

Batalin-Fradkin-Tyutin method for mixed constrained systems and Chern-Simons theoryM. Monemzadeh^{1,*} and A. Shirzad^{1,2,†}¹*Department of Physics, Isfahan University of Technology (IUT), Isfahan, Iran,*²*Institute for Studies in Theoretical Physics and Mathematics (IPM), P. O. Box: 19395-5531, Tehran, Iran*

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We show that the BFT embedding method is problematic for mixed systems (systems possessing both first and second class constraints). The Chern-Simons theory as an example is worked out in detail. We give two methods to solve the problem leading to two different types of finite order BFT embedding for Chern-Simons theory.

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I. INTRODUCTION

Canonical quantization of constrained systems is fully established in the framework of Dirac theory [1]. As is well-known, in the case of second class systems one should convert Dirac brackets to quantum commutators; while for first class systems one constructs the quantum space of states as some representation of all quantized operators (i.e. phase space coordinates) and then imposes the conditions $\Phi_a|\text{phys}\rangle = 0$, where Φ_a are first class constraints and $|\text{phys}\rangle$ means physical states.

Working with first class systems seems to be appealing for some reasons; firstly, because the symmetries and covariance of the classical theory are manifestly demonstrated; secondly, since converting Dirac brackets to quantum commutators sometimes implies factor ordering problem and quantization of these models is not formal; thirdly, because inverting the matrix of Poisson brackets of constraints, which is necessary for writing the Dirac brackets, is not generally an easy task; and finally the most important reason is that the construction of a BRST charge is possible only for first class systems [2,3]. Therefore, there are some efforts to convert a second class system to a first class one [4–6]. The method, recognized as the BFT method, is based on extending the phase space to include a set of new variables and then writing the constraints, as well as the physical quantities, as power series in terms of these added variables.

However, as we will explain in the following, the traditional BFT method is formulated only for pure second class systems [7], while in the general case both first and second class constraints may emerge in the same model. An important example of this case, i.e. *mixed constrained systems*, is the Chern-Simons theory (abelian and nonabelian). After a brief review in the next section of the finite order BFT method, as proposed in [8] for a pure second class system, we will show in Sec. III that in fact it is not possible to embed the second class constraints in a larger space separately. That is, when one tries to convert the second class constraints into first class ones via embed-

ding, the algebra of the original first class constraints may change; in other words, they will not necessarily remain first class.

We will investigate the origin of this violence and search for conditions that can guarantee the embedding of second class constraints without violating the involuting algebra of first class ones. We show that the nonabelian Chern-Simons theory is a special example which exhibits this violence. In Sec. IV we propose two distinct methods that help us to solve the problem. The first method concerns redefining the constraints so that their algebra fulfill the required condition. In the next method we suggest that at first stage one may convert the first class constraints into second class ones by means of adding some auxiliary variables, and then one is able to run the procedure of the usual BFT method. We will show that this suggestions enables us to construct BFT embedding for Chern-Simons theory.

II. FINITE ORDER BFT EMBEDDING

Consider a pure second class constrained system described by the Hamiltonian H_0 in some phase space with coordinates (q^i, p_i) where $i = 1, 2, \dots, K$. Assume we are given a set of second class constraints, $\tau_\alpha^{(0)} \alpha = 1, \dots, m$, satisfying the algebra

$$\Delta_{\alpha\beta} = \{\tau_\alpha^{(0)}, \tau_\beta^{(0)}\} \quad (1)$$

where $\{\cdot\}$ means Poisson bracket and $\Delta_{\alpha\beta}$ is an invertible matrix. To convert this second class system into a gauge system, i.e. a first class system, one should extend the phase space by introducing the same number of auxiliary variables as that of second class constraints. We denote these variables by η_α and assume that they obey the following algebra;

$$\{\eta_\alpha, \eta_\beta\} = \omega_{\alpha\beta} \quad (2)$$

where $\omega_{\alpha\beta}$ is an antisymmetric invertible matrix which may be proposed arbitrarily. The first class constraints in the extended phase space $(q, p) \oplus \eta$ are defined as

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$$\tau_\alpha(q, p, \eta) = \sum_{n=0}^{\infty} \tau_\alpha^{(n)} \quad \alpha = 1, 2, \dots, m \quad (3)$$

where $\tau_\alpha^{(n)}$ is of order n with respect to η_α 's and

$$\tau_\alpha^{(0)} = \tau_\alpha(q, p, 0). \quad (4)$$

In the abelian BFT embedding method one demands that these extended constraints be strongly involuting:

$$\{\tau_\alpha, \tau_\beta\} = 0. \quad (5)$$

Substituting Eq. (3) into Eq. (5) leads to a set of recursive relations. Vanishing of the term independent of η gives:

$$\{\tau_\alpha^{(0)}, \tau_\beta^{(0)}\} + \{\tau_\alpha^{(1)}, \tau_\beta^{(1)}\}_{(\eta)} = 0; \quad (6)$$

and vanishing of the term of order n with respect to η_α 's for $n \geq 1$ gives

$$\{\tau_{[\alpha}^{(1)}, \tau_{\beta]}^{(n+1)}\}_{(\eta)} + B_{\alpha\beta}^{(n)} = 0 \quad n \geq 1 \quad (7)$$

where

$$B_{\alpha\beta}^{(1)} \equiv \{\tau_{[\alpha}^{(0)}, \tau_{\beta]}^{(1)}\}, \quad (8)$$

$$B_{\alpha\beta}^{(n)} \equiv \frac{1}{2} B_{[\alpha\beta]}^{(n)} \equiv \sum_{m=0}^n \{\tau_\alpha^{(n-m)}, \tau_\beta^{(m)}\} + \sum_{m=0}^{n-2} \{\tau_\alpha^{(n-m)}, \tau_\beta^{(m+2)}\}_{(\eta)} \quad n \geq 2. \quad (9)$$

The suffix η in the above equations means that the Poisson brackets must be evaluated with respect to η variables only, otherwise they are calculated in the basis (q, p) . The above equations are used iteratively to obtain the correction terms $\tau^{(n)}$. Since $\tau^{(1)}$ is linear with respect to η we may write

$$\tau_\alpha^{(1)} = \chi_\alpha^\beta(q, p) \eta_\beta. \quad (10)$$

Substituting this expression into Eq. (6) and using Eqs. (1) and (2) we obtain:

$$\Delta_{\alpha\beta} + \chi_\alpha^\gamma \omega_{\gamma\lambda} \chi_\beta^\lambda = 0. \quad (11)$$

This equation contains two sets of unknown elements; χ_α^β and $\omega_{\alpha\beta}$. One should at first assume a suitable antisymmetric matrix for $\omega_{\alpha\beta}$ and then solve Eq. (11) to determine the coefficients χ_α^β . Since $\Delta_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are antisymmetric matrices, there exist totally $\frac{m(m-1)}{2}$ independent equations for χ_α^β , while the number of χ_α^β 's is m^2 . Therefore, an infinite number of solutions for χ_α^β can be found and we are allowed to chose any solution we wish. Using this possibility, χ_α^β 's can be chosen such that the process of determining the correction terms $\tau^{(n)}$ terminates at this stage, i.e. $\tau^{(2)}$ vanishes. We will come to this point later. It can be shown [7,9] that the general solution of Eq. (7) is given by

$$\tau_\alpha^{(n+1)} = -\frac{1}{n+2} \eta_\mu \omega^{\mu\nu} \chi_\nu^{-1\rho} B_{\rho\alpha}^{(n)}; \quad n \geq 1 \quad (12)$$

where $\omega^{\alpha\beta}$ and $\chi_\alpha^{-1\beta}$ are inverse of $\omega_{\alpha\beta}$ and χ_α^β respectively.

To construct the corresponding Hamiltonian $H(q, p, \eta)$ in the extended phase space we demand

$$H = \sum_{n=0}^{\infty} \tilde{H}^{(n)} \quad (13)$$

such that

$$H(q, p, 0) = H_0(q, p) \quad \{\tau_\alpha, H\} = 0, \quad (14)$$

where $H^{(n)}$ is of order n with respect to η_α 's. Substituting from Eqs. (3) and (13) in the second line of Eq. (14) gives:

$$\{\tau_\alpha^{(1)}, H^{(n+1)}\}_{(\eta)} + G_\alpha^{(n)} = 0; \quad n \geq 0 \quad (15)$$

where $G_\alpha^{(n)}$ as the generators of the $H^{(n+1)}$ are defined as the following

$$G_\alpha^{(0)} \equiv \{\tau_\alpha^{(0)}, H^{(0)}\} \quad (16)$$

$$G_\alpha^{(1)} \equiv \{\tau_\alpha^{(1)}, H^{(0)}\} + \{\tau_\alpha^{(0)}, H^{(1)}\} + \{\tau_\alpha^{(2)}, H^{(1)}\}_{(\eta)} \quad (17)$$

$$G_\alpha^{(n)} \equiv \sum_{m=0}^n \{\tau_\alpha^{(n-m)}, H^{(m)}\} + \sum_{m=0}^{n-2} \{\tau_\alpha^{(n-m)}, H^{(m+2)}\}_{(\eta)} + \{\tau_\alpha^{(n+1)}, H^{(1)}\}_{(\eta)}; \quad n \geq 2. \quad (18)$$

It can be shown that the general expression for $H^{(n)}$ is

$$H^{(n+1)} = -\frac{1}{n+1} \eta_\mu \omega^{\mu\nu} \chi_\nu^{-1\lambda} G_\lambda^{(n)}. \quad (19)$$

This completes the BFT method of converting a second class system to a strongly involuting first class one. As can be seen the correction terms $\tau_\alpha^{(n)}$ and $H^{(n)}$ are derived iteratively from Eqs. (12) and (19). Generally, there is no guarantee that the series terminate at some definite order. However, the series will terminate if $B_{\alpha\beta}^{(n)}$ and $G_\alpha^{(n)}$ vanish for a certain order n . If the Δ -matrix in (1) is constant this goal can be reached simply. In this case it is easily seen that the choice

$$\omega = -\Delta \quad \chi = \mathbf{1} \quad (20)$$

solves the basic Eq. (11). With this choice we have $\tau_\alpha^{(1)} = \eta_\alpha$ and $B_{\alpha\beta}^{(1)} = 0$ (see Eq. (8)). Then from Eq. (9) all other $B_{\alpha\beta}^{(n)}$ for $n > 1$ vanish. This leads to the following finite order embedding for the constraints

$$\tau_\alpha = \tau_\alpha + \eta_\alpha. \quad (21)$$

One can show that in this case the embedding series for Hamiltonian will also truncate provided that $H^{(0)}$ be a polynomial function of phase space coordinates [8].

III. THE PROBLEM WITH MIXED SYSTEMS

Consider a mixed constrained system which is described by the Hamiltonian $H^{(0)}(q, p)$. Suppose the system possesses a set of first class constraints ϕ_i as well as the second class ones $\tau_\alpha^{(0)}$. The problem is to find an embedding in such a way that the extended Hamiltonian H , and the extended constraints τ_α and $\tilde{\phi}_i$ have vanishing Poisson brackets altogether. In other words in addition to Eqs. (5) and (14) we expect that

$$\{H, \tilde{\phi}_i\} = 0. \quad (22)$$

The set of Eqs. (5), (14), and (22) should be solved simultaneously. It may seem that an embedding for the second class constraints suffices; i.e. one may consider $\tilde{\phi}_i$ the same as ϕ_i and extend only $\tau_\alpha^{(0)}$ and $H^{(0)}$ into τ_α and H respectively. The point is that in general there is no guarantee that the first class constraints remain still first class. In other words, the constraints ϕ_i may no longer have vanishing Poisson brackets with the embedded Hamiltonian. To see this better, suppose in the original theory the secondary first class constraints, ϕ_s , have been emerged from the consistency of some primary first class constraints, ϕ_p . Since in the embedded model some terms should be added to the Hamiltonian, it is possible that the Poisson brackets $\{\phi_p, H\}$ may no more give the same ϕ_s . They may have been changed to $\tilde{\phi}_s$ such that the new set of constraints ϕ_p and $\tilde{\phi}_s$ are second class. Therefore, the process of embedding may destroy the gauge symmetry generated by the set of first class constraints ϕ_p and ϕ_s .

Now let us go through the details to see when this may happen. We know from Eq. (13) that $\tilde{\phi}_i = \phi_i$ will solve Eq. (22) if

$$\{H^{(n)}, \phi_i\} = 0. \quad (23)$$

Considering Eq. (19) for a finite order BFT embedding in which $\omega_{\alpha\beta}$ and χ_β^ν are chosen as in Eqs. (20), shows that Eq. (23) will be satisfied if

$$\{G_\alpha^{(n)}, \phi_i\} = 0. \quad (24)$$

For $n = 0$ we have from Eq. (16)

$$\{G_\alpha^{(0)}, \phi_i\} = \{\{\tau_\alpha^{(0)}, H_c\}, \phi_i\}. \quad (25)$$

In a second class system the Poisson brackets of constraints with the canonical Hamiltonian vanish weakly except for the constraints of last level. This may be better understood in chain by chain approach [10], where the constraints are collected as chains and within each chain the consistency of every constraint gives the next one, i.e.

$$\{\tau_{\alpha-1}^{(0)}, H_c\} = \tau_\alpha^{(0)} \quad \alpha = 1, \dots, A. \quad (26)$$

Since $\tau_\alpha^{(0)}$ are second class, at the last level $\tau_A^{(0)}$ should have nonvanishing Poisson bracket at least with one of the

primary constraints. However, nothing can be said about $\{\tau_A^{(0)}, H_c\}$; it may vanish, may be constant or may be any function of phase space coordinates which may or may not commute with first class constraints ϕ_i . Therefore, one way to guarantee Eq. (24) for $n = 0$ is to demand that

$$\{\tau_A^{(0)}, H_c\} = \text{constant} \quad (27)$$

where $\tau_A^{(0)}$ is the terminating element of any constraint chain.

Returning to Eq. (24) for $n = 1$, the generator $G_\alpha^{(1)}$ is defined in Eq. (17). From Eq. (20) the first and third terms in Eq. (17) vanish in a simple way. According to Eq. (19), the remaining term $\{\tau_\alpha^{(0)}, H^{(1)}\}$ is proportional to a summation of terms $\{\tau_\alpha^{(0)}, \tau_\beta^{(0)}\} = \Delta_{\alpha\beta}$. Remembering that we have considered systems with constant Δ -matrix, we see that the condition (27) results that $G_\alpha^{(1)}$ are constants and $H^{(2)}$ is a function of η 's only. Hence, Eq. (24) is also valid for $n = 1$. Looking carefully at different terms in Eq. (18) shows that under the considered conditions the subsequent terms $G_\alpha^{(n)}$ for $n > 2$ vanish, giving finally

$$H = H^{(0)} + H^{(1)} + H^{(2)} \quad (28)$$

We see that the constancy of Poisson brackets of the second class constraints and the Hamiltonian is sufficient to have an elegant truncation of the embedded Hamiltonian. Moreover, it help's to construct the embedding in such a way that the involuting algebra of first class constraints with other constraints and with the Hamiltonian is not violated. It should be noted that this conclusion remains valid for BFT embedding with chain structure [11], since it differs with abelian embedding only in additional terms $\tau_{\alpha+1}^{(n)}$ in the definitions of $G_\alpha^{(n)}$ which commute with first class constraints.

On the other hand, if in a certain model Eq. (27) does not hold, then there is no guarantee that the embedding of second class constraints is possible without violating the involuting algebra of first class constraints. The problem is: what should we do to satisfy (27)? We will give our propositions to solve this problem in the next section, specially for the Chern-Simons theory. Before that let us take a look at this theory, its constraint structure and the problem of its embedding.

The nonabelian Chern-Simons theory in $(1 + 2)$ dimensions is governed by the Lagrangian density [12]

$$\mathcal{L} = \frac{1}{2} k \varepsilon^{\mu\nu\rho} \left(A_\mu^a \partial_\nu A_\rho^a + \frac{1}{3} f^{abc} A_\mu^a A_\nu^b A_\rho^c \right) \quad (29)$$

where A_μ^a are dynamical fields, f^{abc} are the structure constants of some nonabelian Lie algebra, $\varepsilon^{\mu\nu\rho}$ refer to the totally antisymmetric tensor and k is a constant. From the definition of canonical momenta three (sets of) primary constraints emerge as follows

$$\begin{aligned}\Phi^{a0} &\equiv \pi^{a0} \approx 0 \\ \Phi^{ai} &\equiv \pi^{ai} - \frac{1}{2} k \epsilon^{ij} A_j^a \approx 0 \quad i = 1, 2.\end{aligned}\quad (30)$$

The canonical Hamiltonian can be written as

$$H_c = -k \int d^2 \mathbf{x} \left(A_0^a \epsilon^{ij} \partial_i A_j^a + \frac{1}{6} \epsilon^{\mu\nu\rho} f^{abc} A_\mu^a A_\nu^b A_\rho^c \right). \quad (31)$$

The consistency condition of Φ^{a0} gives the following secondary constraint

$$\Phi^{a3} \equiv k \epsilon^{ij} \partial_i A_j^a + \frac{k}{2} \epsilon^{ij} f^{abc} A_i^b A_j^c \approx 0. \quad (32)$$

No additional constraint is obtained from the consistency of the constraints Φ^{ai} and Φ^{a3} . It seems that there exist 3 s class constraints Φ^{ai} and Φ^{a3} , but one can combine the constraints to find 2 s class and two first class constraints as follows

$$\begin{aligned}\Lambda^{a0} &= \Phi^{a0} & \Lambda^{a1} &= \Phi^{a1} & \Lambda^{a2} &= \Phi^{a2} \\ \Lambda^{a3} &= \Phi^{a3} + \partial_i \Phi^{ai}\end{aligned}\quad (33)$$

In Eqs. (33) Λ^{a1} and Λ^{a2} are second class and Λ^{a0} and Λ^{a3} are first class constraints. To find the redefinitions explained in Eqs. (33) systematically we could first determine the unknown Lagrange multipliers λ_i^a in the total Hamiltonian

$$H_T = H_c + \lambda_0^a \Phi^{a0} + \lambda_i^a \Phi^{ai} \quad (34)$$

and then use it for the consistency of the remaining constraint Φ^{a0} . In this way we find

$$\lambda_i^a = \partial_i A_0^a + \frac{1}{2} f^{abc} A_i^b A_0^c. \quad (35)$$

Inserting λ_i^a in the total Hamiltonian (34) gives

$$H_T = H^{(0)} + \lambda_0^a \Phi^{a0}, \quad (36)$$

where

$$H^{(0)} = H_c + \left(\partial_i A_0^a + \frac{1}{2} f^{abc} A_i^b A_0^c \right) \Phi^{ai}. \quad (37)$$

Now the consistency of the primary constraint Φ^{a0} , using this modified H , gives

$$\Phi^{*a3} = \{\Phi^{a0}, H\} = \frac{k}{2} \epsilon^{ij} \partial_i A_j^a + \partial_i \pi^{ai} + f^{abc} A_i^b \pi^{ci} \quad (38)$$

which is the same as Λ^{a3} in the definitions (33). Since $\{\Phi^{*a3}, H\} = 0$, no more constraints would emerge. As is demonstrated in Eq. (33), there are three constraint chains, one first class (including two elements Λ^{a0} and Λ^{a3}) and 2 s class, each containing just one element. In fact, Λ^{a1} and Λ^{a2} are the first and last elements of the corresponding chains.

Suppose we want to construct an embedding for Chern-Simons theory. The Δ -matrix of second class constraints reads

$$\Delta^{ai,bj}(\mathbf{x}, \mathbf{y}) \equiv \{\Lambda^{ai}(\mathbf{x}, t), \Lambda^{bj}(\mathbf{y}, t)\} = \delta^{ab} \epsilon^{ij} \delta(\mathbf{x}, \mathbf{y}). \quad (39)$$

Since the Δ -matrix is constant we can choose the finite order embedding (21) as

$$\tau^{ai} = \Lambda^{ai} + \eta^{ai} \quad i = 1, 2. \quad (40)$$

The embedded Hamiltonian can also be found (see Eqs. (13)–(19)) as

$$H = H^{(0)} + H^{(1)} + H^{(2)} \quad (41)$$

where

$$\begin{aligned}H^{(1)} &= \epsilon^{ij} \eta_i^a \partial_j A_0^a + \frac{1}{2} f^{abc} \epsilon^{ij} \eta_i^a A_j^b A_0^c \\ H^{(2)} &= -\frac{1}{4} f^{abc} \eta_i^a \eta^{bi} A_0^c\end{aligned}\quad (42)$$

As it is apparent the constraints Λ^{a0} and Λ^{a3} have no more vanishing Poisson brackets with the embedded Hamiltonian even weakly. In other words, assuming $\Lambda^{a0} = \pi^{a0}$ (for all a) as the primary constraints, we will find some chains of second class constraints, due to additional terms $H^{(1)}$ and $H^{(2)}$ in the Hamiltonian. This shows that the initial gauge symmetry $A \rightarrow A + dA$ generated by the first class constraints Λ^{a0} and Λ^{a3} is no more present in the embedded model. Technically this has happened since Λ^{a1} and Λ^{a2} as the last elements of the corresponding chains have nonvanishing Poisson brackets with the Hamiltonian (36). Therefore the requirement of vanishing the expression given in Eq. (25) is not fulfilled. In fact, one may see that $\{\Lambda^{ai}, H\}$ contain terms with one or two A -fields; hence, they do not commute with first class constraints Λ^{a0} and Λ^{a3} . Direct investigation of the embedded Hamiltonian (60) also shows that it no more commutes with the first class constraints of the model. This is really the origin of the problem of BFT method for some of the mixed constraint systems such as Chern-Simons.

IV. SOLUTION

In this section we give two different methods to overcome the problem which lead to two different types of embedding for Chern-Simons model.

i) In Ref. [10] some technics are given which may help us satisfy the desired condition (27). The main point is that, by adding terms which vanish on the constraint surface one can redefine the constraints as well as the Hamiltonian to satisfy Eq. (27). Suppose we are given 2 s class chains terminating at noncommuting elements Θ_1 and Θ_2 respectively, such that

$$\{\Theta_1, \Theta_2\} = \delta \quad \{\Theta_1, H\} = \varrho \quad \{\Theta_2, H\} = \gamma. \quad (43)$$

Assume the following redefinitions

$$\hat{\Theta}_1 = \gamma\Theta_1 - \varrho\Theta_2 \quad \hat{\Theta}_2 = (\gamma)^{-1}\Theta_2. \quad (44)$$

It is easy to observe that

$$\{\hat{\Theta}_1, \hat{\Theta}_2\} \approx \delta \quad \{\hat{\Theta}_1, H\} \approx 0 \quad \{\hat{\Theta}_2, H\} \approx 1. \quad (45)$$

In other words, the above redefinitions do not change the algebra of second class constraints, while their Poisson brackets with Hamiltonian turn to be constants. The remainder of the problem is straightforward. For nonabelian Chern-Simons theory this method gives the following redefined constraints

$$\hat{\Lambda}^{a1} = \left\{ \frac{1}{2}k^2 A_2^a (\partial_1 A_0^a + f^{abc} A_0^c A_1^b) - 1 \leftrightarrow 2 \right\} \quad (46)$$

$$\begin{aligned} & -\{k\partial_1 A_0^a \pi^{a1} + k\pi^{a1} f^{abc} A_0^c A_1^b + 1 \leftrightarrow 2\} \hat{\Lambda}^{a2} \\ & = -\left(\pi^{a2} + \frac{1}{2}kA_1^a \right) / (k\partial_1 A_0^a + k f^{abc} A_0^c A_1^b), \end{aligned} \quad (47)$$

where no summation on the repeated index a is assumed. Instead of Eq. (40), the embedded constraints are

$$\tau^{ai} = \hat{\Lambda}^{ai} + \eta^{ai} \quad i = 1, 2. \quad (48)$$

Finally according to Eq. (28), the embedded Hamiltonian is

$$H = H^{(0)} + \frac{1}{k} \sum_a \eta^{a1} \quad (49)$$

where $H^{(0)}$ is given in Eqs. (37) and (31).

ii) By adding some auxiliary fields one can first convert the first class constraints to second class ones and then the traditional BFT method can be applied to the whole system. These new auxiliary fields are different from those of the formal BFT formalism. To see how this is possible, suppose we are given K first class two-level chains originated from K primary first class constraints $\phi_i^{(0)}$; $i = 1, \dots, K$. We can assume that $\phi_i^{(0)}$ are principally emerged, in some suitable coordinates, from the definition of the momenta

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}. \quad (50)$$

One can easily see that the following extensions convert first class constraints to second class ones:

$$p_i \rightarrow p_i + \xi_i \quad H_c \rightarrow H_c + \frac{1}{2} \sum_i p_{\xi_i}^2 \quad (51)$$

where ξ_i and p_{ξ_i} are auxiliary conjugate variables. In the Lagrangian formalism this can be done by the replacement

$$L \rightarrow L - \xi_i \dot{q}_i + \frac{1}{2} \sum_i \xi_i^2. \quad (52)$$

It can be shown that the replacement (52) is in fact a gauge fixing term inserted in the gauge invariant Lagrangian L . In other words the new Lagrangian, or equivalently the new

Hamiltonian (51), gives the same equations of motion for the gauge invariant quantities while destroys the arbitrariness of the gauge dependent variables.

Fortunately most physical models fall in the category of two-level systems. However, for more complicated systems it is not too difficult to add suitable variables to convert first class constraints to second class ones. For example, if there are four levels of constraints in a given first class chain beginning with the momentum p , then by adding two conjugate pairs (ξ, p_ξ) and (η, p_η) and the replacements

$$p \rightarrow p + \xi \quad H_c \rightarrow H_c + \frac{1}{2} \eta^2 + p_\eta p_\xi, \quad (53)$$

one can convert the system to a second class one. In fact it is not needed to give a detailed procedure for different cases which may occur, since the process of constructing a second class system from a first class one can be done easily for distinct models.

To apply this method to Chern-Simons theory one can make the following replacement in the original Lagrangian (29)

$$\mathcal{L} \rightarrow \mathcal{L} - \xi^a \dot{A}_0^a + \frac{1}{2} \sum_a (\xi^a)^2 \quad (54)$$

where ξ^a are auxiliary fields in the configuration space. It is obvious that the gauge symmetry $A \rightarrow A + df$ is lost in the Lagrangian (54), while it can be shown that the gauge invariant quantities are remained invariant. The Hamiltonian (31) would consequently admit the following replacement

$$H_c \rightarrow H_c + \frac{1}{2} \sum_a (p_{\xi^a})^2. \quad (55)$$

By these replacements the primary and secondary constraints would change to

$$\begin{aligned} \hat{\Lambda}^{a0} &\equiv \Lambda^{a0} + \xi^a & \hat{\Lambda}^{a1} &\equiv \Lambda^{a1} & \hat{\Lambda}^{a2} &\equiv \Lambda^{a2} \\ \hat{\Lambda}^{a3} &\equiv \Lambda^{a3} + p_{\xi^a}. \end{aligned} \quad (56)$$

In this way we have a pure second class system for which the ordinary finite order BFT method is applicable. The Δ -matrix now reads

$$\Delta^{a\mu, b\nu}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & k & 0 \\ 0 & -k & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \delta^{ab} \delta(\mathbf{x} - \mathbf{y}) \quad (57)$$

where $\mu, \nu = 0, \dots, 3$ are row and column indices of the above 4×4 matrix, respectively. Again the Δ -matrix is constant and one may write the following finite extensions for the constraints

$$\tau^{a\mu} = \hat{\Lambda}^{a\mu} + \eta^{a\mu} \quad \mu = 0, \dots, 3. \quad (58)$$

The embedded Hamiltonian can also be found (see

Eqs. (13)–(19)) as

$$H = H^{(0)} + \frac{1}{2} \sum_a (p_\xi^a)^2 + H^{(1)} + H^{(2)} + H^{(3)} \quad (59)$$

where $H^{(0)}$ is defined in Eq. (37) and

$$\begin{aligned} H^{(1)} &= \frac{-1}{k} \eta_1^a G_2^{a(0)} + \frac{1}{k} \eta_2^a G_1^{a(0)} + \eta_3^a G_0^{a(0)} \\ H^{(2)} &= \frac{-1}{k} \eta_1^a G_2^{a(1)} + \frac{1}{2k} \eta_2^a G_1^{a(1)} + \frac{1}{2} \eta_3^a G_0^{a(1)} \\ H^{(3)} &= \frac{1}{3} f^{abc} (\eta_1^a \eta_2^b \eta_3^c + \eta_2^a \eta_3^b \eta_1^c) \end{aligned} \quad (60)$$

in which

$$\begin{aligned} G_0^{a(0)} &= k \epsilon^{ij} \partial_i A_j^a + \frac{k}{2} f^{abc} \epsilon^{ij} A_i^b A_j^c \\ G_i^{a(0)} &= k \epsilon^{ij} \partial_j A_0^a + k \epsilon^{ij} f^{abc} A_j^b A_0^c \quad i = 1, 2 \\ G_0^{a(1)} &= \partial_i \eta_i^a + f^{abc} (A_1^b \eta_1^c + A_2^b \eta_2^c) \\ G_1^{a(1)} &= f^{abc} \eta_1^b A_0^c - k \partial_2 \eta_3^a - k f^{abc} \eta_3^c A_2^b \\ G_2^{a(1)} &= -f^{abc} \eta_2^b A_0^c + k \partial_1 \eta_3^a + k f^{abc} \eta_3^c A_1^b. \end{aligned}$$

One can easily check that this Hamiltonian and the set of constraints (58) construct a first class system.

It is worth noting that the above results are valid for abelian Chern-Simons theory by imposing $f^{abc} = 0$.

V. CONCLUDING REMARKS

We showed that the BFT embedding method although applicable to pure second class systems, is not guaranteed to work well for systems possessing both first and second class constraints. The Chern-Simons theory is a distinguished example in this regard. As we saw, the bottleneck condition is the requirement that at the last level of consistency the second class constraints have constant Poisson brackets with the Hamiltonian. This condition

guarantees that the algebra of first class constraints is not violated during embedding of second class ones. However, we should admit that this condition is actually stronger to some extent than what is needed. In fact in concrete examples one may be able to find different solutions in which the first class constraints commute with the generators of the embedded Hamiltonian (i.e. G_α^n in Eqs. (16)–(18)). So we think that the problem is open in this regard.

However, if one insists that the critical condition (27) should be satisfied in any case, then several methods can be found to redefine the constraints to reach this goal. We suggested just one possibility in Eqs. (44). It may be possible to give other (or better) solutions for this requirement. The problem is also open in this direction. To sum up, in this approach one tries to find the origin of this violation in the involuting algebra of first class constraints and remove it.

As a second approach we gave another solution with a different character. In this method we first convert the first class constraints into second class ones by means of adding suitable variables and then use the ordinary BFT method to embed the resulting pure second class system into a first class one. It is not usually a difficult task to construct a second class system out of a first class one. We think that this will be easily done in each concrete example and thus it is not needed to give general prescriptions for that.

According to these methods, we gave two different types of embedding for nonabelian Chern-Simons theory which includes the abelian case easily by imposing $f^{abc} = 0$. The embedding of the abelian Chern-Simons theory was previously considered in [13] by using an infinite number of auxiliary fields. However, as far as we know, because of the mixed character of its constraint structure, no finite order BFT embedding has been given for Chern-Simons theory so far.

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