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Length uncertainty in a gravity's rainbow formalism

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It is commonly accepted that the combination of quantum mechanics and general relativity gives rise to the emergence of a minimum uncertainty both in space and time. The arguments that support this conclusion are mainly based on perturbative approaches to the quantization, in which the gravitational interactions of the matter content are described as corrections to a classical background. In a recent paper, we analyzed the existence of a minimum time uncertainty in the framework of doubly special relativity. In this framework, the standard definition of the energy-momentum of particles is modified appealing to possible quantum gravitational effects, which are not necessarily perturbative. Demanding that this modification be completed into a canonical transformation determines the implementation of doubly special relativity in position space and leads to spacetime coordinates that depend on the energy-momentum of the particle. In the present work, we extend our analysis to the quantum length uncertainty. We show that, in generic cases, there actually exists a limit in the spatial resolution, both when the quantum evolution is described in terms of the auxiliary time corresponding to the Minkowski background or in terms of the physical time. These two kinds of evolutions can be understood as corresponding to perturbative and nonperturbative descriptions, respectively. This result contrasts with that found for the time uncertainty, which can be made to vanish in all models with unbounded physical energy if one adheres to a nonperturbative quantization.

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I. INTRODUCTION

A standard result in quantum mechanics is that the measurement of the position of a quantum state is affected by an uncertainty that satisfies the Heisenberg relations [1]. In order to diminish the position uncertainty one is thus forced to consider states with increasing momentum uncertainty, achieving an infinite spatial resolution only at the cost of completely delocalizing the momentum. In the presence of gravity, however, the situation becomes more complicated. Via Einstein equations, an uncertainty in the (energy)momentum of the system results in one in the geometry, which implies an additional uncertainty in the position. The total position uncertainty will therefore consist in the combined effect of a purely quantum mechanical contribution and a contribution of gravitational origin [2]. In these circumstances, one should not expect that an infinite spatial resolution can be reached, unless there exists a very specific relation between these types of contributions. Similar conclusions apply to the measurements of length of spatial intervals, determined by the positions of their end points.

The most common approach to analyze the emergence of a minimum spatial (or time) uncertainty when gravity comes into the scene consists in adopting a perturbative scheme. The starting point is a flat background where the matter is inserted. This matter curves the spacetime, producing a deformation of the geometry which in turn modifies the expression of the physical energy and momentum of the system (usually defined in terms of normalized—asymptotic—Killing vectors). The process continues with successive corrections that one assumes to be less and less important. The studies in the literature indicate that a minimum uncertainty is ineluctable in this kind of perturbative quantization (at least in the next-to-leading-order approximation) [2–5]. A different issue, which is still open to debate, is whether the same result holds as well in the context of a nonperturbative quantum description [6,7].

A suitable arena to test some of these issues is provided by doubly special relativity (DSR) [8,9]. In this kind of theory, the definition of the physical energy and momentum of particles is modified with respect to the standard relativistic one in order to encode, at least to some extent, the possible effects of the gravitational interactions, without necessarily adhering to any perturbative interpretation. The modification is such that the system presents an energy and/or momentum scale which is invariant under Lorentz transformations. This is possible because the action of the Lorentz group becomes nonlinear on the physical energy-momentum space [8–12].

Several proposals have been put forward for the realization of DSR in position space [13–15]. In a previous paper [6] we suggested that this realization should be determined by completing into a canonical transformation the nonlinear mapping that relates the original energy-momentum variables of standard relativity in Minkowski spacetime (that we will call pseudovariables from now on) with the physical energy-momentum of the system in DSR [16]. In this framework, the background Minkowski coordinates are mapped to a new set of spacetime coordinates that can be regarded as canonically conjugate to the physical energy-momentum. Those coordinates are linear in the Minkowski ones, but depend in a nontrivial way on the energy and momentum of the particle. Owing to this dependence of the spacetime description, the formalism can be considered a kind of gravity's rainbow [17].

Our discussion in Ref. [6] was focused on the existence of a minimum time uncertainty in quantum theories derived from DSR. In particular, we considered the different possibilities of describing the quantum evolution in terms of a parameter that corresponds either to the original time of the Minkowski background or to the physical time of the system. According to our comments above, we will, respectively, refer to these two types of quantization as perturbative and nonperturbative ones, given the distinct philosophy in the use of background structures. Our analysis proved that, while there always exists a nonvanishing uncertainty in the physical time when a perturbative quantization is adopted, an infinite time resolution can be achieved in certain theories when the quantization is nonperturbative. More precisely, no minimum time uncertainty arises nonperturbatively in DSR theories whose physical energy is unbounded from above. The aim of the present work is to extend this study of the uncertainty from time lapses to the case of spatial intervals.

A particular class of spacetimes in which the commented analysis of the time uncertainty has been carried out in detail is that of the Einstein-Rosen waves [7]. These linearly polarized waves are described by cylindrically symmetric spacetimes in $3 + 1$ dimensions, but can equivalently be described in terms of a massless scalar field coupled to gravity in $2 + 1$ dimensions with axial symmetry [18–20]. In this dimensionally reduced formulation, the system can in fact be viewed as an example of DSR theories, with a physical energy that is bounded from above [21,22]. Therefore, for Einstein-Rosen waves, a nonvanishing quantum time uncertainty emerges both in the perturbative and in the nonperturbative approaches. The study of the spatial uncertainty is not especially interesting in this case, because the associated DSR theory involves no modification in the definition of the momenta nor in the canonically conjugate position variables.

The rest of the paper is organized as follows. In the following section, we review some aspects of the formulation of DSR theories in momentum space and introduce our canonical proposal for their realization in position space. We obtain spacetime coordinates that are conjugate to the physical energy-momentum, arriving at a gravity's rainbow formalism. Next, we study the quantization of this formalism, restricting our considerations to free systems that can be described within a Hamiltonian scheme. Adopting a perturbative approach to the quantization, we analyze in Sec. III the length uncertainty, i.e. the uncertainty in the difference of spatial positions. We show in Sec. IV that this uncertainty cannot vanish in the perturbative case under quite generic assumptions. Furthermore, in Sec. V we prove that the appearance of a minimum length uncertainty persists when the quantum evolution is described in terms of the physical time, i.e., in a nonperturbative quantization. However, we comment on the possibility that in some DSR models one could construct a different type of nonperturbative quantum theory where the physical position operator became explicitly time independent. In this scenario, the resolution in the spatial position could in principle be made as large as desired if the DSR theory does not possess an invariant momentum scale. The uncertainty in the physical length (as well as in the physical time lapse) is studied in Sec. VI in the low-energy sector, approximating the results of the perturbative quantization up to first order corrections. In Sec. VII we consider the *massless* case in this approximation for large values of the Minkowski time T. We show that the uncertainty increases then like the square root of T, just as it occurs in Salecker and Wigner devices [23]. We present our conclusions in Sec. VIII . Finally, two appendices are added. In the following, we will adopt units in which $\hbar = c = 1$ (with $\hbar$ being Planck constant and c the speed of light).

II. DSR IN MOMENTUM AND POSITION SPACES

A characteristic feature of DSR theories is that they possess a Lorentz invariant energy and/or momentum scale, apart from the scale provided in standard relativity by the speed of light [8–12]. The invariance of such a scale is possible only thanks to a nonlinear realization of the Lorentz group in momentum space. A simple way to construct a realization of this kind is by introducing an invertible map U between the physical energy-momentum $P^a = (E, p^i)$ and a standard Lorentz 4-vector $\Pi^a = (\epsilon, \pi^i)$, which we call the pseudoenergy-momentum [16] (lowercase Latin indices from the beginning and the middle of the alphabet represent Lorentz and flat spatial indices, respectively). Denoting the usual linear action of the Lorentz group by $\mathcal{L}$, the nonlinear Lorentz transformations are then given by $L(P) = (U^{-1} \circ \mathcal{L} \circ U)(P)$ [16,24].

The map U must reduce to the identity when energies and momenta are negligibly small compared to the DSR scale, so that the physical variables and the pseudovariables coincide in this limit. In addition, a simplifying assumption that is generally accepted is that the standard action of rotations is preserved; only boosts are modified in DSR [13,24]. So, with the notation $p := |\vec{p}|$ and $\pi := |\vec{\pi}|$, the most general expression for the map U becomes [6,13]

$$\Pi = U(P) \Rightarrow \begin{cases} \epsilon = \tilde{g}(E, p), \\ \pi^i = \tilde{f}(E, p)\frac{p^i}{p}, \end{cases}$$
$$P = U^{-1}(\Pi) \Rightarrow \begin{cases} E = g(\epsilon, \pi), \\ p^i = f(\epsilon, \pi)\frac{\pi^i}{\pi}. \end{cases} \tag{2.1}$$

Since the only invariant energy-momentum scale in standard special relativity is at infinity, the DSR theory admits a Lorentz invariant scale at a finite value of the energy and/or momentum only if the map U has a singularity there [24]. The domain of definition of U (which is assumed to contain the low-energy-momentum sector) is therefore bounded from above by that scale. Consequently, DSR theories can be classified in three types: DSR1 if it is only the physical momentum that is bounded from above, DSR3 if it is the physical energy what is bounded, and DSR2 if both the physical energy and momentum are bounded.

As it is implicit in our discussion, DSR theories are usually formulated in momentum space, mainly owing to the increasing interest in investigating the observational implications of deformed dispersion relations [8,25]. There are different proposals to determine what is the modified spacetime geometry and the corresponding transformation rules in position space that should complement this formulation [13,14]. Among them, one of the most popular consists in abandoning the commutativity of the spacetime coordinates, as it happens e.g. in κ-deformed Minkowski spacetime [12,13].

However, noncommutative geometries are by no means the only way to obtain a consistent realization in position space. The same goal can be achieved without renouncing the conventional framework of commutative spacetimes. In fact, the literature contains several suggestions for realizations of this kind [6,14,15,26]. A particular example was put forward by Magueijo and Smolin [17], who required that the contraction between the energy-momentum and an infinitesimal spacetime displacement were a linear invariant in DSR. This requirement leads to new spacetime coordinates that depend on the energy-momentum. Ultimately, the system adopts a spacetime metric that directly depends on the energy and momentum of its particle content. This explains the name of *gravity's rainbow* that has been given to this class of DSR implementations.

In this work, we will follow a suggestion for the realization of DSR in position space that differs from that of Magueijo and Smolin, although it leads as well to a gravity's rainbow formalism in the sense of the energy dependence of the geometry. We will adopt the proposal of Ref. [6], namely, we will specify the realization by demanding the invariance of the symplectic form $\mathbf{d}q^a \wedge \mathbf{d}\Pi_a$ (where the wedge denotes the exterior product and Lorentz indices are lowered with the Minkowski metric). This assigns to the system new, modified spacetime coordinates x^a that are conjugate to the physical energy-momentum P_a, so that the relation between (q^a, Π_a) and (x^a, P_a) is given by a canonical transformation. Similar proposals for a canonical implementation of DSR theories have been analyzed by other authors [15,26].

By completing the map U into a canonical transformation, one easily derives the following expressions for the new spacetime coordinates [6]:

$$x^i = \frac{1}{J}\left[\partial_\pi g \frac{\pi^i}{\pi} q^0 + \partial_\epsilon g \frac{\pi^i \pi_j}{\pi^2} q^j\right] + \frac{\pi}{f}\left(q^i - \frac{\pi^i \pi_j}{\pi^2} q^j\right),$$
$$x^0 = \frac{1}{J}\left[\partial_\pi f q^0 + \partial_\epsilon f \frac{\pi_i}{\pi} q^i\right]. \tag{2.2}$$

Here, $J = \partial_\epsilon g \partial_\pi f - \partial_\pi g \partial_\epsilon f$ is the determinant of the Jacobian of the transformation U^{-1} between (ϵ, π) and (E, P), and the functions f and g (and therefore J) depend on (ϵ, π). We point out that the transformation (2.2) is linear in the coordinates q^a, but generally depends nontrivially on the energy and the momentum.

We will refer to (x^a, P_a) and (q^a, Π_a) as physical and background (or pseudo) variables, respectively, and will denote q^0 by T and x^0 by t to emphasize the role played by the evolution parameter in our discussion. In addition, we assume in the following that the system admits a Hamiltonian description, so that the values of the physical energy and pseudoenergy are, respectively, given by a physical Hamiltonian H and a background Hamiltonian H_0. With Eq. (2.1), we then get $E \to H = g(H_0, \pi)$ and $\epsilon \to H_0 = \tilde{g}(H, p)$. Finally, since DSR theories are essentially conceived as effective descriptions of free particles that incorporate quantum gravitational phenomena, we will concentrate our analysis on free systems. For such systems, the energy and momentum are constants of motion. The Hamiltonian is hence time independent and commutes with the momentum under Poisson brackets, both for the physical and the background variables.

III. PHYSICAL LENGTH UNCERTAINTY: PERTURBATIVE CASE

In this section, we will consider the perturbative approach to the quantization of the system in which one adopts the background time coordinate $q^0 = T$ as evolution parameter, so that the evolution is generated by the Hamiltonian H_0. We assume that a quantization of this kind is feasible. In such a quantum description, the physical time is represented by a genuine operator $\hat{t}$ [6,7]. We want to study whether the spatial position and length determined by the physical coordinates x^i is affected in this case by a nonvanishing quantum uncertainty. In order to simplify the analysis and deal only with scalar quantities (circumventing the kind of problems derived from the use of vector components and their dependence on choices of fixed background structures, choices which are questionable both from the viewpoint of general relativity and of the fluctuations inherent to quantum mechanics) we will focus our attention exclusively on the projection of the position vector along the direction of motion:

$$X := x^i \frac{p_i}{p} = x^i \frac{\pi_i}{\pi} = \frac{1}{J}\left[\partial_\pi g T + \partial_{H_0} g \frac{\pi_j}{\pi} q^j\right]. \tag{3.1}$$

We recall that g, f, and J are functions of only H_0 and π. Remarkably, this expression is similar to that given in (2.2) for the time coordinate $x^0 = t$ with the exchange of the function f for g and a flip of global sign (so that the determinant of the Jacobian J is preserved under the commented exchange).

Given our restriction to free systems, where the energy and momentum are conserved, the only variable in the expression for X that evolves in time (apart from the parameter T) is

$$s_T := \pi_j q^j. \tag{3.2}$$

The subscript T emphasizes this time dependence. Moreover, since the system is free, the background Hamiltonian H_0 is a function of only the pseudomomentum. Then, from the Hamiltonian equations of motion, the time derivative of s_T equals $\pi H_0'$, which is a constant of motion. Here, the prime denotes the derivative with respect to π. Thus, we conclude that $s_T = s_0 + T\pi H_0'$, where s_0 is the value of s_T at the initial instant of time.

For our quantum analysis we will only consider differences between position variables, avoiding in this way the arbitrariness in the choice of an origin and the conceptual tensions that arise from fixing it classically while allowing quantum fluctuations in the spatial position. The physics of the problem suggests two possible elections of reference for the position, namely, either the physical or the background initial value (of the projection along the direction of motion) of the position vector. In the first case, the position difference determines the physical interval covered by the particle in the background lapse T. In the second case the difference includes as well the effective corrections to the initial background position contained in DSR. We will study both possibilities to show that our conclusions do not depend on the specific choice adopted. To distinguish between the two cases, we introduce a parameter η, with $\eta = 0$ corresponding to the initial physical position and $\eta = 1$ to the background one. Explicitly, the former of these positions is given by Eq. (3.1) with $T = 0$ and $\pi_j q^j$ replaced with s_0, whereas the latter is equal to s_0/π.

From the difference between X and any of these reference positions, we obtain the following length:

$$\begin{aligned} L_\eta &:= \frac{1}{J}\left[\partial_\pi g T + \partial_{H_0} g S_T + \eta \frac{(\partial_{H_0} g - J)}{\pi} s_0\right], \\ S_T &:= \frac{s_T - s_0}{\pi}. \end{aligned} \tag{3.3}$$

We will refer to it as the physical length. To represent it as an operator, we write

$$\hat{L}_\eta := \hat{M}(H_0, \pi)T + \hat{R}_{T,\eta}, \tag{3.4}$$

$$\begin{aligned} \hat{R}_{T,\eta} = \frac{1}{2}(\hat{N}(H_0, \pi)\hat{S}_T + \hat{S}_T\hat{N}(H_0, \pi)) + \frac{\eta}{2}(\hat{O}(H_0, \pi)\hat{s}_0 \\ + \hat{s}_0\hat{O}(H_0, \pi)), \end{aligned} \tag{3.5}$$

where

$$M := \frac{\partial_\pi g}{J}, \qquad N := \frac{\partial_{H_0} g}{J}, \qquad O := \frac{\partial_{H_0} g - J}{\pi J}. \tag{3.6}$$

The subscript T denotes again dependence on time. In Eqs. (3.4) and (3.5) we have symmetrized the products of $\hat{N}$ with $\hat{S}_T$ and $\hat{O}$ with $\hat{s}_0$, and displayed explicitly the arguments of the functions M, N, and O. As we have commented, these functions correspond to constants of motion. Their respective operators can be defined in terms of those for H_0 and π employing the spectral theorem. As for the operator representing s_T (and hence S_T), we will comment on its definition later in this section.

It is worth pointing out that our expressions are to some extent similar to those introduced in Ref. [6] for the physical time operator $\hat{t}$. The differences come from the fact that in the latter case the role of the initial background position variable s_0/π is played by the initial background time ($T = 0$), and that in that work we only analyzed the choice $\eta = 1$ (initial time identified with that of the background time parameter). Our analysis here can be easily applied to the resulting time lapse, t_η, the precise correspondence being the disappearance of the contribution $-1/\pi$ in the function $O(H_0, \pi)$ (and therefore in $\hat{R}_{T,\eta}$), the exchange of the function f for g in the resulting formulas, and a flip of global sign.

In order to calculate the uncertainty in the physical length operator $\hat{L}_\eta$, we will follow the same procedure that was explained in Ref. [6]. Given a quantum state, one can measure the probability densities of any set of observables at any instant of time [27]. In this way, one can determine e.g. the expectation value of those operators. In addition, one can estimate the value of the parameter T at that instant of time by analyzing the evolution of the probability densities of observables in the considered state. This procedure allows to derive a statistical distribution for T with probability density $\rho(T)$ (and mean value $\bar{T}$). Heisenberg relations imply that the uncertainty ΔT of this distribution satisfies the inequality $\Delta T \Delta H_0 \geq 1/2$ (usually called the fourth Heisenberg relation) [1,6]. The double average process involved by the quantum expectation value $\langle\,\rangle$ and by the estimation of the time parameter leads to the following uncertainty:

$$(\Delta L_\eta)^2 = \int dT \rho(T)\langle(\hat{M}T + \hat{R}_{T,\eta} - \langle\hat{M}\rangle\bar{T} - \langle\hat{R}_{\bar{T},\eta}\rangle)^2\rangle. \tag{3.7}$$

Here, $\langle\hat{R}_{\bar{T},\eta}\rangle$ is the mean value of the operator $\hat{R}_{T,\eta}$ computed with the commented double average [6].

At this stage, some remarks are in order about the precise operator representation adopted for s_T when defining $\hat{R}_{T,\eta}$ and how this affects the measurements that are necessary to determine the mean value of this observable. Two cases are worth commenting on. On the one hand, one can represent s_T as an explicitly T-independent operator by simply adopting a symmetrized factor ordering in Eq. (3.2) and directly promoting the canonical background variables (q^i, π_i) to operators. Similarly, we can define $\hat{S}_T$ from its symmetrized classical expression. By performing quantum measurements at the fixed instant of time in which the system is analyzed, one can then determine the probability distribution for s_T at that instant. No estimation of the value of the evolution parameter is needed, so that the average over T becomes spurious. Similar arguments apply to the products of s_T with constants of motion that appear in $\hat{R}_{T,\eta}$. At least in principle, one may hence identify $\langle \hat{R}_{\overline{T},\eta} \rangle$ and $\langle \hat{R}_{T,\eta} \rangle$ in Eq. (3.7), even if the exact value of T in which the measurements are made is not known.

On the other hand, one can instead reflect explicitly all the T dependence of s_T in the definition of its associated operator. Starting with the solution to its evolution equation, one arrives at $\hat{s}_T := \hat{s}_0 + T\hat{\pi}\widehat{H_0'}$. So $\hat{S}_T := T\widehat{H_0'}$. Here, $\widehat{H_0'}$ can be defined in terms of the pseudomomentum using the spectral theorem. Since the operator $\widehat{H_0'}$ corresponds to a constant of motion, its probability density does not evolve in time. Actually, the same happens with $\hat{s}_0$, $\hat{M}$, $\hat{N}$ and $\hat{O}$, appearing in Eqs. (3.4) and (3.5). In particular, the measurements of all of their densities can be performed at an initial instant of time, identified with $T = 0$. For all other instants, the only missing piece of information is the probability density $\rho(T)$, obtained through measurements of distributions of observables that track the passage of time. In this case, obviously, the average with $\rho(T)$ cannot be obviated when calculating the mean value of $\hat{R}_{T,\eta}$.

The two cases can nevertheless be studied in exactly the same way by simply combining all the explicit linear T dependence of $\hat{X}$. In the latter case, one gets

$$\hat{L}_\eta = \hat{Y}(H_0, \pi)T + \hat{Z}_\eta(H_0, \pi, s_0), \tag{3.8}$$

$$\hat{Y}(H_0, \pi) = \hat{M}(H_0, \pi) + \hat{N}(H_0, \pi)\widehat{H_0'}(\pi), \tag{3.9}$$

$$\hat{Z}_\eta(H_0, \pi, s_0) = \frac{\eta}{2}(\hat{O}(H_0, \pi)\hat{s}_0 + \hat{s}_0\hat{O}(H_0, \pi)). \tag{3.10}$$

For computational purposes, expression (3.4) can be considered a particular example of formula (3.8) with $\hat{Y} = \hat{M}$ and $\hat{Z}_\eta = \hat{R}_{T,\eta}$. With the same substitutions in Eq. (3.7), the physical length uncertainty can then be rewritten:

$$(\Delta L_\eta)^2 = [\Delta(Y\overline{T} + Z_\eta)]^2 + \langle \hat{Y} \rangle^2 (\Delta T)^2 + (\Delta T \Delta Y)^2. \tag{3.11}$$

The case of the physical time lapse can be treated in a completely similar way [6], removing the contribution $-1/\pi$ to O in the definition of Z_η, interchanging the functions f and g, and introducing a global change of sign (to preserve that of J).

IV. EXISTENCE OF A MINIMUM UNCERTAINTY IN THE PERTURBATIVE CASE

The physical length uncertainty vanishes if and only if the three positive terms that form the right-hand side (r.h.s.) of Eq. (3.11) are equal to zero. We will show in this section that this cannot generally occur.

In order for the uncertainty to vanish, it must, in particular, do so at large T, times for which the contribution $(\overline{T}\Delta Y)^2$ dominates in (3.11). Therefore, ΔY (which is independent of time) must vanish. Let us assume that the expression of the background Hamiltonian H_0 as a function of π is invertible for the whole range of pseudoenergies, i.e. $\pi = \pi(H_0)$ [6]. One can then define the function $\mathcal{Y}(H_0) := Y[H_0, \pi(H_0)]$. In these circumstances, it suffices that the system satisfies, e.g., one of the following generic sets of hypotheses to prove that the physical length uncertainty is strictly positive.

(i) We first assume that the function $\mathcal{Y}(H_0)$ is strictly monotonic, namely $d\mathcal{Y}/dH_0 \neq 0$, so that it provides a one-to-one map. Then, via the spectral theorem, the eigenstates of the operators $\mathcal{Y}$ and H_0 coincide, and the demand $\Delta Y = \Delta \mathcal{Y} = 0$ implies that $\Delta H_0 = 0$. The fourth Heisenberg relation leads to $\Delta T \to \infty$. Let us then prove that the third term in Eq. (3.11) does not vanish when ΔH_0 tends to zero. Expanding $\mathcal{Y}$ around the mean value of H_0 [28], we find

$$(\Delta \mathcal{Y})^2 = \langle \hat{\mathcal{Y}}^2 - \langle \hat{\mathcal{Y}} \rangle^2 \rangle \approx \left(\frac{d\mathcal{Y}}{dH_0} \bigg|_{\langle \hat{H}_0 \rangle} \Delta H_0 \right)^2, \tag{4.1}$$

$$\lim_{\Delta H_0 \to 0} 2\Delta T \Delta \mathcal{Y} \geq \lim_{\Delta H_0 \to 0} \frac{\Delta \mathcal{Y}}{\Delta H_0} = \left| \frac{d\mathcal{Y}}{dH_0} \bigg|_{\langle \hat{H}_0 \rangle} \right| \neq 0. \tag{4.2}$$

We hence conclude that the physical length uncertainty cannot vanish in this case.

(ii) We suppose instead that $\mathcal{Y}(H_0)$ is positive and, for large pseudoenergies, grows at least like H_0 multiplied by a constant. We analyze first the case in which $\mathcal{Y}$ is strictly positive. Since $\langle \hat{Y} \rangle = \langle \hat{\mathcal{Y}} \rangle$ is then different from zero, the vanishing of the second term in Eq. (3.11) requires $\Delta T = 0$. So, the fourth Heisenberg relation implies that $\Delta H_0 \to \infty$. Let us consider again the third term in (3.11). Our condition on the behavior of $\mathcal{Y}$ for large H_0 can be rephrased by saying that $\lim_{H_0 \to \infty}(\mathcal{Y}/H_0) > r$ for a certain number $r > 0$. As a consequence, one can see that $\lim_{\Delta H_0 \to \infty}(\Delta \mathcal{Y}/\Delta H_0) > r$. Therefore, the product $\Delta T \Delta Y = \Delta T \Delta \mathcal{Y}$ cannot vanish when ΔH_0 tends to infinity, and the physical length uncertainty is strictly positive. On the other hand, in the case that $\mathcal{Y}$ can also take the zero value, $\langle \hat{Y} \rangle = \langle \hat{\mathcal{Y}} \rangle$ may occasionally vanish, but this may only happen if the quantum state is in the kernel of the operator $\hat{\mathcal{Y}}$. We then introduce the additional assumption that this kernel is formed exclusively by the eigenvectors corresponding to a unique eigenvalue $\overline{H}_0$ of $\hat{H}_0$, a result that holds when $\mathcal{Y}(H_0)$ vanishes only at that value of the pseudoenergy. If the system approaches such an eigenvector, the uncertainty of H_0 tends to zero and $\Delta T \to \infty$. Assuming finally that $(d\mathcal{Y}/dH_0)|_{\overline{H}_0} \neq 0$, one arrives at the same conclusion about the third term in Eq. (3.11) that was obtained in inequality (4.2) [28]. Therefore, under this set of hypotheses, it is impossible to achieve an infinite resolution in the physical length.

An important class of DSR theories in which the positivity of $\mathcal{Y}(H_0)$ is satisfied when s_T is represented by an explicitly time-dependent operator is when the physical energy does not depend on the pseudomomentum, i.e., when the function g depends only on H_0. In this case,

$$M = \frac{\partial_\pi g}{J} = 0, \qquad N = \frac{\partial_{H_0} g}{J} = \frac{1}{\partial_\pi f}, \qquad Y = \frac{H_0'}{\partial_\pi f}.$$

As a consequence, $\mathcal{Y}(H_0)$ is nonzero, because both the map U and $H_0(\pi)$ are invertible by assumption (this guarantees that $\partial_\pi f \neq 0$ and $H_0' \neq 0$). Since $\mathcal{Y}(H_0)$ has a definite sign, and $\partial_\pi f \approx 1$ in the sector of small pseudoenergy and pseudomomentum, in the standard situation with a pseudoenergy that increases with π in that sector we conclude that $\mathcal{Y}(H_0)$ is strictly positive [29].

In conclusion, a nonvanishing uncertainty generically affects the physical length in the perturbative quantization of the system. The above discussion can also be applied to the study of the physical time uncertainty considered in Ref. [6]. All the hypotheses can be easily generalized to that case with the due substitution of $\mathcal{Y}$ by the function $\mathcal{V}$ defined in that reference.

V. PHYSICAL POSITION UNCERTAINTY: NONPERTURBATIVE CASE

We turn now to the analysis of the physical length uncertainty when one adopts what we have called a nonperturbative quantization, i.e., when the quantum evolution is described in terms of the physical time.

In principle, one can always construct a nonperturbative quantum theory (in the sense indicated above) starting with the perturbative one, which has been assumed to exist. Employing the spectral decomposition of the pseudomomentum π and recalling that $H_0 = H_0(\pi)$, one can define the physical Hamiltonian $H = g(H_0, \pi)$ as an operator. The parameter of the evolution generated by this Hamiltonian can be identified with the physical time t. By contrast, the background time gets now promoted to an operator. This fact changes the expression of the observable $\hat{L}_\eta$ when regarded as an explicitly time-dependent operator. From Eqs. (3.4) and (2.2) one gets

$$\hat{L}_\eta^{[2]} = \hat{M}^{[2]}(H_0, \pi)t + \hat{R}_{t,\eta}^{[2]}, \tag{5.1}$$

$$\hat{R}_{t,\eta}^{[2]} := \frac{1}{2}(\hat{N}^{[2]}(H_0, \pi)\hat{S}_t + \hat{S}_t\hat{N}^{[2]}(H_0, \pi)) + \frac{1}{2}(\hat{O}_\eta^{[2]}(H_0, \pi)\hat{s}_0 + \hat{s}_0\hat{O}_\eta^{[2]}(H_0, \pi)), \tag{5.2}$$

where

$$M^{[2]} := \frac{\partial_\pi g}{\partial_\pi f}, \qquad N^{[2]} := \frac{1}{\partial_\pi f}, \tag{5.3}$$

$$O_\eta^{[2]} := \eta\frac{\partial_{H_0} g - J}{\pi J} - \frac{\partial_\pi g \partial_{H_0} f}{\pi J \partial_\pi f}. \tag{5.4}$$

The analysis is parallel to that followed in Secs. III and IV, with the caveat that $s_t := \pi_j q^j$ [and therefore also $S_t := (s_t - s_0)/\pi$] must now be considered a variable that evolves in the physical time t, rather than in the background time. In particular, by extracting explicitly all the time dependence of s_t when defining its operator counterpart, one arrives at

$$\hat{L}_\eta^{[2]} = \hat{Y}^{[2]}(H_0, \pi)t + \hat{Z}_\eta^{[2]}(H_0, \pi, s_0), \tag{5.5}$$

with

$$\hat{Y}^{[2]}(H_0, \pi) = (\widehat{H_0'}\,\widehat{\partial_{H_0} g} + \widehat{\partial_\pi g})\hat{N}^{[2]}(H_0, \pi) + \hat{M}^{[2]}(H_0, \pi), \tag{5.6}$$

$$\hat{Z}_\eta^{[2]}(H_0, \pi, s_0) = \frac{\hat{O}_\eta^{[2]}(H_0, \pi)\hat{s}_0 + \hat{s}_0\hat{O}_\eta^{[2]}(H_0, \pi)}{2}. \tag{5.7}$$

Here, the observable $\hat{s}_0$ represents the value of s_t at the initial physical time, which is a constant of motion.

In order to calculate the physical length uncertainty, one has to average now over the time parameter t, instead of averaging over T, as we did in Eq. (3.7). This leads to

$$(\Delta L_\eta^{[2]})^2 = [\Delta(Y^{[2]}\bar{t} + Z_\eta^{[2]})]^2 + (\langle\hat{Y}^{[2]}\rangle\Delta t)^2 + (\Delta t \Delta Y^{[2]})^2, \tag{5.8}$$

where $\bar{t}$ and Δt are the mean value and the uncertainty of the distribution deduced for the parameter t by analyzing the evolution of the probability densities of observables in our quantum state. Obviously, the time uncertainty satisfies the fourth Heisenberg relation $\Delta t \Delta H \geq 1/2$.

Notice that the physical length uncertainty is again given by the sum of three positive terms. The analysis of the previous section can be easily extended to the case considered here. From the behavior of $\Delta L_\eta^{[2]}$ at large times we conclude that $\Delta Y^{[2]}$ must vanish. Moreover, taking into account the assumption that the function $H_0(\pi)$ be invertible, remembering that $H = g(H_0, \pi)$, and using the implicit function theorem, it is possible to define $Y^{[2]}$ as a function of only H—that we denote $\mathcal{Y}^{[2]}(H)$—provided that $H_0'\partial_{H_0} g + \partial_\pi g \neq 0$. One can then introduce the same two sets of hypotheses that were discussed in Sec. IV, but with the role of $\mathcal{Y}(H_0)$ played by $\mathcal{Y}^{[2]}(H)$. In this way one concludes that, under quite generic assumptions, an infinite resolution cannot be reached for the physical length in a nonperturbative quantization of the system constructed from the perturbative quantum theory.

Finally, we want to comment on the possibility that the system might admit a different nonperturbative quantization (with evolution still generated by the physical Hamiltonian) in which the canonically conjugate physical variables (X, p) were promoted to explicitly time-independent operators and such that the quantum spectrum of the physical momentum p were contained in its corresponding classical domain. This is nontrivial in general, and the viability of such a quantization cannot be taken for granted starting from the only assumption of the existence of a perturbative quantum description with the properties that we have discussed. From Eq. (3.1), we see that a situation in which this possibility is realized is when the physical energy does not depend on the pseudomomentum, $\partial_\pi g = 0$. In this case (which includes the example of the Einstein-Rosen waves), the physical position X is independent of the background time. It may then be promoted to an operator that does not display any explicit time dependence, in terms of those for π^i and for the background coordinates q^i, the latter evolving only implicitly in the time parameter. Strictly speaking, nonetheless, the discussion presented in the paragraphs above cannot be applied in these circumstances because, with such an operator representation, $Y^{[2]}(H_0, \pi)$ must be identified with $M^{[2]}(H_0, \pi)$, the latter being identically zero when so is $\partial_\pi g$ [see Eqs. (5.1) and (5.3)]. This vanishing invalidates the sets of hypotheses under which our study was carried out.

When a nonperturbative quantization with those characteristics exists, the Heisenberg uncertainty principle implies that $\Delta X \Delta p \geq 1/2$. As a consequence, the resolution in the physical position is limited if and only if the physical momentum is bounded from above. This happens in DSR1 and DSR2 theories, but not in DSR3. The same phenomenon occurs with the physical length if it is determined by the difference of two uncorrelated position observables. In conclusion, we see that the emergence of a minimum uncertainty in the physical length is unavoidable nonperturbatively as well as perturbatively, except perhaps for DSR3 theories that admit a nonperturbative quantization in which X can be represented as an explicitly time-independent observable.

VI. FIRST ORDER CORRECTIONS IN THE PERTURBATIVE CASE

In this section we will study the physical length uncertainty that arises in the perturbative quantization when the operator $\hat{L}_\eta$ is approximated up to first order corrections in the energy. To obtain this approximation, we expand the functions f and g (which we suppose smooth) in the variables H_0 and π around their minimum values. Motivated by the case of free particles in special relativity, we assume that the minimum magnitude of the pseudomomentum is zero, whereas the minimum of the pseudoenergy μ will be just non-negative [6]. We then denote $\mathcal{H}_0 := H_0 - \mu$ and keep only up to quadratic terms in $\mathcal{H}_0$ and π in the expansions of the two functions; this truncation will suffice for our purposes. In addition, we suppose that μ is small compared with the invariant energy/momentum scale of the DSR theory, so that the leading terms in the region of expansion are $f(H_0, \pi) \approx \pi$ and $g(H_0, \pi) \approx H_0$ (because the map U determined by f and g must approach the identity in the low-energy-momentum sector).

From Eq. (3.6), one then gets

$$\begin{aligned} M(H_0, \pi) &\approx (\partial_{H_0}\partial_\pi g)|_0 \mathcal{H}_0 + (\partial_\pi^2 g)|_0 \pi, \\ N(H_0, \pi) &\approx 1 - (\partial_{H_0}\partial_\pi f)|_0 \mathcal{H}_0 - (\partial_\pi^2 f)|_0 \pi, \end{aligned} \tag{6.1}$$

where the symbol $|_0$ denotes evaluation at $\mathcal{H}_0 = \pi = 0$. Substituting these results and the expression $H_0(\pi)$ of the background Hamiltonian in Eqs. (3.9) and (3.10) [and recalling definitions (3.6)], we deduce the first order approximation for the operators $\hat{Y}$ and $\hat{Z}_\eta$. An extrapolation of the situation found in special relativity [6] leads us to consider the following cases.

(1) *Massive case:* $\mu \neq 0$, *with* $H_0'|_{\pi=0} = 0$.—We obtain $H_0(\pi) \approx \mu + b\pi^2$, where $2b := H_0''|_{\pi=0}$. Assuming that $b > 0$, we have that $\pi \approx \sqrt{\mathcal{H}_0/b}$. Thus, we can neglect terms proportional to $\mathcal{H}_0$ with respect to those linear in π. In this way, one finds

$$\hat{Y} \approx [2b + (\partial_\pi^2 g)|_0]\hat{\pi}, \tag{6.2}$$

$$\hat{Z}_\eta \approx -\eta(\partial_\pi^2 f)|_0 \hat{s}_0, \tag{6.3}$$

where we have employed that $s_0 = \pi_j q^j|_{T=0}$ is of the same order as π.

The function $\mathcal{Y}$, defined in Sec. IV, is given in this approximation by the classical analog of Eq. (6.2) with $\pi = \sqrt{\mathcal{H}_0/b}$. The resulting function is strictly monotonic in H_0 if the constant coefficient $2b + (\partial_\pi^2 g)|_0$ does not vanish, as it must happen if our truncation provides indeed the first order approximation. Therefore, the first set of hypotheses considered in Sec. IV is applicable in this case, leading us to the conclusion that it is impossible to achieve an infinite resolution in the physical length.

(2) *Massless case:* $\mu = 0$, *with* $H_0'|_{\pi=0} = k \neq 0$.—Now $\mathcal{H}_0 = H_0 \approx k\pi$, so that corrections proportional to either H_0 or π are of the same order. We then arrive at

$$\begin{aligned} \hat{Y} &\approx k + \left[\frac{2b}{k} - (\partial_\pi^2 f)|_0 - k(\partial_{H_0}\partial_\pi f)|_0 + \frac{(\partial_\pi^2 g)|_0}{k} \right. \\ &\qquad \left. + (\partial_{H_0}\partial_\pi g)|_0\right]\hat{H}_0, \\ \hat{Z}_\eta &\approx -\eta[k(\partial_{H_0}\partial_\pi f)|_0 + (\partial_\pi^2 f)|_0]\hat{s}_0. \end{aligned} \tag{6.4}$$

The constant b is defined as in the *massive* case. The next-to-leading order approximation to the function $\mathcal{Y}$ is thus given by the classical counterpart of Eq. (6.4). Again, provided that the constant coefficient of the first order correction in H_0 differs from zero, the function $\mathcal{Y}$ is strictly monotonic. The physical length uncertainty is hence greater than zero in this approximation.

VII. FIRST ORDER CORRECTIONS: BEHAVIOR AT LARGE TIMES

In this section, we will analyze in more detail the physical length uncertainty in the perturbative quantization for the *massless* case adopting the next-to-leading order approximation for low energies. We will pay special attention to the behavior displayed at large values of the background time. We will show that this behavior is of the kind that was first discussed by Salecker and Wigner [23]. Since a similar study was not considered in Ref. [6] for the physical time uncertainty, we will carry out our analysis in a way that is also valid for it.

From the results of Ref. [6] and our comments above, the physical time lapse t_η is affected in the perturbative quantization by the uncertainty:

$$(\Delta t_\eta)^2 = [\Delta(V\overline{T} + W_\eta)]^2 + \langle \hat{V}\rangle^2(\Delta T)^2 + (\Delta T \Delta V)^2, \tag{7.1}$$

where the operators $\hat{V}$ and $\hat{W}_\eta$ have these expressions in the considered approximation for the *massless* case:

$$\begin{aligned}\hat{V} \approx 1 + \bigg[k(\partial^2_{H_0} f)|_0 + (\partial_{H_0}\partial_\pi f)|_0 - (\partial^2_{H_0} g)|_0 \\ - \frac{(\partial_{H_0}\partial_\pi g)|_0}{k}\bigg]\hat{H}_0, \\ \hat{W}_\eta \approx \eta[(\partial_{H_0}\partial_\pi f)|_0 + k(\partial^2_{H_0} f)|_0]\hat{s}_0.\end{aligned} \tag{7.2}$$

We introduce the notation $\{L_{\alpha,\eta}\} := \{t_\eta, L_\eta\}$, $\{Y_\alpha\} := \{V, Y\}$, and $\{Z_{\alpha,\eta}\} := \{W_\eta, Z_\eta\}$ to describe simultaneously the formulas for the physical time and length uncertainties. Let us emphasize that $\alpha = 0, 1$ is just an abstract subscript notation.

After a trivial elaboration, we can rewrite Eqs. (3.11) and (7.1) as

$$\begin{aligned}(\Delta L_{\alpha,\eta})^2 = \overline{T}^2(\Delta Y_\alpha)^2 + (\Delta Z_{\alpha,\eta})^2 + \overline{T}\mathrm{cov}(\hat{Y}_\alpha, \hat{Z}_{\alpha,\eta}) \\ + \langle \hat{Y}_\alpha\rangle^2(\Delta T)^2 + (\Delta T\Delta Y_\alpha)^2.\end{aligned} \tag{7.3}$$

No sum over α is implied and

$$\mathrm{cov}\,(\hat{Y}_\alpha, \hat{Z}_{\alpha,\eta}) := \langle \hat{Y}_\alpha\hat{Z}_{\alpha,\eta} + \hat{Z}_{\alpha,\eta}\hat{Y}_\alpha\rangle - 2\langle\hat{Y}_\alpha\rangle\langle\hat{Z}_{\alpha,\eta}\rangle. \tag{7.4}$$

In addition, in the studied approximation for the *massless* case, we can write the operators $\hat{Y}_\alpha$ and $\hat{Z}_{\alpha,\eta}$ in the form $\hat{Y}_\alpha = \kappa_\alpha + \lambda_\alpha\hat{H}_0/E_P$ and $\hat{Z}_{\alpha,\eta} = \eta\delta_\alpha\hat{s}_0/E_P$ [see Eqs. (6.4) and (7.2)], where E_P is the Planck energy (in our units $E_P = 1/\sqrt{G}$, with G being Newton constant), λ_α and δ_α are appropriate constant coefficients that differ from zero, $\kappa_0 := 1$, and $\kappa_1 := k = H_0'|_{\pi=0}$.

The last term in Eq. (7.3) is then

$$(\Delta T\Delta Y_\alpha)^2 = \frac{\lambda_\alpha^2(\Delta T\Delta H_0)^2}{E_P^2} \geq \frac{\lambda_\alpha^2 l_P^2}{4}. \tag{7.5}$$

In the last step, we have used the fourth Heisenberg relation for the background time and energy, and introduced the Planck length $l_P = 1/E_P$ (in our units). Recalling that the other contributions to the physical uncertainty are positive, we conclude that $\Delta L_{\alpha,\eta} \geq |\lambda_\alpha| l_P/2$. Therefore, we see that the uncertainty in both the physical time lapse and the physical length is bounded from below by a contribution of quantum gravitational origin that is of the order of the Planck length [2–4].

From the rest of contributions to the physical uncertainty (7.3), one gets in a similar way the bound

$$\begin{aligned}(\Delta L_{\alpha,\eta})^2 > \lambda_\alpha^2\overline{T}^2\frac{(\Delta H_0)^2}{E_P^2} + \frac{\langle\hat{Y}_\alpha\rangle^2}{4(\Delta H_0)^2} + (\Delta Z_{\alpha,\eta})^2 \\ + \overline{T}\mathrm{cov}(\hat{Y}_\alpha, \hat{Z}_{\alpha,\eta}).\end{aligned} \tag{7.6}$$

The r.h.s. of this inequality can be regarded as a function of the uncertainty in the background energy ΔH_0, once the next-to-leading order expressions for the operators $\hat{Y}_\alpha$ and $\hat{Z}_{\alpha,\eta}$ have been substituted. Hence, for uncertainties ΔH_0 in a certain interval, one can deduce a more general bound for $\Delta L_{\alpha,\eta}$ by minimizing that function. The extrema can be deduced by imposing the vanishing of the first derivative with respect to ΔH_0:

$$\begin{aligned}0 = 2\lambda_\alpha^2\overline{T}^2\frac{(\Delta H_0)^4}{E_P^2} - \frac{\langle\hat{Y}_\alpha\rangle^2}{2} + (\Delta H_0)^3\partial_\Delta(\Delta Z_{\alpha,\eta})^2 \\ + \frac{\Delta H_0\partial_\Delta(\langle\hat{Y}_\alpha\rangle^2)}{4} + \overline{T}(\Delta H_0)^3\partial_\Delta\mathrm{cov}(\hat{Y}_\alpha, \hat{Z}_{\alpha,\eta}).\end{aligned} \tag{7.7}$$

Here, we have introduced the notation ∂_Δ to denote the derivative with respect to ΔH_0.

Provided that $\langle\hat{Y}_\alpha\rangle$ can be considered independent of both ΔH_0 and the (mean value of the) background time $\overline{T}$, the first two terms in the r.h.s. of Eq. (7.6) are in fact the kind of contributions that lead to the emergence of a minimum uncertainty of the Salecker and Wigner type (see Appendix A for details) [23,30]. Namely, we get a contribution that is linear in $(\Delta H_0)^2$ and another one that is proportional to its inverse. If these two terms were the only ones that appeared in our equations, an analysis similar to the standard one for Salecker-Wigner devices would prove that the bound for $\Delta L_{\alpha,\eta}$ is minimized at a value of ΔH_0 that scales with the background time like $\Delta H_0^{\min} \propto 1/\sqrt{\overline{T}}$, whereas the lower bound obtained for the physical uncertainty at $\Delta H_0^{\min}$ increases in time like $\sqrt{\overline{T}}$.

Motivated by these remarks, we will now show that, at least in the region of small ΔH_0 and for large values of the background time $\overline{T}$, the terms in Eqs. (7.6) and (7.7) other than the first two ones do not invalidate the above conclusions about the existence of a (local) minimum and its associated bound. The restriction to small values of ΔH_0 is natural in the context of the low-energy approximation that we are discussing. Moreover, for unboundedly large times $\overline{T}$, the sector of vanishingly small values of ΔH_0 contains the relevant region for the analysis of the Salecker-Wigner bound on the uncertainty, i.e. the region around the minimum $\Delta H_0^{\min} \propto 1/\sqrt{\overline{T}}$.

In this sector of background energy uncertainties and time, one can demonstrate that a set of sufficient conditions to deduce a Salecker-Wigner behavior are

$$\lim_{\Delta H_0 \to 0} \langle \hat{Y}_\alpha \rangle^2 = c_\alpha^{(1)}, \tag{7.8a}$$

$$\lim_{\Delta H_0 \to 0} (\Delta H_0)^2 (\Delta Z_{\alpha,\eta})^2 = c_\alpha^{(2)}, \tag{7.8b}$$

$$\lim_{\Delta H_0 \to 0} (\Delta H_0)^3 \partial_\Delta (\Delta Z_{\alpha,\eta})^2 = c_\alpha^{(3)}, \tag{7.8c}$$

$$\lim_{\Delta H_0 \to 0} \Delta H_0 \partial_\Delta \langle \hat{Y}_\alpha \rangle^2 = 0, \tag{7.8d}$$

$$\lim_{\Delta H_0 \to 0} \mathrm{cov}(\hat{Y}_\alpha, \hat{Z}_{\alpha,\eta}) = 0, \tag{7.8e}$$

$$\lim_{\Delta H_0 \to 0} \Delta H_0 \partial_\Delta \mathrm{cov}(\hat{Y}_\alpha, \hat{Z}_{\alpha,\eta}) = 0, \tag{7.8f}$$

where $c_\alpha^{(n)}$, $n = 1, 2, 3$, are constants (with $c_\alpha^{(1)} - 2c_\alpha^{(3)} \neq 0$ and $c_\alpha^{(1)} + 2c_\alpha^{(2)} - c_\alpha^{(3)} \neq 0$). Conditions (7.8a)–(7.8c) allow one to absorb the third term in the r.h.s. of Eqs. (7.6) and (7.7) just as a modification to $\langle \hat{Y}_\alpha \rangle^2$ and treat this (square) expectation value as a constant when calculating the value of our function around its extrema in the region $\Delta H_0 \ll 1$. In such a calculation and for sufficiently large background times, conditions (7.8d)–(7.8f) guarantee that all but the first three terms in Eqs. (7.6) and (7.7) can be neglected.

Taking into account that $\hat{Z}_{\alpha,\eta}$ vanishes when $\eta = 0$, the only nontrivial requirements in that case are conditions (7.8a) and (7.8b). Regardless of the value of η, we prove in Appendix B that all the above conditions are satisfied at least for quantum states that are described by Gaussian wave packets [31]. Since we are assuming the feasibility of a (perturbative) quantization with canonical variables given by the background flat spatial coordinates and the pseudomomentum, and in addition we have focused our discussion on free systems, it seems reasonable to suppose that such states exist and provide the analog of classical particles in our quantum theory. Besides, the limitation to wave packets is already present in the deduction of the Salecker-Wigner bound for the spacetime uncertainty (in order to justify the assumption that the position and momentum operators have vanishing covariance) [30]. So, it is natural to incorporate the same restriction to our analysis.

Substituting the values of the constants c_n computed in Appendix B (under the simplifying assumption of only one spatial dimension), one obtains the following bounds for large background times from the corresponding minima in the region $\Delta H_0 \ll 1$:

$$(\Delta L_{\alpha,\eta})^2 > d_{\alpha,\eta} l_P \overline{T}, \tag{7.9}$$

where

$$d_{\alpha,\eta} = \lambda_\alpha \left[\eta k^2 \frac{\delta_\alpha^2}{E_P^2} \nu^2 + \left(\kappa_\alpha + k \frac{\lambda_\alpha}{E_P} |\nu| \right)^2 \right]^{1/2}. \tag{7.10}$$

Here, ν denotes the expectation value of the pseudomomentum.

In conclusion, in the perturbative quantization of free *massless* systems in DSR theories and within the low-energy approximation, we have seen that the physical time and length uncertainties are always bounded from below by a quantum gravitational contribution of the order of the Planck length, while for large values of the background time the uncertainties increase like $\sqrt{l_P \overline{T}}$ (at least for wave packets), just like in Salecker-Wigner devices.

VIII. CONCLUSION

In this work, we have analyzed the emergence of a minimum nonvanishing length uncertainty in the framework of a gravity's rainbow formalism, derived from a dual realization of DSR theories in spacetime. This realization leads to a set of spacetime coordinates that are canonically conjugate to the physical energy and momentum. Therefore, the transformation from the background energy-momentum and spacetime coordinates (also called pseudovariables) to those that we consider as physical is provided by a canonical transformation. In particular, the physical spacetime variables are linear in the background ones, but in general depend nonlinearly on the pseudoenergy and pseudomomentum of the particle.

We have specialized our analysis to systems that admit a Hamiltonian formulation, with the energy determined by the value of the Hamiltonian, and concentrated our attention on the case of a free dynamics, motivated by the consideration of DSR theories as (effective) descriptions of free particles in special relativity modified by gravity. In these free systems, the background Hamiltonian is a function of only the (magnitude of the) pseudomomentum. We have studied the behavior of the physical position, understanding as such the scalar obtained by projecting the physical position vector in the momentum direction. More specifically, we have investigated the quantum uncertainty that affects the physical length, defined by the difference between this physical position and the initial value of the position, either in the background or in the physical variables of the system. This study has been carried out in two possible quantization schemes, referred as perturbative and nonperturbative quantizations.

The perturbative approach corresponds to a quantization in which the evolution is generated by the background Hamiltonian, so that the background time T plays the role of evolution parameter. We have assumed that a quantum theory of this kind is feasible. In this quantization, the physical time and length are represented by genuine operators that depend explicitly on the time parameter. We have been able to generalize the analysis of Ref. [6] for the physical time uncertainty, and prove that the uncertainty in the physical length is also strictly positive in this approach.

Rigorously speaking, we have demonstrated this positivity under two different sets of generic assumptions. Both sets contain the more than reasonable hypothesis that the considered quantum state has a finite expectation value of the background energy, $\langle \hat{H}_0 \rangle < \infty$. Besides, the two sets include an assumption about the functional dependence of the background energy on the pseudomomentum, namely, that the function $H_0 = H_0(\pi)$ be invertible. The rest of hypotheses concern the detailed form of the DSR theory, and more concretely the properties of the function $\mathcal{Y}(H_0) := Y[H_0, \pi(H_0)]$ introduced in Sec. IV.

One set of assumptions requires this function to be strictly monotonic, i.e. $\mathcal{Y}'(H_0) \neq 0$ for all values of H_0.

The other set involves several requirements. The most important ones are (i) the positivity of $\mathcal{Y}$, $\mathcal{Y} \geq 0$; and (ii) a linear or faster increase of $\mathcal{Y}$ with H_0 at infinity, $\lim_{H_0 \to \infty}(\mathcal{Y}/H_0) > r$ for a certain constant $r > 0$. In addition, it is demanded that: (iiia) the kernel of $\mathcal{Y}$ be empty, or either (iiib) this kernel consist of a single point $\overline{H}_0$ where the derivative of $\mathcal{Y}$ does not vanish, $(d\mathcal{Y}/dH_0)|_{\overline{H}_0} \neq 0$.

In the nonperturbative approach, on the other hand, the evolution is generated by the physical Hamiltonian, and the physical time t is identified with the evolution parameter. Starting with the perturbative quantization that we have assumed to exist, it is in general possible to construct a nonperturbative quantum theory of this kind, in which the physical length is represented by an operator that depends explicitly on the time parameter t. We have proved that the quantum uncertainty in this operator is strictly positive under similar sets of assumptions to those discussed for the case of the perturbative quantization. Therefore, it is again impossible to reach an infinite resolution in the physical length.

It might also happen that the system admits a different nonperturbative quantization in which the evolution is indeed generated by the physical Hamiltonian, but the physical position variable gets promoted to an operator that is explicitly independent of time and canonically conjugate to the operator which represents the magnitude of the physical momentum. In general, the existence of such a quantum theory is not granted from the sole assumption of the viability of the perturbative quantization. Supposing besides that the quantum spectrum of the physical momentum is contained in its classical domain, Heisenberg principle implies that the uncertainty in the physical position can be made to vanish only if the physical momentum is not bounded from above. The same result holds for the physical length if it is determined by the difference of two uncorrelated physical positions.

The existence of an upper bound for the physical momentum, with the consequent limit in the spatial resolution, occurs only in the DSR1 and DSR2 families, but not in DSR3 theories. Remarkably, for such theories the physical time uncertainty is always bounded away from zero in the nonperturbative quantum theory [6]. As a result, it is never possible to reach an infinite resolution, both in the physical time and position, in the nonperturbative quantization of Hamiltonian free systems within the context of DSR theories.

Finally, we have also analyzed the uncertainty in the perturbative quantization when the operator corresponding to the physical length is approximated up to first order corrections in the energy. The study has lent support to the conclusion that this uncertainty is generically greater than zero. Special attention has been paid to the *massless* case, in which the background energy is proportional to the magnitude of the pseudomomentum in the considered approximation. We have proved that, in that case, the uncertainty is always bounded by a quantity of the order of the Planck length. This bound can be interpreted as a contribution of quantum gravitational origin. In addition we have proved that, in the low-energy regime and for large values of the background time, the uncertainties in the physical time and length admit lower bounds that increase with the square root of time. This is precisely the kind of behavior that was suggested by Salecker and Wigner for spacetime measurements made with quantum devices.

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APPENDIX A: SALECKER-WIGNER DEVICES

In this appendix we will briefly summarize the rationale of Salecker and Wigner about the quantum uncertainty that is inherent to the measurement of spacetime distances [23,30]. The analysis starts with the consideration of a measurement device, regarded as a free system with mass m and uncertainties in its initial position and momentum Δq and $\Delta \pi$. The (square) uncertainty in its position at a later instant of time t is

$$[\Delta q(t)]^2 = \left[\Delta\left(q + \frac{t}{m}\pi\right)\right]^2 = (\Delta q)^2 + \frac{t^2}{m^2}(\Delta \pi)^2 + \frac{t}{m}\mathrm{cov}(\hat{q}, \hat{\pi}),$$

where $\text{cov}(\hat{q}, \hat{\pi}) := \langle \hat{q}\,\hat{\pi} + \hat{\pi}\,\hat{q}\rangle - 2\langle \hat{q}\rangle\langle \hat{\pi}\rangle$. This expression gets simplified when the (initial) position and momentum observables are not correlated. This occurs, for instance, if the states of the system are plane waves modulated by a Gaussian. In that case $\text{cov}(\hat{q}, \hat{\pi}) = 0$. Making use of the fourth Heisenberg relation, one then obtains the inequality

$$[\Delta q(t)]^2 \geq \frac{t^2}{m^2}(\Delta\pi)^2 + \frac{1}{4(\Delta\pi)^2}. \tag{A1}$$

The r.h.s. of this equation can be viewed as a function of $\Delta\pi$. Its extrema can be determined by imposing the vanishing of the first derivative:

$$0 = \frac{4t^2}{m^2}(\Delta\pi)^4 - 1.$$

The minimum value of the uncertainty is hence reached at $\Delta\pi^{\text{min}} = \sqrt{m/(2t)}$. Substituting this value in (A1) one gets a lower bound for the position uncertainty at the instant t:

$$\Delta q(t) \geq \sqrt{\frac{t}{m}}.$$

Therefore, the arguments of Salecker and Wigner imply that the uncertainty increases with the square root of time.

APPENDIX B: CALCULATIONS FOR WAVE PACKETS

This appendix contains the calculation of the mean values, uncertainties and covariance of the operators $\hat{Y}_\alpha$ and $\hat{Z}_{\alpha,\eta}$ introduced in Sec. VII, adopting the next-to-leading order approximation for low energies and restricting the quantum states to be Gaussian wave packets (in the free quantum theory with elementary variables given by the background spatial coordinates and momenta). Moreover, in order to simplify our calculations, we will carry out our analysis not in three, but just in one spatial dimension. We do not expect this reduction to qualitatively affect our results.

Explicitly, we will adopt a standard momentum representation in one dimension, with wave packets given by the following wave functions [31]:

$$\Psi(\pi_1) = \frac{1}{(2\Pi\sigma^2)^{1/4}} e^{-(\pi_1-\nu)^2/(4\sigma^2)} e^{-i\mu\pi_1}.$$

Here, $\nu := \langle \widehat{\pi_1}\rangle$, $\sigma := \Delta\pi_1$, and $\mu := \langle \widehat{q^1}\rangle$, with q^1 being the initial background position (we obviate its subscript 0 to simplify the notation). The number Π is denoted in this appendix with a capital Greek letter in order to distinguish it from the magnitude of the pseudomomentum π. Besides, note that in one dimension $\pi = |\pi_1|$.

From the functional form of the wave packets, it is clear that the quantities that we want to compute will depend on the parameters μ, ν, and σ. So, to calculate the limiting values (7.8), we need to express the limit $\Delta H_0 \to 0$ in terms of those parameters. In the studied approximation, $H_0 = k\pi$ for the *massless* case, and a trivial calculation shows that the uncertainty ΔH_0 for the wave packets is given by

$$(\Delta H_0)^2 = k^2(\Delta\pi)^2 = k^2(\sigma^2 + \nu^2 - \langle\hat{\pi}\rangle^2) := G^2(\sigma, \nu), \tag{B1}$$

$$\langle\hat{\pi}\rangle = |\nu|\text{erf}\left(\frac{|\nu|}{\sqrt{2}\sigma}\right) + \sqrt{2/\Pi}\,\sigma e^{-\nu^2/(2\sigma^2)}. \tag{B2}$$

It is worth emphasizing that $\langle\hat{\pi}\rangle$, the expectation value of the magnitude of the pseudomomentum, differs in general from ν. We have introduced the error function

$$\text{erf}\,(x) = \frac{2}{\sqrt{\Pi}}\int_0^x dy e^{-y^2}, \text{ with} \lim_{x\to\infty}\text{erf}(x) = 1.$$

From the above equations, we see that $\langle\hat{\pi}\rangle \approx |\nu|$ and $\Delta H_0 \approx k\sigma$ for small uncertainties ΔH_0. Via the implicit function theorem, we can then use the relation $\Delta H_0 = G(\sigma, \nu)$ (G being the square root of G^2) to define σ as a function of ΔH_0 in a neighborhood of the origin of these quantities, provided that $\partial_\sigma G$ does not vanish there. Actually, one has that $\lim_{\sigma\to 0}\partial_\sigma G = k \neq 0$. Therefore, one may replace the limit $\Delta H_0 \to 0$ with $\sigma \to 0$. In addition, one can substitute the partial derivative with respect to ΔH_0 (i.e., ∂_Δ) by $\partial_\Delta\sigma\partial_\sigma$, where $\lim_{\sigma\to 0}\partial_\Delta\sigma = 1/k$. These considerations lead to the results given in the rest of this appendix, where we analyze simultaneously the cases of the physical time and length uncertainties.

In the first order approximation for the *massless* case, the operators $\hat{Y}_\alpha$ and $\hat{Z}_{\alpha,\eta}$ adopt expressions of the form [see Eqs. (6.4) and (7.2)]:

$$\hat{Y}_\alpha = \kappa_\alpha + k\frac{\lambda_\alpha}{E_P}\hat{\pi},$$

$$\hat{Z}_{\alpha,\eta} = \eta\frac{\delta_\alpha}{E_P}\hat{s}_0 = \eta\frac{\delta_\alpha}{2E_P}(\widehat{\pi_1}\,\widehat{q^1} + \widehat{q^1}\,\widehat{\pi_1}),$$

where λ_α and δ_α are certain nonvanishing constants, η can take the values 0 or 1, $\kappa_0 = 1$, and $\kappa_1 = k$. We have employed that in this approximation $\hat{H}_0 = k\hat{\pi}$.

A straightforward calculation along the lines explained above shows that for wave packets

$$\lim_{\Delta H_0\to 0}\langle\hat{Y}_\alpha\rangle^2 = \lim_{\sigma\to 0}\langle\hat{Y}_\alpha\rangle^2 = \left(\kappa_\alpha + k\frac{\lambda_\alpha}{E_P}|\nu|\right)^2 := c_\alpha^{(1)}.$$

In the same way, one finds

$$\Delta H_0\partial_\Delta\langle\hat{Y}_\alpha\rangle^2 = 2k\frac{\lambda_\alpha}{E_P}\left(\kappa_\alpha + k\frac{\lambda_\alpha}{E_P}\langle\hat{\pi}\rangle\right)\partial_\sigma\langle\hat{\pi}\rangle\Delta H_0\partial_\Delta\sigma,$$

$$\Delta H_0\partial_\Delta\sigma = \frac{\sigma^2 + \nu^2 - \langle\hat{\pi}\rangle^2}{\sigma - \langle\hat{\pi}\rangle\partial_\sigma\langle\hat{\pi}\rangle}. \tag{B3}$$

From Eq. (B2) one can check that $\partial_\sigma\langle\hat{\pi}\rangle$ tends fast enough to zero when $\sigma \to 0$ ($\Delta H_0 \to 0$) as to guarantee that

$$\lim_{\Delta H_0 \to 0} \Delta H_0 \partial_\Delta \langle \hat{Y}_\alpha \rangle^2 = 0.$$

On the other hand, a similar computation leads to the following uncertainty for the operator $\hat{Z}_{\alpha,\eta}$

$$(\Delta Z_{\alpha,\eta})^2 = \eta \frac{\delta_\alpha^2}{E_P^2} (\langle \hat{s}_0^2 \rangle - \langle \hat{s}_0 \rangle^2) = \eta \frac{\delta_\alpha^2}{E_P^2} \left(\frac{\nu^2}{4\sigma^2} + \mu^2 \sigma^2 + \frac{1}{2} \right).$$

From this and Eqs. (B1) and (B3) it is not difficult to prove that

$$\lim_{\Delta H_0 \to 0} (\Delta H_0)^2 (\Delta Z_{\alpha,\eta})^2 = \eta k^2 \frac{\delta_\alpha^2}{4E_P^2} \nu^2 := c_\alpha^{(2)},$$

$$\lim_{\Delta H_0 \to 0} (\Delta H_0)^3 \partial_\Delta (\Delta Z_{\alpha,\eta})^2 = -\eta k^2 \frac{\delta_\alpha^2}{2E_P^2} \nu^2 := c_\alpha^{(3)}.$$

Finally, the covariance of $\hat{Y}_\alpha$ and $\hat{Z}_{\alpha,\eta}$ is given by

$$\mathrm{cov}\,(\hat{Y}_\alpha, \hat{Z}_{\alpha,\eta}) = \eta k \frac{\lambda_\alpha \delta_\alpha}{E_P^2} (\langle \hat{\pi} \hat{s}_0 + \hat{s}_0 \hat{\pi} \rangle - 2 \langle \hat{\pi} \rangle \langle \hat{s}_0 \rangle)$$

which for wave packets gives

$$\mathrm{cov}\,(\hat{Y}_\alpha, \hat{Z}_{\alpha,\eta}) = 2\eta k \frac{\lambda_\alpha \delta_\alpha}{E_P^2} \mu \sigma^2 \mathrm{sign}(\nu) \mathrm{erf}\left(\frac{|\nu|}{\sqrt{2}\sigma} \right).$$

Therefore, one can check that

$$\lim_{\Delta H_0 \to 0} \mathrm{cov}(\hat{Y}_\alpha, \hat{Z}_{\alpha,\eta}) = 0,$$

$$\lim_{\Delta H_0 \to 0} \Delta H_0 \partial_\Delta \mathrm{cov}(\hat{Y}_\alpha, \hat{Z}_{\alpha,\eta}) = \lim_{\sigma \to 0} \Delta H_0 \partial_\Delta \sigma \partial_\sigma \mathrm{cov}(\hat{Y}_\alpha, \hat{Z}_{\alpha,\eta}) = 0.$$

In conclusion, we see that conditions (7.8) are satisfied.

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[27] To perform quantum measurements at the same instant of time one only needs a notion of simultaneity, and not the exact knowledge of the value of the time parameter.

[28] In expanding $\mathcal{Y}(H_0)$ around the expectation value of H_0, we have implicitly admitted that the system has a finite background energy, namely, that $\langle \hat{H}_0 \rangle < \infty$.

[29] One may relax the condition of strict positivity on H_0' and allow it to vanish, e.g., at the single point $\pi = 0$, with the function $\mathcal{Y}$ vanishing hence at the corresponding pseudoenergy. See in this sense the discussion of our second set of hypotheses in Sec. IV.

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[31] The quantum states display no evolution if we have adopted a Heisenberg picture (e.g., if we have solved the quantum dynamics by introducing the appropriate explicit time dependence in our operators).