Spherically symmetric monopoles in noncommutative space

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We construct a spherically symmetric noncommutative space in three dimensions by foliating the space with concentric fuzzy spheres. We show how to construct a gauge theory in this space, and, in particular, we derive the noncommutative version of a Yang-Mills-Higgs theory. We find numerical monopole solutions of the equations of motion.

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I. INTRODUCTION

Field theories in noncommutative space have received renewed interest since their emergence in certain low-energy limits of string and M-theory. In particular, nonperturbative soliton configurations have been the object of numerous investigations in recent years [1–13]. Concerning monopoles, they have been constructed mainly using an extension of the Nahm equation in noncommutative space [6,10]. Another approach exploits the connection between soliton solutions in four dimensions and monopole configurations defined in a curved space [13]. However, in all of these approaches the obtained configurations are not the natural extension of the well-known 't Hooft-Polyakov monopole solution. The reason is simple: The 't Hooft ansatz has explicit spherical symmetry, while the standard noncommutative three-dimensional algebra

$$[x_i, x_j] = i\theta_{ij}, \qquad \theta_{ij}\text{: constant matrix} \tag{1}$$

is not invariant under rotations, thus breaking explicitly the rotational symmetry of any noncommutative field theory (NCFT) defined on it.

In this article, we will construct an explicit rotationally invariant noncommutative space by deforming adequately the algebra (1). In particular, we will show how to construct a gauge theory in this space by extending the commutative-space theory written in terms of explicit rotationally invariant operators. The evident advantage is that in this formulation the equations of motion accept a spherically symmetric ansatz, resemblant to the 't Hooft form. Moreover, we will show that in the small θ limit the solutions tend to the well-known Prasad-Sommerfield solutions.

The article is organized as follows. In Sec. II we construct a rotationally invariant noncommutative space. We find that this deformation reduces to a foliation of the three-dimensional space with concentric 2-fuzzy spheres. In Sec. III we show how to construct gauge fields in a manner consistent with the rotational symmetry of the space. In Sec. IV we construct a Yang-Mills-Higgs theory and derive the equations of motion. Section V is devoted to the solution of the equations of motion. Finally, in Sec. VI we summarize the paper and present some discussion.

II. ROTATIONALLY INVARIANT NONCOMMUTATIVE SPACE

One of the main problems in finding noncommutative monopole solutions is that the simplest ansatz ('t Hooft) has explicit spherical symmetry, whereas the standard noncommutative space in three dimensions breaks rotational invariance. Of course, spherical symmetry is not essential for the construction of monopole solutions, and, in fact, several nonspherically symmetric solutions have been found explicitly [6,10,13]. However, spherical symmetry greatly simplifies the equations by reducing the number of degrees of freedom. So, in order to take advantage of this simplification, let us modify the noncommutative structure of the space in order to preserve rotational symmetry.

Consider a three-dimensional noncommutative space with coordinates satisfying the commutator algebra

$$[x_i, x_j] = i\theta\varepsilon_{ijk} f(r) x_k, \tag{2}$$

with $f(r)$ a function to be determined and $r^2 = x_i x_i$.

It can be shown that the Jacobi identity imposes the condition $f(r) \propto r$ (r^2 is a Casimir of the algebra), so we have

$$[x_i, x_j] = i\theta r \varepsilon_{ijk} x_k, \tag{3}$$

with θ a dimensionless parameter (here, unlike fuzzy-sphere coordinates, the coordinates x_1, x_2, x_3 are all independent; there is no constraint between them). Then the operators $x_i/(r\theta)$ satisfy the $SU(2)$ algebra. The algebra (3) being invariant under space rotations, it is natural to extend it with angular momentum operators L_i

$$[L_i, L_j] = i\varepsilon_{ijk} L_k, \qquad [L_i, x_j] = i\varepsilon_{ijk} x_k. \tag{4}$$

We can find a representation of (3) and (4) by identifying the coordinate operators with $\theta r L_i$:

$$x_i = \theta r L_i \tag{5}$$

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with L_i, $SU(2)$ operators. We have that $1/\theta^2 = L_i L_i$, and, if we restrict ourselves to finite-dimensional representations, we have $1/\theta^2 = l(l+1)$, $l \in \frac{1}{2}\mathbb{N}$ (in principle, we allow spinor representations). In this representation, r is a continuous commutative variable. Notice that the algebra (3) and (4) for fixed r describes a fuzzy sphere [14–25], so essentially what we are doing is foliating the three-dimensional noncommutative space with concentric fuzzy spheres (a similar construction was done in [19,22]).

Since algebra (3) is not invariant under space translations, it is impossible to define momentum operators satisfying

$$[P_i, x_j] = -i\delta_{ij}. \tag{6}$$

These relations violate the Jacobi identity for three operators $\{P_i, x_j, x_k\}$. This is analogous to the fact that, for constant noncommutative space

$$[x_i, x_j] = i\theta_{ij}, \tag{7}$$

it is not possible to define angular momentum operators satisfying (4) since the algebra (7) is not rotationally invariant (the Jacobi identity fails for the triplet $\{L_i, x_j, x_k\}$).

In order to define a field theory in this noncommutative space, we first define transversal and radial field components and write the appropriate Lagrangian. In commutative space, given a vector field in Cartesian coordinates V_i, $i = 1, 2, 3$, we can define transversal components V_i^T and a radial component V_r as

$$V_i^T = \varepsilon_{ijk} x_j V_k, \qquad V_r = -x_i V_i. \tag{8}$$

The transversal part satisfies the constraint

$$x_i V_i^T = 0. \tag{9}$$

Cartesian coordinates can be recovered from the transversal and radial ones through the identity

$$r^2 V_i = -\varepsilon_{ijk} x_j V_k^T - x_i V_r. \tag{10}$$

Since we are working in a noncommutative space with explicit rotational invariance (3), it is natural to consider the transversal and radial fields (8) as our primary fields and not the Cartesian components V_i. This is crucial, because in noncommutative space there is no mapping as (10) to define Cartesian coordinates.

So a transversal field in the noncommutative space is a field that satisfies the constraint $x_i V_i^T + V_i^T x_i = 0$, or in virtue of representation (5),

$$L_i V_i^T + V_i^T L_i = 0. \tag{11}$$

It is straightforward to check that any vector field of the form

$$[L_i, \Phi] \tag{12}$$

is transversal. We will see that a slight modification has to be done in the case of a gauge theory.

III. GAUGE FIELDS

As we did for arbitrary vector fields, we define transversal and radial gauge fields A_i^T and A_r in analogy with their commutative counterparts (from now on, we will drop the superscript T in A^T). That is, A_i and A_r are fields that in commutative space take the form

$$A_i = \varepsilon_{ijk} x_j \mathcal{A}_k, \qquad A_r = -x_i \mathcal{A}_i, \tag{13}$$

where $\mathcal{A}_i$ are the Cartesian components of the standard vector potential.

These fields transform, under gauge transformations, as follows:

$$\begin{aligned} A_i &\to g^{-1} A_i g - g^{-1}[L_i, g], \\ A_r &\to g^{-1} A_r g - g^{-1}[P, g], \end{aligned} \tag{14}$$

with

$$P = ir\partial_r. \tag{15}$$

Again, we are going to promote to noncommutative space the transversal and radial fields A_i and A_r and not the standard Cartesian gauge field $\mathcal{A}_i$.

We want to stress again that the map (13) between spherical and Cartesian coordinates is possible only in commutative space. In noncommutative space we are forced to work with spherical coordinates, and we cannot recover the Cartesian coordinates. That is, in this space the fundamental fields are the variables A_i, A_r and not $\mathcal{A}_i$.

But now we have a problem trying to impose the constraint (11). Clearly, the constraint is not invariant under gauge transformations and, thus, not well defined for gauge fields. In order to define a gauge invariant transversal constraint, we introduce the gauge covariant distance X_i,

$$X_i = x_i - \theta r A_i = \theta r(L_i - A_i). \tag{16}$$

As its name suggests, this quantity transforms under gauge transformations as

$$X_i \to g^{-1} X_i g. \tag{17}$$

So the correct gauge invariant constraint is given by[1]

$$X_i X_i = x_i x_i = r^2. \tag{18}$$

This can be written as

$$\{x_i, A_i\} = \theta r A_i A_i, \tag{19}$$

which in the limit $\theta \to 0$ coincides with (11). It is useful at this point to introduce the transverse covariant derivative operator

[1]This equation was first proposed in Ref. [17]. See also [18,20,21].

$$D_i = L_i - A_i, \tag{20}$$

so $X_i = \theta r D_i$, and the constraint can be written as

$$D_i D_i = L_i L_i = \kappa \tag{21}$$

[we have defined $\kappa = l(l+1) = 1/\theta^2$] or

$$\{L_i, A_i\} - A_i A_i = 0. \tag{22}$$

The field strength components F_{ij} and F_{ir} are defined in analogy with the commutative case

$$\begin{aligned} F_{ij} &= -i([L_i, A_j] - [L_j, A_i] - [A_i, A_j] - i\varepsilon_{ijk} A_k) \\ &= i([D_i, D_j] - i\varepsilon_{ijk} D_k), \end{aligned} \tag{23}$$

$$F_{ir} = -i([L_i, A_r] - [P, A_i] - [A_i, A_r]) = i[D_i, D_r], \tag{24}$$

where D_r is the radial covariant derivative

$$D_r = P - A_r. \tag{25}$$

For convenience, we will work in the gauge $A_r = 0$, so $D_r = P$.

As usual, the field F is gauge covariant

$$F_{ij} \to g^{-1} F_{ij} g, \qquad F_{ir} \to g^{-1} F_{ir} g, \tag{26}$$

and satisfies the gauge invariant transversality conditions

$$\{D_i, F_{ij}\} = \{D_i, F_{ir}\} = 0. \tag{27}$$

IV. YANG-MILLS-HIGGS THEORY

A. The action

To write an action in this geometry, we simply write the action in the commutative-space case in terms of transversal and radial fields using the definition (13) and then promote the fields to noncommutative space, respecting gauge invariance when needed. For Yang-Mills and Higgs actions we have[2]

$$\begin{aligned} S_{\rm YM} &= \frac{1}{2} \int dx^3 \frac{1}{r^4} \operatorname{tr}(F_{ij} F_{ij} + 2F_{ir} F_{ir}), \\ S_{\rm Higgs} &= -\int dx^3 \bigg(\frac{1}{r^2} \operatorname{tr}([D_i, \phi][D_i, \phi] \\ &\quad + [D_r, \phi][D_r, \phi]) + V[\phi] \bigg). \end{aligned} \tag{28}$$

That is, in commutative space $S_{\rm YM}$ and $S_{\rm Higgs}$ are the usual Yang-Mills and Higgs actions written in term of the transversal and radial fields A_i, A_r. Using Eqs. (13), we can recover the standard form of the actions in terms of the standard gauge potential $\mathcal{A}_i$. However, Eqs. (13) are not valid in noncommutative space and expressions (28) have to be taken as the defining actions in this geometry.

B. Equations of motion

From actions (28) we get the Euler-Lagrange equations of motion

$$\begin{aligned} &[D_r, F_{ir}] - iF_{ir} - [D_j, F_{ji}] - \frac{i}{2} \varepsilon_{ijk} F_{jk} - ir^2 [[D_i, \phi], \phi] \\ &\quad = \{\mu, D_i\}, \end{aligned} \tag{29}$$

with μ a Lagrange multiplier enforcing the constraint (21). The right-hand side cancels the longitudinal part of the left-hand side so the resulting equation is transversal.

The remaining equations of motion are

$$[D_i, F_{ir}] + ir^2 [[D_r, \phi], \phi] = 0, \tag{30}$$

$$[D_i, [D_i, \phi]] + [D_r, [D_r, \phi]] = r^2 \frac{\delta V}{\delta \phi}. \tag{31}$$

We will concentrate on the case $V \equiv 0$.

To eliminate the Lagrange multiplier, we note that, given an arbitrary vector V_i, we can write its transversal part as

$$V_i^T = V_i - \tfrac{1}{2}\{\mu, D_i\} \tag{32}$$

for some function μ. Now imposing on V^T the transversality condition for $\{V_i^T, D_i\} = 0$, we find the following equation for μ:

$$\kappa \mu + D_i \mu D_i = \{V_i, D_i\}. \tag{33}$$

The transversal part is obtained inserting the solution of Eq. (33) in Eq. (32).

Before ending this section, we have to mention possible Bogomol'nyi-Prasad-Sommerfield (BPS) equations of motion. In commutative space, in terms of the radial and transversal fields, the BPS equations read

$$D_r \phi = \mp \frac{1}{2r^2} \varepsilon_{ijk} x^i F_{jk}, \tag{34}$$

$$D_i \phi = \pm \frac{1}{r^2} \varepsilon_{ijk} x^j F_{kr}. \tag{35}$$

However, we have been unable to construct a noncommutative version of them. The obvious modifications, replacing the coordinate x^i by the covariant coordinate operator X^i and the product of x^i and F_{ab} by the Moyal anticommutator $\{X^i, F_{ab}\}$, do not work. For example, after this replacement Eq. (35) reads

$$[D_i, \phi] = \pm \frac{1}{2r^2} \varepsilon_{ijk} \{X^j, F_{kr}\}. \tag{36}$$

But while the left-hand side is transversal with respect to X^i, the right-hand side is not. Even projecting the right-hand side over the transverse components does not reproduce the equations of motion.

[2] The integration is defined as $\int dx^3 = \frac{4\pi}{2l+1} \operatorname{tr} \int r^2 dr$, where the trace is taken over the angular momentum representation indices.

V. MONOPOLE SOLUTIONS

A. Spherically symmetric ansatz

For a $U(2)$ theory, the most general spherically symmetric ansatz can be written using the operators

$$V_i^{(0)} = L_i, \qquad V_i^{(1)} = \sigma^i, \qquad V_i^{(2)} = \{\alpha, L_i\}, \qquad V_i^{(3)} = [\alpha, L_i], \tag{37}$$

where σ^i are the Pauli matrices and

$$\alpha = \sum_i^3 \sigma_i L_i. \tag{38}$$

Although (37) is the most general set of rotationally covariant operators, it can be shown that the set remains consistent if we drop $V_i^{(3)}$. That is, when we expand the fields in the basis $\{V_i^{(0)}, V_i^{(1)}, V_i^{(2)}\}$, the equations of motions do not have components in the direction $V_i^{(3)}$. So from now on we will work with the basis $\{V_i^{(0)}, V_i^{(1)}, V_i^{(2)}\}$.

Then we expand

$$D_i = \sum_{a=0}^2 v_a V_i^{(a)}, \tag{39}$$

with v_a arbitrary functions of the radial coordinate r, $v_a \equiv v_a(r)$.

The constraint (21) implies the following two equations for the coefficients v_0, v_1, and v_2:

$$\begin{gathered} \kappa {v_0}^2 + 3{v_1}^2 + 4\kappa v_1 v_2 + 2\kappa(2\kappa - 1){v_2}^2 = \kappa, \\ (4\kappa - 3){v_2}^2 - 2v_0(v_1 - (2\kappa - 1)v_2) = 0. \end{gathered} \tag{40}$$

That is, the field D_i depends only on one function.

We have for the field strength

$$\begin{aligned} F_{ij} = \varepsilon_{ija}\{&(v_0 - {v_0}^2 + (3 - 4\kappa){v_2}^2)V_a^{(0)} \\ &+ (-2{v_1}^2 + 2\kappa v_0 v_2 - 4\kappa {v_2}^2 + v_1(1 - 4\kappa v_2))V_a^{(1)} \\ &+ (v_2 - 3v_0 v_2 + 2v_1 v_2 + 5{v_2}^2)V_a^{(2)}\} \end{aligned} \tag{41}$$

and

$$F_{kr} = r(v_0' V_k^{(0)} + v_1' V_k^{(1)} + v_2' V_k^{(2)}). \tag{42}$$

To write the equation of motion (29), we have first to solve Eq. (33) to find the Lagrange multiplier μ that projects the solution onto the tangential space. Spherical symmetry imposes that μ has the form

$$\mu = \mu_0 + \alpha\mu_1, \tag{43}$$

so Eq. (33) leads to algebraic equations for the coefficients μ_0 and μ_1. However, we will see later that we can solve explicitly the constraint and thus work with the physical, unconstrained degrees of freedom, making unnecessary the Lagrange multiplier.

The Higgs field can be expanded as

$$\phi = \phi_0(r) + \phi_1(r)\alpha, \tag{44}$$

and the covariant derivatives take the form

$$\begin{aligned} [D_i, \phi] &= -\phi_1(v_0 - 2v_1 - v_2)V_i^{(3)}, \\ [D_r, \phi] &= ir(\phi_0'(r) + \phi_1'(r)\alpha). \end{aligned} \tag{45}$$

It can be checked that the equation of motion (30) is trivially satisfied.

Finally, the last equations (31) take the form

$$\begin{aligned} r\frac{d^2}{dr^2}(r\phi_0) &= 0 \rightarrow \phi_0 = \frac{c_1}{r} + c_0, \\ r\frac{d^2}{dr^2}(r\phi_1) &= 2\phi_1(v_0 - 2v_1 - v_2)^2. \end{aligned} \tag{46}$$

Notice that ϕ_0 is decoupled from the other fields, and, in fact, it is an irrelevant constant (regularity of the solution at the origin implies $c_1 = 0$).

In these variables the Hamiltonian takes the form

$$\begin{aligned} H = 8\pi \int dr \Big\{ & r^2(\phi_0'^2 + \phi_1'^2\kappa) + 2{\phi_1}^2\kappa(-v_0 + 2v_1 + v_2)^2) + \frac{1}{r^2}(3(1 - 2v_1)^2{v_1}^2 \\ & + \kappa(-2{v_0}^3 + {v_0}^2(1 + 8(4\kappa - 3){v_2}^2) - 2(4\kappa - 3)v_0{v_2}^2(3 + 4v_1 + 10v_2) + v_2(32{v_1}^3 + 8{v_1}^2(4\kappa v_2 - 3) \\ & + v_2(-2 + 4\kappa + 4(6\kappa - 5)v_2 + (-41 + 4\kappa(11 + 4\kappa)){v_2}^2) + 4v_1(1 + v_2(-3 + 2(8\kappa - 5)v_2)))) + {v_0}^4) \\ & + (3v_1'^2 + \kappa(v_0'^2 + 2v_2'(2v_1' + (-1 + 2\kappa)v_2')) \Big\}. \end{aligned} \tag{47}$$

B. Small θ expansion

Let us study first the small θ expansion of the monopole equations. First we note that the operator $V_i^{(2)}$ is already of order $1/\theta^2$:

$$V_i^{(2)} = \{\alpha, L_i\} = \frac{1}{\theta^2}\{\hat{X}\cdot\vec{\sigma}, \hat{X}^i\}, \tag{48}$$

so the coefficient v_2 is of order θ^2. (Since the covariant derivative operator starts with L_i at zero order, the coefficient v_0 is order zero.)

To compare with the usual 't Hooft-Polyakov-Julia-Zee Prasad-Sommerfield solutions, we write

$$\begin{aligned} v_0 - 1 &= \theta^2 v_0^{(2)} + \theta^4 v_0^{(4)} + \cdots, \\ v_1 &= v_1^{(0)} + \theta^2 v_1^{(2)} + \cdots, \\ v_2 &= \theta^2 \frac{k-1}{4} + \theta^4 \frac{k_1}{4} + \cdots, \\ \phi_1 &= \frac{\theta}{2r}(h + \theta^2 h_1 + \cdots. \end{aligned} \tag{49}$$

The constraints (40) can be solved perturbatively in θ and we can write the coefficients of v_0 ($v_0^{(2)}, v_0^{(4)}, \cdots$) and v_1 ($v_1^{(0)}, v_1^{(2)}, \cdots$) as functions of the coefficients of v_2 ($k, k_1, \cdots$). We have

$$\begin{aligned} v_0 - 1 &= -\frac{\theta^2}{4}(k-1)^2 - \frac{\theta^4}{32}(k-1) \\ &\quad \times (5 + k - 7k^2 + k^3 + 16k_1) + \cdots, \\ v_1 &= \frac{1-k}{2} + \frac{\theta^2}{8}(k^2 - 1 - 4k_1)\cdots, \\ v_2 &= \theta^2 \frac{k-1}{4} + \theta^4 \frac{1}{4} k_1 + \cdots. \end{aligned} \tag{50}$$

At leading order we recover the standard monopole equations

$$\begin{aligned} r^2 k''(r) &= k(r)(k(r)^2 - 1 + h^2(r)), \\ r^2 h''(r) &= 2k(r)h(r) \end{aligned} \tag{51}$$

with the well-known solutions [26]

$$k(r) = \frac{r}{\sinh(r)}, \qquad h(r) = r\coth(r) - 1. \tag{52}$$

The next order equations read

$$\begin{aligned} &r^2 k_1''(r) + (1 - h(r)^2 - 3k(r)^2)k_1(r) \\ &= \tfrac{1}{4}(-1 + 8h(r)h_1(r)k(r) + 3k(r)^2 + 7k(r)^3 - 4k(r)^4 \\ &\quad - 2k(r)^5 + h(r)^2(1 + k(r) - 2k(r)^3) + 4r^2k'(r)^2 \\ &\quad + k(r)(-3 - 2r^2k'(r)^2 + 2r^2k''(r))), \end{aligned} \tag{53}$$

$$\begin{aligned} r^2 h_1''(r) - 2h_1(r)k(r)^2 &= h(r)k(r)(1 + k(r) - 2k(r)^2 \\ &\quad + 4k_1(r)) \end{aligned} \tag{54}$$

and can be solved numerically.

C. Solving the constraint

Instead of working with the "linear" variables v_0, v_1, v_2 and the constraints (40), we can try to reparametrize the fields and solve the constraint explicitly. Then the resulting fields are the physical degrees of freedom, and the constraint is automatically incorporated in the equations of motion.

In fact, we can see that the replacement

$$\begin{aligned} v_0 &\to (z_0 - z_1)/\sqrt{2}, \\ v_1 &\to (z_0 + z_1)/\sqrt{2} - \frac{2\kappa - 1}{\sqrt{4\kappa - 3}} z_2, \\ v_2 &\to z_2/\sqrt{4\kappa - 3} \end{aligned} \tag{55}$$

diagonalizes the second Eq. (40)

$$z_0^2 - z_1^2 - z_2^2 = 0, \tag{56}$$

which is straightforwardly solved in terms of two functions ρ and u (both are functions of r)

$$z_0 = \rho, \qquad z_1 = -\rho c(u), \qquad z_2 = -\rho u, \tag{57}$$

with

$$c(u) = \sqrt{1 - u^2} \quad \text{and} \quad -1 \le u \le 1 \tag{58}$$

(we chose the branch solution that matches the standard $\theta \to 0$ limit). Replacing this solution into the first of Eqs. (40), we get a quadratic equation for ρ that can be easily solved. Finally, we have a parametrization that solves the constraint

$$\begin{aligned} v_0 &= \frac{\sqrt{\kappa}}{\sqrt{2d(u)}}(1 + c(u)), \\ v_1 &= \frac{\sqrt{\kappa}}{2\sqrt{d(u)}\sqrt{4\kappa - 3}}(2u(2\kappa - 1) + \sqrt{8\kappa - 6}(1 \\ &\quad - c(u))), \\ v_2 &= -\frac{\sqrt{\kappa}}{\sqrt{d(u)}\sqrt{4\kappa - 3}} u, \end{aligned} \tag{59}$$

where

$$\begin{aligned} d(u) &= 3 + \kappa + \tfrac{1}{2}u^2(3\kappa - 5) + (\kappa - 3)c(u) \\ &\quad + \sqrt{8\kappa - 6}(1 + c(u))u. \end{aligned} \tag{60}$$

That is, we have parametrized the gauge fields in terms of only one function u, which together with the Higgs field ϕ_1 are the only nontrivial degrees of freedom. The next step is to write the equations of motion in terms of them. Actually, though we have reduced significantly the number of degrees of freedom, the equations of motion are very complicated in terms of these fields. We show the complete expression of the equations of motions in the appendix.

In this variables the small θ limit can be recovered through the identification

$$u(r) = \frac{\theta}{\sqrt{2}}(1 - k(r)) + O(\theta^3). \tag{61}$$

D. Boundary conditions

In order to get nonsingular, finite energy solutions, we have to impose appropriate boundary conditions. At the origin we have the usual conditions

$$u(0) = 0, \qquad \phi_1(0) = 0. \tag{62}$$

At $r \to \infty$ the situation is different from the commutative case. Notice that in the presence of a potential,

$$V = \lambda(\phi^2 - \eta^2)^2, \tag{63}$$

the Higgs field tends asymptotically to a minimum of the potential. That is, asymptotically, ϕ_0 and ϕ_1 are minima of

$$V = \lambda\Big(\frac{1}{16} - \frac{1}{2}\phi_0^2 + \phi_0^4 - \frac{\kappa}{2}\phi_1^2 + 6\kappa\phi_0^2\phi_1^2 - 4\kappa\phi_0\phi_1^3 + \kappa(\kappa+1)\phi_1^4\Big), \tag{64}$$

$$V = 1 - 2{\phi_0}^2 + {\phi_0}^4 - 2{\phi_1}^2\kappa + 6{\phi_0}^2{\phi_1}^2\kappa - 4\phi_0{\phi_1}^3\kappa + {\phi_1}^4\kappa + {\phi_1}^4\kappa^2 \tag{65}$$

(we have rescaled the fields so $\eta = 1/2$, consistent with the small θ expansion solution). Besides the trivial solution $\phi_0 = 1$, $\phi_1 = 0$ we have the solutions

$$\phi_0 = \frac{1}{4}\Big(1 + \frac{1}{\sqrt{1+4\kappa}}\Big), \qquad \phi_1 = \frac{1}{2\sqrt{1+4\kappa}}, \tag{66}$$

$$\phi_0 = \frac{1}{2\sqrt{1+4\kappa}}, \qquad \phi_1 = \frac{1}{\sqrt{1+4\kappa}} \tag{67}$$

(these correspond to absolute minima of the potential; there are other local minima but those will give infinite energy when integrated over the whole space).

The first of these equations gives a nontrivial $U(1)$ contribution in the $\theta \to 0$ limit, so we discard it. The second one gives the correct small θ behavior, so we take it as the asymptotic boundary condition:

$$\lim_{r\to\infty}\phi_1(r) = \frac{1}{\sqrt{1+4\kappa}}. \tag{68}$$

Of course, this is valid in the presence of a potential. For a vanishing coupling constant, as it happens in commutative space, we can rescale the Higgs fields arbitrarily by rescaling appropriately the radial variable r.

For the gauge field we impose, as usual, that at infinity the Higgs kinetic term vanishes. This gives the behavior

$$\lim_{r\to\infty}u(r) = \frac{4\sqrt{2}}{\sqrt{4\kappa-3} + 3\sqrt{4\kappa+1}}. \tag{69}$$

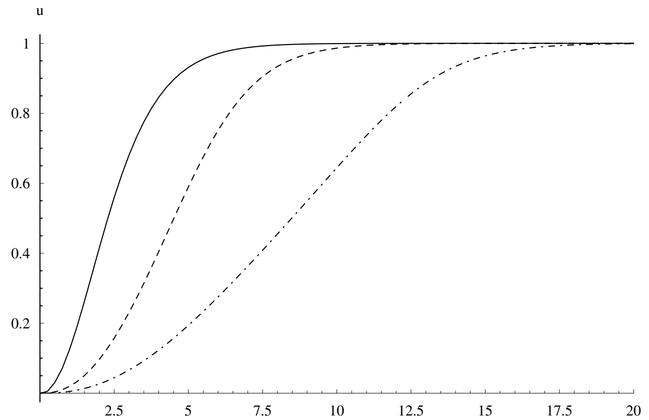

FIG. 1. The field $u(r)$ (normalized to 1 at infinity) for different values of κ. The solid line is for $\kappa = 10$ (indistinguishable from the standard BPS solution), the dashed line is for $\kappa = 0.9$, and the dotted-dashed line is for $\kappa = 0.76$.

E. Numerical solutions

We solved numerically the equations of motion for different values of $\kappa = 1/\theta^2 = l(l+1)$. We found solutions for essentially any value of κ allowed ($\kappa > 3/4$). As expected, for large κ (small θ) the solution tends to the Prasad-Sommerfield (P-S) configurations. Indeed, even for $l = 1$, the profile of the solutions is very similar to the P-S solutions. In order to see the departure of the P-S solutions, we considered continuous values of κ (which correspond to infinite dimensional representation of the noncommutative algebra). It is remarkable that the Higgs field solution is not very sensitive to κ, even for extreme values ($\kappa \sim 3/4$). On the other hand, the gauge field in very sensitive to κ. We show in Figs. 1 and 2 the solutions for the fields u and ϕ_1, respectively, for various values of θ.

We also studied the energy of the monopole solutions as a function of θ. For small values of θ, the energy, in units of $e^2/4\pi$, tends to 1 as expected (BPS bound). As θ increases,

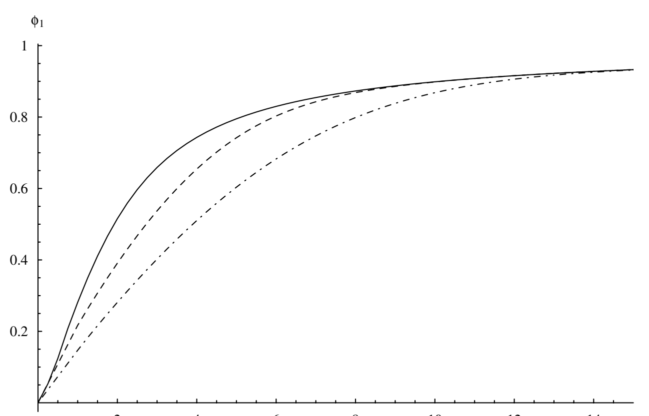

FIG. 2. The field $\phi_1(r)$ (normalized to 1 at infinity) for different values of κ. The solid line is for $\kappa = 10$ (indistinguishable from the standard BPS solution), the dashed line is for $\kappa = 0.9$, and the dotted-dashed line is for $\kappa = 0.76$.

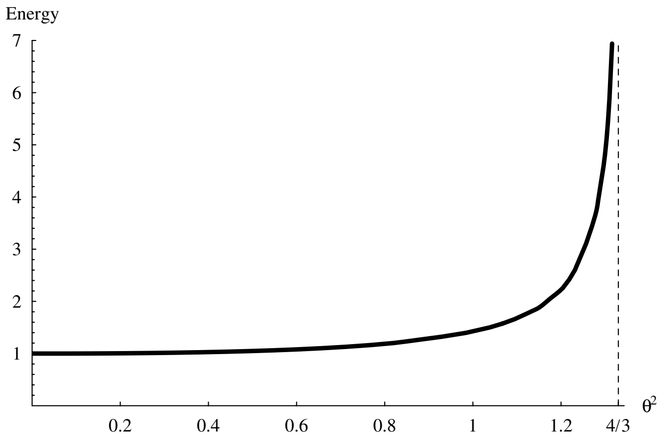

FIG. 3. Energy, in units of $e^2/4\pi$, of the monopole as a function of θ^2. The energy tends to 1 when $\theta^2 \to 0$ (commutative BPS solution) and diverges when $\theta^2 \to 4/3$.

the energy also increases and diverges as θ^2 approaches to 4/3. A plot of the energy as a function of θ is shown in Fig. 3. This behavior is another hint that, for θ different from zero, the solutions obtained are not self-dual, since in that case we expect the energy of the configuration to be equal to some topological number (independent of θ). This situation can be contrasted with the case of self-dual vortex solutions in noncommutative space. In the latter, while the profile of the solutions are dependent of the noncommutative parameter, the energy is θ-independent and, in particular, equal to 1 (in appropriate units), the Bogomolny energy bound [9].

VI. SUMMARY AND CONCLUSIONS

Previous analysis of monopole configuration in noncommutative space were done using the standard noncommutative relations

$$[x_i, x_j] = i\theta_{ij}, \qquad \theta_{ij}\text{: constant.} \tag{70}$$

Although this algebra is invariant under space translations, as is immediate from the definition, commutation relations (70) are not invariant under space rotations. In particular, we cannot benefit from the simplifications, in structure and in number of degrees of freedom, that a spherically symmetric ansatz produces.

In contrast to relations (70), we can construct a different noncommutative algebra which is manifestly rotationally invariant

$$[x_i, x_j] = i\theta r \varepsilon_{ijk} x_k \tag{71}$$

but at the expense of losing translational invariance. In fact, the algebra (71) is incompatible with a momentum operator P_i generating infinitesimal translations. However, this is not an impediment to construct a field theory in this geometry. A representation of this algebra can be constructed by identifying $x_i = \theta r L_i$. In this representation the value θ labels the representation through the relation $1/\theta^2 = \vec{L}^2$. So, although θ can take any positive value, for the special case $1/\theta^2 = l(l+1)$, $l \in \mathbb{N}$, we have finite-dimensional representations (notice, however, that the radial variable r takes continuous values).

In commutative space, a Poincaré invariant Lagrangian can be written in terms of momentum operators P_i, where translational invariance is manifest, or in terms of angular momentum operators L_i (together with a radial scaling operator P), where rotational invariance is obvious. To construct a NCFT with the algebra (70) one chooses the former and promotes the variables (with some prescribed order) to noncommutative operators. Analogously, to construct a NCFT with the algebra (71), we can choose the latter and again promote the variables to noncommutative operators.

In particular, we constructed a Yang-Mills-Higgs Hamiltonian in this space and also derived the equations of motions. A puzzling aspect is that we were unable to derive first order (BPS) equations of motion. Though we do not have a rigorous proof of this statement, there are several hints that suggest this property. Since the theory is manifestly invariant under rotations, we tried a spherically symmetric ansatz, which is nothing but a noncommutative extension of the 't Hooft monopole ansatz. Then, as it happens in the commutative case, the number of degrees of freedom is reduced to just two, one for the gauge field and the other for the Higgs field. The final equations of motion are very complicated in form but not difficult to solve numerically. Moreover, we showed that in the limit $\theta \to 0$ the equations of motion (and, in fact, the whole Hamiltonian) reduce to the standard commutative Yang-Mills-Higgs theory, allowing then a perturbative solution in the noncommutative parameter. Another characteristic of this theory is that it blows up at $\theta^2 = 4/3$, which incidentally is the maximum value of θ for which there is a finite-dimensional representation of the algebra.

We solved numerically the Euler equations of motion for different values of θ. As expected for small values of θ, the solution is indistinguishable from the exact Prasad-Sommerfield solution. As we increase the value of θ, the profile of the solution departs from the P-S solutions, and also the energy increases. In particular, the energy diverges as θ approaches to its maximum value $\theta^2 = 4/3$.

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APPENDIX

In this appendix, we present the Euler-Lagrange equations of motion in terms of the unconstrained variable u and the Higgs field ϕ_1. The equations read:

$$
\begin{aligned}
&u''(r) + ((2r^2\kappa c(u)^4 d(u)^2 \phi_1(r)^2(\sqrt{8\kappa-6} - 3\sqrt{8\kappa-6}c(u) + 2(4\kappa-3)u)(6(4\kappa-3)c(u)^3 \\
&+ c(u)^2(24 - 32\kappa + (21 - 17\kappa)\sqrt{8\kappa-6}u) + u((3-\kappa)\sqrt{8\kappa-6} - 6\sqrt{8\kappa-6}d(u) - 2(12 - 19\kappa + 4\kappa^2)u \\
&+ 2\sqrt{2}(4\kappa-3)^{3/2}u^2) + 2c(u)(4\kappa - 3 + (6-8\kappa)d(u) + (7\kappa-10)\sqrt{8\kappa-6}u + (24 - 41\kappa + 12\kappa^2)u^2)))(4\kappa-3)^{-1} \\
&+ c(u)^4 d(u)(\sqrt{2}\kappa(-(\sqrt{2}\kappa(1+c(u))^3) + 3\sqrt{\kappa}(1+c(u))^2\sqrt{d(u)} - \sqrt{2}(1+c(u))(d(u) + 8\kappa u^2) \\
&+ (2\sqrt{\kappa}u^2(-2\sqrt{2}\sqrt{\kappa}\sqrt{4\kappa-3}c(u) + 3\sqrt{4\kappa-3}\sqrt{d(u)} + 2\sqrt{\kappa}(\sqrt{8\kappa-6} + (-7+4\kappa)u)))(4\kappa-3)^{-1/2})(-2d(u)u \\
&+ (1+c(u))(\sqrt{8\kappa-6}c(u)^2 - c(u)(\sqrt{8\kappa-6} + (3\kappa-5)u) + u(\kappa - 3 - \sqrt{8\kappa-6}u))) + (2\kappa u(-2\sqrt{4\kappa-3}c(u)^3 \\
&+ c(u)^2(4\sqrt{4\kappa-3} + \sqrt{2}(3\kappa-5)u) + u(\sqrt{2}(\kappa-3) + 2\sqrt{2}d(u) - 2\sqrt{4\kappa-3}u) + 2c(u)(-\sqrt{4\kappa-3} \\
&- 2\sqrt{2}(\kappa-2)u + \sqrt{4\kappa-3}u^2))(24\kappa(4\kappa-3)c(u)^2 + (8\kappa-6)d(u) - 3\sqrt{\kappa}\sqrt{d(u)}(\sqrt{2}(4\kappa-3) + 2(7-4\kappa)\sqrt{4\kappa-3}u) \\
&+ \kappa u((27-20\kappa)\sqrt{8\kappa-6} + 2(71 - 64\kappa + 16\kappa^2)u) + c(u)(15\sqrt{2}(3-4\kappa)\sqrt{\kappa d(u)} + 3\kappa(8(4\kappa-3) \\
&+ (29-12\kappa)\sqrt{8\kappa-6}u))))((4\kappa-3)^{3/2})^{-1} + ((-6(4\kappa-3)c(u)^3 + \sqrt{4\kappa-3}c(u)^2(12\sqrt{4\kappa-3} + \sqrt{2}(-21+17\kappa)u) \\
&+ u(3(\kappa-3)\sqrt{8\kappa-6} + 6\sqrt{8\kappa-6}d(u) + 2(18 - 27\kappa + 4\kappa^2)u + 2(3-4\kappa)\sqrt{8\kappa-6}u^2) + 2c(u)(9 - 12\kappa \\
&+ (8\kappa-6)d(u) - 5(2\kappa-3)\sqrt{8\kappa-6}u + (-24 + 41\kappa - 12\kappa^2)u^2))(4\sqrt{2}\kappa(4\kappa-3)^{3/2}c(u)^3 - (4\kappa-3)d(u)(\sqrt{8\kappa-6} \\
&+ 2(2\kappa-1)u) + 6\sqrt{\kappa}\sqrt{4\kappa-3}\sqrt{d(u)}(4\kappa - 3 + (-2+3\kappa)\sqrt{8\kappa-6}u + 2u^2) - 2\kappa(2\sqrt{2}(4\kappa-3)^{3/2} + 4(3-4\kappa)^2 u \\
&+ 3\sqrt{8\kappa-6}(4 - 7\kappa + 4\kappa^2)u^2 - 2(4 - 43\kappa + 48\kappa^2 - 16\kappa^3)u^3) + 6c(u)^2((\sqrt{\kappa}(4\kappa-3)^{3/2}\sqrt{d(u)}) \\
&- 2\kappa(4\kappa-3)(\sqrt{8\kappa-6} + 2(-1+2\kappa)u)) + \sqrt{4\kappa-3}c(u)(\sqrt{2}(4\kappa-3)d(u) - 6\sqrt{\kappa}\sqrt{d(u)}(8\kappa - 6 + (\kappa-2)\sqrt{8\kappa-6}u) \\
&+ 6\kappa(\sqrt{8}(4\kappa-3) + (\kappa-1)\sqrt{4\kappa-3}u + \sqrt{2}(4 - 21\kappa + 12\kappa^2)u^2))))((3-4\kappa))^{-2}) - r^2\kappa(-16\sqrt{8\kappa-6}(12 + \kappa^2 \\
&+ (-12+\kappa^2)c(u)) + 16(117 - 54\kappa - 8\kappa^2 + 2\kappa^3 + (-117 + 58\kappa - 8\kappa^2 + 2\kappa^3)c(u))u + 4\sqrt{8\kappa-6}(339 - 58\kappa \\
&+ 17\kappa^2 + (-315 + 38\kappa + 15\kappa^2)c(u))u^2 - 4(1101 - 884\kappa - 55\kappa^2 + 30\kappa^3 - (859 - 836\kappa + 103\kappa^2 - 26\kappa^3)c(u))u^3 \\
&- \sqrt{8\kappa-6}(1519 - 596\kappa + 93\kappa^2 + (-977 + 432\kappa + 65\kappa^2)c(u))u^4 + (2265 - 2612\kappa + 213\kappa^2 + 34\kappa^3 \\
&- (385 - 600\kappa + 141\kappa^2 + 14\kappa^3)c(u))u^5 + \sqrt{8\kappa-6}(-41 + 84\kappa - 43\kappa^2 + (109 + 48\kappa - 69\kappa^2)c(u))u^6 \\
&+ (267 - 60\kappa - 305\kappa^2 + 54\kappa^3 - 2(485 - 872\kappa + 361\kappa^2 - 18\kappa^3)c(u))u^7 \\
&+ 4\sqrt{8\kappa-6}(99 - 112\kappa + 21\kappa^2)u^8)u'(r)^2)(2r^2\kappa c(u)^3 d(u)(8(6 - \kappa + 2\kappa^2 + (-6 - \kappa + 2\kappa^2)c(u)) \\
&+ 8\sqrt{8\kappa-6}(2 + \kappa + (\kappa-2)c(u))u + 4(-4 + 5\kappa + 2(\kappa-2)\kappa c(u))u^2 + 2\sqrt{8\kappa-6}(3\kappa + 5(\kappa-2)c(u))u^3 \\
&+ (-49 + 57\kappa - 6\kappa^2)u^4))^{-1} = 0, \qquad \text{(A1)}
\end{aligned}
$$

$$
r\frac{d^2}{dr^2}(r\phi_1) - (\kappa((3c(u) - 1) - \sqrt{8\kappa-6}u)^2\phi_1)d(u)^{-1} = 0, \qquad \text{(A2)}
$$

where the functions $c(u)$ and $d(u)$ are defined in Eqs. (58) and (60), respectively.

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