

Convergence towards a self-similar solution for a nonlinear wave equation: A case studyPiotr Bizoń¹ and Tadeusz Chmaj^{2,3}¹*M. Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, 30-059 Kraków, Poland*²*H. Niewodniczanski Institute of Nuclear Physics, Polish Academy of Sciences, Kraków, Poland*³*Cracow University of Technology, Warszawska 24, 31-155 Kraków, Poland*

(Received 6 July 2005; published 16 August 2005)

We consider the problem of asymptotic stability of a self-similar attractor for a simple semilinear radial wave equation which arises in the study of the Yang-Mills equations in $5 + 1$ dimensions. Our analysis consists of two steps. In the first step we determine the spectrum of linearized perturbations about the attractor using a method of continued fractions. In the second step we demonstrate numerically that the resulting eigensystem provides an accurate description of the dynamics of convergence towards the attractor.

DOI: [10.1103/PhysRevD.72.045013](https://doi.org/10.1103/PhysRevD.72.045013)

PACS numbers: 11.10.Lm, 11.10.Jj

I. INTRODUCTION

Self-similar solutions of evolution equations often appear as attractors in a sense that solutions of an initial value problem starting from generic initial data evolve asymptotically into a self-similar form. In such cases one would like to describe the process of convergence to the self-similar solution and understand the mechanism responsible for this phenomenon. These kind of problems are relatively well-understood for diffusion equations where the global dissipation of energy is the mechanism of convergence to an attractor (see [1] for a recent survey), however very little is known for conservative wave equations where the local dissipation of energy is due to dispersion. In this paper we report on analytical and numerical studies of this problem for a semilinear radial wave equation of the form

$$u_{tt} - u_{rr} - \frac{2}{r}u_r + \frac{f(u)}{r^2} = 0, \quad (1)$$

where r is the radial variable, $u = u(t, r)$, and $f(u) = -3u(1 - u^2)$. This equation appears in the study of the spherically symmetric Yang-Mills equations in $5 + 1$ dimensions (see [2] for the derivation). We remark in passing that our results hold for more general nonlinearities, in particular, for $f(u) = \sin(2u)$ which corresponds to the equivariant wave maps from the $3 + 1$ dimensional Minkowski spacetime into the three-sphere.

It was proved in [3] and later found explicitly in [2] that Eq. (1) has a self-similar solution

$$u(t, r) = U_0(\rho) = \frac{1 - \rho^2}{1 + \frac{3}{5}\rho^2}, \quad (2)$$

where $\rho = r/(T - t)$ is a similarity variable and $T > 0$ is a constant [actually, U_0 is the ground state of a countable family of self-similar solutions U_n ($n = 0, 1, \dots$) but since all $n > 0$ solutions are unstable they do not appear as attractors for generic initial data]. Since

$$\partial_r^2 U_0(\rho)|_{r=0} \sim \frac{1}{(T - t)^2}, \quad (3)$$

the solution $U_0(\rho)$ becomes singular at the center when $t \rightarrow T$. By the finite speed of propagation, one can truncate this solution in space to get a smooth solution with compactly supported initial data which blows up in finite time.

In fact, the self-similar solution U_0 is not only an explicit example of singularity formation, but more importantly it appears as an attractor in the dynamics of generic initial data. We conjectured in [4] that generic solutions of Eq. (1) starting with sufficiently large initial data do blow up in a finite time in the sense that $u_{rr}(t, 0)$ diverges at $t \nearrow T$ for some $T > 0$ and the asymptotic profile of blowup is universally given by U_0 , that is

$$\lim_{t \nearrow T} u(t, (T - t)r) = U_0(r). \quad (4)$$

Figure 1 shows the numerical evidence supporting this conjecture.

The goal of this paper is to describe in detail how the limit (4) is attained. To this end, in Sec. II we first study the linear stability of the solution U_0 . This leads to an eigenvalue problem which is rather unusual from the standpoint of spectral theory of linear operators. We solve this problem in Sec. III using the method of continued fractions. Then, in Sec. IV we present the numerical evidence that the deviation of the dynamical solution from the self-similar attractor is asymptotically well described by the least damped eigenmode.

II. LINEAR STABILITY ANALYSIS

The role of the self-similar solution U_0 in the evolution depends crucially on its stability with respect to small perturbations. In order to analyze this problem it is convenient to define the new time coordinate $\tau = -\ln(T - t)$ and rewrite Eq. (1) in terms of $U(\tau, \rho) = u(t, r)$

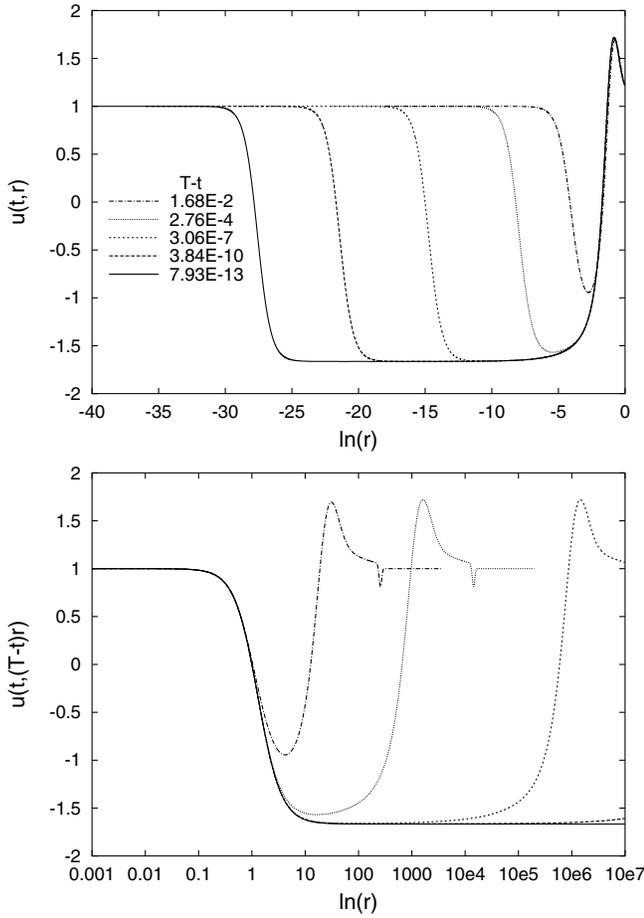


FIG. 1. The upper plot shows the late time evolution of some large initial data which blow up at time T . As the blowup progresses, the inner solution gradually attains the form of the stable self-similar solution $U_0(r/(T-t))$. The outer solution appears frozen on this time scale. In the lower plot the rescaled solutions $u(t, (T-t)r)$ are shown to collapse to the profile $U_0(r)$ (solid line).

$$U_{\tau\tau} + U_{\tau} + 2\rho U_{\rho\tau} - (1 - \rho^2) \left(U_{\rho\rho} + \frac{2}{\rho} U_{\rho} \right) + \frac{f'(U)}{\rho^2} = 0. \quad (5)$$

In these variables the problem of finite time blowup is converted into the problem of asymptotic convergence for $\tau \rightarrow \infty$ towards the stationary solution $U_0(\rho)$. Following the standard procedure we seek solutions of Eq. (5) in the form $U(\tau, \rho) = U_0(\rho) + w(\tau, \rho)$. Neglecting the $O(w^2)$ terms we obtain a linear evolution equation for the perturbation $w(\tau, \rho)$

$$w_{\tau\tau} + w_{\tau} + 2\rho w_{\rho\tau} - (1 - \rho^2) \left(w_{\rho\rho} + \frac{2}{\rho} w_{\rho} \right) + \frac{f''(U_0)}{\rho^2} w = 0. \quad (6)$$

Substituting $w(\tau, \rho) = e^{\lambda\tau} v(\rho)/\rho$ into (6) we get the eigenvalue equation

$$-(1 - \rho^2)v'' + 2\lambda\rho v' + \lambda(\lambda - 1)v + \frac{V(\rho)}{\rho^2}v = 0, \quad (7)$$

where

$$V(\rho) = f'(U_0) = \frac{6(25 - 90\rho^2 + 33\rho^4)}{(5 + 3\rho^2)^2}. \quad (8)$$

We consider Eq. (7) on the interval $0 \leq \rho \leq 1$, which corresponds to the interior of the past light cone of the blowup point ($t = T, r = 0$). Since a solution of the initial value problem for Eq. (1) starting from smooth initial data remains smooth for all times $t < T$, we demand the solution $v(\rho)$ to be analytic at the both end points $\rho = 0$ (the center) and $\rho = 1$ (the past light cone). Such a globally analytic solution of the singular boundary value problem can exist only for discrete values of the parameter λ , hereafter called eigenvalues. In order to find the eigenvalues we apply the method of Frobenius [5].

The indicial exponents at the regular singular point $\rho = 0$ are 3 and -1 , hence the solution which is analytic at $\rho = 0$ has the power series representation

$$v_0(\rho) = \sum_{n=0}^{\infty} a_n(\lambda)\rho^{2n+3}, \quad a_0 \neq 0. \quad (9)$$

Since the nearest singularity in the complex ρ plane is at $\rho = 1$, the series (9) is absolutely convergent for $0 \leq \rho < 1$. At the second regular singular point, $\rho = 1$, the indicial exponents are 0 and $1 - \lambda$ so, as long as λ is not an integer, the two linearly independent solutions have the power series representations

$$v_1(\rho) = \sum_{n=0}^{\infty} b_n^{(1)}(\lambda)(1 - \rho)^n, \quad (10)$$

$$v_2(\rho) = \sum_{n=0}^{\infty} b_n^{(2)}(\lambda)(1 - \rho)^{n+1-\lambda}.$$

These series are absolutely convergent for $0 < \rho \leq 1$. If λ is not an integer, only the solution $v_1(\rho)$ is analytic at $\rho = 1$. From the theory of linear ordinary differential equations we know that the three solutions $v_0(\rho)$, $v_1(\rho)$, and $v_2(\rho)$ are connected on the interval $0 < \rho < 1$ by the linear relation¹

$$v_0(\rho) = A(\lambda)v_1(\rho) + B(\lambda)v_2(\rho). \quad (11)$$

The requirement that the solution which is analytic at $\rho = 0$ is also analytic at $\rho = 1$ serves as the quantization condition for the eigenvalues $B(\lambda) = 0$. Unfortunately,

¹If $1 - \lambda$ is a positive integer N , then the solution which is analytic at $\rho = 1$ behaves as $v_1 \sim (1 - \rho)^{n+N}$ while the second solution v_2 involves the logarithmic term $C_N v_1 \ln(1 - \rho)$. By a straightforward but tedious calculation one can check that the coefficient C_N is nonzero for all N .

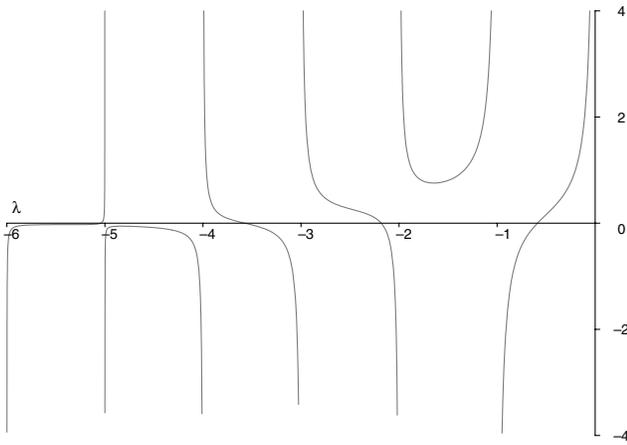


FIG. 2. The Wronskian $W[v_0, v_1](\rho = 1/2, \lambda)$.

the explicit expressions for the connection coefficients $A(\lambda)$ and $B(\lambda)$ are not known for equations with more than three regular singular points. There are, however, other indirect methods of solving the equation $B(\lambda) = 0$. One of them is a shooting method which goes as follows. One approximates the solutions $v_0(\rho)$ and $v_1(\rho)$ by the power series (9) and (10), respectively, truncated at some

$$\begin{aligned}
 p_3(n) &= -100n^2 - 850n - 1650, & p_2(n) &= -20n^2 + (100\lambda - 130)n + 25\lambda^2 + 325\lambda - 750, \\
 p_1(n) &= 84n^2 + (120\lambda + 378)n + 30\lambda^2 + 270\lambda + 618, & p_0(n) &= 36n^2 + (36\lambda + 90)n + 9\lambda^2 + 45\lambda + 54.
 \end{aligned}$$

For $n = -2$ we have

$$-350a_1 + (25\lambda^2 + 125\lambda - 570)a_0 = 0, \quad (13)$$

and for $n = -1$

$$\begin{aligned}
 -900a_2 + (25\lambda^2 + 225\lambda - 640)a_1 \\
 + (30\lambda^2 + 150\lambda + 324)a_0 = 0. \quad (14)
 \end{aligned}$$

The series (9) is absolutely convergent for $0 \leq \rho < 1$ and in general is divergent for $\rho > 1$. In order to determine the analyticity properties of the solution $v_0(\rho)$ at $\rho = 1$ we need to find the large n behavior of the expansion coefficients a_n . The four-term recurrence relation (12) can be viewed as the third order difference equation so it has three linearly independent asymptotic solutions for $n \rightarrow \infty$. Following standard methods (see, for example, [6]) we find

$$\begin{aligned}
 a_n^{(1)} &\sim n^{\lambda-2} \sum_{s=0}^{\infty} \frac{\alpha_s^{(1)}}{n^s}, & a_n^{(2)} &\sim \left(-\frac{3}{5}\right)^n n \sum_{s=0}^{\infty} \frac{\alpha_s^{(2)}}{n^s}, \\
 a_n^{(3)} &\sim \left(-\frac{3}{5}\right)^n n^{-4} \sum_{s=0}^{\infty} \frac{\alpha_s^{(3)}}{n^s}. \quad (15)
 \end{aligned}$$

Thus, in general, the solution of the recurrence relation (12) behaves asymptotically as

sufficiently large n , and then computes the Wronskian of these solutions at a midpoint, say $\rho = 1/2$. The zeros of the Wronskian correspond to the eigenvalues (see Fig. 2). Although this method gives the eigenvalues with reasonable accuracy, it is computationally very costly, especially for large negative values of λ , because the power series (9) and (10) converge very slowly. We point out that shooting towards $\rho = 1$ fails completely for large negative λ because the solution $v_2(\rho)$ is subdominant at $\rho = 1$, that is, it is negligible with respect to the analytic solution $v_1(\rho)$.

III. THE CONTINUED FRACTIONS METHOD

In this section we shall solve the eigenvalue problem (7) using a method continued fractions. The key idea is to determine the analyticity properties of the power series solution $v_0(\rho)$ from the asymptotic behavior of the expansion coefficients a_n . Substituting the power series (9) into Eq. (7) we get the four-term recurrence relation (with the initial conditions $a_0 = 1$ (normalization) and $a_n = 0$ for $n < 0$)

$$p_3(n)a_{n+3} + p_2(n)a_{n+2} + p_1(n)a_{n+1} + p_0(n)a_n = 0, \quad (12)$$

where

$$a_n \sim c_1(\lambda)a_n^{(1)} + c_2(\lambda)a_n^{(2)} + c_3(\lambda)a_n^{(3)}. \quad (16)$$

If the coefficient $c_1(\lambda)$ is nonzero then

$$\frac{a_{n+1}}{a_n} \sim \frac{a_{n+1}^{(1)}}{a_n^{(1)}} \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty, \quad (17)$$

hence the power series (9) is divergent for $\rho > 1$ (in fact it has a branch point singularity at $\rho = 1$). On the other hand, if $c_1(\lambda) = 0$ then the solution $v_0(\rho)$ can be continued analytically through $\rho = 1$. The advantage of replacing the quantization condition $B(\lambda) = 0$ in the connection formula (11) by the equivalent condition $c_1(\lambda) = 0$ follows from the fact that $c_1(\lambda)$ is the coefficient of the dominant solution in (16), in contrast to $B(\lambda)$ which is the coefficient of the subdominant solution in (11).

Now, we shall find the zeros of the coefficient $c_1(\lambda)$ using the method of continued fractions. This method is based on an intimate relationship between three-term recurrence relations and continued fractions. It goes as follows. Suppose that we have a three-term recurrence relation (a second order difference equation)

$$b_{n+2} + A_n b_{n+1} + B_n b_n = 0. \quad (18)$$

Let $r_n = b_{n+1}/b_n$. Then, from (18) we have

TABLE I. The first 12 eigenvalues.

n	0	1	2	3	4	5
λ_n	1	-0.588 904	-2.181 597	-3.570 756	-5.043 294	-6.486 835
n	6	7	8	9	10	11
λ_n	-7.912 777	-9.298 265	-9.907 103	-10.792 456	-12.153 033	-13.164 487

$$r_n = -\frac{B_n}{A_n + r_{n+1}},$$

and applying this formula repeatedly we get the continued fraction representation of r_n

$$r_n = -\frac{B_n}{A_n} - \frac{B_{n+1}}{A_{n+1}} - \frac{B_{n+2}}{A_{n+2}} \dots \quad (19)$$

A theorem due to Pincherle [6] says that the continued fraction on the right hand side of Eq. (19) converges if and only if the recurrence relation (18) has a minimal solution b_n^{\min} , i.e. the solution such that $\lim_{n \rightarrow \infty} b_n^{\min}/b_n = 0$ for any other solution b_n . Moreover, in the case of convergence, Eq. (19) holds with $r_n = b_{n+1}^{\min}/b_n^{\min}$ for each n .

We cannot yet apply this theorem because our recurrence relation (12) has four terms. However, let us observe that

$$a_n = a_n^{\text{exact}} = \left(-\frac{3}{5}\right)^n \left[n + 1 + \frac{5}{16}(\lambda - 1)\right] \quad (20)$$

is the exact solution of our four-term recurrence relation (12) (although it does not satisfy the initial conditions). Using this solution we can reduce the order by the substitution (cf. [7] for an alternative way of reducing the order)

$$b_n = a_{n+1} + \frac{3}{5} \frac{n + 2 + \frac{5}{16}(\lambda - 1)}{n + 1 + \frac{5}{16}(\lambda - 1)} a_n \quad (21)$$

to get the three-term recurrence relation

$$q_2(n)b_{n+2} + q_1(n)b_{n+1} + q_0(n)b_n = 0, \quad (22)$$

where (using the abbreviation $\gamma = 5(\lambda - 1)/16$)

$$\begin{aligned} q_0(n) &= p_1(n) - \frac{3}{5} \frac{n + 3 + \gamma}{n + 2 + \gamma} p_2(n) \\ &\quad + \left(\frac{3}{5}\right)^2 \frac{n + 4 + \gamma}{n + 2 + \gamma} p_3(n), \\ q_1(n) &= p_2(n) - \frac{3}{5} \frac{n + 4 + \gamma}{n + 3 + \gamma} p_3(n), \quad q_2(n) = p_3(n). \end{aligned}$$

The two linearly independent asymptotic solutions of the recurrence relation (22) are

$$b_n^{(1)} \sim n^{\lambda-2} \sum_{s=0}^{\infty} \frac{\beta_s^{(1)}}{n^s}, \quad a_n^{(2)} \sim \left(-\frac{3}{5}\right)^n n^{-5} \sum_{s=0}^{\infty} \frac{\beta_s^{(2)}}{n^s}, \quad (23)$$

so in general

$$b_n \sim C_1(\lambda)b_n^{(1)} + C_2(\lambda)b_n^{(2)}. \quad (24)$$

Now, our quantization condition $c_1(\lambda) = 0$ is equivalent to $C_1(\lambda) = 0$ which is nothing else but the condition for the existence of a minimal solution for Eq. (22). Thus, we can use Pincherle's theorem to find the eigenvalues.

In our case $A_n = q_1(n)/q_2(n)$ and $B_n = q_0(n)/q_2(n)$. From (13) and (21) we get

$$b_0 = \left(\frac{25\lambda^2 + 125\lambda - 570}{350} + \frac{96 + 15(\lambda - 1)}{80 + 25(\lambda - 1)}\right)a_0, \quad (25)$$

$$b_{-1} = a_0.$$

Using Pincherle's theorem and setting $n = -1$ in (19) we obtain the eigenvalue equation

$$\begin{aligned} \frac{b_0}{b_{-1}} &= \frac{25\lambda^2 + 125\lambda - 570}{350} + \frac{96 + 15(\lambda - 1)}{80 + 25(\lambda - 1)} \\ &= -\frac{B_{-1}(\lambda)}{A_{-1}(\lambda)} - \frac{B_0(\lambda)}{A_0(\lambda)} - \frac{B_1(\lambda)}{A_1(\lambda)} \dots \end{aligned} \quad (26)$$

The continued fraction in (26), which by Pincherle's theorem is convergent for any λ , can be approximated with essentially arbitrary accuracy by downward recursion starting from a sufficiently large $n = N$ and some (arbitrary) initial value r_N . The roots of the transcendental Eq. (26) are then found numerically (see Table I).

We recall from [2] that the eigenvalue $\lambda_0 = 1$ is due to the freedom of changing the blowup time T . Although this eigenvalue is positive, it should not be interpreted as the physical instability of the solution U_0 —it is an artifact of introducing the similarity variables and does not show up in the dynamics for $u(t, r)$. Note that, strangely enough, all the eigenvalues are real.

IV. CONVERGENCE TO THE ATTRACTOR

According to the linear stability analysis presented above the convergence of the solution $u(t, r)$ towards the self-similar attractor U_0 should be described by the formula

$$\begin{aligned} u(t, r) &= U_0(\rho) + \sum_{k=1} c_k e^{\lambda_k \tau} v_k(\rho)/\rho \sim U_0(\rho) \\ &\quad + c_1 e^{\lambda_1 \tau} v_1(\rho)/\rho \quad \text{as } \tau \rightarrow \infty, \end{aligned} \quad (27)$$

where $v_k(\rho)/\rho$ are the eigenmodes corresponding to the eigenvalues λ_k and c_k are the expansion coefficients. In order to verify the formula (27) we solved Eq. (1) numeri-

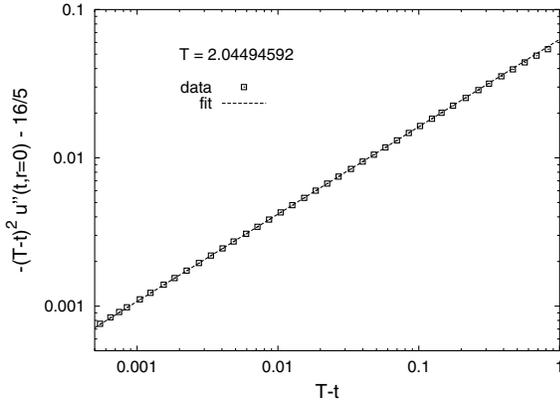


FIG. 3. In this figure we determine the coefficient c_1 in the formula (27) as follows. Taking the second derivative of (27) at $r = 0$ we obtain $(T - t)^2 u_{rr}(t, r = 0) + 16/5 = 2c_1(T - t)^{-\lambda_1}$. For small $T - t$ we plot the left hand side of this equation in the log-log scale and fit the parameters c_1 and λ_1 . We get $c_1 = -0.031436$ and $\lambda_1 = -0.58803$ (in good agreement with Table I).

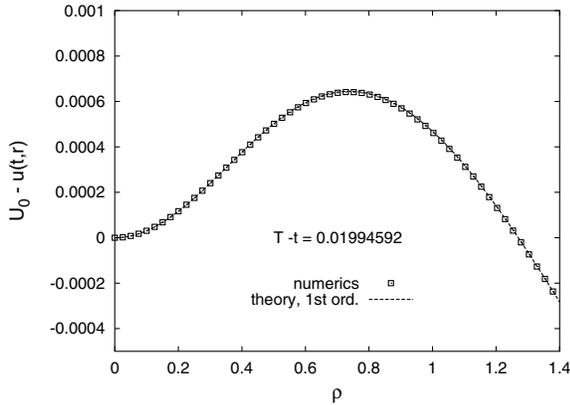


FIG. 4. We plot the deviation of the dynamical solution $u(t, r)$ from the self-similar solution U_0 at some moment of time close to the blowup time. The solid line shows the least damped eigenmode $c_1(T - t)^{-\lambda_1} v_1(\rho)/\rho$ with the coefficient c_1 obtained in Fig. 3.

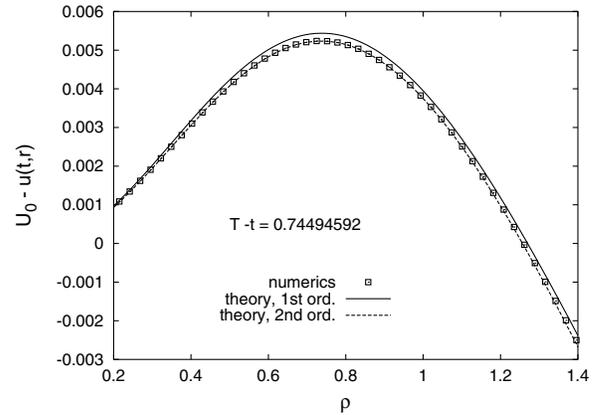


FIG. 5. The same plot as in Fig. 4 but at an earlier moment of time. The solid line showing the least damped eigenmode (with the same coefficient c_1 as before) does not fit well the numerical data because the other modes have not yet decayed. Including the suitably normalized second eigenfunction in the expansion (27) we get a much better fit (dashed line). We remark that a few first eigenvalues can be obtained directly from (27) by solving numerically the linearized evolution Eq. (6) [8].

cally for large initial data leading to blowup, expressed the solution in the similarity variables, and computed the deviation from U_0 for $t \nearrow T$. Figures 3 and 4 show that for small $T - t$ the deviation from U_0 is very well described by the least damped eigenmode v_1 , in agreement with the formula (27). For larger $T - t$ the contribution of higher modes has to be included (see Fig. 5).

ACKNOWLEDGMENTS

This research was supported in part by the Polish Ministry of Science Grant No. 1P03B01229. P.B. acknowledges the friendly hospitality of the Albert Einstein Institute during part of this work.

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