of the four possibilities [25]. By itself, this does not impose any constraints on the parameters, but it plays a role when combining it with the vacuum Cerenkov effect. Such a bound is based on a set of very reasonable assumptions on how some of the standard results of synchrotron radiation extend to the Lorentz noninvariant situation. This certainly implies some dynamical assumptions, besides the purely kinematical ones embodied in (1). In this paper we examine these assumptions in the light of a particular model, which we choose to be the classical version of the Myers-Pospelov (MP) effective theory, that parameterizes LIV using dimension five operators [13]. Furthermore, a complete calculation of synchrotron radiation in the context of this model is presented. This constitutes an interesting problem on its own whose resolution subsequently will allow the use of additional observational information to put bounds upon the correction parameters. For example we have in mind the polarization measurements from cosmological sources. The case of gamma ray bursts (GRB) recently has become increasingly relevant [42], although it is still at a controversial stage [43]. Our calculations rest heavily upon the work by Schwinger *et al.* on synchrotron radiation, reported in Refs. [44–46]. In this work we restrict ourselves to classical theories, and therefore we do not address the fine-tuning problems arising from quantum corrections associated with the coexistence of space granularity and preferred reference frames [9]. A partial list of previous studies of electrodynamics incorporating LIV via dimension three and four operators is given in Refs. [40,47]. A summary of the present work has already been reported in Ref. [48].

The paper is organized as follows: In Sec. II we obtain the equations of motion for the charge and electromagnetic sectors of the MP model. The polarized electromagnetic fields for arbitrary sources in the radiation approximation are subsequently calculated in Sec. III using the standard Green function approach. In Sec. IV we obtain the general expression for the angular distribution of the power spectrum. This is subsequently applied in the next section to the case of the synchrotron radiation produced by a charged particle moving in a plane perpendicular to a constant magnetic field. The corresponding angular distribution of the radiated power in each harmonic is obtained, together with the corresponding total power. Also the associated Stokes parameters are calculated in this section. The expansions of such exact results to leading order in the LIV electromagnetic parameter are contained in Sec. VI, where we highlight the dominating effects of the Lorentz violation. A rough estimation of the relative contributions arising from dimension six operators is included in Sec. VII. Motivated by the estimated parameters of the astrophysical objects under consideration we discuss the large harmonic expansion of the previously found results in Sec. VIII. We analyze some phenomenological consequences of our results in Sec. IX. Finally, Sec. X contains a summary of the main results obtained in the paper together with a discussion of accessible regions for the LIV parameters in the cases of Crab Nebula, Mk 501, and the γ -ray burst GRB021206. There are also two appendices containing frequently used material.

II. THE MYERS-POSPELOV ELECTRODYNAMICS

The Myers-Pospelov approach is based on an effective field theory that describes Lorentz violations generated by dimension five operators, parameterized by the velocity V^μ of a preferred reference frame which is not taken as a dynamical field. These operators are assumed to be suppressed by a factor M_P^{-1} , and hence can be considered as small perturbations at the classical level. In the following we analyze the electromagnetic radiation of a classical charged particle in this framework. As a first step in this approach we characterize the particle and the electromagnetic field dynamics, introducing the charge-electromagnetic field interaction via the minimal coupling, in such a way that the usual gauge symmetry is maintained in this theory.

A. Charged particle dynamics

The dynamics of a classical charged particle can be obtained from the action for a scalar charged field. In this case the Myers-Pospelov action is

$$S_{\text{MP}} = \int d^4x [\partial_\mu \varphi^* \partial^\mu \varphi - \mu^2 \varphi^* \varphi + i\tilde{\eta} \varphi^* (V \cdot \partial)^3 \varphi], \quad (2)$$

with the notation $V \cdot \partial = V^\mu \partial_\mu$. In momentum space, where we write $\varphi(x) = \varphi_0 \exp(i(p^0 t - \mathbf{p} \cdot \mathbf{x}))$, and in the reference frame where $V^\alpha = (1, \mathbf{0})$, the dispersion relation becomes

$$(p^0)^2 + \tilde{\eta}(p^0)^3 = \mathbf{p}^2 + \mu^2. \quad (3)$$

To compare with Jacobson *et al.* results [25–27], the parameter $\tilde{\eta}$ can be written

$$\tilde{\eta} = -\frac{\eta}{M_P}, \quad \eta < 0, \quad (4)$$

where M_P is the Planck mass and η is a dimensionless constant. The Eq. (3) is an exact relation in $\tilde{\eta}$. From here we obtain the Hamiltonian for a massive particle to second order in $\tilde{\eta}$

$$p^0 = H = (\mathbf{p}^2 + \mu^2)^{1/2} - \frac{1}{2} \tilde{\eta}(\mathbf{p}^2 + \mu^2) + \frac{5}{8} \tilde{\eta}^2(\mathbf{p}^2 + \mu^2)^{3/2} + O(\tilde{\eta}^3). \quad (5)$$

In the following we will consider the interaction of a particle of mass μ and charge q with a static magnetic field [$\phi = 0, \mathbf{A}(\mathbf{r})$]. The standard minimal coupling produces the Hamiltonian

$$H = \left[\left(\mathbf{p} - \frac{q}{c} \mathbf{A} \right)^2 + \mu^2 \right]^{1/2} - \frac{1}{2} \tilde{\eta} \left[\left(\mathbf{p} - \frac{q}{c} \mathbf{A} \right)^2 + \mu^2 \right] + \frac{5}{8} \tilde{\eta}^2 \left[\left(\mathbf{p} - \frac{q}{c} \mathbf{A} \right)^2 + \mu^2 \right]^{3/2} + O(\tilde{\eta}^3). \quad (6)$$

Here $c = 3 \times 10^{10} \text{ cm s}^{-1}$ denotes the uncorrected velocity of light in vacuum. In the sequel we set $c = 1$. Observe that the dispersion relation (3) provides the exact inversion

$$(\mathbf{p} - q\mathbf{A})^2 = (1 + \tilde{\eta}E)E^2 - \mu^2, \quad (7)$$

with E being the energy of the particle, which together with the Hamilton equations yield

$$\dot{x}_i = v_i = \frac{\partial H}{\partial p_i} = \frac{1}{E} \left(1 - \frac{3}{2} \tilde{\eta}E + \frac{9}{4} \tilde{\eta}^2 E^2 \right) (p_i - qA_i), \quad (8)$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = q\dot{x}_j \frac{\partial A_j}{\partial x_i}, \quad (9)$$

up to second order in $\tilde{\eta}$. The acceleration results

$$\ddot{\mathbf{r}} = \frac{q}{E} \left(1 - \frac{3}{2} \tilde{\eta}E + \frac{9}{4} \tilde{\eta}^2 E^2 \right) (\mathbf{v} \times \mathbf{B}). \quad (10)$$

As in the usual case, this means that the magnitude $|\mathbf{v}|$ is constant and that the projection of the orbit in a plane orthogonal to $\mathbf{B} = \nabla \times \mathbf{A}$ is a circular orbit with a Larmor frequency

$$\omega_0 = \frac{|q|B}{E} \left(1 - \frac{3}{2} \tilde{\eta}E + \frac{9}{4} \tilde{\eta}^2 E^2 \right). \quad (11)$$

In general the motion is a helix with pitch angle (the angle between the velocity and the magnetic field) α . Let us define

$$\beta_\perp = \beta \sin \alpha, \quad \beta_\parallel = \beta \cos \alpha, \quad (12)$$

where we emphasize that we are using the standard definition $\beta = |\mathbf{v}|/c$. The solution to the equations of motion can be written as

$$\mathbf{r}(t) = \left(\frac{\beta_\perp}{\omega_0} \cos \omega_0 t, \frac{\beta_\perp}{\omega_0} \sin \omega_0 t, \beta_\parallel t \right), \quad (13)$$

and hence

$$\mathbf{v}(t) = (-\beta_\perp \sin \omega_0 t, \beta_\perp \cos \omega_0 t, \beta_\parallel). \quad (14)$$

The orbit Eq. (13) identifies

$$R = \frac{\beta_\perp}{\omega_0} \quad (15)$$

as the radius of the projection of the helix in the plane perpendicular to \mathbf{B} . From Eqs. (7) and (8) we obtain

$$1 - \beta^2 = \frac{\mu^2}{E^2} \left[1 + 2 \frac{\tilde{\eta}E^3}{\mu^2} - \frac{15}{4} \frac{\tilde{\eta}^2 E^4}{\mu^2} + O(\tilde{\eta}^3) \right], \quad (16)$$

where the considered range of energies is such that

$$\frac{\mu}{E} \ll 1, \quad \tilde{\eta}E \ll 1. \quad (17)$$

According to the relation among the quantities appearing in Eq. (17), we can consider different energy ranges to write approximate expressions for the Lorentz factor

$$\gamma = (1 - \beta^2)^{-1/2}. \quad (18)$$

In Eq. (16), the ratio between the second order term in $\tilde{\eta}$ and the first order one is proportional to $\tilde{\eta}E \ll 1$. Therefore, assuming $\tilde{\eta} \geq 10^{-19} \text{ GeV}^{-1}$, the last term is negligible compared with the second one at least for $E \leq 10^{19} \text{ GeV}$. The last term is of order one when $\tilde{\eta}^2 E^4 / \mu^2 \simeq 1$, i.e. when $E \simeq (\mu / \tilde{\eta})^{1/2}$. For electrons this means $E \simeq 7 \times 10^7 \text{ GeV} \simeq 7 \times 10^4 \text{ TeV}$, and an even larger energy for more massive particles. Therefore, in the range of energies expected in astrophysical objects and when $E < 7 \times 10^7 \text{ GeV}$, the γ factor can be written as

$$\gamma \simeq \frac{E}{\mu} \left(1 + 2 \frac{\tilde{\eta}E^3}{\mu^2} \right)^{-1/2}. \quad (19)$$

Hence we have the two possible approximate expressions

$$\gamma \simeq \frac{E}{\mu} \left(1 + \frac{\tilde{\eta}E^3}{\mu^2} \right), \quad \frac{\tilde{\eta}E^3}{\mu^2} \ll 1, \quad (20)$$

$$\gamma \simeq \sqrt{\frac{1}{2\kappa E} \left(1 - \frac{\mu^2}{4\tilde{\eta}E^3} \right)}, \quad \frac{\tilde{\eta}E^3}{\mu^2} \gg 1. \quad (21)$$

The limiting condition between these two energy ranges is $E \simeq 10^4 \text{ GeV}$ for electrons and $E \simeq 10^6 \text{ GeV}$ for protons.

In the case of electrons in a SNR of the type of Crab Nebula, the maximum energy of the electrons is of the order of 10^3 TeV, and therefore the approximation (20) still holds.

According to the preceding analysis, the current for a charged particle moving in the general helical motion is

$$\mathbf{j}(t, \mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}(t))\mathbf{v}(t), \quad (22)$$

where $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are given in Eqs. (13) and (14), respectively, with β determined by the Lorentz factor (20). In the following we will consider only circular motion, i.e. $\alpha = \pi/2$.

B. The electromagnetic field

The Myers-Pospelov action for the electromagnetic field is

$$S_{\text{MP}} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 4\pi j^\mu A_\mu + \tilde{\xi}(V^\alpha F_{\alpha\delta})(V \cdot \partial)(V_\beta \tilde{F}^{\beta\delta}) \right], \quad (23)$$

where $\tilde{\xi} = \xi/M_P$, with ξ being a dimensionless parameter. We are using the conventions

$$\begin{aligned} \tilde{F}^{\beta\delta} &= \frac{1}{2} \epsilon^{\beta\delta\rho\sigma} F_{\rho\sigma}, & \epsilon^{0123} &= +1, & \epsilon_{123} &= +1, \\ \eta &= (+, -, -, -), & \epsilon^{0ijk} &= \epsilon_{ijk}. \end{aligned} \quad (24)$$

As usual we have $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $j^\mu = (\rho, \mathbf{j})$, with the electric and magnetic fields identified according to

$$F_{0i} = E_i, \quad F_{ij} = -\epsilon_{ijk} B_k, \quad B_k = -\frac{1}{2} \epsilon_{kij} F_{ij}, \quad (25)$$

$$\tilde{F}_{0i} = B_i, \quad \tilde{F}_{ij} = \epsilon_{ijk} E_k, \quad E_k = \frac{1}{2} \epsilon_{kij} \tilde{F}_{ij}. \quad (26)$$

These lead to the standard homogeneous Maxwell equations

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0. \quad (27)$$

The equations of motion obtained from the action (23) are

$$\begin{aligned} \partial_\mu F^{\mu\nu} + \tilde{\xi}(V \cdot \partial)(\epsilon^{\beta\nu\delta\rho} V_\beta (V^\alpha \partial_\rho F_{\alpha\delta}) - (V^\alpha \partial_\alpha) \\ \times (V_\beta \tilde{F}^{\beta\nu})) = 4\pi j^\nu. \end{aligned} \quad (28)$$

For simplicity, we keep working in the rest frame $V^\alpha = (1, \mathbf{0})$. If required we can boost to any other reference frame with a given V^α by means of an observer (passive) Lorentz transformation. Rewriting Eqs. (28) in vector form we get

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad (29)$$

$$-\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} + \tilde{\xi} \frac{\partial}{\partial t} \left(-\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) = 4\pi \mathbf{j}. \quad (30)$$

Thus, in terms of the standard potential fields $A^\mu = (\phi, \mathbf{A})$ we are left with

$$\nabla^2 \phi + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -4\pi\rho, \quad (31)$$

$$\frac{\partial}{\partial t} \left(\nabla \phi + \frac{\partial \mathbf{A}}{\partial t} \right) + \nabla \times (\nabla \times \mathbf{A}) + 2\tilde{\xi} \nabla \times \frac{\partial^2 \mathbf{A}}{\partial t^2} = 4\pi \mathbf{j}, \quad (32)$$

which can be conveniently expressed in the radiation (Coulomb) gauge, $\nabla \cdot \mathbf{A} = 0$, as

$$\phi = -4\pi \frac{1}{\nabla^2} \rho, \quad (33)$$

$$\begin{aligned} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} + 2\tilde{\xi} \nabla \times \frac{\partial^2 \mathbf{A}}{\partial t^2} &= 4\pi \left(\mathbf{j} - \nabla \frac{1}{\nabla^2} \nabla \cdot \mathbf{j} \right) \\ &\equiv 4\pi \mathbf{j}_T, \end{aligned} \quad (34)$$

where the electric and magnetic fields reduce to

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad (35)$$

in the corresponding large distance approximation.

The energy momentum tensor $T_{\mu\nu}$ for this modified electrodynamics is given by

$$T_0^0 = \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2) - \frac{\tilde{\xi}}{4\pi} \mathbf{E} \cdot \frac{\partial \mathbf{B}}{\partial t}, \quad (36)$$

$$\mathbf{S} = \frac{1}{4\pi} \mathbf{E} \times \mathbf{B} - \frac{\tilde{\xi}}{4\pi} \mathbf{E} \times \frac{\partial \mathbf{E}}{\partial t}, \quad (37)$$

which are exact expressions in $\tilde{\xi}$. These components satisfy the usual conservation equation

$$\frac{\partial T^{00}}{\partial t} + \nabla \cdot \mathbf{S} = 0, \quad (38)$$

outside the sources.

To solve the equation of motion for \mathbf{A} , Eq. (34), we can go to the momentum space with the convention

$$F(t, \mathbf{r}) = \int \frac{d^4k}{(2\pi)^4} e^{-i\omega t + i\mathbf{k} \cdot \mathbf{r}} F(\omega, \mathbf{k}). \quad (39)$$

The different types of Fourier transforms are specified by the corresponding arguments. For example, if $F(t, \mathbf{r})$ denotes the function in space-time, $F(\omega, \mathbf{r})$ denotes the Fourier transformed function to frequency space, while $F(\omega, \mathbf{k})$ denotes the Fourier transformed function to frequency and momentum spaces. In this way Eq. (34) reduces to

$$(-\omega^2 + k^2 - 2i\tilde{\xi}\omega^2 \mathbf{k} \times) \mathbf{A}(\omega, \mathbf{k}) = 4\pi \mathbf{j}_T(\omega, \mathbf{k}). \quad (40)$$

This equation can be diagonalized using the basis of circular polarization $\lambda = \pm 1$, defined in Appendix A, giving

$$(-\omega^2 + k^2 \mp 2\tilde{\xi}\omega^2 k)\mathbf{A}^\pm(\omega, \mathbf{k}) = 4\pi\mathbf{j}_T^\pm(\omega, \mathbf{k}). \quad (41)$$

III. GREEN FUNCTIONS AND FIELDS

The simplest way to proceed is by introducing the retarded Green functions with definite polarization

$$G_{\text{ret}}^\lambda(\omega, \mathbf{k}) = \frac{1}{k^2 - \lambda 2\tilde{\xi}\omega^2 k - \omega^2} \Big|_{\omega \rightarrow \omega + i\epsilon}, \quad (42)$$

together with the total retarded Green function $[G_{\text{ret}}(\omega, \mathbf{k})]_{ik} = \sum_\lambda P_{ik}^\lambda G_{\text{ret}}^\lambda(\omega, \mathbf{k})$ and to calculate $[G_{\text{ret}}(\omega, \vec{r} - \vec{r}')]_{ik}$. Here P_{ik}^λ is the helicity projector defined in Eq. (A4). The angular integration can be performed immediately and the remaining integration over dk gets contributions only from the upper half plane in the complex variable k . Finally one can identify the polarization components of the total Green function as

$$\begin{aligned} G_{\text{ret}}^\lambda(\omega, \mathbf{r} - \mathbf{r}') &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} G_{\text{ret}}^\lambda(\omega, \mathbf{k}) \\ &= \frac{1}{4\pi R} \frac{n(\lambda z)}{\sqrt{1+z^2}} e^{in(\lambda z)\omega R}, \end{aligned} \quad (43)$$

where $R = |\mathbf{r} - \mathbf{r}'|$. Here we introduce the polarization-dependent refraction index $n(\lambda z)$

$$n(\lambda z) = \sqrt{1+z^2} + \lambda z, \quad z = \tilde{\xi}\omega. \quad (44)$$

In this way, the fields \mathbf{A}^λ in Eq. (41) have well-defined phase velocities $v_\lambda = 1/n(\lambda z)$, and this situation can be described as the propagation of photons in a dispersive birefringent medium.

The Green functions (43) determine the corresponding potentials with the standard replacements $1/R \simeq 1/|\mathbf{r}| \equiv 1/r$ in the denominator and $R \simeq r - \hat{\mathbf{n}} \cdot \mathbf{r}'$ in the phase, with $\hat{\mathbf{n}} = \mathbf{r}/r$ being the direction of observation. Note that in the Green function phase

$$\begin{aligned} n(\lambda z)\omega|\mathbf{r} - \mathbf{r}'| &\simeq \omega r \left[1 - \frac{\hat{\mathbf{n}} \cdot \mathbf{r}'}{r} + \lambda \tilde{\xi}\omega - \lambda \tilde{\xi}\omega \frac{\hat{\mathbf{n}} \cdot \mathbf{r}'}{r} \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{r'}{r} \right)^2 \right], \end{aligned} \quad (45)$$

we are taking the radiation approximation, which means that the subdominant terms of order $(r'/r)^2$ or higher are neglected. Consistency demands that terms proportional to the LIV parameter $\tilde{\xi}$ are larger than the neglected one in order to properly include them in this phase. Our general results are presented in this full far-field approximation.

Using (43) we obtain

$$\begin{aligned} \mathbf{A}^\lambda(\omega, \mathbf{r}) &= 4\pi \int d^3\mathbf{r}' G_{\text{ret}}^\lambda(\omega, \mathbf{r} - \mathbf{r}') \mathbf{j}_T^\lambda(\omega, \mathbf{r}') \\ &= \frac{1}{R} \frac{n(\lambda z)}{\sqrt{1+z^2}} e^{in(\lambda z)\omega r} \\ &\quad \times \int d^3\mathbf{r}' e^{-i[n(\lambda z)\omega \hat{\mathbf{n}}] \cdot \mathbf{r}'} \mathbf{j}_T^\lambda(\omega, \mathbf{r}'), \end{aligned} \quad (46)$$

and thus we finally get

$$\mathbf{A}^\lambda(\omega, \mathbf{r}) = \frac{1}{r} \frac{n(\lambda z)}{\sqrt{1+z^2}} e^{in(\lambda z)\omega r} \mathbf{j}^\lambda(\omega \mathbf{k}_\lambda) \quad (47)$$

in the radiation approximation. The fields $\mathbf{A}^+(\omega, \mathbf{r})$ and $\mathbf{A}^-(\omega, \mathbf{r})$ correspond to right and left circular polarization, respectively. Let us emphasize that the momenta

$$\mathbf{k}_\lambda = n(\lambda z)\omega \hat{\mathbf{n}} \quad (48)$$

in Eq. (47) are fixed in terms of the frequency and the direction of observation. As usual, the integration over $d^3\mathbf{r}'$ has been conveniently written as the Fourier transform in momentum space $\mathbf{j}_T^\lambda(\omega, \mathbf{k}_\lambda)$ of the function $\mathbf{j}_T^\lambda(\omega, \mathbf{r}')$ in coordinate space. In the right-hand side of Eq. (47) we have used the relation (A1) for the transverse current. The full vector potential is given by the superposition

$$\mathbf{A}(\omega, \mathbf{r}) = \frac{1}{r} \frac{1}{\sqrt{1+z^2}} \sum_{\lambda=\pm 1} n(\lambda z) e^{in(\lambda z)\omega r} \mathbf{j}^\lambda(\omega, \mathbf{k}_\lambda). \quad (49)$$

Hence the electric and magnetic fields are

$$\begin{aligned} \mathbf{B}(\omega, \mathbf{r}) &= \nabla \times \mathbf{A}(\omega, \mathbf{r}) \\ &= \frac{1}{r} \frac{\omega}{\sqrt{1+z^2}} [n^2(z) e^{in(z)\omega r} \mathbf{j}^+(\omega, \mathbf{k}_+) \\ &\quad - n^2(-z) e^{in(-z)\omega r} \mathbf{j}^-(\omega, \mathbf{k}_-)], \end{aligned} \quad (50)$$

$$\begin{aligned} \mathbf{E}(\omega, \mathbf{r}) &= i\omega \mathbf{A}(\omega, \mathbf{r}) \\ &= \frac{1}{r} \frac{i\omega}{\sqrt{1+z^2}} [n(z) e^{in(z)\omega r} \mathbf{j}^+(\omega, \mathbf{k}_+) \\ &\quad + n(-z) e^{in(-z)\omega r} \mathbf{j}^-(\omega, \mathbf{k}_-)]. \end{aligned} \quad (51)$$

Note that, contrary to the standard case, where $\hat{\mathbf{n}} \times \mathbf{E} \propto \mathbf{B}$, here we have

$$\hat{\mathbf{n}} \times \mathbf{E}(\omega, \mathbf{r}) = \frac{1}{\sqrt{1+z^2}} [\mathbf{B}(\omega, \mathbf{r}) + iz\mathbf{E}(\omega, \mathbf{r})]. \quad (52)$$

IV. THE ANGULAR DISTRIBUTION OF THE POWER SPECTRUM

This is defined as $\frac{d^2 P(T)}{d\omega d\Omega}$, where $P(T)$ is the radiated power at time T into the solid angle $d\Omega$ in a given radiation problem. We can compute the total energy emitted in terms of the Poynting vector (37)

$$E = \int_{-\infty}^{\infty} dt \mathbf{m} \cdot \mathbf{S}(t, \mathbf{r}) r^2 d\Omega. \quad (53)$$

This last expression can be rewritten introducing the Fourier transform of the Poynting vector,

$$\begin{aligned} E &= \int_0^{\infty} d\omega \int d\Omega \frac{d^2 E}{d\Omega d\omega} \\ &= \int_0^{\infty} \frac{d\omega}{2\pi} [\mathbf{n} \cdot \mathbf{S}(\omega, \mathbf{r}) + \mathbf{n} \cdot \mathbf{S}(-\omega, \mathbf{r})] r^2 d\Omega, \end{aligned} \quad (54)$$

and allows us to obtain the angular distribution of the energy spectrum

$$\frac{d^2 E}{d\Omega d\omega} = \frac{r^2}{2\pi} [\mathbf{n} \cdot \mathbf{S}(\omega, \mathbf{r}) + \mathbf{n} \cdot \mathbf{S}(-\omega, \mathbf{r})], \quad (55)$$

from where the angular distribution of the power spectrum can be identified as

$$\frac{d^2 E}{d\Omega d\omega} = \int_{-\infty}^{+\infty} dT \frac{d^2 P(T)}{d\omega d\Omega}. \quad (56)$$

The Poynting vector is given by Eq. (37), from which results

$$\begin{aligned} \mathbf{S}(\omega, \mathbf{r}) &= \frac{1}{4\pi} (\mathbf{E}(-\omega, \mathbf{r}) \mathbf{B}(\omega, \mathbf{r}) \\ &\quad + i\omega \tilde{\xi} \mathbf{E}(-\omega, \mathbf{r}) \mathbf{E}(\omega, \mathbf{r})). \end{aligned} \quad (57)$$

This expression, exact in $\tilde{\xi}$, also can be written in terms of the potentials, in which case we have

$$\begin{aligned} \mathbf{S}(\omega, \mathbf{r}) &= -\frac{i}{4\pi} \{ \omega^2 \mathbf{A}(-\omega, \mathbf{r}) [n(\tilde{\xi}\omega) \mathbf{A}_+(\omega, \mathbf{r}) \\ &\quad - n(-\tilde{\xi}\omega) \mathbf{A}_-(\omega, \mathbf{r})] \\ &\quad - \tilde{\xi} \omega^3 \mathbf{A}(-\omega, \mathbf{r}) \mathbf{A}(\omega, \mathbf{r}) \}. \end{aligned} \quad (58)$$

Finally, using the relations (A3) of Appendix A we get

$$\begin{aligned} \mathbf{n} \cdot \mathbf{S}(\omega, \mathbf{r}) &= \frac{\omega^2}{4\pi} \sqrt{1+z^2} [\mathbf{A}_-(-\omega, \mathbf{r}) \cdot \mathbf{A}_+(\omega, \mathbf{r}) \\ &\quad + \mathbf{A}_+(-\omega, \mathbf{r}) \cdot \mathbf{A}_-(\omega, \mathbf{r})]. \end{aligned} \quad (59)$$

Introducing Eq. (59) in Eq. (55) we finally have

$$\begin{aligned} \frac{d^2 E}{d\Omega d\omega} &= \frac{r^2 \omega^2}{4\pi^2} \sqrt{1+z^2} [\mathbf{A}_-(-\omega, \mathbf{r}) \cdot \mathbf{A}_+(\omega, \mathbf{r}) \\ &\quad + \mathbf{A}_-(\omega, \mathbf{r}) \cdot \mathbf{A}_+(-\omega, \mathbf{r})]. \end{aligned} \quad (60)$$

Now we need to express the products $\mathbf{A}_{\mp}(-\omega, \mathbf{r}) \cdot \mathbf{A}_{\pm}(\omega, \mathbf{r})$ in terms of the current $\mathbf{j}(\omega, \mathbf{k})$ via the relation (47). To this end it is convenient to introduce the projectors (A4) that satisfy the relations (A5)–(A7). Using these results together with the general relation

$$j_k(-\omega, -\mathbf{k}) = j_k^*(\omega, \mathbf{k}), \quad (61)$$

we obtain

$$\begin{aligned} \mathbf{A}_-(\mp\omega, \mathbf{r}) \cdot \mathbf{A}_+(\pm\omega, \mathbf{r}) &= \frac{1}{r^2} \frac{n^2(z)}{1+z^2} j_l^*(\omega, \omega n(\pm z) \hat{\mathbf{n}}) \\ &\quad \times P_{lk}^{\pm} j_k(\omega, \omega n(\pm z) \hat{\mathbf{n}}), \end{aligned} \quad (62)$$

which leads to

$$\begin{aligned} \frac{d^2 E}{d\Omega d\omega} &= \frac{1}{4\pi^2} \frac{\omega^2}{1+z^2} [n^2(z) j_l^*(\omega, \mathbf{k}_+) P_{lk}^+ j_k(\omega, \mathbf{k}_+) \\ &\quad + n^2(-z) j_k^*(\omega, \mathbf{k}_-) P_{kl}^- j_l(\omega, \mathbf{k}_-)]. \end{aligned} \quad (63)$$

In order to identify the angular distribution of the power spectrum we need to go back to Eq. (56). Each contribution in Eq. (63) is of the type

$$\begin{aligned} C(\omega) &= j_k^*(\omega, \mathbf{k}) X_{kr} j_r(\omega, \mathbf{k}) \\ &= \int_{-\infty}^{+\infty} dt dt' e^{-i\omega(t-t')} j_k^*(t, \mathbf{k}) X_{kr} j_r(t', \mathbf{k}). \end{aligned} \quad (64)$$

Introducing the new time variables $\tau = t - t'$ and $T = (t + t')/2$ we get

$$\begin{aligned} C(\omega) &= \int_{-\infty}^{+\infty} dT \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} j_k^*(T + \tau/2, \mathbf{k}) \\ &\quad \times X_{kr} j_r(T - \tau/2, \mathbf{k}). \end{aligned} \quad (65)$$

Inserting this last relation in Eq. (63) and comparing with Eq. (56) we obtain the final expression for the angular distribution of the radiated power spectrum

$$\begin{aligned} \frac{d^2 P(T)}{d\omega d\Omega} &= \frac{1}{4\pi^2} \frac{\omega^2}{\sqrt{1+z^2}} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} [n^2(z) j_k^*(T + \tau/2, \mathbf{k}_+) \\ &\quad \times P_{kr}^+ j_r(T - \tau/2, \mathbf{k}_+) + n^2(-z) j_k^*(T + \tau/2, \mathbf{k}_-) \\ &\quad \times P_{kr}^- j_r(T - \tau/2, \mathbf{k}_-)], \end{aligned} \quad (66)$$

as the sum of the contributions of both circular polarizations.

V. SYNCHROTRON RADIATION

Let us rewrite Eq. (66) in the form

$$\begin{aligned} \frac{d^2 P(T)}{d\omega d\Omega} &= \sum_{\lambda=\pm 1} \frac{\omega^2}{4\pi^2} \frac{n^2(\lambda z)}{\sqrt{1+z^2}} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} J_k^*(T + \tau/2, \mathbf{k}_{\lambda}) \\ &\quad \times P_{kr}^{\lambda} j_r(T - \tau/2, \mathbf{k}_{\lambda}), \end{aligned} \quad (67)$$

where the current associated to the motion of a particle in an external magnetic field was already obtained in Eq. (22). Its spatial Fourier transform is

$$\mathbf{j}(t, \mathbf{k}) = \int d^3 y(t, \mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{y}} = q \mathbf{v}(t) e^{-i\mathbf{k} \cdot \mathbf{r}(t)}, \quad (68)$$

and thus we get

$$\frac{d^2 P(T)}{d\omega d\Omega} = \sum_{\lambda=\pm 1} \frac{\omega^2}{2\pi^2} \frac{n^2(\lambda z)}{\sqrt{1+z^2}} J(T, \omega, \lambda, \hat{\mathbf{n}}), \quad (69)$$

where

$$J(T, \omega, \lambda, \mathbf{n}) = \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} j_k^*(T + \tau/2, \mathbf{k}_\lambda) \times P_{kr}^\lambda j_r(T - \tau/2, \mathbf{k}_\lambda). \quad (70)$$

This integral can be expressed as

$$J(T, \omega, \lambda, \mathbf{n}) = q^2 \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} e^{i\omega n(\lambda z) \hat{\mathbf{n}} \cdot (\mathbf{r}(T+\tau/2) - \mathbf{r}(T-\tau/2))} \mathfrak{S}, \quad (71)$$

with

$$\begin{aligned} \mathfrak{S} &= v_k(T + \tau/2) P_{kr}^\lambda v_r(T - \tau/2) \\ &= \frac{1}{2} [\mathbf{v}(T + \tau/2) \cdot \mathbf{v}(T - \tau/2) - \hat{\mathbf{n}} \cdot \mathbf{v}(T + \tau/2) \hat{\mathbf{n}} \\ &\quad \cdot \mathbf{v}(T - \tau/2) - i\lambda \hat{\mathbf{n}} \cdot \mathbf{v}(T + \tau/2) \mathbf{v}(T - \tau/2)]. \end{aligned} \quad (72)$$

Here it is useful to rewrite the terms containing $\hat{\mathbf{n}} \cdot \mathbf{j} \approx \hat{\mathbf{n}} \cdot \mathbf{v}$ in terms of the charge density. To this end we use current conservation in momentum space

$$\omega \rho(\omega, \mathbf{k}) - \mathbf{k} \cdot \mathbf{j}(\omega, \mathbf{k}) = 0, \quad (73)$$

obtaining

$$\hat{\mathbf{n}} \cdot \mathbf{j}(\omega, \mathbf{k}_\lambda) = \frac{1}{n(\lambda z)} \rho(\omega, \mathbf{k}_\lambda), \quad (74)$$

which leads to

$$\hat{\mathbf{n}} \cdot \mathbf{j}(t, \mathbf{k}_\lambda) = \frac{1}{n(\lambda z)} \rho(t, \mathbf{k}_\lambda). \quad (75)$$

In this way we have

$$\begin{aligned} &[\hat{\mathbf{n}} \cdot \mathbf{j}^*(T + \tau/2, \mathbf{k}_\lambda)][\hat{\mathbf{n}} \cdot \mathbf{j}(T - \tau/2, \mathbf{k}_\lambda)] \\ &= \frac{1}{n^2(\lambda z)} \rho^*[T + \tau/2, \mathbf{k}_\lambda] \rho[T - \tau/2, \mathbf{k}_\lambda]. \end{aligned} \quad (76)$$

In other words, in the expression of \mathfrak{S} , Eq. (72), we can make the replacement $\hat{\mathbf{n}} \cdot \mathbf{v}(\mathbf{t}) \rightarrow 1/n(\lambda z)$, leading to

$$\begin{aligned} \mathfrak{S} &= \frac{1}{2} [\mathbf{v}(T + \tau/2) \cdot \mathbf{v}(T - \tau/2) - n^{-2}(\lambda z) \\ &\quad - i\lambda \hat{\mathbf{n}} \cdot \mathbf{v}(T + \tau/2) \mathbf{v}(T - \tau/2)]. \end{aligned} \quad (77)$$

In the following we will consider the simplest case, when the orbit of the charged particle is in a plane perpendicular to the magnetic field, i.e. the case where $\beta_{\parallel} = 0$ and $\beta_{\perp} = \beta$. To continue with the calculation we introduce the direction of observation

$$\hat{\mathbf{n}} = (\sin\theta, 0, \cos\theta), \quad (78)$$

in the same coordinate system where $\mathbf{r}(t)$ and $\mathbf{v}(t)$ were defined in Eqs. (13) and (14). Recalling the corresponding expressions we calculate

$$\mathbf{v}(T + \tau/2) \cdot \mathbf{v}(T - \tau/2) = \beta^2 \cos\omega_0\tau, \quad (79)$$

$$\mathbf{n} \cdot \mathbf{v}(T + \tau/2) \mathbf{v}(T - \tau/2) = -\beta^2 \cos\theta \sin\omega_0\tau, \quad (80)$$

and substituting in Eq. (77) we are left with

$$\mathfrak{S} = \frac{1}{2} [\beta^2 \cos\omega_0\tau - n^{-2}(\lambda z) + i\lambda \beta^2 \cos\theta \sin\omega_0\tau]. \quad (81)$$

Now we can compute the cycle average over the macroscopic time T , of the angular distribution of the radiated power spectrum

$$\left\langle \frac{d^2 P(T)}{d\omega d\Omega} \right\rangle = \sum_{\lambda=\pm 1} \frac{\omega^2}{4\pi^2} \frac{n^2(\lambda z)}{\sqrt{1+z^2}} \langle J(T, \omega, \lambda, \hat{\mathbf{n}}) \rangle. \quad (82)$$

The dependence on T is in the exponential factor of Eq. (71), where we have

$$\begin{aligned} &\hat{\mathbf{n}} \cdot [\mathbf{r}(T + \tau/2) - \mathbf{r}(T - \tau/2)] \\ &= -2R \sin\theta \sin(\omega_0 T) \sin(\omega_0 \tau/2). \end{aligned} \quad (83)$$

Thus the required averaged value is

$$\langle e^{i\omega n(\lambda z) \hat{\mathbf{n}} \cdot (\mathbf{r}(T+\tau/2) - \mathbf{r}(T-\tau/2))} \rangle = J_0(2Y \sin\theta), \quad (84)$$

with

$$Y = \omega n(\lambda z) R \sin(\omega_0 \tau/2). \quad (85)$$

From here we get

$$\begin{aligned} \langle J(T, \omega, \lambda, \mathbf{n}) \rangle &= \frac{q^2}{2} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} J_0(2Y \sin\theta) [\beta^2 \cos\omega_0\tau \\ &\quad - n^{-2}(\lambda z) + i\lambda \beta^2 \cos\theta \sin(\omega_0\tau)]. \end{aligned} \quad (86)$$

Now we apply the addition theorem for Bessel functions [49]

$$\begin{aligned} &J_0\left(k\sqrt{\rho^2 + \sigma^2 - 2\rho\sigma \cos(\phi - \phi')}\right) \\ &= \sum_{m=-\infty}^{\infty} J_m(k\rho) J_m(k\sigma) e^{im(\phi - \phi')}, \end{aligned} \quad (87)$$

with

$$\rho = \sigma = R, \quad k = \omega n(\lambda z) \sin\theta, \quad (\phi - \phi') = \omega_0\tau, \quad (88)$$

which yields

$$J_0(2Y \sin\theta) = \sum_{m=-\infty}^{\infty} J_m^2[\omega n(\lambda z) R \sin\theta] e^{im\omega_0\tau}. \quad (89)$$

The next step is to rewrite the trigonometric functions depending upon $\omega_0\tau$ in Eq. (81) as exponentials and to perform the τ integration. In such a way we finally arrive at the expression for the observed time average of the power angular distribution, given by

$$\left\langle \frac{d^2 P(T)}{d\omega d\Omega} \right\rangle = \sum_{\lambda=\pm 1} \sum_{m=0}^{\infty} \delta(\omega - m\omega_0) \frac{dP_{m,\lambda}}{d\Omega}, \quad (90)$$

with

$$\begin{aligned} \frac{dP_{m,\lambda}}{d\Omega} = & \frac{\omega_m^2 q^2}{8\pi} \frac{n^2(\lambda z_m)}{\sqrt{1+(z_m)^2}} \{ \beta^2 [(1 + \lambda \cos\theta) J_{m-1}^2(W_{m\lambda}) \\ & + (1 - \lambda \cos\theta) J_{m+1}^2(W_{m\lambda}) \\ & - 2[n(\lambda z_m)]^{-2} J_m^2(W_{m\lambda}) \}, \end{aligned} \quad (91)$$

where $\omega_m = m\omega_0$, $z_m = \tilde{\xi}\omega_m$ and

$$W_{m\lambda} = \omega_m n(\lambda z_m) R \sin\theta. \quad (92)$$

Using the relation $\omega_0 R = \beta$ we can rewrite $W_{m\lambda}$ as

$$W_{m\lambda} = mn(\lambda z_m) \beta \sin\theta. \quad (93)$$

We notice that in the limit $\tilde{\xi} = 0$ the unpolarized angular distribution of the power obtained from Eq. (91) coincides with the standard result given in Eq. (38.37) of Ref. [45]. Let us observe that to first order in the LIV parameters we can take

$$\tilde{\xi}\omega_0 = \tilde{\xi} \frac{|q\mathbf{B}|}{\mu\gamma}, \quad (94)$$

according to Eqs. (11) and (19), since $\tilde{\xi}\tilde{\eta}$ is of second order in the correction parameters.

Using additional properties of the Bessel functions and following Ref. [45] we can recast Eq. (91) as

$$\begin{aligned} \frac{dP_{m\lambda}}{d\Omega} = & \frac{\omega_m^2 q^2}{4\pi} \frac{n^2(\lambda z_m)}{\sqrt{1+z_m^2}} \{ [\beta J'_m(W_{m\lambda})]^2 \\ & + [n^{-1}(\lambda z_m) \cot\theta J_m(W_{m\lambda})]^2 \} + \lambda \frac{\omega_m^2 q^2 \beta}{4\pi} \\ & \times \frac{n(\lambda z_m)}{\sqrt{1+z_m^2}} \cot\theta \frac{dJ_m^2(W_{m\lambda})}{dW_{m\lambda}}. \end{aligned} \quad (95)$$

It is interesting to remark that the above expression can be written as a perfect square

$$\begin{aligned} \frac{dP_{m\lambda}}{d\Omega} = & \frac{\omega_m^2 q^2}{4\pi} \frac{1}{\sqrt{1+z_m^2}} [\lambda \beta n(\lambda z_m) J'_m(W_{m\lambda}) \\ & + \cot\theta J_m(W_{m\lambda})]^2. \end{aligned} \quad (96)$$

A. Total spectral distribution

To calculate the integrated power spectrum it is convenient to start from Eqs. (82) and (86). Integrating over the solid angle Ω we have

$$\begin{aligned} \left\langle \frac{dP(T)}{d\omega} \right\rangle = & 2\pi \sum_{\lambda=\pm 1} \frac{\omega^2}{4\pi^2} \frac{n^2(\lambda z)}{\sqrt{1+z^2}} \\ & \times \int \sin\theta d\theta \langle J(T, \omega, \lambda, \mathbf{n}) \rangle, \end{aligned} \quad (97)$$

where the required integral is

$$\begin{aligned} \int \sin\theta d\theta \langle J(T, \omega, \lambda, \mathbf{n}) \rangle = & \frac{q^2}{2} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \{ [\beta^2 \cos\omega_0\tau \\ & - n^{-2}(\lambda z)] I_1 \\ & + \lambda \beta^2 (i \sin\omega_0\tau) I_2 \}. \end{aligned} \quad (98)$$

Here

$$I_1 = \int_0^\pi d\theta \sin\theta J_0(2Y \sin\theta) = \frac{\sin 2Y}{Y}, \quad (99)$$

$$I_2 = \int_0^\pi d\theta \sin\theta \cos\theta J_0(2Y \sin\theta) = 0, \quad (100)$$

where Y is given in Eq. (85). The last integral, related to parity violation, is zero by symmetry. Substituting in Eq. (97) we get

$$\left\langle \frac{dP(T)}{d\omega} \right\rangle = \frac{q^2}{4\pi R} \sum_{\lambda=\pm 1} \frac{\omega n(\lambda z)}{\sqrt{1+z^2}} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} f_\lambda(\omega_0\tau), \quad (101)$$

where

$$f_\lambda(\phi) = \frac{\sin[2\omega R n(\lambda z) \sin(\phi/2)]}{\sin(\phi/2)} (\beta^2 \cos\phi - n^{-2}(\lambda z)). \quad (102)$$

Following the method of Ref. [45] we introduce the Fourier decomposition of the periodic function $f_\lambda(\phi)$

$$\begin{aligned} f_\lambda(\phi) = & \sum_{m=-\infty}^{\infty} e^{im\phi} f_{\lambda m}, \\ f_{\lambda m} = & \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{-im\phi} f_\lambda(\phi), \end{aligned} \quad (103)$$

and performing the τ integration we have

$$\left\langle \frac{dP(T)}{d\omega} \right\rangle = \frac{q^2 \omega_0}{2R} \sum_{\lambda=\pm 1} \sum_{m=0}^{\infty} \delta(\omega - m\omega_0) \frac{mn(\lambda z)}{\sqrt{1+z^2}} f_{\lambda m}. \quad (104)$$

It only remains to compute $f_{\lambda m}$

$$\begin{aligned} f_{\lambda m} = & \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{-im\phi} \frac{\sin[2\omega n(\lambda z) R \sin(\phi/2)]}{\sin(\phi/2)} \\ & \times [\beta^2 \cos\phi - n^{-2}(\lambda z)]. \end{aligned} \quad (105)$$

The only contribution from the exponential comes from the symmetric part under $\phi \rightarrow -\phi$. After rewriting $\cos\phi$ in terms of the $\phi/2$ we are left with

$$\begin{aligned} f_{\lambda m} = & \int_0^\pi \frac{d\phi}{\pi} \cos m\phi \frac{\sin[2\omega n(\lambda z) R \sin(\phi/2)]}{\sin(\phi/2)} \\ & \times [\beta^2 - n^{-2}(\lambda z) - 2\beta^2 \sin^2(\phi/2)]. \end{aligned} \quad (106)$$

Using the following representations of the Bessel functions [45]

$$J_{2m}(x) = \int_0^\pi \frac{d\phi}{\pi} \cos m\phi \cos(x \sin(\phi/2)), \quad (107)$$

$$J'_{2m}(x) = - \int_0^\pi \frac{d\phi}{\pi} \sin(\phi/2) \cos m\phi \sin(x \sin(\phi/2)), \quad (108)$$

$$\int_0^x dy J_{2m}(y) = \int_0^\pi \frac{d\phi}{\pi} \cos m\phi \frac{\sin(x \sin(\phi/2))}{\sin(\phi/2)}, \quad (109)$$

with $x = 2R\omega n(\lambda z) = 2mR\omega_0 n(\lambda z)$, and recalling that $z = \tilde{\xi}\omega = \tilde{\xi}m\omega_0$ and $R\omega_0 = \beta$, we arrive at

$$f_{\lambda m} = 2\beta^2 J'_{2m}(2m\beta n(\lambda z)) - [n^{-2}(\lambda z) - \beta^2] \times \int_0^{2mn(\lambda z)\beta} dx J_{2m}(x). \quad (110)$$

Thus the total averaged and integrated power radiated in the m th harmonic is

$$P_m = \frac{q^2 m \omega_0}{2R\sqrt{1+z^2}} \left\{ n(z) \left[2\beta^2 J'_{2m}(2m\beta n(z)) - [n^{-2}(z) - \beta^2] \int_0^{2mn(z)\beta} dx J_{2m}(x) \right] + n(-z) \times \left[2\beta^2 J'_{2m}(2m\beta n(-z)) - [n^{-2}(-z) - \beta^2] \times \int_0^{2mn(-z)\beta} dx J_{2m}(x) \right] \right\}. \quad (111)$$

This result is exact in $z = \tilde{\xi}\omega$. Let us observe that each separate contribution in Eq. (111) has the form given in Eq. (2.22b) of Ref. [46], after the appropriate change $e^2/4\pi \rightarrow q^2$ is made, multiplied by the additional factor

$$\frac{1}{2} \frac{n(\lambda z)}{\sqrt{1+z^2}} = \frac{n(\lambda z)}{n(z) + n(-z)} = \frac{n^2(\lambda z)}{1 + n^2(\lambda z)}, \quad (112)$$

respectively. The calculation performed in this reference is for a constant index of refraction but, as mentioned in footnote 3, the corresponding results hold for a ω and H dependence of n and μ . At this stage, the additional factors (112) together with the specific form of the refraction indices are the remainders of that we are considering a different electrodynamics.

B. Polarization of the synchrotron radiation

In general, the electric field (51), which is naturally written in terms of the circular polarization basis \mathbf{e}_\pm introduced in Appendix B, will exhibit elliptic polarization. To describe the polarization we will use the Stokes parameters according to the definitions of Ref. [50]

$$I = |(\mathbf{e}_+^* \cdot \mathbf{E}(\omega, \mathbf{r}))|^2 + |(\mathbf{e}_-^* \cdot \mathbf{E}(\omega, \mathbf{r}))|^2, \quad (113)$$

$$V = |(\mathbf{e}_+^* \cdot \mathbf{E}(\omega, \mathbf{r}))|^2 - |(\mathbf{e}_-^* \cdot \mathbf{E}(\omega, \mathbf{r}))|^2, \quad (114)$$

$$Q = 2 \operatorname{Re}[(\mathbf{e}_+^* \cdot \mathbf{E}(\omega, \mathbf{r}))^*(\mathbf{e}_-^* \cdot \mathbf{E}(\omega, \mathbf{r}))], \quad (115)$$

$$U = 2 \operatorname{Im}[(\mathbf{e}_+^* \cdot \mathbf{E}(\omega_m, \mathbf{r}))^*(\mathbf{e}_-^* \cdot \mathbf{E}(\omega_m, \mathbf{r}))]. \quad (116)$$

It is convenient to define the reduced Stokes parameters by dividing each one by the intensity I in such a way that

$$v = \frac{V}{I}, \quad q = \frac{Q}{I}, \quad u = \frac{U}{I}, \quad (117)$$

satisfying the relation

$$1 = v^2 + q^2 + u^2. \quad (118)$$

As usual, purely circular polarization is described by $v = 1$, $q = u = 0$, while purely linear polarization corresponds to $v = 0$.

A first step in the description of the polarization is the explicit calculation of the electric field, which is given in terms of the currents $\mathbf{j}^+(\omega, \mathbf{k}_+)$ and $\mathbf{j}^-(\omega, \mathbf{k}_-)$. We need to calculate the time Fourier transform of the current (68) for each polarization λ and subsequently project into the circular basis. We start from

$$\mathbf{j}(t, \mathbf{k}_\lambda) = q\mathbf{v}(t)e^{-i\mathbf{k}_\lambda \cdot \mathbf{r}(t)} = q\beta(-\sin\omega_0 t, \cos\omega_0 t, 0)e^{-i\mathbf{k}_\lambda \cdot \mathbf{r}(t)}, \quad (119)$$

where we can write

$$e^{-i\mathbf{k}_\lambda \cdot \mathbf{r}(t)} = e^{-i(\omega/\omega_0)n(\lambda z)\beta \sin\theta \cos\omega_0 t}. \quad (120)$$

The Fourier transform is

$$\mathbf{j}(\omega, \mathbf{k}_\lambda) = \int_{-\infty}^{\infty} dt e^{i\omega t} \mathbf{j}(t, \mathbf{k}_\lambda) = q\beta \int_{-\infty}^{\infty} dt e^{i\omega t} (-\sin\omega_0 t, \cos\omega_0 t, 0) e^{-iW_\lambda \cos\omega_0 t}, \quad (121)$$

with $W_\lambda = (\omega n(\lambda z)\beta \sin\theta)/\omega_0$. Using the generating function for the Bessel functions of integer order (see, for example, Eq. (38.27) in Ref. [45]) we obtain

$$e^{-iW_\lambda \cos\omega_0 t} = \sum_{m=-\infty}^{\infty} (-i)^m e^{-im\omega_0 t} J_m(W_\lambda), \quad (122)$$

and in this way we have

$$\mathbf{j}(\omega, \mathbf{k}_\lambda) = q\beta \sum_{m=-\infty}^{\infty} (-i)^m J_m(W_\lambda) \times \int_{-\infty}^{\infty} dt e^{it(\omega - m\omega_0)} (-\sin\omega_0 t, \cos\omega_0 t, 0) dt. \quad (123)$$

Now we can perform the integration with respect to t . The final result can be written in terms of each mode contributing to the current

$$\mathbf{j}(\omega, \mathbf{k}_\lambda) = \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_0) \mathbf{j}_m(\omega_m, \mathbf{k}_\lambda), \quad (124)$$

where

$$\begin{aligned} \mathbf{j}_m(\omega_m, \mathbf{k}_\lambda) &= 2\pi(-i)^m q\beta \\ &\times \left[\frac{m}{W_{\lambda m}} J_m(W_{m\lambda}), i \frac{d}{dW_{\lambda m}} J_m(W_{m\lambda}), 0 \right], \end{aligned} \quad (125)$$

with $W_{m\lambda}$ given by Eq. (92).

The projection $\mathbf{j}_m^\pm(\omega_m, \mathbf{k}_\lambda)$ of the currents into the circular basis is given by Eqs. (A10) of the appendix. Using these relations together with the definitions (51) we arrive at the final expressions for the circularly polarized electric fields

$$\begin{aligned} \mathbf{E}_m^+(\omega_m, \mathbf{k}_+) &= -\sqrt{2}\pi(-i)^m \frac{q\beta}{r} \frac{\omega_m n(z_m)}{\sqrt{1+z_m^2}} e^{in(z_m)\omega_m r} \\ &\times (B_m^+ + A_m^+ \cos\theta) \mathbf{e}_+, \end{aligned} \quad (126)$$

$$\begin{aligned} \mathbf{E}_m^-(\omega_m, \mathbf{k}_-) &= -\sqrt{2}\pi(-i)^m \frac{q\beta}{r} \frac{\omega_m n(-z_m)}{\sqrt{1+z_m^2}} e^{in(-z_m)\omega_m r} \\ &\times (B_m^- - A_m^- \cos\theta) \mathbf{e}_-, \end{aligned} \quad (127)$$

$$\mathbf{E}_m(\omega_m, \omega_m \mathbf{n}) = 2\pi(-i)^m q\omega_m \frac{e^{i\omega_m r}}{r} \left[\frac{\cos^2\theta}{\sin\theta} J_m(W_{m\lambda}), i\beta \frac{dJ_m(W_{m\lambda})}{dW_{m\lambda}}, -\cos\theta J_m(W_{m\lambda}) \right], \quad (131)$$

which is clearly orthogonal to $\hat{\mathbf{n}}$ and parallel to the corresponding electric field given in Eq. (6.65) of Ref. [51] for the case $\beta_{\parallel} = 0$.

Next we calculate the Stokes parameters, Eqs. (116), starting from Eqs. (127). The required projections are

$$\begin{aligned} (\mathbf{e}_+^* \cdot \mathbf{E}_m(\omega_m, \mathbf{r})) &= -\sqrt{2}\pi(-i)^m \frac{q}{r} \frac{\omega_m n(z_m)}{\sqrt{1+z_m^2}} e^{in(z_m)\omega r} \\ &\times (B_m^+ + D_m^+), \end{aligned} \quad (132)$$

$$\begin{aligned} (\mathbf{e}_-^* \cdot \mathbf{E}_m(\omega_m, \mathbf{r})) &= -\sqrt{2}\pi(-i)^m \frac{q}{r} \\ &\times \frac{\omega_m n(-z_m)}{\sqrt{1+z_m^2}} e^{in(-z_m)\omega r} (B_m^- - D_m^-), \end{aligned} \quad (133)$$

and we also need

$$\begin{aligned} &(\mathbf{e}_+^* \cdot \mathbf{E}_m(\omega_m, \mathbf{r}))^* (\mathbf{e}_-^* \cdot \mathbf{E}_m(\omega_m, \mathbf{r})) \\ &= 2 \left(\frac{\pi q \omega_m}{r} \right)^2 \frac{n(z_m)n(-z_m)}{1+z_m^2} e^{i[n(-z_m)-n(z_m)]\omega r} \\ &\times (B_m^+ + D_m^+) (B_m^- - D_m^-). \end{aligned} \quad (134)$$

In this way we arrive at the reduced combinations

where the notation is

$$\mathbf{j}_m(\omega_m, \mathbf{k}_\lambda) = 2\pi(-i)^m q [A_m^\lambda, iB_m^\lambda, 0], \quad (128)$$

with

$$A_m^\lambda = \beta \frac{m}{W_{\lambda m}} J_m(W_{m\lambda}), \quad B_m^\lambda = \beta \frac{dJ_m(W_{m\lambda})}{dW_{m\lambda}}. \quad (129)$$

The combination $A^\lambda \cos\theta$ reduces to

$$A_m^\lambda \cos\theta = D_m^\lambda = \frac{J_m(W_{m\lambda})}{\tan\theta}. \quad (130)$$

In the standard situation, where $\tilde{\xi} = 0$, the above expressions produce the result

$$\begin{aligned} v_m &= \frac{1 - R_m^2}{1 + R_m^2}, \\ q_m &= \frac{2R_m}{1 + R_m^2} \cos\{[n(-z_m) - n(z_m)]\omega_m r\}, \\ u_m &= \frac{2R_m}{1 + R_m^2} \sin\{[n(-z_m) - n(z_m)]\omega_m r\}, \end{aligned} \quad (135)$$

which completely characterize the polarization of the wave in terms of the polarization index

$$R_m = \frac{n(-z_m) (B_m^- - D_m^-)}{n(z_m) (B_m^+ + D_m^+)}. \quad (136)$$

From the above relations the verification of the condition (118) is direct. The purely circular polarization corresponds to $R_m = 0$, while the purely linear polarization is given by $R_m^2 = 1$. Substituting the corresponding expressions for B_m^\pm and D_m^\pm we obtain the general expression for the polarization index

$$R_m = \frac{n(-z_m) (\beta \sin\theta J_m'(W_{m-}) - \cos\theta J_m(W_{m-}))}{n(z_m) (\beta \sin\theta J_m'(W_{m+}) + \cos\theta J_m(W_{m+}))}. \quad (137)$$

In the limit $\xi = 0$ we have $B_m^\pm = B_m$ and $D_m^\pm = D_m$ and Eq. (136) reduces to

$$R_m = \frac{B_m - D_m}{B_m + D_m} = \frac{1 - \frac{D_m}{B_m}}{1 + \frac{D_m}{B_m}}. \quad (138)$$

The additional definition

$$r_m = \frac{D_m}{B_m} = \frac{J_m(m\beta \sin\theta)}{\beta \tan\theta J'_m(m\beta \sin\theta)} \quad (139)$$

leads to

$$v_m = \frac{2r_m}{1+r_m^2}, \quad q_m = \frac{1-r_m^2}{1+r_m^2}, \quad u_m = 0, \quad (140)$$

which coincides, up to the sign of r_m , with the parameters introduced in Ref. [51].

VI. DOMINATING EFFECTS OF THE LORENTZ VIOLATION IN PHOTON DYNAMICS

Several characteristics of the synchrotron radiation are affected by the Lorentz-violating parameter $\tilde{\xi}$. From the phenomenological point of view the more relevant are the power angular distribution, the total power, and the polarization of the radiation. In what follows we discuss these effects.

A. Angular distribution of the power radiated in the unpolarized m th harmonic

In this subsection we expand the distribution (95) to first order in $\tilde{\xi}$. Notice that its general structure is given by

$$\frac{d\hat{P}_{m,\lambda}}{d\Omega} = F_m(\lambda\tilde{\xi}) + \lambda G_m(\lambda\tilde{\xi}), \quad (141)$$

and thus we get

$$\frac{dP_{m,\lambda}}{d\Omega} = F_m(0) + \tilde{\xi}G'_m(0) + \lambda[\tilde{\xi}F'_m(0) + G_m(0)], \quad (142)$$

where the derivatives are with respect to the argument $\lambda\tilde{\xi}$ of the functions. When we sum over polarizations it is evident that the term linear in λ vanishes, but nevertheless there remains a contribution proportional to $\tilde{\xi}$. Here we identify the unpolarized average angular distribution of the power radiated into the m th harmonic

$$\frac{dP_m}{d\Omega} = \frac{dP_{m+}}{d\Omega} + \frac{dP_{m-}}{d\Omega} = 2[F_m(0) + \tilde{\xi}G'_m(0)]. \quad (143)$$

From Eq. (95), and eliminating the second derivative of the Bessel function via its differential equation we get

$$F_m(0) = \frac{(q\omega_m)^2}{4\pi} \{[\beta J'_m(W_m)]^2 + [\cot\theta J_m(W_m)]^2\}, \quad (144)$$

$$G'_m(0) = \frac{(q\omega_m)^2 m\omega_m}{2\pi} \cos\theta \left\{ [\beta J'_m(W_m)]^2 + (1 - \beta^2 \sin^2\theta) \times \left[\frac{J_m(W_m)}{\sin\theta} \right]^2 \right\}, \quad (145)$$

where

$$W_m = W_{m\lambda}|_{\tilde{\xi}=0} = m\beta \sin\theta. \quad (146)$$

After some rearrangement and use of trigonometric identities, we obtain from Eq. (143) our final result

$$\begin{aligned} \frac{dP_m}{d\Omega} = & \frac{(q\omega_m)^2}{2\pi} (1 + 2(m\tilde{\xi})\omega_m \cos\theta) \left\{ [\beta J'_m(W_m)]^2 \right. \\ & \left. + \left[\frac{J_m(W_m)}{\tan\theta} \right]^2 \right\} + \frac{(m\tilde{\xi})(q\omega_m)^2 \omega_m}{\pi\gamma^2} \cos\theta [J_m(W_m)]^2, \end{aligned} \quad (147)$$

in terms of the observed frequency $\omega_m = m\omega_0$. This distribution shows a parity violation contribution proportional to $\cos\theta$. The standard result for the angular distribution of the spectral power given in Ref. [45] is directly recovered when $\tilde{\xi} = 0$. Let us observe that the expansion parameter $\tilde{\xi}$ appears amplified by the factor m , which turns out to be very large in some astrophysical settings, as we will discuss in Sec. VIII.

B. Total power radiated in the m th harmonic

In the previous subsection we computed the unpolarized angular distribution of the averaged power radiated into the m th harmonic, up to first order in $\tilde{\xi}$. Such linear terms vanish when we integrate on the whole solid angle, as it is evident by symmetry arguments. In this subsection we are interested in the leading contribution to the unpolarized (angular) integrated spectral averaged power distribution, which is of second order in $z = \tilde{\xi}\omega$. We expect that such second order contribution will still be amplified by factors having powers of m . In order to perform the expansion let us introduce the notation

$$\begin{aligned} g(z) = & \frac{n(z)}{\sqrt{1+z^2}} \left\{ 2\beta^2 J'_{2m}[2m\beta n(z)] - [n^{-2}(z) - \beta^2] \right. \\ & \left. \times \int_0^{2mn(z)\beta} dx J_{2m}(x) \right\}, \end{aligned} \quad (148)$$

in such a way that we can write

$$P_m = \frac{q^2 m\omega_0}{2R} [g(z_m) + g(-z_m)] = P_{m,+} + P_{m,-}. \quad (149)$$

Expanding in powers of z we have

$$P_m = \frac{q^2 m\omega_0}{2R} (2g(0) + z_m^2 g''(0)). \quad (150)$$

The term $g(0)$ is readily obtained as

$$g(0) = \left\{ 2\beta^2 J'_{2m}(2m\beta) - \gamma^{-2} \int_0^{2m\beta} dx J_{2m}(x) \right\}. \quad (151)$$

The computation of $g''(0)$ is more laborious. It can be written as

$$\frac{d^2 g(z)}{dz^2} \Big|_{z=0} = \frac{dg(n)}{dn} \Big|_{n=1} + \frac{d^2 g(n)}{dn^2} \Big|_{n=1}. \quad (152)$$

Thus we need to calculate the first and second derivatives of $g(n)$ evaluated at $n = 1$. Our results are

$$\left. \frac{dg}{dn} \right|_{n=1} = (1 + \beta^2) \int_0^{2m\beta} dx J_{2m}(x) + 2m\beta(1 - \beta^2) J_{2m}(2m\beta), \quad (153)$$

$$\left. \frac{d^2g}{dn^2} \right|_{n=1} = -2\beta^2 J'_{2m}(2m\beta) - (1 + \beta^2) \int_0^{2m\beta} dx J_{2m}(x) + \left(\frac{2m\beta}{\gamma} \right)^2 J'_{2m}(2m\beta), \quad (154)$$

which lead to

$$\left. \frac{d^2g(z)}{dz^2} \right|_{z=0} = 2m\beta\gamma^{-2} J_{2m}(2m\beta) + (4m^2\gamma^{-2} - 2)\beta^2 J'_{2m}(2m\beta). \quad (155)$$

Going back to Eq. (150) giving the total power in the m th harmonic, we obtain our final result

$$P_m = \frac{q^2 \omega_m}{R} \left\{ 2\beta^2 J'_{2m}(2m\beta) - \gamma^{-2} \int_0^{2m\beta} dx J_{2m}(x) + (\tilde{\xi} \omega_m)^2 \beta [\gamma^{-2} [m J_{2m}(2m\beta) + 2m^2 \beta J'_{2m}(2m\beta)] - \beta J'_{2m}(2m\beta)] \right\}. \quad (156)$$

When $\tilde{\xi} = 0$ we recover the well-established result for the total power radiated in the m th harmonic [45]. Again we notice the dominant amplifying factor m^2 in the fourth term in the right-hand side of Eq. (156).

C. Polarization index

Now we make an expansion to first order in $\tilde{\xi}$ of the polarization index in Eq. (137). To this order

$$W_{m\lambda} \simeq (1 + \lambda \tilde{\xi} \omega) m \beta \sin \theta, \quad (157)$$

together with

$$J_m(W_{m\lambda}) \simeq J_m(m\beta \sin \theta) + \lambda \tilde{\xi} \omega m \beta \sin \theta J'_m(m\beta \sin \theta), \quad (158)$$

$$J'_m(W_{m\lambda}) \simeq J'_m(m\beta \sin \theta) - \lambda \tilde{\xi} \omega \left[J'_m(m\beta \sin \theta) + m \frac{\beta^2 \sin^2 \theta - 1}{\beta \sin \theta} J_m(m\beta \sin \theta) \right], \quad (159)$$

where we have substituted the second derivative of the Bessel function arising in Eq. (159) via its differential equation. Substituting in Eq. (137) we obtain

$$R_m = \left(\frac{1 - \tilde{\xi} \omega}{1 + \tilde{\xi} \omega} \right) \frac{\beta J'_m(W_{m-}) - \cot \theta J_m(W_{m-})}{\beta J'_m(W_{m+}) + \cot \theta J_m(W_{m+})}. \quad (160)$$

The basic building blocks can be rewritten as

$$\beta J'_m(W_{m-}) - \cot \theta J_m(W_{m-}) = P_- + \tilde{\xi} \omega Q_-, \quad (161)$$

$$\beta J'_m(W_{m+}) + \cot \theta J_m(W_{m+}) = P_+ - \tilde{\xi} \omega Q_+, \quad (162)$$

where

$$P_- = \beta J'_m(m\beta \sin \theta) - \cot \theta J_m(m\beta \sin \theta), \quad (163)$$

$$Q_- = \beta(1 + m \cos \theta) J'_m(m\beta \sin \theta) + m \frac{\beta^2 \sin^2 \theta - 1}{\sin \theta} J_m(m\beta \sin \theta), \quad (164)$$

$$P_+ = \beta J'_m(m\beta \sin \theta) + \cot \theta J_m(m\beta \sin \theta), \quad (165)$$

$$Q_+ = \beta(1 - m \cos \theta) J'_m(m\beta \sin \theta) + m \frac{\beta^2 \sin^2 \theta - 1}{\sin \theta} J_m(m\beta \sin \theta). \quad (166)$$

In terms of the above quantities, and to first order in $\tilde{\xi}$, we arrive at

$$R_m = \tilde{R}_m \left[1 - \tilde{\xi} \omega \left(2 - \frac{Q_-}{P_-} - \frac{Q_+}{P_+} \right) \right]. \quad (167)$$

Here we have introduced

$$\tilde{R}_m = \frac{P_-}{P_+} = \frac{\beta J'_m(m\beta \sin \theta) - \cot \theta J_m(m\beta \sin \theta)}{\beta J'_m(m\beta \sin \theta) + \cot \theta J_m(m\beta \sin \theta)}, \quad (168)$$

which corresponds to the polarization index in the case $\tilde{\xi} = 0$. Using Eqs. (163)–(166) we can evaluate the first order correction in $\tilde{\xi}$ to R_m

$$2 - \frac{Q_-}{P_-} - \frac{Q_+}{P_+} = \frac{2}{\sin^2 \theta} \frac{\cos^2 \theta J_m(m\beta \sin \theta) - m\beta \gamma^{-2} \sin^3 \theta J'_m(m\beta \sin \theta)}{[\cot \theta J_m(m\beta \sin \theta)]^2 - [\beta J'_m(m\beta \sin \theta)]^2}. \quad (169)$$

The polarization index determines the Stokes parameters, given by Eqs. (135). Using Eqs. (160), (168), and (169), we get in particular

$$u_m = -\tilde{\xi} \omega \frac{4\tilde{R}_m}{1 + \tilde{R}_m^2} = -2\tilde{\xi} \omega \frac{[\beta \sin \theta J'_m(m\beta \sin \theta)]^2 - [\cos \theta J_m(m\beta \sin \theta)]^2}{[\beta \sin \theta J'_m(m\beta \sin \theta)]^2 + [\cos \theta J_m(m\beta \sin \theta)]^2}, \quad (170)$$

which gives a neat signature of the Lorentz violation.

VII. CONTRIBUTIONS FROM DIMENSION SIX OPERATORS

Up to this point we have considered the Myers-Pospelov model as an exact one, which allows us to make quantitative predictions of the ensuing effects. The inclusion of higher dimension LIV operators in the action will certainly modify the results obtained for the higher order contributions in the corresponding parameters, i.e. $\tilde{\xi}^{n+1}$ and $\tilde{\eta}^{n+1}$ with $n > 0$. The investigation of additional dimension six operators is an interesting problem of its own, but it is beyond the scope of this paper and we do not consider it in any detail here. Nevertheless, it is important to emphasize that the contributions of these operators do not produce birefringent effects, and thus this type of effects by dimension five operators can be clearly disentangled from dimension six contributions by polarization measurements, for example, as indicated in Eq. (170).

We can perform a rough estimation of the induced effects due to the addition of dimension six operators in the Maxwell action for the total power radiated into the m th harmonic, on the basis of the following considerations. In the first place, since the propagation of the electromagnetic field is basically characterized by the refraction index, the addition of dimension six operators leads to a modification dominated by a term proportional to ω^2

$$n(\lambda z) \rightarrow \bar{n}(\lambda z) = n(\lambda z) + \tilde{\xi}_2 \omega^2, \quad (171)$$

where $\tilde{\xi}_2 = \xi_2/M^2$ is the parameter associated to dimension six operators, and $n(\lambda z)$ is given by Eq. (44). The total radiated power into the m th harmonic remains given by Eq. (149), but with modified $g(\lambda z_m)$ functions

$$g(\lambda z_m) \rightarrow \bar{g}(\lambda z_m) = g(\bar{n}(\lambda z_m))F(\lambda \omega_m), \quad (172)$$

where the real function $F(\lambda \omega_m) = F(\lambda \tilde{\xi} \omega_m, \tilde{\xi}_2 \omega_m^2)$ depends upon m only through ω_m , and satisfies $F(\tilde{\xi} = \tilde{\xi}_2 = 0) = 1$. Thus we have

$$P_m = \frac{q^2 \omega_m}{2R} [g(\bar{n}(z_m))F(\omega_m) + g(\bar{n}(-z_m))F(-\omega_m)], \quad (173)$$

instead of Eq. (149). The corresponding expansions are now

$$F(\lambda \omega_m) = 1 + A\lambda \tilde{\xi} \omega_m + (B\tilde{\xi}^2 + C\tilde{\xi}_2)\omega_m^2 + \dots, \quad (174)$$

where we assume that A , B , and C are of order one together with

$$g(\bar{n}(z_m)) = g(z_m) + \tilde{\xi}_2 \omega_m^2 \left[\frac{dg(n)}{dn} \right]_{\tilde{\xi}_2=0} + O(M^{-3}). \quad (175)$$

Since we already have a factor $\tilde{\xi}_2 \propto M^{-2}$ in front of the above derivative we can further set there $\tilde{\xi} = 0$, which means that it is evaluated at $n = 1$, and thus

$$g(\bar{n}(z_m)) = g(z_m) + \tilde{\xi}_2 \omega_m^2 \left[\frac{dg(n)}{dn} \right]_{n=1} + O(M^{-3}). \quad (176)$$

Replacing in Eq. (173) and recalling that $g(z) \simeq g(0) + zg'(0) + \frac{1}{2}z^2g''(0)$, together with $g'(0) = [dg(n)/dn]_{n=1}$, we finally obtain for the total radiated power

$$P_m = \frac{q^2 \omega_m}{2R} \left\{ 2g(0)[1 + (\tilde{\xi}^2 + \tilde{\xi}_2)\omega_m^2] + \left[\tilde{\xi}^2 g''(0) + 2(2\tilde{\xi}^2 + \tilde{\xi}_2) \left(\frac{dg(n)}{dn} \right)_{n=1} \right] \omega_m^2 \right\}. \quad (177)$$

The factor of $g(0)$ in the right-hand side shows corrections of order ω_m^2/M^2 that we can safely neglect. The second square bracket also contains contributions arising from dimension five and six operators. Using Eqs. (151), (153), and (155) the corresponding magnitudes can be written as

$$X_{m5} = 4\omega^2 \tilde{\xi}^2 \left| \left[m^2 \gamma^{-2} J'_{2m}(2m\beta) + \int_0^{2m\beta} dx J_{2m}(x) \right] \right|, \quad (178)$$

$$X_{m6} = 4\omega^2 \left| \tilde{\xi}_2 \left[\int_0^{2m\beta} dx J_{2m}(x) + m\gamma^{-2} J_{2m}(2m\beta) \right] \right|. \quad (179)$$

Using the corresponding expressions for large m given in Eqs. (B2), and the further approximation (B6) for the case $m/m_c \lesssim 1$ we obtain

$$\frac{X_{m6}}{X_{m5}} = \frac{|\tilde{\xi}_2| [1 + 0.36m^{2/3}\gamma^{-2}]}{\tilde{\xi}^2 [1 + 0.26m^{4/3}\gamma^{-2}]}. \quad (180)$$

Thus, the effects of dimension six operators are negligible in the whole range of the spectrum if $|\tilde{\xi}_2|/\tilde{\xi}^2 < 1$. On the other hand, if $|\tilde{\xi}_2|/\tilde{\xi}^2 \gtrsim \gamma^2$ they are significant even in the high energy limit, near the cutoff frequency where $m = m_c = 3\gamma^3/2$. At the moment there is not enough experimental evidence to set this point. For example, Ref. [39] gives the boundaries $|\tilde{\xi}| \lesssim 10^{-12}$ and $-10^{-6} \lesssim \tilde{\xi}_2 \lesssim 10^{-4}$. In the following we restrict ourselves to the Myers-Pospelov model with no admixture of dimension six operators.

VIII. HIGH ENERGY LIMIT

In Table I we present a rough estimation of the relevant parameters associated to some observed cosmological objects corresponding to SNR and GRB. There $r[\text{ly}]$ is the distance of the object to the earth, γ is the Lorentz factor of the charged particles, $B[\text{Gauss}]$ is the average magnetic field producing the synchrotron radiation, $\omega_c[\text{GeV}]$ is the

TABLE I. Typical parameters of some relevant astrophysical objects.

	r	γ	B	ω_c	ω_0	m	m/γ
CRAB	10^4	10^9	10^{-3}	10^{-1}	10^{-30}	10^{29}	10^{20}
(Mkn 501) _p	10^8	10^{11}	10^2	10^4	10^{-29}	10^{33}	10^{22}
(Mkn 501) _e	10^8	10^{11}	10^{-1}	10^4	10^{-29}	10^{33}	10^{22}
GRB021206	10^{10}	10^5	10^4	10^{-3}	10^{-18}	10^{15}	10^{10}

cutoff frequency, and ω_0 [GeV] is the Larmor frequency. In all cases the cutoff frequency ω_c has been estimated from the radiation spectrum fitted by a synchrotron model in the corresponding reference. This, together with the magnetic field B allows us to estimate the Lorentz factor

$$\gamma = 2.36 \times 10^8 \sqrt{\frac{\omega_c[\text{GeV}]}{B[\text{Gauss}]} \frac{M}{m_e}}, \quad (181)$$

where M is the mass of the charged particle. From the above we further obtain the Larmor frequency

$$\omega_0[\text{GeV}] = 0.6 \times 10^{-17} \left(\frac{2m_e}{\gamma M} \right) B[\text{Gauss}]. \quad (182)$$

In the case of Crab Nebula we adopt the typical values given in Ref. [52]. For Mkn 501 we consider two possible models for synchrotron radiation where the emitter particles are either protons [53] or electrons [54]. In the latter case we use the radius of the orbit $r' = 1/\omega_0 = 1.5 \times 10^{15}$ cm and the magnetic field to obtain the Lorentz factor. Finally we consider the GRB021206 ($z \approx 1.25$). According to Ref. [55] this object has a very compact core with a radius of the order of 1 km and a magnetic field $\approx 10^{12}$ Gauss. The synchrotron emission region is about 10^8 km from the core [56], so that we estimate the magnetic field to be 10^4 Gauss using the transport law $B \times r = \text{const}$. From Ref. [57] we take the cutoff frequency as $\omega_c = 1$ MeV.

As indicated in Table I, the radiation of interest is dominated by very high harmonics $10^{15} \leq m \leq 10^{30}$ exhibiting also high ratios of m/γ , typically in the range $10^{10} \leq m/\gamma \leq 10^{22}$. The corresponding values for γ imply also $\beta \approx 1$. In this way, the high harmonics present in the synchrotron radiation of these astrophysical sources and the γ factor of the corresponding radiating charges highlight the relevance of the large m and γ limit with the constraint $(m/\gamma)^2 \gg 1$ to study Lorentz-violating effects. In the following we present the most interesting results of this limit, from the phenomenological point of view.

A. Large m approximation

We start from Eq. (111) by recalling the expressions for each helicity contribution to the integrated power spectrum

$$P_{m\lambda} = \frac{q^2 \omega_m}{R} \frac{1}{1 + n^2(\lambda z_m)} \left\{ 2\beta^2 n^2(\lambda z_m) J'_{2m} [2m\beta n(\lambda z_m)] - [1 - \beta^2 n^2(\lambda z_m)] \int_0^{2mn(\lambda z_m)\beta} dx J_{2m}(x) \right\}, \quad (183)$$

where $n(\lambda z)$ are given in Eq. (44). According to Ref. [46], the important point here is the presence of two different regimes of what is called synergic synchrotron-Cerenkov radiation, depending upon $(1 - \beta^2 n^2(\lambda z_m)) \leq 0$. Using expressions (16) and (44), this basic combination can be written in terms of the electron E and photon ω_m energies as

$$1 - \beta^2 n^2 = (\mu^2/E^2)(1 + 2\omega_m^2 \tilde{\xi}^2) - 2\omega_m^2 \tilde{\xi}^2 + 2\tilde{\eta}E - \frac{15}{4} \tilde{\eta}^2 E^2 - 2\lambda \omega_m \tilde{\xi} [1 - 2\tilde{\eta}E - (\mu^2/E^2)] + O(\tilde{\eta}^2 \tilde{\xi}) + O(\tilde{\eta}^3) + O(\tilde{\xi}^3) + O(\tilde{\eta} \tilde{\xi}^2), \quad (184)$$

which contains the correction terms in which we are interested.

Moreover, in our case we have

$$\beta^2 n^2(z) > \beta^2 n^2(-z), \quad (185)$$

so that we have three different possibilities for the total power $P_m = P_{m+} + P_{m-}$: (i) $n^2(\lambda z)\beta^2 < 1$ for $\lambda = \pm 1$, (ii) $n^2(\lambda z)\beta^2 > 1$ for $\lambda = \pm 1$, and (iii) $n^2(+z)\beta^2 > 1$ and $n^2(-z)\beta^2 < 1$. They are combinations of two possible situations. Let us recall the corresponding expressions of the integrated power spectrum (183) for these basic situations.

1. Case $1 > \beta^2 n^2(\lambda z)$

Introducing the notation

$$1 - n^2(\lambda z_m)\beta^2 = \left(\frac{3}{2\tilde{m}_{\lambda c}} \right)^{2/3}, \quad (186)$$

together with the large m expressions given in Eq. (B2) of Appendix B we obtain

$$P_{m\lambda} = \frac{q^2 \omega_m}{\sqrt{3}\pi R(1 + n^2(\lambda z_m))} \left(\frac{3}{2\tilde{m}_{\lambda c}} \right)^{2/3} \left[\int_{m/\tilde{m}_{\lambda c}}^{\infty} dx K_{5/3}(x) - 2 \left(\frac{3}{2\tilde{m}_{\lambda c}} \right)^{2/3} K_{2/3}(m/\tilde{m}_{\lambda c}) \right], \quad (187)$$

where the required values of $n = n(\lambda z_m)$ should be substituted in the sequel. Using the bremsstrahlung function defined in (B4) and the asymptotic limits given in Eqs. (B5)–(B7), we obtain

$$P_{m\lambda} = \frac{q^2 \omega_0 m^{1/2}}{2R\sqrt{\pi}(1 + n^2)} \left(\frac{3}{2\tilde{m}_{\lambda c}} \right)^{1/6} \times \left[1 - 2 \left(\frac{3}{2\tilde{m}_{\lambda c}} \right)^{2/3} \right] e^{-m/\tilde{m}_{\lambda c}}, \quad (188)$$

for $m/\tilde{m}_{\lambda c} \gg 1$, and

$$P_{m\lambda} = \frac{q^2 \omega_0 (3)^{1/6} \Gamma(2/3)}{\pi R (1+n^2)} \left[1 - \left(\frac{3}{2\tilde{m}_{\lambda c}} \right)^{2/3} \right] m^{1/3}, \quad (189)$$

for $m/\tilde{m}_{\lambda c} \ll 1$. These expressions are similar to the standard ones up to the factor $1/(1+n^2)$ which might contribute to the corrections in the LIV parameters. Equation (188) exhibits $\tilde{m}_{\lambda c}$ as the cutoff harmonic, which also depends on the LIV parameters.

2. Case $1 < \beta^2 n^2 (\lambda z)$

Using the results of Eqs. (B9) in the appendix we obtain the expression analogous to Eq. (187)

$$P_m(n) = \frac{q^2 m \omega_0 (n^2 \beta^2 - 1)}{R (1+n^2)} \times \left[\Lambda(x) - 2(n^2 \beta^2 - 1) \frac{Ai'(-x)}{x} \right]. \quad (190)$$

As shown in [46] the expressions (187) and (190) are continuous in $n\beta = 1$. The Airy functions of negative argument involved in this sector are oscillating functions in the large x limit.

B. Angular distribution of the m th harmonic to first order in ξ

To compute the unpolarized average angular distribution of the power radiated into the m th harmonic in the high energy limit we can start from Eq. (147). By performing the required expansions we get

$$\frac{dP_m}{d\Omega} = \frac{\omega_m^2 q^2}{2\pi} (1 - m\tilde{\xi}\omega_m \cos\theta) \{ [J'_m(m\beta \sin\theta)]^2 + \cot^2\theta [J_m(m\beta \sin\theta)]^2 \}. \quad (191)$$

This expression shows an anisotropy, which is an effect of first order in $\tilde{\xi}$. The emission is suppressed for $m\tilde{\xi}\omega_m \cos\theta \approx 1$ and enhanced for $m\tilde{\xi}\omega_m \cos\theta \approx -1$. Note that both effects are amplified by the presence of factor $m = \omega/\omega_0$ which can be very large in some regimes. When m is large, and since we have already expanded around $\tilde{\xi}$, we can use the standard result that the radiation becomes confined to a small angular range $\Delta\theta \approx m^{-1/3}$ around $\theta = \pi/2$ [45]. Thus, if we consider the radiation in the frontiers of the beam, i.e. $\theta \approx \pi/2 \pm m^{-1/3}$, the anisotropy becomes significant when

$$\omega_m \approx (\omega_0^2 / \tilde{\xi}^3)^{1/5}. \quad (192)$$

C. Total power radiated into the m th harmonic for large m to second order in $\tilde{\xi}$

The total power radiated into the m th harmonic in the large m limit can be computed directly from Eq. (156), which gives

$$P_m \approx \frac{q^2 \omega_m}{R} \left\{ 2\beta^2 J'_{2m}(2m\beta) - \gamma^{-2} \int_0^{2m\beta} dx J_{2m}(x) + 2\tilde{\xi}^2 \left(\frac{m\omega_m \beta}{\gamma} \right)^2 J'_{2m}(2m\beta) \right\}, \quad (193)$$

where we only keep the dominant term proportional to m^2 in the fourth term of the right-hand side in Eq. (156). Substituting in Eq. (193) the asymptotic representations (B2) for the Bessel functions for $n = 1$, with m_c being the corresponding critical value

$$m_c = \frac{3}{2} \gamma^3, \quad (194)$$

we get

$$P_m \approx \frac{q^2 \omega_m}{\sqrt{3} \pi R \gamma^2} \left[\frac{m_c}{m} \kappa\left(\frac{m}{m_c}\right) - \frac{2}{\gamma^2} K_{2/3}\left(\frac{m}{m_c}\right) + 2\tilde{\xi}^2 \left(\frac{m\omega_m \beta}{\gamma} \right)^2 K_{2/3}\left(\frac{m}{m_c}\right) \right]. \quad (195)$$

For large m and $(1 - \beta^2) > 0$ the behavior of the power radiated in the m th harmonic can be obtained using the asymptotic expressions (B5)–(B7). For the case $m/m_c \gg 1$ and $\gamma \gg 1$ the result is

$$P_m \approx \frac{q^2 \omega_0}{2\sqrt{\pi} R} \left(\frac{m}{\gamma} \right)^{1/2} \left[1 - \frac{2}{\gamma^2} + 2\tilde{\xi}^2 \left(\frac{m\omega_m \beta}{\gamma} \right)^2 \right] e^{-2m/3\gamma^3}. \quad (196)$$

Since the regime of interest has $m/\gamma \gg 1$, we keep the contribution from the LIV part that is amplified by the factor m^2/γ^2 , obtaining

$$P_m \approx \frac{1}{2} \left(\frac{m}{\pi\gamma} \right)^{1/2} \left(\frac{q^2 B}{E} \right)^2 \left(1 - 3\tilde{\eta}E + \frac{27}{4} \tilde{\eta}^2 E^2 \right) \times \left[1 + 2\tilde{\xi}^2 \left(\frac{m\omega_m}{\gamma} \right)^2 \right] e^{-2m/3\gamma^3}, \quad (197)$$

where we have rewritten ω_0/R in terms of the electron energy E using (15) together with (11). We also set $\beta = 1$ in all factors where no divergence arises.

The complementary range $m/m_c \ll 1$ yields

$$P_m \approx \frac{\sqrt{9}\Gamma(2/3)}{\sqrt{3}\pi} \left(\frac{q^2 B}{E} \right)^2 \left(1 - 3\tilde{\eta}E + \frac{27}{4} \tilde{\eta}^2 E^2 \right) \times \left[1 + \tilde{\xi}^2 \left(\frac{m\omega_m}{\gamma} \right)^2 \right] m^{1/3}. \quad (198)$$

In Eqs. (197) and (198) the Lorentz factor γ is given either by Eq. (20) and (21) according to the value of $\tilde{\eta}E^3/\mu^2$.

D. Polarization in the large m limit

The general expression for the polarization index R_m , from which we can calculate the reduced Stokes parameters, is given in Eq. (167). Since the radiation is mostly

concentrated around the plane of the orbit, we will examine the different limits of

$$R_m(\theta = \pi/2) = 1 - 2 \frac{\tilde{\xi} m \omega_m}{\beta \gamma^2} \frac{J_m(m\beta)}{J'_m(m\beta)}. \quad (199)$$

In the large m approximation and in the case $1 - \beta^2 > 0$, one can write the Bessel function and its derivative in terms of the Macdonald functions as stated in Eqs. (B2) of the appendix. Using (B6) and (B7) the latter can be further approximated according to $\frac{m}{m_c} \ll 1$, where m_c is given by Eq. (194). When $m/m_c \gg 1$ the power of the radiation is exponentially damped and we have

$$\frac{J_m(m\beta)}{J'_m(m\beta)} = \gamma, \quad (200)$$

which leads to

$$R_m(\theta = \pi/2) = 1 - \frac{2\tilde{\xi}\omega}{\beta} \frac{m}{\gamma}. \quad (201)$$

On the other hand, when $m/m_c \ll 1$, we have

$$\frac{J_m(m\beta)}{J'_m(m\beta)} = \left(\frac{1}{4}\right)^{1/3} \frac{\Gamma(1/3)}{\Gamma(2/3)} m^{1/3}, \quad (202)$$

yielding

$$R_m(\theta = \pi/2) = 1 - \frac{2\tilde{\xi}\omega}{4^{1/3}\beta} \frac{\Gamma(1/3)}{\Gamma(2/3)} \frac{m^{4/3}}{\gamma^2}. \quad (203)$$

In both cases there appear amplifying factors such as m and $m^{4/3}$, whose effects have to be evaluated in each separate situation. Let us also remark that the limit $\tilde{\xi} = 0$ reproduces a linear polarization as dictated by the value $R_m^2 = 1$.

E. Averaged degree of circular polarization for large m

We can often assume that the relativistic electrons ($\beta \approx 1$) have an energy distribution of the type [50]

$$N(E)dE = CE^{-p}dE, \quad (204)$$

in some energy range $E_1 < E < E_2$, where typically $p \approx 2 - 3$. Let us define the circular analog of the degree of linear polarization introduced in Ref. [50] as

$$\Pi_{\circ} = \frac{\langle P_+(\omega) - P_-(\omega) \rangle}{\langle P_+(\omega) + P_-(\omega) \rangle}, \quad (205)$$

where $P_{\pm}(\omega)$ is the total power distribution per unit frequency and polarization $\lambda = \pm 1$. The average $\langle \dots \rangle$ on the energy distribution of the charged particles is calculated as

$$\langle Q(E) \rangle = C \int_0^{\infty} E^{-p} Q(E) dE. \quad (206)$$

In (111) we have introduced the power emitted into the m th harmonic $P_{m\lambda}$, such that the total power emitted with polarization λ is

$$P_{\lambda} = \sum_m P_{m\lambda} = \int P_{m\lambda} dm = \int \frac{P_{m\lambda}}{\omega_0} d\omega, \quad (207)$$

which produces

$$P_{\lambda}(\omega) = \frac{P_{m\lambda}}{\omega_0}. \quad (208)$$

We will carry the calculation only to first order in the LIV parameters. To this end we start from Eq. (187) and set $\beta = 1 = n(\lambda z_m)$ everywhere, except in the critical terms involving $\tilde{m}_{\lambda c}$ from where the corrections arise. We should also take into account that most of the radiation comes from the terms with $m \approx \tilde{m}_{\lambda c} \gg 1$, where $K_{2/3}(1) = 0.49$, $\kappa(1) = 0.65$. Thus the dominant term is

$$P_{\lambda}(\omega) = D \left(\frac{1}{\tilde{m}_{\lambda c}} \right)^{2/3} \frac{\tilde{m}_{\lambda c}}{m} \kappa \left(\frac{m}{\tilde{m}_{\lambda c}} \right), \quad (209)$$

where D is a constant. Starting from the definition (205), making an expansion in $u = \lambda \tilde{\xi} \omega$ and recalling that $\tilde{m}_{\lambda c} = \tilde{m}_{\lambda c}(u)$, we find

$$\Pi_{\circ} = \tilde{\xi} \omega \frac{\langle [\frac{dP_{\lambda}(\omega)}{du}]_{\tilde{\xi}=0, \tilde{\eta}=0} \rangle}{\langle [P_{\lambda}(\omega)]_{\tilde{\xi}=0, \tilde{\eta}=0} \rangle} + O(\tilde{\xi}^2, \tilde{\xi} \tilde{\eta}, \dots). \quad (210)$$

Since $n(u) = \sqrt{1+u^2} + u$, we can rewrite d/du in terms of d/dn . Besides, $\tilde{m}_{\lambda c}$ depends only on the combination $n\beta$ so that we can further go to $d/d\beta$ obtaining

$$\left. \frac{dP_{\lambda}(\omega)}{du} \right|_{\tilde{\xi}=0, \tilde{\eta}=0} = D \left(\frac{2n^2}{1+n^2} \frac{\beta}{n} \right)_{\tilde{\xi}=0, \tilde{\eta}=0} \times \frac{d}{d\beta} \left[\left(\frac{1}{m_c} \right)^{2/3} \frac{m_c}{m} \kappa(m/m_c) \right], \quad (211)$$

where the first parenthesis on the right-hand side gives 1 upon evaluation. In the second parenthesis we have already taken the limit $n = 1$, $\tilde{\eta} = 0$. In this case we also have $m_c = 3\gamma^3/2$ and $E = \mu\gamma$. Therefore, the argument of the bremsstrahlung function $\kappa(x)$ becomes

$$x = \frac{m}{m_c} = \frac{2}{3} \frac{\omega}{\omega_0} \gamma^{-3} = A^2 \gamma^{-2}, \quad (212)$$

where $A^2 = 2\mu\omega/(3qB)$. The above leads to $\gamma = Ax^{-1/2}$. Again, a successive change of independent variables yields

$$d/d\beta = \beta \gamma^3 d/d\gamma = -2A^2 d/dx, \quad (213)$$

where we have set $\beta = 1$ in the corresponding factor. Finally, we have to analyze the term $xm_c^{2/3}$ that multiplies $\kappa(x)$ inside the square bracket of Eqs. (209) and (211). Substituting $m_c = 3\gamma^3/2 = 3A^3 x^{-3/2}/2$ we find that $xm_c^{2/3} = (3A^3/2)^{2/3}$ is a constant that cancels when taking the ratio in (210). Thus we are left with

$$\Pi_{\circ} = -\frac{4}{3} \tilde{\xi} \omega \left(\frac{\mu\omega}{qB} \right) \frac{\langle \frac{d\kappa(x)}{dx} \rangle}{\langle \kappa(x) \rangle}, \quad (214)$$

where the average $\langle \dots \rangle$ over E has been replaced by one over x , via the change of variables $E = \mu A x^{-1/2}$, with all the constant factors cancelling in the ratio of the two integrals. It should be pointed out that the factor A^2 arising from the derivative in the numerator survives after taking this ratio. For the purpose of making an estimation of $\Pi(p)$ we take the energy range to be $0 \leq E \leq \infty$. We then obtain

$$\Pi_{\odot} = -\frac{4}{3} \tilde{\xi} \omega \left(\frac{\mu \omega}{qB} \right) \frac{\int_0^{\infty} x^{(p-3)/2} \frac{d\kappa(x)}{dx} dx}{\int_0^{\infty} x^{(p-3)/2} \kappa(x) dx}. \quad (215)$$

Using the expression [50]

$$\int_0^{\infty} x^{\mu} \kappa(x) dx = \frac{2^{\mu+1}}{\mu+2} \Gamma\left(\frac{\mu}{2} + \frac{7}{3}\right) \Gamma\left(\frac{\mu}{2} + \frac{2}{3}\right),$$

$$\mu + 1/3 > -1, \quad (216)$$

and comparing with Eq. (217), we finally get

$$\Pi_{\odot} = \tilde{\xi} \omega \left(\frac{\mu \omega}{qB} \right) \Pi(p), \quad (217)$$

with

$$\Pi(p) = \frac{(p-3)(3p-1)}{3(3p-7)} \frac{(p+1)}{(p-1)} \frac{\Gamma(\frac{p}{4} + \frac{13}{12}) \Gamma(\frac{p}{4} + \frac{5}{12})}{\Gamma(\frac{p}{4} + \frac{19}{12}) \Gamma(\frac{p}{4} + \frac{11}{12})},$$

$$p > 7/3. \quad (218)$$

The constraint $p > 7/3$ is required to avoid the divergence of the integral in the numerator of Eq. (215) at $x = 0$. The above result for Π_{\odot} is the analogous for the averaged linear degree of polarization $\Pi = (p+1)/(p+7/3)$ which is independent of the frequency [50]. On more realistic grounds, one should avoid the infinite upper limit of E , (the zero lower limit of x). In this case the divergence of the integral at $x = 0$, and hence the mathematical constraint on p , disappears, but the expression for $\Pi(p)$ becomes more complicated.

In any case, the most important feature of the result in Eq. (217) is the presence of the amplifying factor $(\mu \omega / qB)$, which is independent of such details. An estimation of this factor in the zeroth-order approximation ($\tilde{\xi} = 0 = \tilde{\eta}$), which is appropriate in Eq. (217), yields $(\mu \omega / qB) = \omega / (\omega_0 \gamma) = m / \gamma$, which is not necessarily a small number. According to the values that we have mentioned at the beginning of this section, this amplifying factor could be as large as $m / \gamma \simeq 10^{22}$.

IX. PHENOMENOLOGICAL CONSEQUENCES

In our analysis of the synchrotron radiation we have found that a careful perturbative expansion in the Lorentz-violating parameters produced not only the naïvely expected factors, but also nontrivial large amplifying factors for some Lorentz violation effects. These unexpectedly large amplification factors open the possibility of observing such effects in the radiation of astrophysical

sources where these factors become important. In this section we explore different aspects of the synchrotron radiation from astrophysical objects such as SNRs and GRBs and the constraints upon the Lorentz-violating parameters imposed by their measurement.

A. Radiation angular distribution

This subsection is devoted to the study of the preferred directions together with the opening angles in the Lorentz-violating circularly polarized synchrotron radiation. The standard Lorentz covariant results (see Ref. [45] for example) are that the radiation is confined to the forward particle direction with an opening angle $\delta\theta \simeq \gamma^{-1}$. The recent bounds on the LIV parameters, derived from the synchrotron radiation observed from the Crab nebula, heavily rest upon extrapolating some of these results, based on the assumption that the LIV violations should only slightly modify the standard ones [25]. This has been the subject of some controversy in the literature [58]. For this reason it is relevant to provide some answers arising from a specific model calculation in which the analysis of Refs. [25] can be embedded.

Our starting point is the angular distribution of the power emitted in the m th harmonic with polarization λ given in Eq. (96), which we rewrite here

$$\frac{dP_{m\lambda}}{d\Omega} = \frac{\omega_0^2 q^2}{4\pi} \frac{1}{\sqrt{1+z_m^2}} [\lambda m \beta n(\lambda z_m) J'_m(W_{\lambda m}) + m \cot\theta J_m(W_{\lambda m})]^2. \quad (219)$$

Recalling that we are dealing with large values of m it is convenient to use the corresponding approximations in (B2). By doing this Eq. (96) leads to

$$\frac{dP_{m\lambda}}{d\Omega} = \frac{\omega_0^2 q^2}{12\pi^3} \frac{1}{\sqrt{1+z_m^2}} \left\{ m \lambda \beta n(\lambda z_m) (1 - (\beta n(\lambda z_m) \sin\theta)^2) K_{2/3} \left[\frac{m}{3} (1 - (\beta n(\lambda z_m) \sin\theta)^2)^{3/2} \right] + m \cot\theta (1 - (\beta n(\lambda z_m) \sin\theta)^2)^{1/2} K_{1/3} \times \left[\frac{m}{3} (1 - (\beta n(\lambda z_m) \sin\theta)^2)^{3/2} \right] \right\}^2. \quad (220)$$

Using further the approximation corresponding to $m/\tilde{m}_c \gg 1$, we can make explicit the asymptotic behavior of the Macdonald functions which yields

$$\frac{dP_{m\lambda}}{d\Omega} = \frac{\omega_0^2 q^2}{8\pi^2} \frac{m}{\sqrt{1+z_m^2}} e^{-2m[1-(\beta n(\lambda z_m) \sin\theta)^2]^{3/2}/3} \times [\lambda \beta n(\lambda z_m) (1 - (\beta n(\lambda z_m) \sin\theta)^2)^{1/4} + \cot\theta (1 - (\beta n(\lambda z_m) \sin\theta)^2)^{-1/4}]^2. \quad (221)$$

The above corresponds to the use of the approximation (B7) of Appendix B. In fact this expression turns out to be a reasonable approximation for values of the Macdonald

function argument as low as $\vartheta = 0.2$ ($m \approx 1$). In this case it gives an error of the order of 15%, which is acceptable for our purposes.

To visualize the geometry of the radiation beam we can rewrite this angular distribution in terms of the angle $\alpha = \pi/2 - \theta$. First we estimate the direction of the maximum radiated power in the m th harmonic given by α_{\max} . Recalling the definition (186) for $\tilde{m}_{\lambda c}$ and introducing the variables

$$\mu = \frac{m}{\tilde{m}_{\lambda c}}, \quad y = \lambda \left(\frac{2\tilde{m}_{\lambda c}}{3} \right)^{1/3} \sin \alpha, \quad (222)$$

we can write the extremum condition for (221) as

$$3\mu y((1+y^2)^2 + y(1+y^2)^{3/2}) - y\sqrt{1+y^2} - (1+y^2) - 1 \approx 0. \quad (223)$$

We assume the above equation to be valid for $\mu \leq 1$, for reasons to be explained *a posteriori*. Also, numerical estimations show that $y > 1$ ($y < 1$) when $\mu < 1$ ($\mu > 1$), respectively. These considerations lead to the solutions

$$\sin \alpha_{\max} \approx \alpha_{\max} \approx \lambda \left(\frac{2\tilde{m}_{\lambda c}}{3m} \right)^{2/3} m^{-1/3}, \quad \mu \gg 1, \quad (224)$$

$$\sin \alpha_{\max} \approx \alpha_{\max} \approx \lambda 0.64 (2m_c)^{-1/3}, \quad \mu = 1, \quad (225)$$

$$\sin \alpha_{\max} \approx \alpha_{\max} \approx \lambda (2m)^{-1/3}, \quad \mu \ll 1. \quad (226)$$

The result (226) justifies the use of expression (221) for $\mu < 1$. The argument ϑ_{\max} of the corresponding Macdonald functions for α_{\max} is

$$\vartheta_{\max} = \frac{m}{3} \left[\sin^2 \alpha_{\max} + \left(\frac{3}{2\tilde{m}_{\lambda c}} \right)^{2/3} \cos^2 \alpha_{\max} \right]_{\mu \ll 1}^{3/2} \approx \frac{1}{6}. \quad (227)$$

As explained after Eq. (221), we are still very close to the range where we have considered the approximation (B7) to be acceptable.

Finally we estimate the opening angle of the radiation $\delta\theta$, defined with respect to the angle of maximum radiation. To do this we only consider the exponential factor in (221) and determine the cutoff angle α_c , defined to be the angle where the power decreases by a factor $1/e$ with respect to its maximum value. This leads to the equation

$$\frac{2}{3} m (1 - (\beta n \cos \alpha_c)^2)^{3/2} = 1 + \frac{2}{3} m (1 - (\beta n \cos \alpha_{\max})^2)^{3/2}, \quad (228)$$

together with

$$\delta\theta = 2(\alpha_c - \alpha_{\max}). \quad (229)$$

The radiation occurs mostly in the sector $\mu \leq 1$, thus by solving (228) to lowest order and using the expressions (225) and (226) we find

$$\begin{aligned} \delta\theta &= 0.98 m^{-1/3}, & \mu &\ll 1, \\ \delta\theta_c &= 1.25 m_c^{-1/3}, & \mu &= 1, \end{aligned} \quad (230)$$

which effectively means $\delta\theta \approx m^{-1/3}$.

When it is possible to ignore the photon contribution to the LIV that enters only through $n(z_m)$, that is to say when we can set $n = 1$ as it is the case of Ref. [25], we recover the result

$$\delta\theta_c \approx (1 - \beta^2)^{1/2} \approx \gamma^{-1},$$

for the opening angle, which was one of the starting assumptions in this reference.

B. Birefringence

One of the most evident features of this Lorentz-violating electrodynamics is directly related with the propagation of the electromagnetic field and manifests itself as a vacuum birefringence effect, i.e. circular polarizations propagate with different velocities

$$v_{\pm}^{-1} = \sqrt{1 + \omega^2 \tilde{\xi}^2} \pm \omega \tilde{\xi}. \quad (231)$$

In principle this difference produces a shift in the arrival time of the different polarizations radiated by a given source, but in fact this time delay is very small to be observed even for a propagation in cosmological distances. A more sensible manifestation of this difference in the propagation velocities is the polarization of the field, which changes with the distance.

C. Radiation anisotropy

Another interesting effect produced by the Lorentz violation is the anisotropy in the emitted radiation. According to Eq. (192) this anisotropy becomes apparent for $\omega \sim (\omega_0^2 / \tilde{\xi}^3)^{1/5}$, where the radiation is suppressed in the region $0 \leq \theta < \pi/2$ ($\pi/2 < \theta \leq \pi$) and enhanced in $\pi/2 < \theta \leq \pi$ ($0 \leq \theta < \pi/2$) for $\tilde{\xi} > 0$ ($\tilde{\xi} < 0$), respectively. This effect gives a clear signature of the Lorentz violation, and can be used to set a bound for $\tilde{\xi}$. In this respect, the most favorable test in laboratory conditions could be provided by an electron accelerator such as LEP. There the energy of the electrons is $E \approx 55$ GeV, the radius of the orbit is $R \approx 4.25$ km, yielding the Larmor and a cutoff frequencies of $\omega_0 \approx 5 \times 10^{-20}$ GeV and $\omega_c \approx 5 \times 10^{-5}$ GeV, respectively. The absence of asymmetry in the radiation up to the critical frequency should imply $|\tilde{\xi}| \leq 10^{-6}$ GeV $^{-1}$. In fact this is a very weak bound. A more stringent one could be obtained from astrophysical systems such as GRBs, provided that the electromagnetic radiation of these objects actually corresponds to synchrotron radiation.

D. Cutoff frequency

The exact high energy limit for the integrated power spectrum is given in Eq. (188). At this level we can identify the modifications induced by the Lorentz-violating effects in the cutoff frequency through the term in the exponential $e^{-\omega/\omega_c}$, where

$$\omega_c = \frac{3}{2} \omega_0 (1 - n^2 \beta^2)^{-3/2}. \quad (232)$$

The corresponding factors expressed in terms of the electron energy, given by Eqs. (11) and (184) are rewritten here to first order in the small dimensionless parameters ξ , η previously introduced

$$\omega_0 = \frac{|q|B}{E} \left(1 + \frac{3}{2} \eta \frac{E}{M} \right), \quad (233)$$

$$1 - \beta^2 n^2 = \frac{\mu^2}{E^2} - 2\eta \frac{E}{M} - 2\lambda \xi \frac{\omega}{M}. \quad (234)$$

Since the terms in $(1 - \beta^2 n^2)^{-3/2}$ are already very large it is appropriate to neglect the very small correction appearing in ω_0 . In this way we have

$$\omega_c = \frac{3}{2} \frac{|q|B}{E} \left(\frac{\mu^2}{E^2} - 2\eta \frac{E}{M} - 2\lambda \xi \frac{\omega}{M} \right)^{-3/2}. \quad (235)$$

The relation of the above result with that given in Eq. (4) of Ref. [25] can be found easily. Changing \tilde{v}/c by $\tilde{v}/c(\omega)$ in our definition (18) of the Lorentz factor we get

$$\gamma_J = \{1 - [v(E)/c(\omega)]^2\}^{-1/2} = (1 - n^2 \beta^2)^{-1/2}, \quad (236)$$

where we have introduced the index of refraction $n = c/c(\omega)$. Let us rewrite

$$\begin{aligned} \gamma_J &= c(\omega) \{ [c(\omega) - v(E)][c(\omega) + v(E)] \}^{-1/2} \\ &= \{ 2[c(\omega) - v(E)] \}^{-1/2} \end{aligned} \quad (237)$$

to the leading order. The dispersion relations (1) give

$$2(c(\omega) - v(E)) = -2\lambda \xi \frac{\omega}{M} + \frac{\mu^2}{E^2} - 2\eta \frac{E}{M}, \quad (238)$$

in such a way that Eq. (235) can be written as

$$\omega_c = \frac{3}{2} \frac{|q|B}{E} \gamma_J^3, \quad (239)$$

which is precisely Eq. (4) of Ref. [25], up to the polarization factor multiplying the photon frequency which has been subsequently taken into account in later publications. Nevertheless, in the considered energy range the photon contribution is negligible and the following maximum energy is obtained

$$E_{\max} = \left(-\frac{2}{5\eta} \mu^2 M \right)^{1/3}. \quad (240)$$

It is possible to verify that

$$(1 - \beta^2 n^2)_{E=E_{\max}} = \frac{9}{5} \frac{\mu^2}{E_{\max}^2} > 0, \quad (241)$$

so that one stays in the allowed range of radiation emission.

E. Additional corrections

Another factor that modifies the power spectrum and tests the dynamics of the electromagnetic field, according to Eq. (197), is $1 + 2(\tilde{\xi} m^2 \omega_0 / \gamma)^2$. If this correction is not observed at a given frequency ω we infer that $\tilde{\xi} \omega^2 / (\omega_0 \gamma) \ll 1$, or equivalently

$$\tilde{\xi} \lesssim \gamma \omega_0 / \omega^2 = qB / (\mu \omega^2). \quad (242)$$

This bound is independent of the energy of the electrons. It is proportional to ω_0 / ω^2 , as in the parity violating effect already discussed after Eq. (191), but now we have an additional factor of γ , which increases the resulting upper limit, thus making the boundary much less stringent.

It is interesting to compare the bounds for $\tilde{\xi}$ provided by the first order anisotropic effect in (197) and the second order suppression in the power spectrum in (191). According to Eq. (192), the first case requires

$$\tilde{\xi} \lesssim (\omega_0^2 / \omega^5)^{1/3} = (\omega / \omega_0)^{1/3} \omega_0 / \omega^2, \quad (243)$$

while in the second case Eq. (242) demands

$$\tilde{\xi} \lesssim \gamma \omega_0 / \omega^2. \quad (244)$$

Assuming that the Lorentz violation is not observed up to a frequency $\omega < \omega_c = \gamma^3 \omega_0$, the boundary (243) leads to

$$\begin{aligned} \tilde{\xi} &\lesssim (\omega / \omega_0)^{1/3} \omega_0 / \omega^2 \\ &= (\omega / \omega_c)^{1/3} (\gamma \omega_0 / \omega^2) < \gamma \omega_0 / \omega^2. \end{aligned} \quad (245)$$

If $\omega \simeq \omega_c$ both relations (242) and (243) give the same bound for $\tilde{\xi}$.

Finally, Eq. (197) contains two additional frequency independent factors including η -dependent corrections: $(1 + \tilde{\eta} E^3 / 2\mu^2)$ that comes from $\gamma^{-1/2}$ through Eq. (20) and $(1 - 3\tilde{\eta} E + 27\tilde{\eta}^2 E^2 / 4)$. Both test the dynamics of the charge. The first one dominates the linear corrections at energies that we can find in astrophysical systems, and if they are not observed for electrons of energy E this implies

$$|\tilde{\eta}| \lesssim \mu^2 / E^3. \quad (246)$$

X. ACCESSIBLE PARAMETER REGIONS AND GENERAL OUTLOOK

To close our analysis, we will determine the regions of LIV parameters that can be explored by means of the synchrotron radiation of the already considered astrophysical sources: Crab Nebula, Mkn 501, and GRB021206. The Crab Nebula is a well-known synchrotron radiation emitter, while the other two are on a more conjectural level.

Let us first consider the two LIV contributions to the power spectrum Eq. (197) discussed in Sec. IX E. The one containing $\tilde{\eta}$ is a frequency independent factor which simply renormalizes the power spectrum. Since we do not have an independent determination of the density of the emitter particles we are not able to evaluate it. The second one, which contains ξ and depends on the frequency, distorts the shape of the power spectrum thus being in principle measurable. If this distortion is not observed in the power spectrum we can set the bound (242). In the particular case of the CRAB Nebulae this leads to $\xi < 5 \times 10^{-1}$. Other strong sources of high energy electromagnetic radiation are Mkn 501 and GRB021206. Equation (242) shows that these objects could allow us to explore the region $\xi \gtrsim 10^{-6}$.

The values of the different factors contributing to the Green function phase (45) for the astrophysical objects under consideration are given in Table II. Here we take the Larmor radius as an estimation of the size of the radiating source r' . The last column gives the subdominant term in the phase of the Green function, $(r'/r)^2$, that allows us to estimate if a given LIV contribution is significant for the radiation field, according to the discussion in Sec. III. In the Crab Nebulae case Table II tells us that $\tilde{\xi}\omega = \xi\omega/M_p \ll (r'/r)^2$, even for $\xi \approx 1$. In this way, consistency requires that the phase of the Green function (45) reduces to $\omega(r - \hat{n} \cdot \mathbf{r}')$. This is equivalent to set $\tilde{\xi} = 0$, $n = 1$ in all the arguments of the Bessel functions. This leads to

$$\delta\theta \approx \gamma^{-1}(E), \quad \omega_c = qB\gamma^3(E)/E, \quad (247)$$

where $\gamma(E)$ still includes corrections depending upon the fermionic parameter $\tilde{\eta}$ according to Eq. (16). In other words, the results in Ref. [25] are recovered for this particular situation. On the other hand, the corresponding phase in the cases of Mkn 501 and the GRB021206, leading to modified expressions for the cutoff frequency could, in principle, give information about first order contributions in $\tilde{\xi}$. Since these corrections are not affected by significant amplifying factors, their observation would require an extremely precise determination of the cutoff frequency. In this analysis we have considered the case of a charged particle in a circular orbit orthogonal to the magnetic field, and we have used our results to obtain some rough estimations of the regions where the LIV parameters can be explored. In realistic systems such as SNRs or

TABLE II. Factors contributing to the Green function phase for the astrophysical objects.

	r'/r	ω_c/M_p	$(r'/r)(\omega_c/M_p)$	$(r'/r)^2$
CRAB	10^{-6}	10^{-20}	10^{-26}	10^{-12}
Mkn 501	10^{-11}	10^{-15}	10^{-26}	10^{-22}
GRB021206	10^{-24}	10^{-22}	10^{-46}	10^{-48}

GRBs, we actually have a population of charged particles with a certain energy distribution and different pitch angles. Thus, in order to derive more reliable bounds it is necessary to take into account these distributions and perform the corresponding averages. This process involves a rather detailed model for the astrophysical source and is beyond the scope of the present article.

To summarize, we have presented a complete analysis of the synchrotron radiation in the rest frame of the Myers-Pospelov model for a charged particle in circular motion perpendicular to a constant magnetic field. The process can be visualized as a standard electrodynamics radiation in a dispersive parity violating media, with helicity-dependent refraction indices that encode the Lorentz-violating corrections to the electromagnetic sector. The charged particle sector also presents corrections arising from the Lorentz violation, which mainly modify the dependence of the β factor upon the energy of the particle. The calculation includes exact expressions for the angular distribution of the power (96) and the total power (111) radiated into the m th harmonic for each polarization, together with their corresponding expansions to lowest order in the LIV parameter $\tilde{\xi}$ given in Eqs. (147) and (156), respectively. The physical situations under consideration correspond to large Lorentz factors γ together with harmonics in the range $m \approx 10^{15} - 10^{30}$, but in such a way that the ratio m/γ is very large also. Because of this fact, a large m expansion has been performed in the total power radiated into the m th harmonic for the exact polarized case, Eqs. (188)–(189), and in the unpolarized $\tilde{\xi}$ -expanded situation, Eqs. (197)–(198). The $\tilde{\xi}$ -exact case allows us to identify the cutoff frequency according to Eq. (186). Special emphasis has been given to the polarization of the radiation which exact Stokes parameters are obtained in terms of Eq. (137), with their corresponding expansion written in Eq. (167). We also have calculated the averaged degree of circular polarization for a standard electron energy distribution in the large m approximation in Eq. (217). This quantity starts linearly in $\tilde{\xi}$ which, unexpectedly, is multiplied by the amplifying factor $(\mu\omega)/(qB) \approx m/\gamma$. Such amplifying factors occur also in other $\tilde{\xi}$ -expanded quantities. Similar amplifying factors have been obtained in calculations of the synchrotron radiation spectra in the context of non-commutative electrodynamics [59]. A study of the observational relevance of these factors is outside the scope of the present work, but certainly it is a matter that deserves further investigation. As in the standard case, the radiation in the large m limit is concentrated in the forward direction. Finally, the angles for maximum radiation together with the corresponding opening angles are calculated in Eqs. (224)–(226) and (230), respectively.

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APPENDIX A: CIRCULAR POLARIZATION BASIS

Given a wave propagating in the direction \mathbf{k} we define the \pm polarization components of a vector $\mathbf{j} = \{j_i(\omega, \mathbf{k})\}$ as

$$\mathbf{j}^\pm = \frac{1}{2}[\mathbf{j} - (\hat{\mathbf{k}} \cdot \mathbf{j})\hat{\mathbf{k}} \pm i(\hat{\mathbf{k}} \times \mathbf{j})], \quad (\text{A1})$$

where we have introduced the unit vector $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$. These components satisfy the following relations

$$\mathbf{j} = j^+ + j^-, \quad \mathbf{j}^\pm \cdot \mathbf{l}^\pm = 0, \quad (\text{A2})$$

$$\hat{\mathbf{k}} \times \mathbf{j}^+ = -ij^+, \quad \hat{\mathbf{k}} \times \mathbf{j}^- = ij^-, \quad (\text{A3})$$

with \mathbf{l} being another arbitrary vector.

Sometimes it is convenient to rewrite (A1) in terms of projectors P_{ik}^\pm defined as

$$j_i^\pm = P_{ik}^\pm j_k, \quad P_{ik}^\pm = \frac{1}{2}(\delta_{ik} - \hat{k}_i \hat{k}_k \pm i\epsilon_{ijk} \hat{k}_j). \quad (\text{A4})$$

The algebra of these projectors is very useful and is summarized in the relations

$$P_{ik}^+ = P_{ki}^-, \quad (\text{A5})$$

$$P_{ki}^\pm P_{il}^\pm = P_{kl}^\pm, \quad P_{ki}^\mp P_{il}^\pm = 0, \quad (\text{A6})$$

$$\epsilon_{pqr} \hat{k}_p P_{qs}^+ P_{rt}^\pm = 0, \quad \epsilon_{pqr} \hat{k}_p P_{qs}^\pm P_{rt}^\mp = \mp i P_{st}^\mp. \quad (\text{A7})$$

Let us state that in the frame we have previously chosen we have the following basis

$$\hat{\mathbf{k}} = (\sin\theta, 0, \cos\theta), \quad \mathbf{e}_\parallel = (0, 1, 0), \quad (\text{A8})$$

$$\mathbf{e}_\perp = (-\cos\theta, 0, \sin\theta),$$

with the associated circular basis

$$\mathbf{e}_\pm = \frac{1}{\sqrt{2}}(\mathbf{e}_\parallel \pm i\mathbf{e}_\perp), \quad (\mathbf{e}_\pm)^* \cdot \mathbf{e}_\pm = 1, \quad (\text{A9})$$

$$(\mathbf{e}_\pm)^* \cdot \mathbf{e}_\mp = 0.$$

Thus a direct calculation shows that for an arbitrary complex current $\mathbf{j} = (j_x, j_y, j_z)$ the corresponding expressions for the currents defined in Eq. (A1) are

$$\mathbf{j}^\pm = \frac{1}{\sqrt{2}}[j_y \pm i(j_x \cos\theta - j_z \sin\theta)]\mathbf{e}_\pm. \quad (\text{A10})$$

APPENDIX B: LARGE m EXPANSION OF SOME REQUIRED BESSEL FUNCTIONS

Using the Nicholson-Olber [60] asymptotic representations for the Bessel functions in terms of the Macdonald functions $K_{n/m}$, we can write, in the large m approximation [46,61]:

1. The case $1 - n^2\beta^2 > 0$

$$(1 - n^2\beta^2) = \left(\frac{2}{3}\tilde{m}_c\right)^{-2/3} = \left(\frac{3}{2\tilde{m}_c}\right)^{2/3}, \quad (\text{B1})$$

$$J_{2m}(2mn\beta) \simeq \frac{1}{\sqrt{3}\pi} \left(\frac{3}{2\tilde{m}_c}\right)^{1/3} K_{1/3}\left(\frac{m}{\tilde{m}_c}\right),$$

$$J'_{2m}(2mn\beta) \simeq \frac{1}{\sqrt{3}\pi} \left(\frac{3}{2\tilde{m}_c}\right)^{2/3} K_{2/3}\left(\frac{m}{\tilde{m}_c}\right), \quad (\text{B2})$$

$$\int_0^{2mn\beta} dx J_{2m}(x) \simeq \frac{1}{\sqrt{3}\pi} \int_{m/\tilde{m}_c}^\infty dx K_{1/3}(x),$$

where the cutoff number \tilde{m}_c signals different regimes in the behavior of the Bessel functions K_ν of imaginary argument.

A useful relation among the Macdonald functions is

$$2K'_{2/3}(x) = -[K_{5/3}(x) + K_{1/3}(x)]. \quad (\text{B3})$$

The bremsstrahlung function $\kappa(z)$ is defined as

$$\kappa(z) = z \int_z^\infty dx K_{5/3}(x), \quad (\text{B4})$$

and has the following asymptotic behavior [62],

$$\kappa(z) \simeq 2^{2/3} \Gamma(2/3) z^{1/3}, \quad z \ll 1, \quad (\text{B5})$$

$$\kappa(z) \simeq \sqrt{\frac{\pi}{2}} z^{1/2} e^{-z}, \quad z \gg 1.$$

We also need the asymptotic behavior of $K_\nu(z)$

$$K_\nu(z) \simeq \frac{\Gamma(\nu)}{2} \left(\frac{z}{2}\right)^{-\nu}, \quad z \ll 1, \quad (\text{B6})$$

$$K_\nu(z) \simeq \sqrt{\frac{\pi}{2}} z^{-1/2} e^{-z}, \quad z \gg 1, \quad (\text{B7})$$

where the last relation is independent of ν .

2. The case $1 - n^2\beta^2 < 0$

Here the notation is

$$x = m^{2/3}(n^2\beta^2 - 1), \quad (\text{B8})$$

and we have

$$J'_{2m}(2mn\beta) \simeq -m^{-2/3} Ai'(-x), \quad \int_0^{2mn\beta} J_{2m}(t) dt = \frac{1}{3} + \int_0^x dt Ai(-t) \Lambda(x) = -\frac{2Ai'(-x)}{x} + \frac{1}{3} + \int_0^x dt Ai(-t), \quad (\text{B9})$$

where we have introduced the analogous $\Lambda(x)$ of the bremsstrahlung function $\kappa(x)$.

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