Anomalous axion interactions and topological currents in dense matter

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Recently an effective Lagrangian for the interactions of photons, Nambu-Goldstone bosons and superfluid phonons in dense quark matter has been derived using anomaly matching arguments. In this paper we illuminate the nature of certain anomalous terms in this Lagrangian by an explicit microscopic calculation. We also generalize the corresponding construction to introduce the axion field. We derive an anomalous axion effective Lagrangian describing the interactions of axions with photons and superfluid phonons in the dense matter background. This effective Lagrangian, among other things, implies that an axion current will be induced in the presence of magnetic field. We speculate that this current may be responsible for the explanation of neutron star kicks.

I. INTRODUCTION

It has been recently realized [1] that some very unusual effects may take place in dense matter QCD in the presence of topological defects and/or external magnetic field. It is known that at large baryon density many global symmetries of QCD are spontaneously broken [2,3]. In particular, it is expected that the chiral symmetry will be spontaneously broken. This leads to the appearance of pseudoscalar Goldstone bosons, which are generated out of the vacuum by axial chiral currents. The spontaneous breaking of global symmetries also leads to the existence of topological defects: domain walls and strings in dense QCD [4,5]. In a recent paper [1] it was shown that the effective Lagrangian for the interaction of Goldstone bosons with the electromagnetic field in the presence of chemical potential μ contains terms, which imply that topological defects such as axial vortices and domain walls in dense QCD carry electric current and magnetization, respectively. The corresponding effective Lagrangian was derived in [1] in a formal way by treating the fermion chemical potential as the zeroth component of a fictitious vector gauge field V_μ and appealing to chiral anomalies induced by V_μ.

The main goal of this paper is twofold. First, we observe that one of the terms in the effective Lagrangian derived in [1] implies that flux tubes in dense quark matter carry an axial current. We investigate the microscopic origin of this current. Second, we generalize the derivation [1] to include the axion field which, if exists, may play an important role in astrophysics and cosmology.

This paper is organized as follows. In Sec. II, we will be studying axial currents on flux tubes in dense matter. We will confirm the existence of these currents in three ways: (a) by appealing to the effective Lagrangian derived in [1], (b) directly from the chiral anomaly due to the fictitious field V_μ, (c) from a microscopic calculation. These three methods agree.

In Sec. III, we derive the anomalous axion effective Lagrangian using previously established methods. The corresponding effective Lagrangian describes the interactions of axions with photons and superfluid phonons in the dense matter background. We speculate regarding some phenomenological implications of the obtained results.

II. AXIAL CURRENT ON MAGNETIC FLUX TUBES

A. Short overview

The appearance of fermion zero modes on topological defects is a mathematically rich phenomenon with links to index theory of elliptic operators and quantum anomalies, see original papers [6] and review [7]. In particular, it is known [6–8] that flux tubes in $2 + 1$ dimensional QED possess fermion zero modes, whose existence can be proved using trace identities and 2-dimensional Euclidean chiral anomaly equations. In the context of $2 + 1$ dimensional QED these zero modes lead to a degeneracy of the flux tube ground state and induction of a Chern-Simons term in the effective bosonic action. However, when we consider massless QED in $3 + 1$ dimensions, the zero modes are free to move up or down the flux tube, depending on their chirality and polarization. We show in this paper, that for massless Dirac fermions in $3 + 1$ dimensions at finite fermion chemical potential μ and temperature T in the background of a magnetic flux tube, the zero modes generate an axial current of $J = (e\mu/2\pi^2)\Phi$ along the flux tube, where e is the fermion charge and Φ is the total magnetic flux. The current is topological in nature as it depends only on the total flux and not on the particular details of distribution of the magnetic field. This result is *exact* (at least if the magnetic flux does not fluctuate) as the contribution of all, but the zero modes to the axial current vanishes.

As already noted, the appearance of axial current on flux tubes can be derived by using a somewhat different aspect of chiral anomalies. Here the anomaly resides directly in $3 + 1$ dimensions and appears when one thinks of the fermion chemical potential as the zeroth component of a

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fictitious vector gauge field V_μ. Such a gauge field, as well as the ordinary electromagnetic field A_μ, contributes to the anomalous nonconservation of axial current. If the chiral symmetry is spontaneously broken in the system under consideration (say in dense QCD) and a corresponding Goldstone boson η appears, one requires the effective Lagrangian for η to reproduce the axial current nonconservation. As will be shown below, the effective Lagrangian for η, originally derived in [1], then implies the appearance of axial current on flux tubes, which agrees with the microscopic zero mode result. We also show that, alternatively, one can obtain this result starting directly from the axial current nonconservation equation, without appealing to the effective Lagrangian for the η boson. This later method, as well as the microscopic derivation, imply that axial current on flux tubes appears even if the chiral symmetry is not broken spontaneously.

We note that although the presence of zero modes on flux tubes in QED and the symmetry of higher energy states, which makes the microscopic calculations of this paper possible, have been previously known [8], the problem of calculation of axial current on flux tubes at finite chemical potential and temperature, to our knowledge, has not been considered before. Moreover, the computation of the axial current at finite chemical potential using fictitious anomalies in $3 + 1$ dimensions is certainly a new trick. This work confirms the validity of this trick by an explicit microscopic calculation, thus, supporting the original derivation of the anomalous effective Lagrangian [1]. We note the similarity of computation of axial current on flux tubes in QED undertaken in the present paper to the recent computation of electric current on cosmic strings at finite chemical potential [9], which was also motivated by the anomalous terms in the effective Lagrangian for Goldstones in dense QCD. Both calculations rely on the idea of zero modes with fixed quantum numbers running along a 2-dimensional topological object uniform in the 3rd direction and yield similar results for the current.

B. Anomalous terms

Here we briefly review the anomaly based arguments of [1], which indicate that when an axial-like symmetry is spontaneously broken at finite fermion chemical potential, the effective Lagrangian for the corresponding Goldstone mode receives a correction, topological in nature. We show that in the presence of a background magnetic field, this correction leads to the appearance of axial current on magnetic flux tubes.

Consider QCD at large baryon chemical potential. As is well known, such a system spontaneously breaks various global symmetries of QCD [2,3], leading to the existence of Goldstone bosons, whose number and form depends on the number of light (massless) quark flavors N_f. Here, we

consider a general neutral Goldstone boson η,[1] with the following transformation properties under one of the diagonal axial symmetries of QCD[2]:

$$\psi_a \rightarrow e^{iQ_a\theta\gamma^5}\psi_a, \qquad \eta \rightarrow \eta + \theta. \qquad (1)$$

Here Q_a denotes the flavor content of the Goldstone boson, $a = 1 \ldots N_f$, and η is created out of the vacuum by the current,

$$j^\mu = \sum_a Q_a \bar{\psi}_a \gamma^\mu \gamma^5 \psi_a. \qquad (2)$$

As is well known, it is useful to represent quark chemical potentials as the zeroth components of a fictitious vector gauge field $V_\mu = (1, \vec{0})$. Then the coupling of quarks to V_μ and to the usual electromagnetic gauge field A_μ takes the form

$$\mathcal{L} = \sum_a (\mu_a V_\mu - e_a A_\mu)\bar{\psi}_a \gamma^\mu \psi_a, \qquad (3)$$

where μ_a and e_a are quark chemical potentials and electromagnetic charges, respectively. The anomaly equation for the current j^μ in the background of fields V_μ and A_μ takes the form

$$\partial_\mu j^\mu = \epsilon^{\mu\nu\lambda\sigma}(C_{\eta AA}F_{\mu\nu}F_{\lambda\sigma} + C_{\eta AV}V_{\mu\nu}F_{\lambda\sigma}$$
$$+ C_{\eta VV}V_{\mu\nu}V_{\lambda\sigma}), \qquad (4)$$

where the field tensors $F_{\lambda\sigma}$, $V_{\lambda\sigma}$ are defined as $F_{\lambda\sigma} \equiv \partial_\lambda A_\sigma - \partial_\sigma A_\lambda$, $V_{\lambda\sigma} \equiv \partial_\lambda V_\sigma - \partial_\sigma V_\lambda$, and the coefficients,

$$C_{\eta AA} = -N_c \sum_a \frac{e_a^2 Q_a}{16\pi^2}, \qquad C_{\eta AV} = N_c \sum_a \frac{e_a \mu_a Q_a}{8\pi^2},$$
$$C_{\eta VV} = -N_c \sum_a \frac{\mu_a^2 Q_a}{16\pi^2}. \qquad (5)$$

The anomalous current nonconservation (4) must be reproduced in the effective Lagrangian for the neutral Goldstone boson η. Thus, as was shown in [1], the effective Lagrangian for η picks up the following anomalous term describing its interaction with the fields A_μ and V_μ:

$$L_\eta = L_\eta^0 + 2\partial_\mu\eta\,\epsilon^{\mu\nu\lambda\sigma}(C_{\eta AA}A_\nu F_{\lambda\sigma} + C_{\eta AV}V_\nu F_{\lambda\sigma}$$
$$+ C_{\eta VV}V_\nu V_{\lambda\sigma}). \qquad (6)$$

Here L_η^0 is the standard, nonanomalous part of the effective Lagrangian for η, which transforms as $L^0 \rightarrow L^0 - j^\mu \partial_\mu \theta$ under (1). We now restore the fictitious field V_μ to its true

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[1]This need not be *the* "η" boson of QCD.

[2]As was shown in [4], all instanton effects are suppressed in high-density QCD and at very large baryon chemical potential the explicit breaking of the formally anomalous symmetry $U(1)_A$ becomes very weak.

value $V_\mu = (1, \vec{0})$. Then the last term in (6) vanishes[3], and we are left with

$$L_\eta = L_\eta^0 - C_{\eta AA}\eta\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} + 4C_{\eta AV}\nabla\eta \cdot \vec{B}, \quad (7)$$

where $\vec{B}$ is the magnetic field. The first anomalous term in Eq. (7) describes the usual anomalous decay of a Goldstone boson to two photons, and is absent on the classical level, if there is no background electric field present. We now concentrate on the second anomalous term in (7), which does not occur in vacuum (at $\mu = 0$):

$$L_\eta = L_\eta^0 + 4C_{\eta AV}\nabla\eta \cdot \vec{B}. \quad (8)$$

One effect of the new anomaly term originally discussed in [1] is the magnetization of the domain walls formed by the η field [such domain walls are possible, say, if η is associated with spontaneous breaking of the $U(1)_A$ symmetry, which is also explicitly slightly broken by instantons [4]]. Here we discuss a different consequence of this term. Let us vary the action obtained from Lagrangian (8), with respect to $\eta \to \eta + \theta$, to derive the classical equations of motion. By construction, $L^0 \to L^0 - j^\mu\partial_\mu\theta$, hence,

$$\delta S = \int d^4x(-j^\mu\partial_\mu\theta + 4C_{\eta AV}B^i\partial_i\theta)$$
$$= \int d^4x\partial_\mu j^\mu\theta + \int dt\int_{\partial R} dS_i(-j^i + 4C_{\eta AV}B^i)\theta. \quad (9)$$

Here the surface integral is over the boundary of the region R where our dense matter is realized. So, as $\nabla \cdot \vec{B} = 0$, the anomalous term does not contribute to the equation of motion $\partial_\mu j^\mu = 0$. However, if we do not restrict θ to vanish on the boundary, we also obtain a boundary condition,

$$\vec{j} \cdot d\vec{S} = 4C_{\eta AV}\vec{B} \cdot d\vec{S}. \quad (10)$$

Now, in the steady state situation, there is no buildup of axial charge, and we have $\nabla \cdot \vec{j} = 0$. Hence for any cross section S of the region R let S_b be the part of ∂R such that $\partial S = \partial S_b$. Then,

$$\int_S d\vec{S} \cdot \vec{j} = \int_{S_b} d\vec{S} \cdot \vec{j} = 4C_{\eta AV}\int_{S_b} d\vec{S} \cdot \vec{B}$$
$$= 4C_{\eta AV}\int_S d\vec{S} \cdot \vec{B} = N_c\sum_a \frac{e_a\mu_a Q_a}{2\pi^2}\Phi, \quad (11)$$

where Φ is the total magnetic flux through the cross section

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[3]However, as shown in [1] this term can become important if the quark matter is rotating and/or superfluid vortices appear.

S. So we see that the anomalous term in Eq. (8) implies the existence of an axial current flowing through the dense matter which is proportional to the magnetic flux.

At this point we make the following important remark regarding the formula (11): the final expression for the current does not depend on the specific properties of the pseudo-Goldstone boson η, such as its coupling constant f_η. This is not due to our choice of units, and this is not a typo, so it leads us to weaken our assumption of spontaneous chiral symmetry breaking and existence of the η Goldstone.

Indeed, the result (11) can be derived in the following way for a generic system of massless fermions at finite chemical potential in the magnetic field, without appealing to the effective Lagrangian for the η Goldstone boson. Let us return to the anomaly Eq. (4). We can think of the fictitious field V_0, as taking the value "1" inside the region R where the quark matter is realized and "0" outside (if we have a quark star, this is actually rather close to reality, since the interface between quark matter and vacuum is very narrow). Then, again, if no electric field is present, and if the axial charge density is time independent, Eq. (4) takes the form

$$\nabla \cdot \vec{j} = 4C_{\eta AV}\nabla \cdot (V_0\vec{B}). \quad (12)$$

The right-hand side of Eq. (12) vanishes both inside and outside R, yielding $\nabla \cdot \vec{j} = 0$. However, integrating (12) over a small Gaussian pillbox on the boundary of R, and recalling that in the vacuum outside R, $\vec{j} = 0$, we obtain on the inner boundary of R,

$$\vec{j} \cdot d\vec{S} = 4C_{\eta AV}\vec{B} \cdot d\vec{S}. \quad (13)$$

This is the same result (10) that we obtained by minimizing the action for the Goldstone η. From this, we readily obtain the expression (11) for the total current across any cross section of R.

The last derivation does not use anywhere the spontaneous breaking of chiral symmetry and the existence of the η Goldstone, and relies solely on chiral anomalies. Thus, when the dynamics of our problem are such that a Goldstone boson η exists, the appearance of axial current on flux tubes can be extracted from the effective Lagrangian (8) for the η field. However, the existence of such axial current does not depend on spontaneous chiral symmetry breaking, but rather on the mere existence of chiral symmetry. This observation is confirmed microscopically in the next subsection, where the result (11) is reproduced in a simple QED-like system.

C. Microscopic arguments

We will show in this section that the appearance of current on magnetic flux tubes at finite chemical potential derived in the previous section using the anomalous effective Lagrangian for η, can be understood very simply

microscopically within the following model. As we already saw at the end of the previous section, the existence of axial currents on flux tubes does not rely on spontaneous chiral symmetry breaking. Thus, we choose to ignore all the strong interaction effects leading to formation of the Goldstone field η, and consider only the following Lagrangian,

$$\mathcal{L} = \overline{\psi} i (\partial_\mu + ie A_\mu) \gamma^\mu \psi - m \overline{\psi} \psi + \mu \overline{\psi} \gamma^0 \psi, \quad (14)$$

which describes the interactions of a single light quark ψ of mass m with a background electromagnetic field A_μ, at finite baryon chemical potential μ. Hence, the discussion in this section actually applies to any QED-like system at finite chemical potential.

We are interested in the case of magnetic flux tubes, i.e. A_μ is static and the magnetic field $\vec{B} = \nabla \times \vec{A} = B(x, y)\hat{z}$ is uniform in the third direction z. Our goal is to compute the total axial current $J_5^3 = \int d^2x \langle \overline{\psi} \gamma^3 \gamma^5 \psi \rangle$ along the flux tube. The Dirac Hamiltonian is,

$$H = -i(\partial_i + ie A_i)\gamma^0 \gamma^i + m\gamma^0, \quad (15)$$

and the Dirac equation becomes

$$-H_R \psi_L + m\psi_R = E\psi_L, \quad (16)$$

$$m\psi_L + H_R \psi_R = E\psi_R, \quad (17)$$

where we use the conventions of Peskin and Schroeder and,

$$H_R = (-i\partial_i + e A_i)\sigma^i. \quad (18)$$

So, $\psi_L = (1/m)(E - H_R)\psi_R$, where

$$(H_R^2 + m^2)\psi_R = E^2 \psi_R. \quad (19)$$

Hence every eigenstate ψ_R of H_R with eigenvalue ϵ generates two solutions of the Dirac equation with energies $E = \pm\sqrt{\epsilon^2 + m^2}$ and,

$$\psi_\pm = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}_\pm$$
$$= (4(m^2 + \epsilon^2))^{-1/4} \begin{pmatrix} \pm((m^2 + \epsilon^2)^{\frac{1}{2}} \mp \epsilon)^{\frac{1}{2}} \psi_R \\ ((m^2 + \epsilon^2)^{\frac{1}{2}} \pm \epsilon)^{\frac{1}{2}} \psi_R \end{pmatrix}. \quad (20)$$

Now we concentrate on the right sector $H_R \psi_R = \epsilon \psi_R$. Because of invariance with respect to translation in the z direction, we go to momentum eigenstates $-i\partial_3 \psi_R = p_3 \psi_R$ (we take the third direction to be periodic of length L, and take the limit $L \to \infty$ at the end of the calculation). In each momentum sector, the operator H_R takes the form

$$H_R = p_3 \sigma^3 + H^\perp, \quad (21)$$

$$H^\perp = (-i\partial_a + e A_a)\sigma^a, \quad a = 1, 2. \quad (22)$$

We note that $\{\sigma^3, H^\perp\} = 0$. Hence, if $|\lambda\rangle$ is a properly normalized eigenstate of $H^\perp$ with eigenvalue λ then $\sigma^3|\lambda\rangle$ is a properly normalized eigenstate of $H^\perp$ with eigenvalue

$-\lambda$. So, all eigenstates of $H^\perp$ with nonzero eigenvalues are of the form $|\lambda\rangle, |-\lambda\rangle = \sigma^3|\lambda\rangle$, where $\lambda > 0$. Also, σ^3 maps zero eigenstates of $H^\perp$ to zero eigenstates of $H^\perp$ and hence we can classify all zero modes of $H^\perp$ by their eigenvalue under σ^3.

The eigenstates of H_R can now be expressed in terms of eigenstates of $H^\perp$. Clearly, $[H_R, H^{\perp 2}] = 0$, so H_R only mixes states $|\lambda\rangle, |-\lambda\rangle$. For $\lambda > 0$, we write

$$\psi_R = c_1|\lambda\rangle + c_2 \sigma^3|\lambda\rangle \quad (23)$$

where c_1, c_2 satisfy:

$$\begin{pmatrix} \lambda & p_3 \\ p_3 & -\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \epsilon \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \quad (24)$$

Hence $\epsilon = \pm\sqrt{\lambda^2 + p_3^2}$ and,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}_\pm = (4(\lambda^2 + p_3^2))^{-1/4}$$
$$\times \begin{pmatrix} \pm\text{sgn}(p_3)((\lambda^2 + p_3^2)^{1/2} \pm \lambda)^{1/2} \\ ((\lambda^2 + p_3^2)^{1/2} \mp \lambda)^{1/2} \end{pmatrix}. \quad (25)$$

So each eigenstate of $H^\perp$ with an eigenvalue $\lambda > 0$ generates two eigenstates of H_R.[4]

The zero modes of $H^\perp$ are simultaneously eigenstates of H_R with eigenvalue,

$$\epsilon = p_3 \sigma^3. \quad (26)$$

Hence, when the mass $m \to 0$, zero modes of $H^\perp$ become gapless modes of H capable of traveling up or down the flux tube depending on the sign of σ^3 and on the chirality. We will shortly see, that at finite chemical potential, precisely these modes carry an axial current along the flux tube.

The following quantity will be of particular importance to us: $N = N_+ - N_-$, where N_+ and N_- are the numbers of zero modes of $H^\perp$ with $\sigma^3 = 1$ and $\sigma^3 = -1$ respectively. Observe, that if $|\lambda\rangle$ is a zero mode of $H^\perp$ with $|\lambda\rangle = (u, v)$ then,

$$\mathcal{D}v = 0, \qquad \mathcal{D}^\dagger u = 0, \quad (27)$$

where,

$$\mathcal{D} = -i\partial_1 - \partial_2 + e(A_1 - iA_2). \quad (28)$$

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[4]There are known examples [7], such as fermion number appearing on domain walls, when this is not strictly speaking true. Indeed, some of the energy levels of H_R are continuous rather than discreet and the correspondence discussed above between the eigenstates of $H^\perp$ and H_R need not preserve the density of states. However, for the particular Hamiltonian H_R, it can be shown that if $B(x) \to 0$ as $x \to \infty$ sufficiently fast, this problem does not arise.

Hence $N_+ = \dim(\ker(\mathcal{D}^\dagger))$, $N_- = \dim(\ker(\mathcal{D}))$, and,

$$N = \text{index}(H^\perp) = N_+ - N_-$$
$$= \dim(\ker(\mathcal{D}^\dagger)) - \dim(\ker(\mathcal{D})). \qquad (29)$$

The index of the elliptic operator $H^\perp$ has been computed in numerous works [6–8] using two types of methods: (i) complex analysis methods, (ii) trace identities and axial Euclidean anomaly in 2 dimensions (this is particularly interesting in the light of our using 4 dimensional anomalies above to derive axial currents on flux tube at finite μ). The zero modes have also been computed exactly for some simple configurations of the gauge field [8]. In general the index is given by

$$\text{index}\,(H^\perp) = \frac{e\Phi}{2\pi}, \qquad (30)$$

$$\Phi = \int d^2x B^3(x). \qquad (31)$$

Hence the index measures the number of flux quanta through the xy plane, which is in essence a topological quantity.

Now let us proceed to compute the axial fermion current induced at finite chemical potential μ. For further generality, we also include the effects of nonzero temperature T. The axial current density in the third direction is given by

$$j_5^3(x) = \overline{\psi}(x)\gamma^3\gamma^5\psi(x) = \psi_L^\dagger\sigma^3\psi_L(x) + \psi_R^\dagger\sigma^3\psi_R(x). \qquad (32)$$

We wish to compute the expectation value of the total current along the flux tube, $J_5^3 = \int d^2x\langle j_5^3(x)\rangle$. At finite chemical potential and temperature we have,

$$\langle j_5^3(x)\rangle = \sum_E n(E)\psi_E^\dagger(x)\gamma^0\gamma^3\gamma^5\psi_E(x)$$
$$= \sum_\epsilon (n((\epsilon^2 + m^2)^{1/2})$$
$$+ n(-(\epsilon^2 + m^2)^{1/2}))\psi_{R\epsilon}^\dagger(x)\sigma^3\psi_{R\epsilon}(x). \qquad (33)$$

Here, $n(E) = [\text{sgn}(E)/(e^{\beta(E-\mu)\text{sgn}(E)} + 1)]$ is the usual Fermi-Dirac distribution, ψ_E are eigenstates of H with energy E, $\psi_{R\epsilon}$ are eigenstates of H_R with eigenvalue ϵ, and we have used Eq. (20). The explicit form of $\psi_{R\epsilon}$ in terms of eigenstates of $H^\perp$ implies,

$$\langle J_5^3\rangle = \frac{1}{L}\sum_{p_3}\sum_{\lambda>0}\sum_{s=\pm}(n((\lambda^2 + p_3^2 + m^2)^{1/2})$$
$$+ n(-(\lambda^2 + p_3^2 + m^2)^{1/2}))$$
$$\times \langle\psi_R^s(\lambda, p_3)|\sigma^3|\psi_R^s(\lambda, p_3)\rangle \qquad (34)$$

$$+ \frac{1}{L}\sum_{p_3}\sum_{\lambda=0}(n((p_3^2 + m^2)^{1/2})$$
$$+ n(-(p_3^2 + m^2)^{1/2}))\langle\lambda|\sigma^3|\lambda\rangle. \qquad (35)$$

Here $\lambda > 0$ label eigenstates of $H^\perp$, which generate eigenstates $\psi_R^\pm(\lambda, p_3)$ of H_R with momentum p_3 and eigenvalue $\epsilon_\pm = \pm\sqrt{\lambda^2 + p_3^2}$, while $\lambda = 0$ label the zero modes of $H^\perp$. Now, let us evaluate the matrix element $\langle\psi_R^s(\lambda, p_3)\times|\sigma^3|\psi_R^s(\lambda, p_3)\rangle$ for $\lambda > 0$. Using Eq. (23) and dropping the subscripts λ, p_3, s, we obtain, $\langle\psi_R|\sigma^3|\psi_R\rangle = (|c_1|^2 + |c_2|^2)\langle\lambda|\sigma^3|\lambda\rangle + (c_2^*c_2 + c_1c_2^*)$. Noting, $\langle\lambda|\sigma^3|\lambda\rangle = \langle\lambda| - \lambda\rangle = 0$ for $\lambda > 0$, and using the explicit formula (25) for c_1, c_2, we obtain, $\langle\psi_R|\sigma^3|\psi_R\rangle = sp_3(\lambda^2 + p_3^2)^{-1/2}$. This matrix element is odd in both p_3 and s, hence the sum over all $\lambda > 0$ in Eq. (34) vanishes , and only the zero modes of $H^\perp$ contribute to J_5^3. The zero modes carry a definite value of σ_3, so that $\langle\lambda|\sigma^3|\lambda\rangle = \sigma^3$. Thus, we are left with,

$$J_5^3 = (N_+ - N_-)\frac{1}{L}\sum_{p_3}(n((p_3^2 + m^2)^{1/2})$$
$$+ n(-(p_3^2 + m^2)^{1/2})) = \frac{e\Phi}{2\pi}n_m(T, \mu), \qquad (36)$$

$$n_m(T, \mu) = \int\frac{dp_3}{2\pi}(n((p_3^2 + m^2)^{1/2})$$
$$+ n(-(p_3^2 + m^2)^{1/2})). \qquad (37)$$

Here, $n_m(T, \mu)$ is just the number density of one-dimensional two-component (Dirac) fermions of mass m at finite temperature T and chemical potential μ. Hence, our final result (36) is topological in nature, since for each value of T and μ, it is sensitive only to the total magnetic flux and is independent of the particular distribution of the magnetic field.

Several limits of the result (36) are noteworthy. First of all, in the massless limit $m \to 0$, one has $n(T, \mu) = \frac{\mu}{\pi}$ and,

$$J_5^3 = \frac{e\mu}{2\pi^2}\Phi. \qquad (38)$$

If there are several species of quarks present, we can sum Eq. (38) over quark flavors and colors to obtain the current J^3 of Eq. (2), which in the true dense QCD creates the η boson,

$$J^3 = N_c\sum_a\frac{e_aQ_a\mu_a}{2\pi^2}\Phi. \qquad (39)$$

This agrees with our result (11) of the previous subsection, where we explicitly used the fact $m = 0$ (and, hence, chiral symmetry) in assuming that the axial current conservation is violated only by anomalies. So, we see that the appearance of axial current on flux tubes, which was derived somewhat mysteriously in the previous section using the trick of fictitious chiral anomalies, is microscopically due to fermion zero modes. Our microscopic approach supports

the validity of the fictitious chiral anomaly trick and serves as a check of the anomalous effective Lagrangian derived in [1].

Let us note that the result (38) is also independent of temperature for $m = 0$, which is a quite natural feature of a truly topological phenomenon. More explicitly, this fact is due to the special property of massless one-dimensional fermions, namely, their density at finite chemical potential is temperature independent.

For arbitrary mass $m \neq 0$, the density of one-dimension fermions $n(T, \mu)$ is generally temperature dependent, so for simplicity we consider the limit $T = 0$. Then, $n(0, \mu) = \sqrt{\mu^2 - m^2}/\pi$ and,

$$J_5^3 = \frac{e\sqrt{\mu^2 - m^2}}{2\pi^2} \Phi. \qquad (40)$$

It is instructive to take the nonrelativistic limit of Eq. (40). Writing, $\mu = m + \mu_{nr}$, where the nonrelativistic chemical potential $\mu_{nr} \ll m$,

$$J_5^3 \approx \frac{e\sqrt{2m\mu_{nr}}}{2\pi^2} \Phi. \qquad (41)$$

In the nonrelativistic setting, J_5^3 is just the spin S^3, and for the case of uniform magnetic field, our result stems from the familiar fact that all Landau levels are doubly degenerate with respect to spin, except the lowest Landau level. It is amazing that this simple fact has such deep connections to chiral anomalies in 2 and 4 dimensions.

III. AXION

This section is devoted to the derivation of the anomalous effective Lagrangian including the axion field using some previously developed methods. While the axion is considered to be one of the best dark matter candidates (see original papers [10–13] and reviews [14]), it has not been discovered yet. We derive novel low energy terms, which describe the interaction of the axion field with other light particles: photons and superfluid phonons in dense matter background. These terms may lead to phenomenologically important effects related to the axion astrophysics, which were not discussed previously.

We define the θ term in the fundamental QCD Lagrangian in the standard way, $L_\theta = (g^2\theta/32\pi^2)\tilde{G}^{\mu\nu a}G_{\mu\nu a}$. The existence of the θ term implies a violation of P, CP and T symmetries. However, there is no experimental evidence for P or CP violation in strong interactions. For example, CP violation in QCD would induce electric dipole moments of strongly interacting particles and there are stringent experimental limits on those quantities. Thus the absence of CP violating effects in QCD indicates a very small value for the parameter θ: why is θ so small?

The most elegant resolution was proposed by Peccei and Quinn who assumed that the strong interactions

Lagrangian has a global $U(1)_{\mathrm{PQ}}$ chiral symmetry [10]. Weinberg and Wilczek [11] analyzed the consequences of the Peccei-Quinn symmetry and noticed that the spontaneous breaking of a global chiral symmetry $U(1)_{\mathrm{PQ}}$ leads to a light pseudoscalar pseudo-Goldstone boson, called an axion, that will interact with topological charge density, $(g^2/32\pi^2)\tilde{G}^{\mu\nu a}G_{\mu\nu a}$. In papers [12,13] two different types of the invisible axion models were suggested where it was demonstrated that the strong CP problem in QCD can be successfully solved with arbitrarily weak axion coupling constant.

The only information which is relevant for us in what follows is the transformation properties of quarks under $U(1)_{\mathrm{PQ}}$ chiral symmetry,

$$\psi_a \rightarrow e^{i\alpha Q_a^{\mathrm{PQ}}\gamma^5}\psi_a, \qquad (42)$$

where Q_a^{PQ} is the PQ charge for quark species ψ_a. We should note that leptons and Higgs bosons have also non-trivial transformation properties under $U(1)_{\mathrm{PQ}}$ symmetry, however this part is not essential for the present paper. We note that the axion interacts with gluons through a triangle diagram, $L_{agg} = \frac{a}{f_a}\frac{g^2}{32\pi^2}\tilde{G}^{\mu\nu a}G_{\mu\nu a}$. It also interacts with photons $L_{a\gamma\gamma} \sim g_{a\gamma\gamma}a\tilde{F}^{\mu\nu}F_{\mu\nu}$ with a coupling constant $g_{a\gamma\gamma} \sim 1/f_a\sum_a Q_a^{\mathrm{PQ}}e_a^2$ expressed in terms of electric e_a and PQ charges Q_a^{PQ} of all quarks and leptons. In what follows, we identify the axion field with the dimensionless phase $\alpha(x)$ (42). Physical, dimensional field $a \sim f_a\alpha$.

A. Anomalous axion Lagrangian

The key observation here is as follows. The relevant for this work transformation properties of quarks under chiral rotations (1) and under PQ rotations (42) are very similar, $Q_a \rightarrow Q_a^{\mathrm{PQ}}$. Therefore, we can literally follow our previous calculations (Sec. II) in order to derive the anomalous effective Lagrangian for the axion field (which replaces the Goldstone field η) in the presence of chemical potential μ. We use the same trick [1] in this derivation by representing quark chemical potential as the zeroth component of a fictitious field V_μ. The result of this calculation is almost identical to (6),

$$L_\alpha = L_\alpha^0 + 2\partial_\mu\alpha\epsilon^{\mu\nu\lambda\sigma}(C_{\alpha\gamma\gamma}A_\nu F_{\lambda\sigma} + C_{\alpha\gamma V}V_\nu F_{\lambda\sigma}$$
$$+ C_{\alpha VV}V_\nu V_{\lambda\sigma}), \qquad (43)$$

where L_α^0 describes all nonanomalous terms including the axion kinetic term $f_a^2(\partial_\mu\alpha)^2$, as well as different interaction terms of the axion with quarks and leptons. Coefficients $C_{\alpha\gamma\gamma}$, $C_{\alpha\gamma V}$, $C_{\alpha VV}$ can be easily extracted from the calculation of the triangle diagram and are given by

$$C_{\alpha\gamma\gamma} = -\sum_a \frac{e_a^2 Q_a^{\text{PQ}}}{16\pi^2}, \qquad C_{\alpha\gamma V} = \sum_a \frac{e_a \mu_a Q_a^{\text{PQ}}}{8\pi^2},$$

$$C_{\alpha VV} = -\sum_a \frac{\mu_a^2 Q_a^{\text{PQ}}}{16\pi^2}, \tag{44}$$

where label a runs over all species (colors) of particles with nonzero PQ charges including quarks and leptons. The first term $\sim C_{\alpha\gamma\gamma}\alpha\tilde{F}^{\mu\nu}F_{\mu\nu}$ in Eq. (43) is the well known interaction between the axion and photons.[5] In particular, this term describes the axion decay into two photons. It also describes the axion $\leftrightarrow \gamma$ transitions in the presence of the magnetic field B. The corresponding effect plays a crucial role in most axion search experiments. We shall not discuss this term in the present paper.

Two other terms are new, and as far as we know, these terms have never been discussed in the literature. The last term in Eq. (43) vanishes in the topologically trivial background. However, in the presence of superfluid vortices similar to the case discussed in [1], this term, among other things, describes the axion $\leftrightarrow$ superfluid phonon transitions. It might be phenomenologically important in rotating neutron stars.

B. Possible applications

Let us concentrate on the middle term in brackets in Eq. (43). We rewrite it in terms of the physical fields in the following way:

$$L_\alpha = 4C_{\alpha\gamma V}\vec{\nabla}\alpha \cdot \vec{B}. \tag{45}$$

This interaction explicitly shows that a Peccei-Quinn current corresponding to the Peccei-Quinn symmetry will be induced in the presence of an external magnetic field. Indeed, one can literally follow the derivation (11) to get the following expression for the Peccei-Quinn current,

$$J^{\text{PQ}} = \int_S d\vec{S} \cdot \vec{j} = 4C_{\alpha\gamma V}\int_S d\vec{S} \cdot \vec{B} = \sum_a \frac{e_a \mu_a Q_a^{\text{PQ}}}{2\pi^2}\Phi, \tag{46}$$

where Φ is the total magnetic flux through the cross section S, and we have neglected the lepton chemical potentials. So we see that the anomalous term in Eq. (45) implies the existence of a Peccei-Quinn current (46) flowing through the dense matter, which is proportional to the magnetic flux.

A few remarks are in order. First, as we noticed previously, the result (46) does not depend on f_a similar to the previous case (11) when the result did not depend on f_η. It may look very suspicious because one can make f_a arbitrarily large, which corresponds to an arbitrarily small

interaction of the original fermions with the axion. However, the point is that the current (46) corresponds to the equilibrium state in an infinitely large bulk of matter. The relevant question for the present case is: what is the formation time for such a current? It is obvious that with f_a increasing, the formation time also increases, such that there is no contradiction with Eq. (46) being independent of f_a.

Our next remark is that the Peccei-Quinn (46) as well as the axial current (11) are unique due to their topological nature. Indeed, even in the strongly interacting theory the axial current (11) is persistent and nondissipating. It means that even in such an unfriendly environment as the dense quark/nuclear matter in neutron stars the current does not dissipate due to rescattering and can be effectively used to deliver information across the bulk of the star. Therefore, there is a unique opportunity here to use our topological currents (11) and (46) for delivering the asymmetry produced in the bulk of the star to solve the problem of neutron star kicks [15,16].

The problem can be explained as follows. As is known, pulsars exhibit rapid proper motions characterized by a mean birth velocity of 450 ± 90 km/s. Their velocities range from 100 to 1600 km/s [15], while their distribution leans toward the high-velocity end, with about 15% of all pulsars having speeds over 1000 km/s [16]. Pulsars are born in supernova explosions; therefore, it would be natural to look for an explanation in the internal dynamics of the supernova. However, three-dimensional numerical simulations [17] show that even the most extreme asymmetric explosions do not produce pulsar velocities greater than 200 km/s. Therefore, a different explanation should be found. The origin of these motions has been the subject of intense study and several possible explanations have been proposed. Many of the suggested mechanisms are capable ("in principle") to produce the required asymmetry. Indeed, in the presence of an external magnetic field, the produced neutrinos are automatically asymmetric with respect to the direction of $\vec{B}$. However, the main common problem suffered by most suggested mechanisms is the difficulty of delivering the produced asymmetry to the surface of the star. Only in this case, the asymmetry may result in producing the proper motion of the entire star. To overcome the difficulty with delivery of the produced asymmetry to the surface, some proposals, for example, are based on new particles (such as a sterile neutrino), which could escape from the bulk of the neutron star and deliver the asymmetry to the surface, see e.g. [18,19].[6] Our

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[5]In the literature devoted to the axion [14] one typically includes in $C_{\alpha\gamma\gamma}$ the effects due to the mixing of the axion with Goldstone fields such as the pion. We ignore this mixing for all qualitative discussions in what follows.

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[6]It is interesting to note that the interaction (45) is a precise realization of the idea [19] that the magnetic field may be correlated with the momentum of a very weakly interacting particle, which can easily escape the star (Majoron or sterile neutrino as suggested in [19]). Dynamics, more precisely, anomaly, does the job of correlating the magnetic field with the weakly interacting axion current.

main observation here is that due to their topological nature, the currents (11) and (46) may be capable of delivering the required asymmetry (produced in the interior of the star) to the surface without dissipation [20].

IV. CONCLUSION AND FUTURE DIRECTIONS.

In this paper we have discussed the appearance of axial current on magnetic flux tubes at finite fermion chemical potential using several approaches. All of these approaches are weaved together by chiral anomalies. Microscopically, the current can be understood in terms of fermion zero modes on the flux tube. These fermion zero modes are in a certain sense themselves due to anomaly in 2 dimensional Euclidean field theory and have implications for the $2 + 1$ dimensional QED. Thus, we see that our trick with fictitious anomalies at finite chemical potential in $3 + 1$ dimensions, in a sense continues the propagation of anomaly from 2 to 3 to 4 dimensions. This is a common pattern in the study of anomalies.

We anticipate a number of different applications of the derived anomalous effective low energy Lagrangian for the Goldstone bosons and axion in dense matter. Some of them were mentioned in the original paper [1], others were mentioned in the present text. One specific application, which we believe deserves further study is the explanation of neutron star kicks [20]. In addition, novel anomalous effective Lagrangian including the axion field might be quite important for analysis of a number of astrophysical problems, which would be the subject of a future work.

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