

On wormholes with arbitrarily small quantities of exotic matter

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Recently several models of traversable wormholes have been proposed which require only arbitrarily small amounts of negative energy to hold them open against self-collapse. If the exotic matter is assumed to be provided by quantum fields, then quantum inequalities can be used to place constraints on the negative energy densities required. In this paper, we introduce an alternative method for obtaining constraints on wormhole geometries, using a recently derived quantum inequality bound on the *null*-contracted stress-energy averaged over a *timelike* worldline. The bound allows us to perform a simplified analysis of general wormhole models, not just those with small quantities of exotic matter. We then use it to study, in particular, the models of Visser, Kar, and Dadhich (VKD) and the models of Kuhfittig. The VKD models are constrained to be either submicroscopic or to have a large discrepancy between throat size and curvature radius. A recent model of Kuhfittig is shown to be nontraversable. This is due to the fact that the throat of his wormhole flares outward so slowly that light rays and particles, starting from outside the throat, require an infinite lapse of affine parameter to reach the throat.

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I. INTRODUCTION

Recent years have seen much progress in our understanding of the physical laws governing negative energy densities associated with quantum fields. It has been known for some time that quantum field theory allows violations of all the classical pointwise energy conditions. In particular, quantum fields violate both the weak energy condition (WEC), which requires the stress-energy tensor T_{ab} to obey

$$T_{ab}u^a u^b \geq 0, \quad (1)$$

for all timelike vectors u^a , and the null energy condition (NEC), which requires

$$T_{ab}k^a k^b \geq 0, \quad (2)$$

for all null vectors k^a [1]. Examples are squeezed vacuum states of light [2] and the Casimir effect [3], both of which can be realized in the laboratory. It is also known that negative energy is required for the Hawking evaporation of black holes [4], in which an outgoing flux of positive energy seen at infinity is paid for by a negative energy flux through the horizon [5].

With this in mind, it is worth considering weaker variants of the WEC and NEC, based on averages along timelike and null geodesics, respectively. Two such conditions are the averaged weak energy condition (AWEC):

$$\int_{-\infty}^{\infty} T_{ab}u^a u^b d\tau \geq 0, \quad (3)$$

where u^a is the tangent vector to an inextendible timelike geodesic parametrized by proper time τ , and the averaged null energy condition (ANEC):

$$\int_{-\infty}^{\infty} T_{ab}k^a k^b d\lambda \geq 0, \quad (4)$$

where k^a is the tangent vector to an inextendible null geodesic and λ is an affine parameter. Violation of the ANEC is known to be a necessary condition for the maintenance of traversable wormholes [6].

The fact that quantum field theory allows the existence of states violating the classical energy conditions raises various concerns. If arbitrarily large negative energy densities could persist for arbitrarily long times, gross macroscopic effects might occur, including violations of the second law of thermodynamics or the formation of exotic spacetime structures. The latter includes “designer spacetimes” such as traversable wormholes [7–9], warp drives [10,11], and time machines [12]. In two seminal papers, Ford [13,14] introduced the notion of what have come to be called “quantum inequalities” (QIs) [15], which are restrictions derived from quantum field theory on the magnitude and duration of negative energy. More specifically, Ford’s original papers were primarily concerned with negative energy fluxes. His work was subsequently extended and generalized by himself and others to constraints on negative energy densities (see Sec. II and Refs. [16,17] for recent reviews and more extensive references).

In this paper we will apply QIs to place constraints on a class of wormholes introduced Morris and Thorne [7] and to some particular instances recently advanced by Visser, Kar, and Dadhich (VKD) [18] and Kuhfittig [19–21]. In so doing, we will make some improvements to the arguments

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originally set out in Ref. [22] to constrain traversable wormhole geometries. The VKD and Kuhfittig models we study are of interest because they are claimed to use “arbitrarily small” quantities of exotic matter. In particular, VKD propose a “volume integral quantifier” which they suggest is a good measure for the amount of exotic matter required to maintain a traversable wormhole. Using this measure, they have shown that the amount of exotic matter can be made arbitrarily small, even though the ANEC integral along radial null geodesics passing through the wormhole is shown to be finite and negative. Kuhfittig has also proposed several wormhole models [19–21] with similar properties, the last of which he claims to be macroscopic and traversable, to require arbitrarily small amounts of exotic matter, and to be consistent with the QI bounds.

If one assumes that the exotic matter required to maintain these wormholes comes from quantum matter fields, then we show, using techniques related to those in Ref. [22], that the geometry of these wormholes is severely constrained. However, our analysis differs from that presented in Ref. [22], in that we introduce an alternative method for obtaining constraints on wormhole geometries, using a recently derived QI bound on the *null*-contracted stress-energy averaged over a *timelike* worldline. The bound allows us to perform a simplified analysis of general wormhole models, not just those with small quantities of exotic matter. We then use it to study, in particular, the models of Visser, Kar, and Dadhich, and the models of Kuhfittig.

The VKD wormholes are constrained by the QI bound to be either submicroscopic in size (e.g., a few orders of magnitude above the Planck length), or to have a very small ratio of minimum curvature radius to throat size. An examination of Kuhfittig’s models shows that a confusion between proper and coordinate distances in fact renders the model proposed in Ref. [21] *non*-traversable. In particular, we explicitly show that radially infalling light rays and particles reach the throat of this wormhole only after an infinite lapse of affine parameter. Lastly, we provide a further justification for our bound using a “difference inequality” argument.

II. QUANTUM INEQUALITY CONSTRAINTS ON EXOTIC SPACETIMES

A. Quantum inequalities

We begin with a short review of quantum inequalities, both to explain their nature and to set out the extent of the known results and the classes of states for which they hold.

To start, consider D -dimensional Minkowski space, and let $\xi(\tau)$ be the worldline of an inertial observer with proper time parameter τ and velocity $u = d\xi/d\tau$. If $q(\tau)$ is a smooth non-negative function peaked around $\tau = 0$, with unit characteristic width [23] and normalized so that $\int_{-\infty}^{\infty} q(\tau)d\tau = 1$, then the integrals

$$\int_{-\infty}^{\infty} \langle T_{ab}u^a u^b \rangle_{\omega}(\xi(\tau)) \frac{1}{\tau_0} q(\tau/\tau_0) d\tau \quad (5)$$

are local averages of the expected renormalized energy density seen over a timescale τ_0 about $\tau = 0$ when the field is in state ω . By decreasing τ_0 , we can “zoom in” on the region around $\tau = 0$; we may use the freedom to move the zero of proper time along ξ to zoom in on different regions of the worldline.

Quantum inequalities are constraints on these local averages. An example would be a statement of the form: *there exists a dimensionless positive constant C (depending on q and D , but not τ_0 or ω) such that*

$$\int_{-\infty}^{\infty} \langle T_{ab}u^a u^b \rangle_{\omega}(\xi(\tau)) \frac{1}{\tau_0} q(\tau/\tau_0) d\tau \geq -\frac{C}{\tau_0^D} \quad (6)$$

for all physically reasonable states ω and all sampling times $\tau_0 > 0$. (Here, as elsewhere, we employ units with $\hbar = c = 1$.) A variety of such bounds have been established in varying levels of generality and rigor and with varying conditions on the sampling functions and the class of states involved. The original QIs (see, e.g., Ref. [14,24]), were established for the Lorentzian sampling function $q(\tau) = 1/(\pi(\tau^2 + 1))$, and provided the constant $C = 3/(32\pi^2)$ if $D = 4$. A subsequent generalization [25,26] permitted all q of the form $q(\tau) = g(\tau)^2$ where g is smooth, real-valued and of compact support [i.e., vanishing outside a compact set] or of sufficiently rapid decay at infinity. In this case, for $D = 4$,

$$C = \frac{\int_{-\infty}^{\infty} g''(\tau)^2 d\tau}{16\pi^2 \int_{-\infty}^{\infty} g(\tau)^2 d\tau}, \quad (7)$$

which has a minimum value of around 5 [27] if g is supported in an interval of unit length. Note that it is crucial that g be smooth enough to have a square-integrable second derivative [28].

The arguments so far mentioned, i.e., those of Refs. [24–26], utilize formal manipulations in Fock space and so are limited—in the first instance—to a class of states arising as vectors in (or density matrices on) the Fock space built on the Minkowski vacuum state. These limitations were removed by the first fully rigorous bound applicable in general dimension $D \geq 2$ [29]. This bound applies to a general smooth timelike curve $\xi(\tau)$ in a general globally hyperbolic spacetime and asserts: *given any Hadamard state ω_0 as a “reference state”, the bound*

$$\int_{-\infty}^{\infty} [\langle T_{ab}u^a u^b \rangle_{\omega}(\xi(\tau)) - \langle T_{ab}u^a u^b \rangle_{\omega_0}(\xi(\tau))] g(\tau)^2 d\tau \geq -\mathcal{Q}[g] \quad (8)$$

holds for all Hadamard states ω and smooth, real-valued, compactly supported g , where $g \mapsto \mathcal{Q}[g]$ is a quadratic form depending on the spacetime, the trajectory ξ , a choice of D -bein near ξ , and the reference state ω_0 . It must be emphasized that $\mathcal{Q}[g]$ is finite for all g in our class, that

there is a closed-form expression for $\mathcal{Q}[g]$ in terms of the two-point function of ω_0 and, most importantly, that ω and ω_0 are *arbitrary* Hadamard states: there is no assumption that ω can be represented as a vector or density matrix in the same Hilbert space representation as ω_0 [i.e., ω and ω_0 may belong to different “folia”]. This is because the argument used in [29] is formulated within the algebraic approach to quantum field theory in curved spacetimes, and does not require the theory to be formulated in Hilbert space.

The QI, Eq. (8), is an example of a “difference QI”; that is, it bounds the difference of the expectation values of the energy density in an arbitrary quantum state and in some reference state. However, Eq. (8) is easily converted into an “absolute QI”

$$\int_{-\infty}^{\infty} \langle T_{ab} u^a u^b \rangle_{\omega}(\xi(\tau)) g(\tau)^2 d\tau \geq -\mathcal{Q}'[g] \quad (9)$$

where

$$\mathcal{Q}'[g] = \mathcal{Q}[g] - \int_{-\infty}^{\infty} \langle T_{ab} u^a u^b \rangle_{\omega_0}(\xi(\tau)) g(\tau)^2 d\tau \quad (10)$$

is also finite for all g in our class, because the integrand on the right-hand side is smooth and compactly supported. Again, it must be emphasized that—contrary to the mistaken view recently expressed in Ref. [30]—this bound holds for *any* Hadamard state ω , not simply those in the folium of ω_0 . If we apply the above bounds to the case of an inertial worldline in four-dimensional Minkowski space, with ω_0 chosen to be the Minkowski vacuum state then a bound of the form Eq. (6) is obtained if we put $q(\tau) = g(\tau)^2$ and with C given by Eq. (7).

Let us note a separate strand of work [31–33], initiated by Flanagan, which treats massless fields in two-dimensional curved spacetimes for general worldlines and arbitrary Hadamard states. There are also various extensions of the QI bounds to free Dirac [32,34,35], Maxwell [36–38], Proca [38] and Rarita-Schwinger [39] fields. In addition, it has recently been proved (by extending the argument of Ref. [31]) that all unitary, positive energy conformal field theories in two-dimensional Minkowski space obey QI bounds [40], thus providing the first examples of QIs for interacting quantum field theories. No general results are known for other interacting quantum field theories, although Olum and Graham have provided an example in four-dimensions with two coupled scalar fields in which a static negative energy density is created [41]. This suggests that worldline quantum inequalities might not hold for general interacting quantum field theories without some further qualification. However, several important caveats must be entered: first, the Olum-Graham example does not exclude the possibility that QIs might hold for local averages over suitable spacetime volumes. We expect that this is indeed the case—as it is for conformal fields [40]—and moreover that this would not substantially modify the results of our analysis in any

significant way. Second, the Olum-Graham example is effectively hard-wired into the Lagrangian, which has been engineered to produce a domain wall configuration of the required type. It is not clear to us that a *single* choice of Lagrangian (including specific values for any parameters it contains) could produce *arbitrarily* negative static energy densities. If not, then one might well be able to apply worldline quantum inequalities on scales shorter than the length scales [implicit in the Lagrangian] which fix the magnitude of any static negative energy density configurations. However definite statements on these issues must await more progress on interacting theories.

B. Constraints on exotic spacetimes

Quantum inequalities have been used to place constraints on several different “designer spacetimes”, such as traversable wormhole and warp drive spacetimes [22,42–44]. In each case, the basic idea is to obtain the stress-energy tensor required to support a given spacetime and then test it for consistency with the QI bounds, leading to constraints on various parameters arising in the metric. A problem which must be confronted is that the QI, Eq. (9), requires explicit knowledge of the two-point function of a reference state ω_0 to compute the right-hand side. Such knowledge is not at hand for general wormhole models and hinders attempts to use Eq. (9) directly. Instead, we will follow Ref. [22] in assuming that the *flat spacetime* QI bounds should also be applicable in curved spacetimes and/or spacetimes with boundaries, in the “short-sampling time limit.” Specifically, we restrict the sampling time to be $\tau_0 = f\ell_{\min}$, where $f \ll 1$ and ℓ_{\min} is the smallest proper radius of curvature or the smallest proper distance to any boundary of the spacetime, and apply QI bounds for averaging along timelike *geodesics* [45].

Strictly speaking, this is an *assumption*, but it is one for which good justification can be provided. Three arguments may be given (see also the discussion in Ref. [22]): first, the equivalence principle leads us to expect that physics “in the small” should be approximately Minkowskian as far as freely falling observers are concerned; second, it is borne out by specific examples in four dimensions by taking the short-sampling time limit of various curved spacetime QIs [26,46]; third, one of us (CJF) has recently established the validity of this assumption for massless scalar fields in general two-dimensional spacetimes [47]. Further support is provided by a new argument sketched in Sec. VIII. It is also expected that a more general proof may be given, and work is in progress on this question.

The use of flat spacetime QIs in the above fashion suffices to put fairly strong constraints on “designer spacetimes,” such as traversable wormhole and warp drive spacetimes [22,42,43]. These analyses were based on QIs using Lorentzian sampling functions, but the more recent QIs based on compactly supported sampling functions remove worries that the infinite “tails” of a noncompactly

supported sampling function might invalidate the analysis by picking up large nonlocal effects. (This could also be dealt with by making the width of the sampling function small enough to make the sampling function drop off sufficiently fast, at the expense of weakening the QI bounds.)

III. SOME WORMHOLE GEOMETRY

We study a class of four-dimensional traversable wormholes introduced by Morris and Thorne [7] in which two spacetime regions, referred to as the “upper universe” and the “lower universe” are joined by a throat. The wormhole models are static, spherically symmetric, and, for simplicity, the upper and lower universes are taken to be isometric. The parameter ℓ measures the signed proper radial distance from the wormhole throat, running from $-\infty$ in the asymptotic region of the lower universe to $+\infty$ in the asymptotic region of the upper universe, with $\ell = 0$ at the wormhole throat itself. The general form of the wormhole metric is

$$ds^2 = -e^{2\Phi(r(\ell))} dt^2 + d\ell^2 + r^2(\ell)(d\theta^2 + \sin^2\theta d\phi^2), \quad (11)$$

where $2\pi r(\ell)$ is the proper circumference of a circle of fixed ℓ in the equatorial plane $\theta = \pi/2$ (with t constant), and Φ is called the “red-shift function”. The function $r(\ell)$ is assumed to be even, twice continuously differentiable, and to possess a global minimum at the throat, where $r(0) = r_0 > 0$, and no other stationary points. It is also assumed that $0 < dr/d\ell \leq 1$ for all $\ell > 0$, which ensures that the wormhole “flares out” when seen in an embedding diagram, such as the one shown in Fig. 1. We also require that $r(\ell)/|\ell| \rightarrow 1$ as $|\ell| \rightarrow \infty$, fast enough to ensure asymptotic flatness.

The red-shift function $\Phi(r)$ is defined on $[r_0, \infty)$. We require $\Phi(r(\ell))$ to be twice continuously differentiable and to satisfy $\Phi(r(\ell)) \rightarrow 1$ as $|\ell| \rightarrow \infty$ fast enough to ensure asymptotic flatness. Although symmetry between the

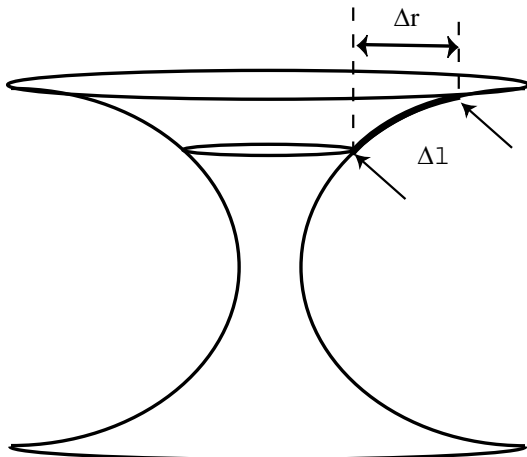


FIG. 1. Proper versus coordinate distance in a wormhole.

upper and lower universes requires

$$\left. \frac{d\Phi(r(\ell))}{d\ell} \right|_{\ell=0} = 0, \quad (12)$$

we note that $\Phi'(r)$ and $\Phi''(r)$ [i.e., the derivatives with respect to r] may be divergent as $r \rightarrow r_0$, a point which we will discuss later. For the wormhole to be traversable it must have no horizons, which implies that $g_{tt} = -e^{2\Phi(r)}$ must never be allowed to vanish, and hence $\Phi(r)$ must be everywhere bounded from below; it must also be bounded from above by virtue of continuity and its behavior as $r \rightarrow \infty$.

The restrictions that $\Phi(r(\ell))$ and $r(\ell)$ be twice continuously differentiable ensure that the stress-energy tensor (obtained from Einstein’s equations) is continuous. It is sometimes useful to weaken this condition at isolated values of ℓ so as to allow the inclusion of thin shells of matter.

It is also convenient to introduce a radial coordinate r , with range $[r_0, \infty)$, on the upper universe (or, equally, on the lower universe) so that the metric now takes the form

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{(1 - b(r)/r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (13)$$

Here, $b(r)$, defined on $[r_0, \infty)$, is called the “shape function” and is related to the function $r(\ell)$ by

$$\left(\frac{dr}{d\ell} \right)^2 = 1 - \frac{b(r)}{r}, \quad (14)$$

or, equivalently,

$$\ell = \int_{r_0}^{r(\ell)} \frac{dr'}{(1 - b(r')/r')^{1/2}}. \quad (15)$$

for $\ell > 0$. [In the lower universe we would insert an overall minus sign on the right-hand side.] From Eq. (14) we see that $b(r(\ell))$ must be continuously differentiable (with a one-sided derivative at r_0); differentiating Eq. (14) and dividing by $2dr(\ell)/d\ell$ gives

$$\frac{d^2 r(\ell)}{d\ell^2} = \frac{b(r)}{2r^2} - \frac{b'(r)}{2r}. \quad (16)$$

In particular, as $\ell \rightarrow 0$, $b'(r)$ tends to a finite limit $b'_0 = b'(r_0) = 1 - 2r_0 d^2 r(\ell)/d\ell^2|_{\ell=0}$. Note that $b'_0 \leq 1$, because $r(\ell)$ has a minimum at the throat, and therefore $d^2 r(\ell)/d\ell^2|_{\ell=0} \geq 0$.

Like $r(\ell)$, the shape function $b(r)$ determines the outward flaring of the wormhole throat as viewed, for example, in an embedding diagram; the geometry is completely specified by Φ together with either $r(\ell)$ or $b(r)$. Since $0 \leq dr/d\ell \leq 1$ we have $0 \leq 1 - b(r)/r \leq 1$; since $r(\ell)$ has a unique minimum at $r = r_0$ we see that this is also the unique solution to the equation $b(r) = r$. Thus g_{rr} diverges at the throat, but this is clearly only a coor-

dinate singularity as the metric, Eq. (11), is regular there. We also emphasize that the proper distance is greater than or equal to the coordinate distance: $|\ell| \geq r - r_0$.

Substitution of the metric Eq. (13) into the Einstein equations $G_{ab} = 8\pi T_{ab}$ gives the stress-energy tensor required to generate the wormhole geometry. In this section we will use units in which the Planck length is set to unity. It is also convenient to work in the static orthonormal frame given by the basis:

$$\begin{aligned} \mathbf{e}_{\hat{t}} &= e^{-\Phi} \mathbf{e}_t, & \mathbf{e}_{\hat{r}} &= (1 - b/r)^{1/2} \mathbf{e}_r, \\ \mathbf{e}_{\hat{\theta}} &= r^{-1} \mathbf{e}_\theta, & \mathbf{e}_{\hat{\phi}} &= (r \sin\theta)^{-1} \mathbf{e}_\phi, \end{aligned} \quad (17)$$

where $\mathbf{e}_t = \partial/\partial t$ etc. (These definitions are extended to $r = r_0$ and $\theta = 0, \pi$ by continuity.) This basis represents the proper reference frame of an observer who is at rest relative to the wormhole. In this frame the stress tensor components are given by

$$T_{\hat{t}\hat{t}} = \rho = \frac{b'}{8\pi r^2}, \quad (18)$$

$$T_{\hat{r}\hat{r}} = p_r = -\frac{1}{8\pi} \left[\frac{b}{r^3} - \frac{2\Phi'}{r} \left(1 - \frac{b}{r}\right) \right], \quad (19)$$

$$\begin{aligned} T_{\hat{\theta}\hat{\theta}} &= T_{\hat{\phi}\hat{\phi}} = P \\ &= \frac{1}{8\pi} \left[\frac{1}{2} \left(\frac{b}{r^3} - \frac{b'}{r^2} \right) + \frac{\Phi'}{r} \left(1 - \frac{b}{2r} - \frac{b'}{2}\right) \right. \\ &\quad \left. + \left(1 - \frac{b}{r}\right) (\Phi'' + (\Phi')^2) \right]. \end{aligned} \quad (20)$$

The quantities ρ , p_r , and P are the mass-energy density, radial pressure, and transverse pressure, respectively, as measured by a static observer. At the throat of the wormhole, $r = r_0$, these reduce to

$$\rho_0 = \frac{b'_0}{8\pi r_0^2}, \quad (21)$$

$$p_0 = -\frac{1}{8\pi r_0^2}, \quad (22)$$

$$P_0 = \frac{1 - b'_0}{16\pi r_0} \left(\Phi'_0 + \frac{1}{r_0} \right), \quad (23)$$

where $b'_0 = b'(r_0)$ and $\Phi'_0 = \Phi'(r_0)$. Note that in taking the limit in Eq. (23), we have implicitly assumed that Φ' does not diverge at the throat.

Given the above definitions, we may now see why the wormhole must violate the NEC. Let \mathbf{k} be the null vector $\mathbf{k} = \mathbf{e}_{\hat{t}} + \mathbf{e}_{\hat{r}}$. Then, arguing as in Sec. 11.4 of Ref. [9],

$$T_{ab} k^a k^b = \rho + p_r = -\frac{e^{2\Phi}}{8\pi r} \frac{d}{dr} \left(e^{-2\Phi} \left[1 - \frac{b}{r} \right] \right), \quad (24)$$

which reduces to

$$T_{ab} k^a k^b = \frac{b'_0 - 1}{8\pi r_0^2}, \quad (25)$$

at the throat. Since $b'_0 \leq 1$, we see that the NEC is violated at the throat unless $b'_0 = 1$. If $b'_0 = 1$, we may argue as follows: the quantity inside the parentheses in Eq. (24) vanishes at $r = r_0$, but is strictly positive for any $r > r_0$. Therefore, by the mean value theorem, there must a point in (r_0, r) at which the derivative in Eq. (24) is strictly positive, and for which the NEC is therefore violated. Since r was arbitrary, we have proved that the NEC is violated arbitrarily close to the throat [48].

The curvature tensor components are given by

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = \left(1 - \frac{b}{r}\right) [\Phi'' + (\Phi')^2] + \frac{\Phi'}{2r^2} (b - b'r), \quad (26)$$

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = \frac{\Phi'}{r} \left(1 - \frac{b}{r}\right), \quad (27)$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = \frac{1}{2r^3} (b'r - b), \quad (28)$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{b}{r^3}. \quad (29)$$

All other components of the curvature tensor vanish, except for those related to the above by symmetry. At the throat, these components reduce to

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}}|_{r_0} = \frac{\Phi'_0}{2r_0} (1 - b'_0), \quad (30)$$

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}}|_{r_0} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}}|_{r_0} = 0, \quad (31)$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}|_{r_0} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}}|_{r_0} = -\frac{1}{2r_0^2} (1 - b'_0), \quad (32)$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}|_{r_0} = \frac{1}{r_0^2}. \quad (33)$$

The limit Eq. (30) depends on the assumption that Φ' and Φ'' do not diverge at the throat. This will turn out to be an important consideration later in our discussion.

Let the magnitude of the maximum curvature component be R_{\max} . Therefore the smallest proper radius of curvature (which is also the coordinate radius of curvature in an orthonormal frame) is:

$$r_c \approx \frac{1}{\sqrt{R_{\max}}}. \quad (34)$$

We wish to work in a small spacetime volume around the throat of the wormhole such that all dimensions of this volume are much smaller than r_c , the smallest proper radius of curvature anywhere in the region. Thus, in the

absence of boundaries, spacetime can be considered to be approximately Minkowskian in this region, and we can apply a flat spacetime QI-bound, which we now describe.

IV. A ‘NULL-CONTRACTED’ QUANTUM INEQUALITY

The QIs discussed in Sec. II (and most QIs proved to date) are constraints on the energy density as seen by an observer moving along a (not necessarily inertial) worldline. However this is not the only possibility. In Ref. [49], we proved a QI which constrains the *null*-contracted stress tensor $\langle T_{ab}k^ak^b \rangle_\omega$ of a free scalar field along a *timelike* worldline, where k^a is a smooth null vector field. The result takes on a particularly simple form for massless fields in four-dimensional Minkowski space, with averaging conducted along an inertial worldline, and for a constant null vector field k^a . Let the worldline be $\xi(\tau)$, parametrized by proper time τ , and with (constant) four-velocity $u = d\xi/d\tau$. Then, as shown in Ref. [49], we have the QI

$$\begin{aligned} \int_{-\infty}^{\infty} d\tau \langle T_{ab}k^ak^b \rangle_\omega(\xi(\tau))g(\tau)^2 \\ \geq -\frac{(k_a u^a)^2}{12\pi^2} \int_{-\infty}^{\infty} d\tau g''(\tau)^2, \end{aligned} \quad (35)$$

for all Hadamard states ω and any smooth real-valued g which is compactly supported in \mathbb{R} . On the left-hand side, the stress tensor is normal-ordered with respect to the Minkowski vacuum, which is equivalent to renormalization according to the Hadamard prescription in this case. One could also consider noncompactly supported g by taking appropriate limits using sequences of functions with increasing support. We will not do this, partly to avoid technical issues concerning the limits, but mainly because it will not be necessary for our application.

Suppose that we are told that a certain state ω has $\langle T_{ab}k^ak^b \rangle_\omega(\xi(\tau)) \leq \mathcal{E}$ during $0 < \tau < \tau_0$. Applying the QI, we know that

$$\begin{aligned} \mathcal{E} \int_{-\infty}^{\infty} g(\tau)^2 d\tau &\geq \int_{-\infty}^{\infty} \langle T_{ab}k^ak^b \rangle_\omega g(\tau)^2 d\tau \\ &\geq -\frac{(k_a u^a)^2}{12\pi^2} \int_{-\infty}^{\infty} g''(\tau)^2 d\tau, \end{aligned} \quad (36)$$

for any smooth g compactly supported in $(0, \tau_0)$. Thus

$$\mathcal{E} \geq -\frac{(k_a u^a)^2}{12\pi^2} \frac{\int_{-\infty}^{\infty} g''(\tau)^2 d\tau}{\int_{-\infty}^{\infty} g(\tau)^2 d\tau}, \quad (37)$$

(provided g is not identically zero) and we are free to optimize the right-hand side over the class of allowed g 's. The variational problem may be solved by converting it into an eigenvalue problem [27,50] and leads to the conclusion that

$$\mathcal{E} \geq -\frac{C(k_a u^a)^2}{\tau_0^4}, \quad (38)$$

where $C \approx 4.23$. Note that \mathcal{E} scales quadratically with k by definition, which is why the right-hand side of the bound also has this dependence.

Our analysis will be based on the application of this bound to four-dimensional wormhole spacetimes over short timescales. More precisely, let ξ be a timelike geodesic with four-velocity u and suppose that k is parallel-transported along ξ . Then, motivated by the equivalence principle and the above analysis, we assume that: *for any Hadamard state ω , if $\langle T_{ab}k^ak^b \rangle_\omega(\xi(\tau)) \leq \mathcal{E}$ for (at least) a proper duration τ_0 which is short in comparison with the minimum length-scale characterizing the geometry, then \mathcal{E} and τ_0 must be constrained by Eq. (38)*. As mentioned in Sec. II this assumption is supported by various examples and its validity (at least for the close analogue of energy density, rather than the null-contracted stress-energy tensor) has been proved in the two-dimensional case.

In our application, the spacetimes in question are static, and the trajectory ξ will be a static trajectory. If ω_0 is the ground state of the quantum field theory on this spacetime, then $\mathcal{E}_0 = \langle T_{ab}k^ak^b \rangle_{\omega_0}(\xi(\tau))$ will be constant in τ . In examples of static spacetimes where the ground state stress-energy tensor is known, $|\mathcal{E}_0|$ is typically 2 orders of magnitude smaller than the magnitude of the right-hand side of Eq. (38), if τ_0 is comparable with the length-scales characterizing the geometry. Furthermore, one may prove that Casimir energies in locally Minkowskian spacetimes are consistent with this requirement [27]. So our expectation is that our assumption holds for ground states on static spacetimes, with considerable room to spare.

This consistency is clearly a *necessary* condition for the validity of our assumption. In Sec. VIII we will argue that it is also a *sufficient* condition. Thus we have good reason to believe that our assumption will produce reliable results.

V. GENERAL ANALYSIS

We begin by examining what conclusions may be drawn on the general symmetric Morris-Thorne wormhole model, before passing to particular examples. We initially assume only that $b'_0 < 1$, and discuss the case $b'_0 = 1$ separately.

Let $k = e_{\hat{t}} + e_{\hat{r}}$. Then k is everywhere null, and parallel-transported along the trajectory $\xi(\tau) = (e^{-\Phi(r_0)}\tau, r_0, \pi/2, 0)$, which is the worldline of a static observer at the throat. Then $T_{ab}k^ak^b$ takes the constant value

$$\mathcal{E} = \frac{b'_0 - 1}{8\pi r_0^2 l_p^2} < 0, \quad (39)$$

along $\xi(\tau)$, where $b'_0 = b'(r_0)$ as before and we have reinserted the Planck length l_p (keeping $\hbar = c = 1$) for later convenience. On the assumption that the stress-energy tensor is generated by a Hadamard state of a free scalar quantum field, we therefore have

$$\frac{1 - b'_0}{8\pi r_0^2 l_p^2} \leq \frac{C}{\tau_0^4}, \quad (40)$$

from Eq. (38), for all τ_0 small compared to local geometric scales. Note that $k_a u^a = 1$ for this trajectory, and also that the left-hand side is necessarily non-negative. Let ℓ_{\min} be the minimum length scale characterizing the local geometry. Then setting $\tau_0 = f\ell_{\min}$ in Eq. (40) and taking square roots we get

$$\frac{\ell_{\min}^2}{r_0} \sqrt{1 - b'_0} \leq f^{-2} \sqrt{8\pi C} l_p. \quad (41)$$

Although we could easily proceed with a general value of $f \ll 1$, we will take the more concrete path of fixing $f = 0.01$, which is quite a generous interpretation of $\tau_0 \ll \ell_{\min}$. As $\sqrt{8\pi C} \approx 10.3$, and we are only really interested in order of magnitude estimates, this gives

$$\frac{\ell_{\min}^2}{r_0} \sqrt{1 - b'_0} \lesssim 10^5 l_p. \quad (42)$$

Clearly, one or both of ℓ_{\min}^2/r_0 or $\sqrt{1 - b'_0}$ must be small in order for this to be satisfied. In fact our assumptions are quite conservative: as we will see in Sec. VIII, violation of Eq. (42) occurs only if the vacuum stress-energy tensor is 10 orders of magnitude larger than its value in typical static spacetimes. It is therefore likely that the actual constraints on wormholes arising from quantum field theory are yet stronger than those we describe below.

Before considering the consequences of this bound, let us note that our analysis has the following advantage over the one in Ref. [22]. In the case of wormholes with $\rho \geq 0$ for static observers, it was necessary in Ref. [22] to consider the usual QIs applied in the frame of a boosted observer passing through the throat in order to get a bound on energy density. The greater the boost, however, the shorter the proper time the observer will spend near the throat, and one should also consider the transit-time across the region of exotic matter as a relevant timescale in the analysis. This problem is absent from the present approach, because we use a *null*-contracted stress tensor averaged over the timelike worldline of a *static* observer at the throat. Hence we do not have to worry about the observer leaving the region of exotic matter.

We now begin our analysis of Eq. (42). First, assume that ℓ_{\min} is given by the minimum local curvature radius r_c . An examination of the curvature components shows that, ignoring constants of order 1, the three competing curvature radii at the throat are: r_0 , $r_0(1 - b'_0)^{-1/2}$ and $(r_0/(|\Phi'_0|[1 - b'_0]))^{1/2}$. In the last case, we have assumed that Φ' and Φ'' are nondivergent at the throat. Later we will consider what happens if this is not true.

If the minimum radius of curvature is $\ell_{\min} = r_0$, we obtain from Eq. (42) that

$$r_0 \lesssim \frac{10^5 l_p}{\sqrt{1 - b'_0}}. \quad (43)$$

Macroscopic wormholes in this regime therefore require extreme fine-tuning of b'_0 : even to approach a throat radius of 10^{20} Planck lengths ($10^{20} l_p \approx 10^{-15} \text{m} \approx 1 \text{fermi} \approx 1 \text{proton radius}$) one needs $1 - b'_0 \leq 10^{-30}$.

Next consider the case where $\ell_{\min} = r_0(1 - b'_0)^{-1/2}$ is the minimum curvature radius. Equation (42) then becomes

$$r_0 \lesssim 10^5 l_p \sqrt{1 - b'_0}. \quad (44)$$

Even our wormhole with a small 10^{20} Planck length-sized throat clearly requires $b'_0 \sim -10^{30}$. Note that the curvature radius $\ell_{\min} = r_0/\sqrt{1 - b'_0} \lesssim 10^5 l_p$ [from Eq. (44)] so one could arguably exclude these wormholes as unphysical, at least for the purposes of traversability.

Let us now consider the case when $(r_0/(|\Phi'_0|[1 - b'_0]))^{1/2}$ is the smallest local proper radius of curvature, where we continue to assume that Φ'_0 is finite at the throat. From Eq. (42), we obtain the constraint

$$|\Phi'_0|^{-1} \lesssim 10^5 l_p \sqrt{1 - b'_0}, \quad (45)$$

which implies a minimum local radius of curvature

$$\ell_{\min} \approx \sqrt{\frac{r_0}{|\Phi'_0|(1 - b'_0)}} \lesssim \frac{\sqrt{10^5 l_p r_0}}{(1 - b'_0)^{1/4}}, \quad (46)$$

which is roughly $\sqrt{10^5 l_p r_0}$ if b'_0 is not very close to 1. With this assumption, for a ‘‘human-sized’’ wormhole with $r_0 \approx 1 \text{m} \approx 10^{35} l_p$, we have that $\ell_{\min} \lesssim 10^{20} l_p \approx 10^{-15} \text{m}$. However, to be *traversable* for a human traveller, the wormhole must satisfy the radial tidal constraint, $|R_{\hat{r}\hat{r}\hat{r}\hat{r}}| \lesssim 1/(10^8 \text{m})^2$, (see, for example, Eqs. (47a) and (49) of Ref. [7]). At the throat this reduces to

$$|R_{\hat{r}\hat{r}\hat{r}\hat{r}}| = \frac{|\Phi'_0|(1 - b'_0)}{2r_0} \approx \frac{1}{\ell_{\min}^2} \lesssim \frac{1}{(10^8 \text{m})^2}, \quad (47)$$

in our case, which means that we must have $\ell_{\min} \gtrsim 10^8 \text{m}$. Since in the present case, $\ell_{\min} < r_0$ by assumption and $\ell_{\min} \gtrsim 10^8 \text{m}$ for human traversability, let us set

$$r_0 = \sigma 10^8 \text{m}, \quad (48)$$

with $\sigma > 1$. If we combine the last expression with Eq. (46), we get that

$$\sigma \gtrsim 10^{38} (1 - b'_0)^{1/2}, \quad (49)$$

and therefore

$$r_0 \gtrsim 10^{46} (1 - b'_0)^{1/2} \text{m}. \quad (50)$$

Thus we conclude that either r_0 is enormous, e.g., if

$1 - b'_0 \sim 10^{-10}$ we have $r_0 \gtrsim 10^{41} \text{ m} = 10^{25}$ light years, or b'_0 is incredibly fine-tuned, e.g., $1 - b'_0 < 10^{-72}$ for $r_0 = 10^{10} \text{ m}$, i.e., $\sigma = 100$.

We now examine the case where $\Phi'(r)$ or $\Phi''(r)$ may diverge at the throat. Recall that if the wormhole is symmetric, then we must have $d\Phi(r(\ell))/d\ell = 0$ at the throat. However, since

$$\frac{d\Phi(r(\ell))}{d\ell} = \Phi'(r)\sqrt{1 - b(r)/r}, \quad (51)$$

it is possible for $\Phi'(r)$ to diverge at the throat without the occurrence of a curvature singularity, provided that $\Phi'(r)$ diverges no faster than $(1 - b(r)/r)^{-1/2}$. Similarly, since

$$\frac{d^2\Phi(r(\ell))}{d\ell^2} = \left(1 - \frac{b(r)}{r}\right)\Phi''(r) + \frac{1}{2r}\left(\frac{b(r)}{r} - b'(r)\right)\Phi'(r), \quad (52)$$

then at the throat, for $\Phi'(r)$ finite, $\Phi''(r)$ could diverge, provided that it diverges no faster than $(1 - b(r)/r)^{-1}$. Of course both Φ' and Φ'' could diverge provided their contributions cancel in the limit. Therefore, we must be careful in interpreting the derivatives of $d\Phi(r)/dr$ and $d^2\Phi(r)/dr^2$ at r_0 . One can circumvent these worries by writing the curvature tensor components using the metric written in proper radial coordinates, Eq. (11). Then one finds, in particular, that [51]

$$R_{\hat{r}\hat{r}\hat{r}\hat{r}} = -\frac{d^2\Phi(r(\ell))}{d\ell^2} - \left(\frac{d\Phi(r(\ell))}{d\ell}\right)^2, \quad (53)$$

and

$$R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}} = \frac{1}{r(\ell)} \frac{dr(\ell)}{d\ell} \frac{d\Phi(r(\ell))}{d\ell}. \quad (54)$$

Since $d\Phi(r(\ell))/d\ell = 0$ at the throat, $\ell = 0$, and $\Phi(r(\ell))$ is required to have bounded second derivatives with respect to ℓ we have

$$R_{\hat{r}\hat{r}\hat{r}\hat{r}}|_{\ell=0} = -\frac{d^2\Phi(r(\ell))}{d\ell^2} \Big|_{\ell=0}, \quad (55)$$

and

$$R_{\hat{t}\hat{t}\hat{t}\hat{t}}|_{\ell=0} = R_{\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}}|_{\ell=0} = 0. \quad (56)$$

If $R_{\hat{r}\hat{r}\hat{r}\hat{r}}$ is not the largest curvature component then the analysis reduces to one of the cases considered in Eqs. (43) and (44) above. If it is the largest curvature component, then the smallest local proper radius of curvature is $(|d^2\Phi(r(\ell))/d\ell^2|)^{-1/2}$. From Eq. (42), we then obtain the constraint

$$\ell_{\min} \approx \left(\left| \frac{d^2\Phi(r(\ell))}{d\ell^2} \right| \right)^{-1/2} \lesssim \frac{\sqrt{10^5 l_p r_0}}{(1 - b'_0)^{1/4}}, \quad (57)$$

which entails the same fine-tuning constraints on b'_0 discussed after Eq. (46) above.

We conclude this section with some remarks on the case in which $b'_0 = 1$. Since the NEC is not violated at the throat, the above analysis does not apply. However, we have already seen that the NEC is violated arbitrarily close to the throat, and one could modify the analysis by considering static trajectories with $r > r_0$ where NEC is violated. To do so would require more information about the shape and red-shift functions, and we do not pursue this direction further. It seems to us that the $b'_0 = 1$ case is nongeneric, because it corresponds to a $r(\ell)$ having a minimum at $\ell = 0$ with $r''(0) = 0$. One would not expect this nongeneric feature to be stable against small fluctuations in the metric (either due to quantum effects, or the passage of a material body through the wormhole throat). We note that this criticism could be levelled at some of the Kuhfittig models to be considered later; a stronger objection is that they fail to be traversable, as we will see.

VI. THE VISSER-KAR-DADHICH MODELS

Visser, Kar, and Dadhich (VKD) [18] (see also [52]) have recently suggested that a suitable measure of the ‘‘amount of exotic matter required’’ for wormhole maintenance is given by integrating $\rho + p_r$ (to quantify the degree of NEC violation) with respect to the measure $dV = r^2 \sin\theta dr d\theta d\phi$ to obtain

$$\int [\rho + p_r] dV = 2 \int_{r_0}^{\infty} [\rho + p_r] 4\pi r^2 dr. \quad (58)$$

The factor of 2 comes from including both wormhole mouths. The overall form of Eq. (58) and the integration measure are chosen to generalize the mass formula for relativistic stars to wormholes [53]. VKD then argue that for a traversable wormhole, although the ANEC (line) integral must be finite and negative, the volume integral given in Eq. (58) can be made as small as one likes. Therefore they conclude that the amount of exotic matter required to maintain a traversable wormhole can be made arbitrarily small.

A. Spatially Schwarzschild wormhole

VKD introduce two specializations of their model. We will treat each in turn. The first is what they call the ‘‘spatially Schwarzschild (SS)’’ wormhole. They choose $b(r) \equiv 2m = r_0$, so that the spatial metric is exactly Schwarzschild and the energy density (measured by static observers) ρ is zero throughout the spacetime [54]. In particular, VKD consider a wormhole whose metric only differs from Schwarzschild in the region from the throat out to some radius $r = a$ (which would have to be reflected in the structure of $\Phi(r)$, since $b(r) = \text{const}$). They then argue that by considering a sequence of traversable wormholes with suitably chosen a and $\Phi(r)$, with $b(r) \equiv 2m = r_0$, they can take the limit $a \rightarrow 2m$, and construct travers-

able wormholes with arbitrarily small amounts of exotic matter.

Consider a static observer at the throat of the wormhole. For the SS wormhole, $b(r) \equiv 2m = r_0$, and $b' = 0$, so $\rho(r) \equiv 0$. Since the energy density is zero in the static frame, to obtain a bound using the usual QIs, one would need to boost to the frame of a radially moving geodesic observer (see Ref. [22]). The current approach using the *null*-contracted stress energy makes this unnecessary, as the radial pressure term is included in $T_{ab}k^ak^b$. From Eq. (39) in this case we simply have

$$T_{ab}k^ak^b = -\frac{1}{8\pi r_0^2}, \quad (59)$$

at the throat. For this wormhole, the nonzero curvature components are

$$R_{\hat{r}\hat{r}\hat{r}\hat{r}} = \left(1 - \frac{r_0}{r}\right) [\Phi'' + (\Phi')^2] + \frac{\Phi' r_0}{2r^2}, \quad (60)$$

$$R_{\hat{t}\hat{t}\hat{t}\hat{t}} = R_{\hat{t}\hat{t}\hat{r}\hat{r}} = \frac{\Phi'}{r} \left(1 - \frac{r_0}{r}\right), \quad (61)$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{r_0}{2r^3}, \quad (62)$$

$$R_{\hat{t}\hat{t}\hat{t}\hat{t}} = -\frac{r_0}{r^3}. \quad (63)$$

Let us first consider the case where $R_{\max} = |R_{\hat{r}\hat{r}\hat{r}\hat{r}}|$. For the current argument it is simpler to consider this component expressed in terms of proper length. If this is the largest curvature component, then the discussion in the last section and Eq. (55) imply that, at the throat, the smallest local proper radius of curvature is $(|d^2\Phi(r(\ell))/d\ell^2|)^{-1/2}$. Then from Eq. (57), and the fact that $b'(r) = b'_0 = 0$ for SS wormholes, we have

$$\ell_{\min} \approx \left(\left| \frac{d^2\Phi(r(\ell))}{d\ell^2} \right| \right)^{-1/2} \lesssim \sqrt{10^5 l_p r_0}. \quad (64)$$

For a ‘‘human-sized’’ wormhole with $r_0 \approx 1 \text{ m} \approx 10^{35} l_p$, we have that $\ell_{\min} \lesssim 10^{20} l_p \approx 10^{-15} \text{ m}$. Even a wormhole with $r_0 \approx 1 \text{ A.U.} \approx 10^8 \text{ m}$ would have $\ell_{\min} \approx 10^{-11} \text{ m}$, which is about one-tenth the radius of a hydrogen atom. A somewhat larger wormhole, with $r_0 \approx 1 \text{ light year} \approx 10^{16} \text{ m}$ would still have a local radius of curvature $\ell_{\min} \approx 10^{-7} \text{ m}$, which is on the order of a wavelength of light. However, recall that as discussed in the last section, to be *traversable* for a human traveller, the wormhole must satisfy the radial tidal constraint, $|R_{\hat{r}\hat{r}\hat{r}\hat{r}}| \lesssim 1/(10^8 \text{ m})^2$. At the throat this reduces to $|R_{\hat{r}\hat{r}\hat{r}\hat{r}}| = 1/\ell_{\min}^2 \lesssim 1/(10^8 \text{ m})^2$, which means that in our case we must have $\ell_{\min} \gtrsim 10^8 \text{ m}$. Recall that here $\ell_{\min} < r_0$ by assumption, and since $\ell_{\min} \gtrsim 10^8 \text{ m}$ for human traversability, if we set $r_0 = \sigma 10^8 \text{ m}$ with $\sigma > 1$, we get that $\sigma \gtrsim 10^{38}$ and hence $r_0 \gtrsim 10^{46} \text{ m}$. Thus we conclude that our bound implies

that r_0 for an SS wormhole must be enormous in order to be traversable for human travellers, e.g., $r_0 \gtrsim 10^{46} \text{ m} = 10^{30}$ light years, which is about 10^{20} times the radius of the visible universe!

If $R_{\max} \neq |R_{\hat{r}\hat{r}\hat{r}\hat{r}}|$, then recalling Eq. (56), we see that the largest curvature component is

$$R_{\max} = \frac{1}{r_0^2}, \quad (65)$$

and so the smallest local proper radius of curvature is

$$\ell_{\min} = r_c \approx r_0. \quad (66)$$

Applying our QI bound, Eq. (40), with $b'_0 = 0$, we have

$$r_0 \lesssim 10^5 l_p, \quad (67)$$

where we have, as before, chosen $f \sim 0.01$. This is similar to the result obtained for the case discussed at the end of the ‘‘proximal Schwarzschild’’ subsection in Ref. [22]. Therefore, it would seem that macroscopic ‘‘spatially Schwarzschild’’ wormholes are ruled out or highly constrained by the QIs.

B. Piecewise $R = 0$ wormhole

As a further specialization, VKD consider a segment of $R = 0$ wormhole (zero Ricci scalar) truncated and embedded in a Schwarzschild geometry. For $r \in (r_0 = 2m, a)$, they choose

$$\exp[\Phi(r)] = \epsilon + \lambda \sqrt{1 - 2m/r}, \quad (68)$$

and

$$\exp[\Phi(r)] = \sqrt{1 - 2m/r}, \quad (69)$$

for $r \in (a, \infty)$, with $b(r) = 2m$ everywhere. Continuity of the metric coefficients implies that

$$\lambda = 1 - \frac{\epsilon}{\epsilon_s}, \quad (70)$$

where $\epsilon_s = \sqrt{1 - 2m/a}$. There is a thin shell of what VKD call ‘quasinormal’ matter at $r = a$. VKD argue that by taking suitable limits of ϵ, ϵ_s , they can make the amount of exotic matter required to support the wormhole arbitrarily small. Because this is a more detailed example of an SS wormhole, with a specific form given for $\Phi(r)$, we can make an even stronger argument for ruling out macroscopic wormholes of this type.

We can write $\Phi(r)$ on $(2m, a)$ as

$$\Phi(r) = \ln \left[\epsilon + \sqrt{1 - 2m/r} \left(1 - \frac{\epsilon}{\epsilon_s} \right) \right]. \quad (71)$$

Now in this case, although $\Phi(r)$ is well-behaved at the throat, $\Phi'(r)$ diverges. This is due to the fact that r is a bad coordinate at the throat, and because the divergence of Φ' involves factors of $1 - 2m/r$. One can see this by examining the derivative of Φ with respect to proper length,

$d\Phi/d\ell$, which in fact vanishes at the throat. As a result, in this case the limit of $R_{\hat{r}\hat{r}\hat{r}} = (1 - r_0/r)[\Phi'' + (\Phi')^2] + \Phi'/2r^2(b - b'r)$, as $r \rightarrow r_0 = 2m$, is *not* $\Phi'/2r_0$, due to the presence of $\sqrt{1 - r_0/r}$ terms in the derivatives of $\Phi(r)$. These will result in cancellations between terms in $R_{\hat{r}\hat{r}\hat{r}}$. (Similar considerations apply to $T_{\hat{\theta}\hat{\theta}}$ at the throat.) However, an explicit calculation shows that in fact

$$R_{\hat{r}\hat{r}\hat{r}} = R_{\hat{\theta}\hat{\theta}\hat{\theta}} = 0, \quad \text{at the throat,} \quad (72)$$

in this subcase.

A similar calculation to that of the general SS wormhole case yields similar results. We again find that at the throat, $r = r_0 = 2m$, the smallest local proper radius of curvature is

$$r_0 \lesssim 10^5 l_p, \quad (73)$$

and so macroscopic piecewise $R = 0$ wormholes are ruled out.

Quite apart from the QI arguments we have given, it should also be noted that there are some practical difficulties with the VKD models as well. The smaller the amount of exotic matter used in these wormholes, the closer they are to being vacuum Schwarzschild wormholes. Therefore the smaller the amount of exotic matter, the longer it will take an observer to traverse the wormhole as measured by clocks in the external universe. Perhaps one could counter this by moving the wormhole mouths around. In addition, the smaller the amount of exotic matter, the more prone the wormhole is to destabilization by even very small amounts of infalling positive matter, since this matter will be enormously blueshifted by the time it reaches the throat.

Barcelo and Visser [55,56] have proposed classical non-minimally coupled scalar fields as sources of exotic matter for wormhole maintenance. Since such classical fields (if they exist) would not be subject to the QIs, one might hope to circumvent the restrictions derived from them. However, the Barcelo-Visser wormholes have some problems of their own (see Sec. 5 of Ref. [17]).

VII. THE KUHFITTING MODELS

Kuhfittig has written a number of papers which attempt to construct wormholes which both satisfy the QIs and which require arbitrarily small amounts of exotic matter. We will examine three of these papers, which we will denote as KI [19], KII [20], and KIII [21].

A. Kuhfittig I

In his first model, KI [19], Kuhfittig sets $\Phi(r) \equiv 0$ and defines $b(r)$ in three regions: one near the throat, $r_0 \leq r \leq r_\epsilon$, in which $b(r) = kr + \epsilon(r)$, an intermediary region $r_\epsilon \leq r < r_1$ in which $b(r) = kr$ and an outer region $r > r_1$ in which $b(r)$ becomes constant after a smooth transition near r_1 . Here, $k < 1$ is a fixed parameter, while $\epsilon(r)$ is a C^2 non-negative function which obeys

$$\epsilon(r_0) = (1 - k)r_0, \quad \epsilon'(r_0) = 0, \quad (74)$$

so that $b(r_0) = r_0$, and

$$\epsilon(r_\epsilon) = \epsilon'(r_\epsilon) = \epsilon''(r_\epsilon) = 0, \quad (75)$$

so the transition at r_ϵ is C^2 . Kuhfittig's aim in Ref. [19] was to demonstrate the existence of wormhole models in which the exotic matter can be confined to an arbitrarily small region: the interval (r_0, r_ϵ) in this case. We can use our general analysis to see what constraints are put on this class of models by QIs.

Since $\Phi \equiv 0$ and $b'_0 = k$, examination of Eqs. (32) and (33) shows that the smallest curvature radius at the throat is r_0 , and therefore we have the constraint

$$r_0 \lesssim \frac{10^5 l_p}{\sqrt{1 - k}}. \quad (76)$$

As already mentioned, this requires significant fine-tuning. For a wormhole with a 1 m throat, $r_0 = 10^{35}$ Planck lengths, Eq. (76) requires that $1 - k \lesssim 10^{-60}$, for example. Since $b'_0 \leq 1$ for wormhole models, we see that $k = b'_0$ must be tuned to a precision of at least one part in 10^{60} . [We note that Kuhfittig acknowledges that k might need to be taken close to 1, although he does not give estimates.] Fine-tuning of k entails that various coordinate-independent quantities are also fine-tuned. For example, the Ricci scalar is

$$R = \frac{2b'_0}{r_0^2}, \quad (77)$$

at the throat, and is also tuned to within one part in 10^{60} for a 1 m throat which satisfies the bound Eq. (76). The engineering challenge is yet more severe for an Earth-sized throat (one part in 10^{74}) and barely less daunting for a proton-sized throat (one part in 10^{30}).

We also observe that taking k close to unity means that the wormhole has extremely slow flaring at the throat. The proper radial distance may be estimated, for $r < r_1$ by

$$\ell(r) = \int_{r_0}^r \frac{dr'}{\sqrt{1 - k - \epsilon(r')/r'}} \geq \frac{r - r_0}{\sqrt{1 - k}}, \quad (78)$$

and so we see that the coordinate distances r_ϵ and r_1 must be close to r_0 to avoid unfeasibly long traversal times if $\ell(r)$ gets too large. Again, this indicates the necessity for fine-tuning of the model.

B. Kuhfittig II

In this paper [20], Kuhfittig writes his line element as

$$ds^2 = -e^{2\gamma(r)} dt^2 + e^{2\alpha(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (79)$$

The function $\alpha(r)$ is required to have a vertical asymptote at $r = r_0$: $\lim_{r \rightarrow r_0^+} \alpha(r) = +\infty$. By comparing the g_{rr} coefficients in Eqs. (13) and (79), we can express $b(r)$ in

terms of $\alpha(r)$ as follows

$$b(r) = r(1 - e^{-2\alpha(r)}). \quad (80)$$

The choice of the behavior of $\alpha(r)$ is designed to make $b'(r)$ close to 1 near the throat, $r = r_0$, in order to satisfy one of the general QI bounds (Eq. (95) of Ref. [22]). This condition on $b'(r)$ implies that the embedding diagram will flare out very slowly, a fact which Kuhfittig himself recognizes. However, he then claims that this slow flaring need not be fatal—a claim which we will show to be mistaken.

Kuhfittig then modifies his notation (in a rather confusing way), in order to emphasize the behavior at the throat, by rewriting the metric as

$$ds^2 = -e^{-2\alpha(r)} dt^2 + e^{2\alpha(r-r_0)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (81)$$

replacing the original $\alpha(r)$ by $\alpha(r - r_0)$. He also makes the choice $\gamma(r) = -\alpha(r)$ for the red-shift function, and lets this new $\alpha(r)$ diverge at the origin ($r = 0$). This has the effect of allowing $\alpha(r - r_0) \rightarrow \infty$ as $r \rightarrow r_0$, while keeping $\gamma(r) = -\alpha(r)$ finite in the same limit, in order to avoid the appearance of an event horizon. We will use this form of the metric for the remainder of this subsection, but will revert to the form Eq. (79) in the next subsection, in order to there follow the notation given in Ref. [21].

The first indication of trouble comes from the evaluation of the proper radial distance to the throat from any point outside. Since $e^{\alpha(r-r_0)} > \alpha(r - r_0)$ for all $\alpha(r - r_0)$, we have

$$\ell(r) = \int_{r_0}^r e^{\alpha(r'-r_0)} dr' > \int_{r_0}^r \alpha(r' - r_0) dr', \quad (82)$$

which will diverge if $\alpha(r - r_0)$ diverges fast enough as $r \rightarrow r_0$. In particular, ℓ is infinite if $\alpha(r - r_0) > \text{const} \times (r - r_0)^{-1}$ for all r sufficiently close to r_0 . Moreover, a similar argument (using $e^x \geq x^p/p!$ for each $p = 1, 2, 3, \dots$ and any $x > 0$) shows that ℓ is infinite even for a weak divergence such as $\alpha(r - r_0) > \text{const} \times (r - r_0)^{-1/p}$ for some $p > 0$. More can be said if $\alpha(r - r_0)$ is monotonically decreasing in some interval (r_0, r_ϵ) . In this case, $\ell(r)$ is finite and tends to 0 as $r \rightarrow r_0$ only if

$$0 \leq (r - r_0)e^{\alpha(r-r_0)} \leq \int_{r_0}^r e^{\alpha(r'-r_0)} dr' \rightarrow 0, \quad (83)$$

in this limit, where the central inequality holds because $e^{\alpha(r'-r_0)} \geq e^{\alpha(r-r_0)}$ for $r' \in (r_0, r)$. It follows that $\alpha(r - r_0)$ is less than $-\log(r - r_0)$ for all r sufficiently close to r_0 [indeed, $-\log(r - r_0) - \alpha(r - r_0) \rightarrow \infty$ as $r \rightarrow r_0$]. Certainly the particular choices employed in Sec. III.A of KII, namely

$$\alpha(r) = \kappa/r, \quad r_0 = 0, \quad \kappa = 0.00025 \text{ light years}, \quad (84)$$

lead to an infinite proper distance from any point $r_1 > 0$ to the throat $r_0 = 0$. We note that Kuhfittig claims a finite traversal time for this model at the end of his Sec. III.A; this appears to be based on a confusion between coordinate and proper distance.

C. Kuhfittig III

In paper [21], Kuhfittig suggests that good choices for $\gamma(r)$, $\alpha(r)$ in Eq. (79) are:

$$\alpha(r) = \frac{k}{(r - r_0)^n}, \quad n \geq 1, \quad (85)$$

where, in this subsection, k is a (positive) constant with the same units as r^n . The choice of $n \geq 1$ is made in order to obtain $b'(r) \sim 1$ near $r = r_0$. The function $\gamma(r)$ is chosen to be:

$$\gamma(r) = -\frac{L}{(r - r_2)^n}, \quad n \geq 1, \quad (86)$$

where L is another positive constant with the same units as r^n , and $0 < r_2 < r_0$. The condition on r_2 is made to avoid an event horizon at the throat r_0 .

Once again, we see that there is a problem when one evaluates the proper distance to the throat from any point outside. As in KII, since $e^{\alpha(r)} > \alpha(r)$ for all $\alpha(r)$, we have

$$\ell = \int_{r_0}^{r_1} e^{\alpha(r)} dr > \int_{r_0}^{r_1} \alpha(r) dr = \int_{r_0}^{r_1} \frac{k}{(r - r_0)^n} dr, \quad (87)$$

which diverges if $n \geq 1$. Hence the proper distance from any point $r_1 > r_0$ to the throat r_0 is infinite. Now it is known that there are black hole spacetimes where the proper distance to the horizon is infinite, but it nevertheless takes only a *finite* proper time to fall into them. An example is the extreme $Q = M$ Reissner-Nordström black hole [57,58]. In that case, however, there is a horizon in the spacetime, so an infalling observer who crosses the horizon cannot get back out. By contrast, for a traversable wormhole, it should be possible for two static observers on opposite sides of the wormhole to stretch a measuring tape between them and measure their separation in proper distance. If the proper distance from any observer's location to the throat is infinite, this will of course not be possible.

Let us pursue this reasoning further and calculate the proper time for a radially infalling observer to reach the throat. For a radial timelike geodesic in this wormhole metric we have that

$$\frac{d^2 r}{d\tau^2} + e^{-2[\alpha(r)-\gamma(r)]} \gamma'(r) \left(\frac{dt}{d\tau}\right)^2 + \alpha'(r) \left(\frac{dr}{d\tau}\right)^2 = 0, \quad (88)$$

where τ is the observer's proper time. From the four-velocity u^a , we have $u^a u_a = -1$, and thus

$$-e^{2\gamma(r)} \left(\frac{dt}{d\tau}\right)^2 + e^{2\alpha(r)} \left(\frac{dr}{d\tau}\right)^2 = -1. \quad (89)$$

If we solve this equation for $(dt/d\tau)^2$, and substitute back into Eq. (88), we obtain

$$\frac{d^2 r}{d\tau^2} + \gamma'(r)e^{-2\alpha(r)} + [\gamma'(r) + \alpha'(r)]\left(\frac{dr}{d\tau}\right)^2 = 0. \quad (90)$$

This can be solved exactly for any $\alpha(r)$, $\gamma(r)$. It has a first integral

$$\frac{dr}{d\tau} = -e^{-[\gamma(r)+\alpha(r)]}\sqrt{K - e^{2\gamma(r)}}, \quad (91)$$

where $\gamma(r)$, $\alpha(r)$ are evaluated at $r(\tau)$ on the right-hand side, and K is a constant which fixes the initial radial velocity; the overall minus sign on the right-hand side corresponds to initially in-going geodesics. (To check this, simply differentiate both sides with respect to τ and substitute into Eq. (90), using Eq. (91) again to simplify.) Suppose the initial radius is r_1 at time $\tau = 0$. Then

$$\tau = \int_{r_1}^r dr' \frac{e^{[\gamma(r')+\alpha(r')]}}{\sqrt{K - e^{2\gamma(r')}}}, \quad (92)$$

is the proper time of (first) arrival at radius r . In the KIII model, $\gamma(r)$ is well-behaved at $r = r_0$, but $\alpha(r)$ has a nonintegrable singularity there. As $\alpha(r) > 0$ we deduce that $\exp(\alpha(r))$ also has a nonintegrable singularity at the throat. Thus $\tau \rightarrow \infty$ as $r \rightarrow r_0$, and so the proper time for a radially infalling observer to reach the throat is infinite.

Lastly, let us consider radially infalling light rays. Null geodesics obey

$$g_{ab}k^a k^b = -e^{2\gamma(r)}\left(\frac{dt}{d\lambda}\right)^2 + e^{2\alpha(r)}\left(\frac{dr}{d\lambda}\right)^2 = 0, \quad (93)$$

where λ is an affine parameter. Thus

$$\left(\frac{dt}{dr}\right)^2 = e^{2\beta(r)}, \quad (94)$$

where $\beta(r) = \alpha(r) - \gamma(r)$. Define $B(r)$ such that $B'(r) = e^{\beta(r)}$, i.e.,

$$B(r) = \int_{r_i}^r e^{\beta(r')} dr', \quad (95)$$

for some $r_i > r_0$. Now define radial null coordinates by

$$u = t - B(r) \quad (96)$$

$$v = t + B(r). \quad (97)$$

Then we can write

$$dudv = (dt - e^{\beta(r)} dr)(dt + e^{\beta(r)} dr) \quad (98)$$

$$= dt^2 - e^{2\beta(r)} dr^2 \quad (99)$$

$$= -e^{-2\gamma(r)} ds^2, \quad (100)$$

where ds^2 is the wormhole metric in the t, r plane, which may then be written as

$$ds^2 = -e^{2\gamma(r)} dudv, \quad \text{with } r = B^{-1}\left(\frac{v-u}{2}\right). \quad (101)$$

For Kuhfittig's metric, $\int_{r_i}^{r_0} e^{\beta(r')} dr' \rightarrow \infty$ as $r \rightarrow r_0$, so we have $B(r) \rightarrow -\infty$, as $r \rightarrow r_0$. To determine an affine parameter, we use the fact that $(\partial/\partial t)^a$ is a Killing vector, so $E = -g_{ab}k^a(\partial/\partial t)^b$ is a constant along null geodesics [59]. So we have that $E = \exp[2\gamma(r)]dt/d\lambda = (1/2) \times \exp[2\gamma(r)]du/d\lambda$, since $u = 2t - v$, and $v = \text{const}$ on the in-going null rays. Therefore a suitable affine parameter is

$$\lambda(u) = \frac{1}{2E} \int_{u_i}^u \exp\left[2\gamma\left(B^{-1}\left(\frac{v-u}{2}\right)\right)\right] du. \quad (102)$$

As the throat is approached, $u \rightarrow \infty$, $(v-u)/2 \rightarrow -\infty$, so $B^{-1}[(v-u)/2] \rightarrow r_0$. If $\lim_{r \rightarrow r_0} \exp(2\gamma(r))$ is nonzero, as in Kuhfittig's example from Eqs. (85) and (86), then $\lambda(u) \rightarrow \infty$ as $u \rightarrow \infty$, so in-going radial null geodesics do not arrive at the throat at finite affine parameter. Hence even light rays cannot traverse the wormhole.

A common problem in all the Kuhfittig models is the confusion of coordinate distances and proper distances, as, for example, in his estimates of traversability times. For a slowly flaring wormhole, a small difference in coordinate length can correspond to an enormous difference in proper length (see Figs. 1 and 2). As a result, although such a wormhole might have its exotic matter concentrated in a

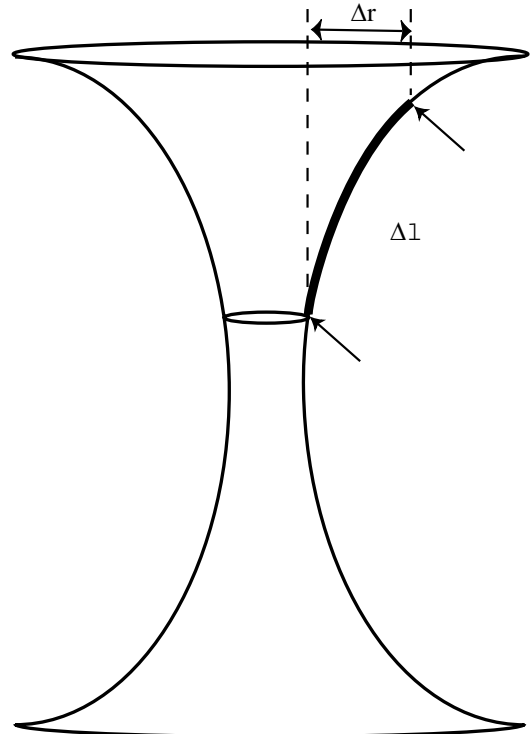


FIG. 2. Proper versus coordinate distance in a slowly flaring wormhole.

small coordinate thickness in radius, the proper volume of the region of exotic matter could in fact be very large.

VIII. A FURTHER JUSTIFICATION FOR OUR WORMHOLE BOUNDS

At the end of Sec. IV, we argued that the Minkowski space bound Eq. (38) could be adapted to curved spacetimes under certain hypotheses. In the static case, we noted that a necessary condition for the validity of this approach is that the bound is satisfied for the expected stress-energy tensor of the static ground state (assuming this is Hadamard), with τ_0 of the order of the minimum length scale characterizing the geometry. Here, we show that this is also a sufficient condition; this may also be regarded as providing a second argument in favor of a bound of the form Eq. (42). Instead of applying the Minkowski bound on sufficiently small scales, we may consider what sort of QI could be derived directly in the wormhole spacetime. In fact our analysis applies to any static globally hyperbolic spacetime, provided the quantum field theory does not have bad infrared behavior. Suppose a scalar quantum field of mass $m \geq 0$ admits a Hadamard static ground state ω_0 . Applying Theorem III.1 in Ref. [49] to averages along a static trajectory $\xi(\tau)$, and using arguments similar to those in Sec. 5 of Ref. [29] we obtain a bound

$$\begin{aligned} & \int \left(\langle T_{ab} k^a k^b \rangle_\omega(\xi(\tau)) - \langle T_{ab} k^a k^b \rangle_{\omega_0}(\xi(\tau)) \right) g(\tau)^2 d\tau \\ & \geq -\frac{1}{\pi} \int_0^\infty Q(y) |\hat{g}(y)|^2 dy, \end{aligned} \quad (103)$$

for any Hadamard state ω of the scalar field, where the hat denotes Fourier transform [60]. As before g is smooth, real-valued and compactly supported in \mathbb{R} . *The advantage here is that we now no longer need to restrict the support of g to be small in relation to curvature scales.* By arguments parallel to those in Ref. [29], the function Q is non-negative, continuous from the left, increasing, and growing no faster than polynomially at infinity. In fact, if the two-point function of the ground state is given by a sum (or integral) of mode functions

$$\langle \varphi(t, x) \varphi(t, x') \rangle_{\omega_0} = \sum_\lambda e^{-i\omega_\lambda(t-t')} U_\lambda(x) \overline{U_\lambda(x')}, \quad (104)$$

(note that the ω_λ will be non-negative if ω_0 is a ground state) then

$$Q(y) = \sum_{\text{a.s.t. } \tilde{\omega}_\lambda < y} |c_\lambda|^2, \quad (105)$$

where $\tilde{\omega}_\lambda = e^{-\Phi(r_0)} \omega_\lambda$ and

$$c_\lambda = k^a \nabla_a e^{-i\omega_\lambda t} U_\lambda(x) |_{(t,x)=\xi(0)}. \quad (106)$$

(See, e.g., the introduction to Ref. [29]. This result could also be obtained using the less rigorous methods of Ref. [26].)

Suppose that the static spacetime geometry is supported by a particular Hadamard state ω . Then $\langle T_{ab} k^a k^b \rangle_\omega$ must be constant on the static trajectory ξ , as must the vacuum energy $\langle T_{ab} k^a k^b \rangle_{\omega_0}$ because ω_0 was assumed to be static. These terms may therefore be taken outside the integral in Eq. (103), and the QI then implies

$$\langle T_{ab} k^a k^b \rangle_\omega \geq \langle T_{ab} k^a k^b \rangle_{\omega_0} - \frac{\int_0^\infty Q(y) |\hat{g}(y)|^2 dy}{\pi \int_{-\infty}^\infty g(t)^2 dt}, \quad (107)$$

where the expectation values are evaluated, for example, at $\xi(0)$.

It is convenient to rewrite the denominator in the last expression in terms of the Fourier transform, using Parseval's theorem. Moreover, $|\hat{g}(y)|^2$ is even in y because g is real-valued, which permits us to write

$$\langle T_{ab} k^a k^b \rangle_\omega \geq \langle T_{ab} k^a k^b \rangle_{\omega_0} - \frac{\int_0^\infty Q(y) |\hat{g}(y)|^2 dy}{\int_0^\infty |\hat{g}(y)|^2 dy}. \quad (108)$$

We may now try to maximize the expression on the right-hand side over the class of g at our disposal. Fix any compactly supported real-valued g for which the denominator above is unity, and then replace g by $\lambda^{-1/2} g(\tau/\lambda)$, and therefore $\hat{g}(y)$ by $\lambda^{1/2} \hat{g}(\lambda y)$. As the sampling time is increased by increasing λ , $\lambda^{1/2} \hat{g}(\lambda y)$ becomes more sharply peaked near $y = 0$ ($\hat{g}(y)$ decays rapidly at infinity because g is smooth and compactly supported). Taking the limit $\lambda \rightarrow \infty$ (cf. an argument in the proof of Theorem 4.7 in Ref. [61]) we obtain

$$\langle T_{ab} k^a k^b \rangle_\omega \geq \langle T_{ab} k^a k^b \rangle_{\omega_0} - Q(0+), \quad (109)$$

where $Q(0+) = \lim_{y \rightarrow 0^+} Q(y)$. Now $Q(0+)$ will vanish unless the quantum field theory has bad infrared behavior, e.g., a square-integrable zero mode. Excluding such pathological cases, we have

$$\langle T_{ab} k^a k^b \rangle_\omega \geq \langle T_{ab} k^a k^b \rangle_{\omega_0}. \quad (110)$$

Thus the ground state yields the lowest constant value of $\langle T_{ab} k^a k^b \rangle$ possible (amongst Hadamard states) along a static trajectory. Accordingly, if the ground state obeys a bound of the form Eq. (38) with τ_0 of the order of the minimum length scale characterizing the geometry, then the bound will hold for all Hadamard states capable of supporting a static geometry.

Let us develop this line of reasoning a bit further, returning to the particular class of Morris-Thorne wormholes. In our $\hbar = c = 1$ units, the right-hand side of Eq. (110) has the dimensions of $(\text{length})^{-4}$, and can be written as $K_0 (k_a u^a)^2 / \ell_{\min}^4$, where ℓ_{\min} is the minimal length scale associated with the geometry. We know that the wormhole models violate the NEC at [or arbitrarily close to] the throat, so if $K_0 > 0$ the wormhole would be inconsistent with QIs. So let us assume that $K_0 < 0$. Inserting the required stress-energy tensor, we now have a bound

$$\frac{b'_0 - 1}{8\pi r_0^2 l_p^2} \geq -\frac{|K_0|}{\ell_{\min}^4} \quad (111)$$

or

$$\frac{\ell_{\min}^2}{r_0} \sqrt{1 - b'_0} \leq \sqrt{8\pi |K_0|} l_p. \quad (112)$$

which should be compared with Eq. (42). Since the dimensionless constant $|K_0|$ is typically of the order of 10^{-2} , this new bound is stronger than that of Eq. (42) by 5 orders of magnitude. This supports our earlier argument and also suggests that the bounds given above are extremely conservative: the Casimir energy would have to be 10 orders of magnitude higher than our typical experience (i.e., $|K_0| \sim 10^8$) in order for Eq. (42) to be violated.

IX. SUMMARY

We have analyzed the recent wormhole models of Visser, Kar, and Dadhich, and of Kuhfittig. In these models only arbitrarily small amounts of exotic matter are required to hold the wormholes open. If the exotic matter is composed of quantum fields, then they are subject to the constraints derived from the quantum inequality bounds on negative energy. In particular, our analysis employs a recently derived quantum inequality bound on the *null*-contracted stress-energy averaged over a *timelike* worldline. The bound allows us to perform a simplified analysis

of general wormhole models, not just those with small quantities of exotic matter.

We showed that our bound implies that *macroscopic* wormholes of the Visser-Kar-Dadhich type are ruled out or severely constrained. For the Kuhfittig models, we showed that a confusion between coordinate lengths and proper lengths in fact disqualifies the model in Ref. [21] from being traversable. It turns out that for this model, radially infalling particles and light rays reach the throat only after an infinite lapse of affine parameter, due to the extremely slow flaring of the wormhole throat. Related constraints were also derived for two of Kuhfittig's earlier models. One lesson to be drawn from our results is that simply concentrating the exotic matter, in a classical analysis, to an arbitrarily small region around the wormhole throat is, by itself, not sufficient to guarantee both traversability and consistency with (or evasion of) the quantum inequality bounds.

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