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Second-order perturbations of a zero-pressure cosmological medium: Proofs of the relativistic-Newtonian correspondence

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The dynamic world model and its linear perturbations were first studied in Einstein's gravity. In the system without pressure, the relativistic equations coincide exactly with the later known ones in Newton's gravity. Here we *prove* that, except for the gravitational wave contribution, even to the second-order perturbations, equations for the relativistic irrotational zero-pressure fluid in a flat Friedmann background *coincide* exactly with the previously known Newtonian equations. Thus, to the second order, we correctly identify the relativistic density and velocity perturbation variables, and we *expand* the range of applicability of the Newtonian medium without pressure to *all* cosmological scales including the superhorizon scale. In the relativistic analyses, however, we do *not* have a relativistic variable which corresponds to the Newtonian potential to the second order. Mixed usage of different gauge conditions is useful to make such proofs and to examine the result with perspective. We also present the gravitational wave equation to the second order. Since our correspondence includes the cosmological constant, our results are relevant to currently favored cosmology. Our result has an important practical implication that one can use the large-scale Newtonian numerical simulation more reliably even as the simulation scale approaches near horizon.

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I. INTRODUCTION

Despite its algebraic and conceptual complexity in Einstein's gravity, the evolving world model and its linear structures were first studied based on Einstein's gravity in the classic works by Friedmann in 1922 [1] and Lifshitz in 1946 [2], respectively. In an interesting sequence, the much simpler and, in hindsight, more intuitive Newtonian studies followed later by Milne in 1934 [3] and Bonnor in 1957 [4], respectively. In the case without pressure, the Newtonian results *coincide* exactly with the previously derived relativistic ones for both the background world model and its first-order (linear) perturbations. The case with pressure *cannot* be handled in the Newtonian context despite several failed attempts in the literature to simulate it, especially for the perturbation. The situation is still well described by Sachs and Wolfe in 1967 [5]: "When these modified equations were perturbed to first order, their solutions did not agree with the relativistic results, even qualitatively." In this work, we will show an additional continuation of relativistic-Newtonian correspondences in the zero-pressure medium by *proving* that the relativistic second-order scalar-type perturbations are described by the same equations known in Newton's theory. That is, the Newtonian equations coincide exactly with the relativistic ones even to the second order in perturbations.

In the relativistic perturbations, due to the covariance of field equation we have freedom to fix the spacetime coordinate system by choosing some of the metric or energy-momentum variables at our disposal: This is often called the gauge choice. The original study of Lifshitz started by choosing the synchronous gauge which is still quite popular in the literature. Other gauge conditions were discovered later [6,7]. It is an ironic situation that, except for the widely used synchronous gauge condition, each of the other gauge conditions fixes the gauge freedom completely. Thus, each has its own unique corresponding gauge-invariant combination. Notice some common algebraic errors (not in Lifshitz's work, though) widespread in the literature, including many textbooks, due to the incomplete gauge fixing nature of the synchronous gauge; see [8].

Although infinitely many gauge conditions are available, it has been common in the literature to fix gauge conditions from the beginning. The importance of using different gauge conditions for different variables and the gauge invariance of such variables were shown by Bardeen in 1980 [7]. Bardeen's work also showed the importance of having access to many different gauge conditions, which become apparent in his work in 1988 [9]. In this work, the importance of having different variables evaluated in different gauges (all correspond to unique gauge-invariant combinations) will become clear as we extend Bardeen's approach to the second-order perturbations.

Recently, we have presented a second-order perturbation formulation of the Friedmann world model considering quite general situations [10]. We have resolved the gauge issue, identifying the variables to use in fixing the gauges and constructing gauge-invariant combinations, which can be easily extended even to the higher order. The basic equations are presented without fixing the temporal gauge condition, thus allowing us to choose or try many available gauge conditions later depending on the situation: We call this a gauge-ready approach; see Eqs. (5)–(11) below. The Newtonian correspondence to the linear order was made by properly arranging the equations using various gauge-invariant variables in Refs. [7,11,12]. Extending such correspondences to the second order is our task in this work. We set $c \equiv 1$.

II. BASIC EQUATIONS

We consider a *scalar-type* perturbation in the *flat* Friedmann background. We will consider the presence of tensor-type perturbation (gravitational waves) in Sec. VI. The vector-type perturbation (rotation) is not important because it always decays in the expanding phase even to the second order; see Sec. VII E of Ref. [10]. Our reason for considering the flat background will be explained below Eq. (4). As the metric we take

$$ds^2 = -a^2(1+2\alpha)d\eta^2 - 2a^2\beta_{,\alpha}d\eta dx^\alpha + a^2[g^{(3)}_{\alpha\beta}(1+2\varphi) + 2\gamma_{,\alpha|\beta}]dx^\alpha dx^\beta, \quad (1)$$

which follows from our convention in Eqs. (49), (175), and (178) of Ref. [10]. Here $a(t)$ is the scale factor, and α, β, γ, and φ are spacetime dependent perturbed-order variables; we take Bardeen's metric convention in Ref. [9] extended to the second order. A vertical bar indicates a covariant derivative based on $g^{(3)}_{\alpha\beta}$, which becomes $\delta_{\alpha\beta}$ if we take Cartesian coordinates in the flat Friedmann background. By taking $\gamma \equiv 0$, which we call the spatial C gauge, the spatial gauge mode is removed completely; thus, all the remaining variables we are using are spatially gauge-invariant to the second order; see Sec. VI B 2 of Ref. [10]. In the following, we will take $\gamma \equiv 0$ as the spatial gauge condition and use $\chi \equiv a\beta + a^2\dot{\gamma}$, which becomes $\chi = a\beta$.

As the energy-momentum tensor, we take

$$\begin{aligned}\tilde{T}^0_0 &= -\mu - \delta\mu + \frac{1}{a}\mu\chi^{,\alpha}v_{,\alpha},\\ \tilde{T}^0_\alpha &= -\mu(1-\alpha)v_{,\alpha}, \\ \tilde{T}^\alpha_\beta &= \delta p \delta^\alpha_\beta + \frac{1}{a^2}\left(\Pi^{,\alpha}{}_{|\beta} - \frac{1}{3}\delta^\alpha_\beta\Delta\Pi\right) - \frac{1}{a}\mu\chi^{,\alpha}v_{,\beta},\end{aligned} \quad (2)$$

which follows from our convention in Eqs. (84), (175), and (178) of Ref. [10]; tildes indicate the covariant quantities. Here μ is the background energy density, and $\delta\mu$, δp, Π, and v are the perturbed-order energy density, isotropic pressure, anisotropic pressure, and the flux, respectively, all based on the normal-frame vector $\tilde{n}_a$, with $\tilde{n}_\alpha \equiv 0$. Although we are considering a zero-pressure system (thus, $p = 0$ and $\delta p = 0 = \Pi$ to the linear order), it is essential to keep the perturbed pressure terms δp and Π, because these do not necessarily vanish to the second order in perturbation depending on the coordinate (gauge) condition we choose. This is because in Ref. [10] we have evaluated the fluid quantities based on the normal frame $\tilde{n}_a$; we will elaborate this point in Sec. III.

To the background order, we have the Friedmann equation [1,3,13]

$$H^2 = \frac{8\pi G}{3}\mu - \frac{\text{const.}}{a^2} + \frac{\Lambda}{3}, \quad (3)$$

with the energy (mass) density μ (ϱ) $\propto a^{-3}$; Λ is the cosmological constant. To the linear-order perturbations we have a second-order differential equation originally derived by Lifshitz [2,4]

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\mu\delta = 0. \quad (4)$$

An overdot indicates a time derivative based on t ($dt \equiv ad\eta$) and $H \equiv \frac{\dot{a}}{a}$. The variable $a(t)$ is the scale factor, and $\delta \equiv \frac{\delta\mu}{\mu} = \frac{\delta\varrho}{\varrho}$, with μ (ϱ) and $\delta\mu$ ($\delta\varrho$) the background and perturbed parts, respectively, of the energy (mass) density field. The "const." part is interpreted as the spatial curvature in Einstein's gravity and the total energy in Newton's gravity [13]. Equation (4) is valid even in the presence of the cosmological constant Λ as well as the background curvature. We will include the Λ term in the following. In the relativistic context, Eq. (4) can be derived in the comoving gauge condition; the original derivation by Lifshitz is based on the synchronous gauge, and in the zero-pressure medium to the linear order the synchronous gauge coincides with the comoving gauge: Further discussion about this point will be made in Sec. IV. Although Eq. (4) is also valid with general spatial curvature, the relativistic-Newtonian correspondence is somewhat *ambiguous* in the case with curvature; for details, see Sec. 3 of Ref. [11]. Thus, we consider the flat background only.

The perturbed parts of equations to the second order are presented in Eqs. (195)–(201) of Ref. [10]. In a flat background with vanishing background pressure, we have

$$\kappa - 3H\alpha + 3\dot{\varphi} + \frac{\Delta}{a^2}\chi = N_0, \quad (5)$$

$$4\pi G\delta\mu + H\kappa + \frac{\Delta}{a^2}\varphi = \frac{1}{4}N_1, \quad (6)$$

$$\kappa + \frac{\Delta}{a^2}\chi - 12\pi G\mu a v = N_2^{(s)}, \quad (7)$$

$$\dot{\kappa} + 2H\kappa - 4\pi G(\delta\mu + 3\delta p) + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\alpha = N_3, \quad (8)$$

$$\dot{\chi} + H\chi - \varphi - \alpha - 8\pi G\Pi = N_4^{(s)}, \quad (9)$$

$$\delta\dot{\mu} + 3H(\delta\mu + \delta p) - \mu\left(\kappa - 3H\alpha + \frac{\Delta}{a}v\right) = N_5, \quad (10)$$

$$\frac{(a^4\mu v)^{\cdot}}{a^4\mu} - \frac{1}{a}\alpha - \frac{1}{a\mu}\left(\delta p + \frac{2}{3}\frac{\Delta}{a^2}\Pi\right) = N_6^{(s)}, \qquad (11)$$

where the pure quadratic-order terms N_i can be read from Eqs. (99)–(105) in Ref. [10]. Δ is a Laplacian operator. Equation (5) is a definition of κ, Eqs. (6)–(9) follow from $\tilde{G}^0_0$, $\tilde{G}^0_\alpha$, $\tilde{G}^\alpha_\alpha - \tilde{G}^0_0$, and $\tilde{G}^\alpha_\beta - \frac{1}{3}\delta^\alpha_\beta \tilde{G}^\gamma_\gamma$ components of Einstein's equation, respectively, and Eqs. (10) and (11) follow from $\tilde{T}^b_{0;b} = 0$ and $\tilde{T}^b_{\alpha;b} = 0$, respectively. To the linear order this set of equations without fixing the temporal gauge was presented by Bardeen in Ref. [9]. All our equations *include* the cosmological constant in the background. These equations are presented without fixing the temporal gauge condition and using only the spatially gauge-invariant variables even to the second order; our choice of the spatial C gauge ($\gamma \equiv 0$) guarantees such invariances of the remaining variables; see Sec. VI B of Ref. [10]. As the proper temporal gauge condition, we can choose any of the following: $\alpha \equiv 0$ (the synchronous gauge), $\chi \equiv 0$ (the zero-shear gauge), $\delta \equiv 0$ (the uniform-density gauge), $\kappa \equiv 0$ (the uniform-expansion gauge), $v \equiv 0$ (the comoving gauge), $\varphi \equiv 0$ (the uniform-curvature gauge), etc. Except for the synchronous gauge, each of the other temporal gauge conditions completely removes the temporal gauge mode. We can also take linear combinations of the above conditions and choose different gauge conditions to different order; see Sec. VI C 2 of Ref. [10]. Thus, we have an infinite number of different temporal gauge choices available to each order in perturbations.

From Eqs. (5)–(11), we can derive the following set of equations expressed using gauge-invariant variables:

$$\alpha_v = -\frac{1}{2}v_\chi{}^{,\alpha}v_{\chi,\alpha} - \frac{1}{\mu}\left(\delta p_v + \frac{2}{3}\frac{\Delta}{a^2}\Pi_v\right), \qquad (12)$$

$$\dot{\delta}_v - \kappa_v = \frac{1}{a}(\delta_v v_\chi{}^{,\alpha})_{,\alpha} - 3\frac{H}{\mu}\delta p_v, \qquad (13)$$

$$\dot{\kappa}_v + 2H\kappa_v - 4\pi G\mu\delta_v = \frac{\Delta}{2a^2}(v_\chi{}^{,\alpha}v_{\chi,\alpha}) + 12\pi G\delta p_v, \qquad (14)$$

$$\begin{aligned}\kappa_v - \frac{\Delta}{a}v_\chi = \frac{1}{a}(v_\chi\Delta\varphi_\chi - 2\varphi_\chi\Delta v_\chi + \varphi_\chi{}^{,\alpha}v_{\chi,\alpha}) \\ + \frac{5}{2}H(2v_\chi\Delta v_\chi + v_\chi{}^{,\alpha}v_{\chi,\alpha}) \\ - \frac{1}{a}\nabla^\alpha(\delta_v v_{\chi,\alpha}) - \frac{3}{a}\Delta^{-1}\nabla^\alpha(v_{\chi,\alpha}\Delta\varphi_\chi),\end{aligned} \qquad (15)$$

$$\begin{aligned}\alpha_\chi + \varphi_\chi = \varphi_\chi^2 - \Delta^{-1}(\varphi_\chi\Delta\varphi_\chi) + 3\Delta^{-2}\nabla^\alpha\nabla^\beta(\varphi_\chi\varphi_{\chi,\alpha\beta}) \\ - 8\pi G\Pi_\chi,\end{aligned} \qquad (16)$$

$$\begin{aligned}4\pi G\mu\delta_v + \frac{\Delta}{a^2}\varphi_\chi = \frac{1}{2}\dot{H}\Delta v_\chi^2 - 3aH\dot{H}\Delta^{-1}\nabla^\alpha(\delta_{v,\alpha}v_\chi) \\ + \frac{1}{a^2}\left(4\varphi_\chi\Delta\varphi_\chi + \frac{3}{2}\varphi_\chi{}^{,\alpha}\varphi_{\chi,\alpha}\right),\end{aligned} \qquad (17)$$

$$\begin{aligned}\dot{v}_\chi + Hv_\chi - \frac{1}{a}\alpha_\chi = -\frac{3}{2}a\dot{H}v_\chi^2 + 3H\varphi_\chi v_\chi - \frac{1}{2a}\varphi_\chi^2 \\ - \frac{1}{a}\Delta^{-1}\nabla^\alpha(\delta_v\varphi_{\chi,\alpha}) \\ + \frac{1}{a\mu}\left(\delta p_\chi + \frac{2}{3}\frac{\Delta}{a^2}\Pi_\chi\right),\end{aligned} \qquad (18)$$

$$\dot{\varphi}_\chi - H\alpha_\chi + 4\pi G\mu a v_\chi = \varphi_\chi\left(\dot{\varphi}_\chi - \frac{3}{2}H\varphi_\chi\right), \qquad (19)$$

$$\dot{\varphi}_v = \frac{1}{2a}\Delta^{-1}\nabla^\alpha(v_\chi{}^{,\beta}\varphi_{v,\alpha\beta} + v_{\chi,\alpha}\Delta\varphi_v). \qquad (20)$$

Equations (12)–(15) follow from Eqs. (11), (10), (8), and (7), respectively, evaluated in the comoving gauge. In Eq. (15) we used $\chi_v + av_\chi = \chi_v^{(q)} + av_\chi^{(q)}$ and $\chi_v^{(q)}|_v = 0$; see Sec. VI C 2 of Ref. [10]. Equation (16) follows from Eq. (9) evaluated in the zero-shear gauge. Equation (17) follows from Eqs. (6) and (7) and using $\delta\mu_v \equiv \delta\mu - \dot{\mu}av + \delta\mu_v^{(q)}$, $\varphi_\chi \equiv \varphi - H\chi + \varphi_\chi^{(q)}$, and $\varphi_\chi^{(q)}|_\chi = 0$. Equation (18) follows from Eq. (11) evaluated in the zero-shear gauge. Equation (19) follows from Eqs. (5) and (7), removing the κ term and evaluating in the zero-shear gauge. Equation (20) follows from Eqs. (5) and (7), removing the κ term and evaluating in the comoving gauge. In this set of equations, we located the pure quadratic terms and the possible second-order pressure terms on the right-hand sides.

Our notation with a perturbed-order variable as a subindex, for example, δ_v, indicates a unique gauge-invariant combination of δ and v which becomes δ under the comoving gauge condition $v = 0$. Thus, δ in the comoving gauge is *equivalent* to a unique gauge-invariant combination δ_v. To the linear order we have $\delta_v \equiv \delta - a(\dot{\mu}/\mu)v$. An explicit form of δ_v to the second order and other gauge-invariant combinations can be found in Eqs. (280)–(284) of Ref. [10]. As we can construct many (in fact, infinitely many) gauge-invariant combinations for δ, our notation apparently has the advantage of showing explicitly which gauge-invariant combination we are considering [14].

Here we briefly discuss a conserved variable to the second order. From Eqs. (20), (18), and (16) we have

$$\frac{1}{a^3}(a^3\dot{\varphi}_v)^{\cdot} = -\frac{1}{2a^2}\Delta^{-1}\nabla^\alpha\nabla^\beta(\varphi_{v,\alpha}\varphi_{v,\beta}). \qquad (21)$$

To the linear order we have

$$\varphi_v = C(\mathbf{x}). \qquad (22)$$

Thus, φ_v remains constant in time. In the large-scale limit (superhorizon scale), ignoring the quadratic-order spatial gradient terms, Eq. (22) remains valid even to the second order; for a more general proof considering the pressure term, see [10,15].

III. ISSUE OF PRESSURE

Now we discuss the role of pressure terms in a medium without pressure. From Eqs. (233) and (235) of Ref. [10], we notice that the gauge (coordinate) transformation to the second order causes pressure (both isotropic and anisotropic) terms to appear even in the case without pressure originally (physically). Such a complication occurs because our fluid quantities introduced in Ref. [10] are based on the normal-frame four-vector $\tilde{n}_a$, which differs from the fluid four-vector $\tilde{u}_a$. In Ref. [10] we have presented the fluid quantities based on $\tilde{u}_a$ separately as well; see Eqs. (87) and (88) of Ref. [10]; by using these equations we can translate fluid quantities in the normal frame to the ones in the fluid frame, and vice versa; the gauge transformation properties of the fluid quantities in the fluid frame are presented in Eq. (238) of Ref. [10]. The isotropic and anisotropic pressures are gauge (coordinate) dependent quantities. To the linear order in the Friedmann background, the anisotropic pressure is gauge-invariant and the perturbed isotropic pressure depends on the coordinate only if we have nonvanishing (and time varying) background pressure. In the normal frame, the pure coordinate transformation to the second and higher orders will cause both pressures (i.e., isotropic and anisotropic pressure like terms in the energy-momentum tensor) generated even in the case of vanishing pressures to the background and to the linear order; see Eq. (233) of Ref. [10]; the frame dependence of fluid quantities was studied in Ref. [16]. This complication does not occur for the fluid quantities based on the fluid frame vector $\tilde{u}_a$; see Eq. (238) in Ref. [10].

For vanishing pressure terms in the background and first-order perturbations, we have the following gauge-invariant combinations of pressure terms (based on $\tilde{n}_a$) [17]:

$$\begin{aligned}\delta p_v &= \delta p - \tfrac{1}{3}\mu v^{,\alpha} v_{,\alpha},\\ \Pi_v &= \Pi - \tfrac{3}{2}\mu a^2 \Delta^{-2}\nabla^\alpha\nabla^\beta\left(v_{,\alpha}v_{,\beta} - \tfrac{1}{3}g^{(3)}_{\alpha\beta}v^{,\gamma}v_{,\gamma}\right).\end{aligned} \tag{23}$$

From this we notice that the gauge-invariant combination δp_v is the same as δp in the comoving gauge. Evaluating Eq. (23) in the zero-shear gauge ($\chi \equiv 0$) and using $v_\chi \equiv v - \frac{1}{a}\chi$ to the linear order, we have

$$\begin{aligned}\delta p_\chi &= \delta p_v + \tfrac{1}{3}\mu v_\chi{}^{,\alpha} v_{\chi,\alpha},\\ \Pi_\chi &= \Pi_v + \tfrac{3}{2}a^2\mu \Delta^{-2}\nabla^\alpha\nabla^\beta\left(v_{\chi,\alpha}v_{\chi,\beta} - \tfrac{1}{3}g^{(3)}_{\alpha\beta}v_\chi{}^{,\gamma}v_{\chi,\gamma}\right).\end{aligned} \tag{24}$$

As the definition of fluid without pressure, we *set* the pressure terms in the comoving gauge equal to be zero; thus,

$$\delta p_v \equiv 0 \equiv \Pi_v, \tag{25}$$

which are gauge-invariant (and physical) zero-pressure conditions. Thus,

$$\begin{aligned}\delta p_\chi &= \tfrac{1}{3}\mu v_\chi{}^{,\alpha} v_{\chi,\alpha},\\ \Pi_\chi &= \tfrac{3}{2}a^2\mu \Delta^{-2}\nabla^\alpha\nabla^\beta\left(v_{\chi,\alpha}v_{\chi,\beta} - \tfrac{1}{3}g^{(3)}_{\alpha\beta}v_\chi{}^{,\gamma}v_{\chi,\gamma}\right).\end{aligned} \tag{26}$$

We set the pressure terms using Eqs. (25) and (26). Thus, for fluid quantities based on the normal frame, in the gauge other than the comoving gauge the physical zero-pressure condition implies presence of pressure terms in the definition of the energy-momentum tensor.

In the comoving gauge without rotation, the two frames $\tilde{u}_a$ and $\tilde{n}_a$ coincide. The normal frame $\tilde{n}_a$ has $\tilde{n}_\alpha \equiv 0$. The fluid quantities are ordinarily defined in the fluid ($\tilde{u}_a$) frame, which differs in general from the normal four-vector $\tilde{n}_a$. In the normal frame information about the fluid motion is present in the flux four-vector $\tilde{q}_a$, with $\tilde{q}_a\tilde{n}^a \equiv 0$. In the energy frame, which takes vanishing flux $\tilde{q}_a \equiv 0$ as the frame condition, the comoving gauge condition takes $\tilde{u}_\alpha \equiv 0$ for the fluid four-vector; here we ignore the vector-type perturbation. Since $\tilde{u}_\alpha = 0$, it coincides with the normal-frame vector. Now in the normal frame, which takes $\tilde{n}_\alpha \equiv 0$ as the frame condition, the comoving gauge condition without rotation implies $\tilde{q}_a \equiv 0$. Thus, as long as we take the comoving gauge without rotation, in either frame we have $\tilde{q}_a \equiv 0$ and $\tilde{u}_\alpha = 0 = \tilde{n}_\alpha$; i.e., the fluid four-vector coincides with the normal four-vector.

IV. A PROOF

Now we come to our main point proving the relativistic-Newtonian correspondence to the second order. Combining Eqs. (13) and (14), we can derive [18]

$$\ddot{\delta}_v + 2H\dot{\delta}_v - 4\pi G\mu\delta_v = \frac{1}{a^2}\frac{\partial}{\partial t}\left[a(\delta_v v_\chi{}^{,\alpha})_{,\alpha}\right] + \frac{\Delta}{2a^2}(v_\chi{}^{,\alpha}v_{\chi,\alpha}). \tag{27}$$

Equations (13), (14), (17), and (27) can be compared with the Newtonian perturbation equations.

The mass conservation, the momentum conservation, and the Poisson's equation in Newtonian context give [19]

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u}), \tag{28}$$

$$\dot{\mathbf{u}} + H\mathbf{u} + \frac{1}{a}\nabla\delta\Phi = -\frac{1}{a}\mathbf{u}\cdot\nabla\mathbf{u}, \tag{29}$$

$$\frac{1}{a^2}\nabla^2\delta\Phi = 4\pi G\varrho\delta. \tag{30}$$

From these we have

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\varrho\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}[a\nabla\cdot(\delta\mathbf{u})] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}). \quad (31)$$

In the Newtonian context, Eqs. (28)–(31) are valid to fully nonlinear order; i.e., the zero-pressure Newtonian fluid equations are exact in quadratic-order nonlinearity. Equation (31) has been analyzed extensively in the Newtonian context; see [20,21].

To the *linear order* it is well known that δ_v, $-\nabla v_\chi$, and $-\varphi_\chi$ (or α_χ) correspond to a density perturbation ($\delta \equiv \frac{\delta\varrho}{\varrho}$, with $\tilde{\varrho} \equiv \varrho + \delta\varrho$ and $\tilde{\varrho}$ the mass density), a velocity perturbation ($\mathbf{u}$), and a perturbation of the gravitational potential ($\delta\Phi$), respectively [7,11,12]. To the linear order we may identify [11]

$$\delta = \delta_v, \qquad \delta\Phi = -\varphi_\chi = \alpha_\chi, \qquad \mathbf{u} \equiv -\nabla v_\chi, \qquad -\frac{1}{a}\nabla\cdot\mathbf{u} = \frac{\Delta}{a}v_\chi = \kappa_v. \quad (32)$$

As we *identify*

$$\delta_v = \delta, \qquad \kappa_v \equiv -\frac{1}{a}\nabla\cdot\mathbf{u}, \quad (33)$$

to the *second order*, Eq. (27) coincides exactly with Eq. (31). Equation (13) becomes

$$\dot{\delta}_v + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta_v\mathbf{u}), \quad (34)$$

which coincides with Eq. (28). Equation (14) gives

$$\nabla\cdot(\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\mu a\delta_v = -\frac{1}{a}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}), \quad (35)$$

which also follows from Eqs. (29) and (30) in the Newtonian context. This completes our proof of the correspondence. Such identifications of density and velocity perturbations imply that we *cannot* identify $-\varphi_\chi$ (or α_χ) with $\delta\Phi$ to the second order. This conclusion follows from a close examination of Eqs. (12)–(20). In fact, using the intrinsic three-space curvature in Eq. (265) of Ref. [10]

$$R^{(h)} = \frac{2}{a^2}[-2\Delta\varphi + 8\varphi\Delta\varphi + 3\varphi^{,\alpha}\varphi_{,\alpha}], \quad (36)$$

Eq. (17) becomes

$$4\pi G\mu\delta_v - \tfrac{1}{4}R^{(h)}_\chi = \tfrac{1}{2}\dot{H}\Delta v_\chi^2 - 3aH\dot{H}\Delta^{-1}\nabla^\alpha(\delta_{v,\alpha}v_\chi), \quad (37)$$

which still differs from the Newtonian Poisson's equation. Thus, we conclude that we do not have a relativistic variable which corresponds to the Newtonian potential to the second order. Apparently, it is essentially important to employ mixed gauge conditions, i.e., take different gauge conditions for different variables, to make correspondence with the Newtonian system: In this way, correct identifications of (gauge-invariant) variables are important to show the relativistic-Newtonian correspondence.

At this point, let us clarify the meaning of the quantities involved in Eqs. (32) and (33). Variables α, χ, and φ are defined in the metric in Eq. (1). Variables χ and φ can be further identified as the perturbed shear and perturbed three-space curvature of the normal hypersurface, respectively. From Eq. (36) we find that the intrinsic scalar curvature $R^{(h)}$ vanishes for $\varphi = 0$. From Eq. (264) of Ref. [10] we find that the trace-free part of the extrinsic curvature tensor $\bar{K}_{\alpha\beta}$ (equivalently, shear tensor of the normal-frame vector with a minus sign) vanishes for $\chi = 0$. The variable κ can be interpreted as the perturbed expansion with a minus sign. From Eqs. (57), (99), and (179) of Ref. [10] we have $K = -3H + \kappa$, where K is a trace of the extrinsic curvature tensor $K_{\alpha\beta}$ (equivalently, the expansion scalar $\tilde{\theta} \equiv \tilde{n}^a{}_{;a}$ with a minus sign). Variables δ and v are defined in Eq. (2) and can be interpreted as the perturbed energy density ($\delta \equiv \frac{\delta\mu}{\mu}$, with $\tilde{\mu} = \mu + \delta\mu$) and the flux of the normal frame, respectively. In the normal frame, from Eqs. (4), (76), and (175) of Ref. [10], we have that the flux vector becomes $J_\alpha \equiv -\tilde{n}_b\tilde{T}^b_\alpha = -a\mu v_{,\alpha}$.

Here we discuss the relation between the comoving and the synchronous gauge to the second order. Equation (12) shows that α_v, which is the same as α in the comoving gauge ($v \equiv 0$), does not vanish to the second order. This means that the comoving gauge does not imply our synchronous gauge to the second order in a zero-pressure medium. At this point, it is important to remember that we already have fixed the spatial gauge condition using $\gamma \equiv 0$. The original synchronous gauge used by Lifshitz fixes $\delta g_{00} \equiv 0 \equiv \delta g_{0\alpha}$; thus, $\alpha \equiv 0$ for the temporal gauge and $\beta \equiv 0$ for the spatial gauge condition. We prefer to fix $\gamma \equiv 0$ (spatial C gauge) as the spatial gauge condition instead of $\beta \equiv 0$ (spatial B gauge) because the latter condition fails to fix the spatial gauge degree of freedom completely, whereas the first one fixes it completely; this is true even to the second order and, in fact, to all orders, in perturbations; see Secs. VI B 2 and VI C of Ref. [10]. We can show that the comoving temporal gauge ($v \equiv 0$) together with spatial B gauge ($\beta = 0$) implies $\alpha = 0$ even to the second order; for a proof, see [22]. By imposing the comoving ($v \equiv 0$) and the synchronous ($\alpha \equiv 0$) gauge conditions simultaneously, Kasai [23] has derived a different equation compared with ours: Such a redundant choice is allowed as one takes $\beta = 0$ as the spatial gauge condition. However, in that gauge condition (the spatial B gauge), the spatial gauge mode is incompletely fixed, and the comparison with the Newtonian analyses is *not* available.

V. FULLY NONLINEAR EQUATIONS

By extending our comoving gauge condition to be valid to all orders, we can formally derive the *completely nonlinear* equations for the density and velocity perturbations. We will present two methods to reach such nonlinear equations. These are based on the Arnowitt-Deser-Misner (ADM) (3 + 1) equations and the covariant (1 + 3) equations summarized in Secs. II A and II B, respectively, of Ref. [10]. With the hindsight gained from our second-order perturbations in previous sections, it is best to take the comoving gauge condition to all orders. In the normal-frame context, only the comoving gauge allows the zero-pressure conditions to be, by definition, vanishing pressure terms to all orders. To the second order, all the equations we need to derive Eqs. (27), (34), and (35) are Eqs. (12)–(14), which follow from Eqs. (8), (10), and (11); these are the Raychaudhury, the energy-conservation, and the momentum-conservation equations, respectively. We have presented a redundant set of equations in (12)–(20) in order to show the relativistic-Newtonian correspondences with some perspective.

The complete set of ADM (3 + 1) equations is presented in Eqs. (8)–(13) of Ref. [10]; see [24] for original studies. We only need Eqs. (10), (12), and (13) of Ref. [10], which are the trace of the ADM propagation equation, and the energy- and momentum-conservation equations, respectively. We take the comoving gauge condition to all orders which makes the flux four-vector vanish; i.e., $J_\alpha \equiv 0$; here we *assume* vanishing vector-type perturbation, thus irrotational, which could contribute to J_α. Under such conditions, the zero-pressure conditions (in our normal frame) imply $S \equiv 0 \equiv \bar{S}_{\alpha\beta}$ to all orders; S and $\bar{S}_{\alpha\beta}$ are the trace and trace-free parts, respectively, of the spatial part of energy-momentum tensor. Equation (13) of Ref. [10] gives

$$N_{,\alpha} = 0, \tag{38}$$

where N is defined as $\tilde{g}^{00} \equiv -N^{-2}$. Thus, we may set $N \equiv a(t)$ to all orders. In this case we have, for example, $\dot{E} \equiv E_{,0}N^{-1}$. Now, Eqs. (12) and (10) of Ref. [10] become, respectively,

$$\hat{\dot{E}} - KE = 0, \tag{39}$$

$$\hat{\dot{K}} - \tfrac{1}{3}K^2 - \bar{K}^{\alpha\beta}\bar{K}_{\alpha\beta} - 4\pi GE + \Lambda = 0, \tag{40}$$

where $\hat{\dot{E}} \equiv \dot{E} - E_{,\alpha}N^\alpha N^{-1}$, etc.; E is the energy density based on the normal-frame vector, and K and $\bar{K}_{\alpha\beta}$ are the trace and trace-free parts, respectively, of the extrinsic curvature; N_α is defined as $\tilde{g}_{0\alpha} \equiv N_\alpha$. The spatial indices in ADM formulation are based on the spatial metric $h_{\alpha\beta}$ defined as $h_{\alpha\beta} \equiv \tilde{g}_{\alpha\beta}$. By combining these equations, we have

$$\left(\frac{\hat{\dot{E}}}{E}\right)^{\hat{\cdot}} - \frac{1}{3}\left(\frac{\hat{\dot{E}}}{E}\right)^2 - \bar{K}^{\alpha\beta}\bar{K}_{\alpha\beta} - 4\pi GE + \Lambda = 0. \tag{41}$$

Notice again that Eqs. (39)–(41) are valid to all orders; i.e., these equations are fully nonlinear. From Eqs. (39)–(41), using

$$E \equiv \mu + \delta\mu \tag{42}$$

and the quantities presented in Ref. [10], we can easily derive Eqs. (34), (35), and (27), respectively; see the next section.

The complete set of covariant (1 + 3) equations is presented in Eqs. (26)–(37) of Ref. [10]; see [25] for original studies. We need only Eqs. (26)–(28) of Ref. [10], which are the energy and momentum conservations and the Raychaudhury equation, respectively. We take the energy frame which sets the energy flux term to vanish, i.e., $\tilde{q}_a \equiv 0$. In this frame the frame four-vector $\tilde{u}_a$ is the fluid four-vector. The zero-pressure conditions imply $\tilde{p} \equiv 0 \equiv \tilde{\pi}_{ab}$ to all orders; $\tilde{\pi}_{ab}$ is the covariant anisotropic stress based on $\tilde{u}_a$. Equation (27) of Ref. [10] gives a vanishing acceleration vector, i.e., $\tilde{a}_a \equiv \tilde{u}_{a;b}\tilde{u}^b = 0$, to all orders. Thus, Eqs. (26) and (28) of Ref. [10] become, respectively,

$$\tilde{\dot{\tilde{\mu}}} + \tilde{\mu}\,\tilde{\theta} = 0, \tag{43}$$

$$\tilde{\dot{\tilde{\theta}}} + \tfrac{1}{3}\tilde{\theta}^2 + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} - \tilde{\omega}^{ab}\tilde{\omega}_{ab} + 4\pi G\tilde{\mu} - \Lambda = 0, \tag{44}$$

where $\tilde{\theta} \equiv \tilde{u}^a{}_{;a}$ is an expansion scalar, and $\tilde{\sigma}_{ab}$ is the shear tensor. An overdot with tilde is a covariant derivative along the $\tilde{u}_a$ vector, e.g., $\tilde{\dot{\tilde{\mu}}} \equiv \tilde{\mu}_{,a}\tilde{u}^a$. By combining these equations, we have

$$\left(\frac{\tilde{\dot{\tilde{\mu}}}}{\tilde{\mu}}\right)^{\tilde{\cdot}} - \frac{1}{3}\left(\frac{\tilde{\dot{\tilde{\mu}}}}{\tilde{\mu}}\right)^2 - \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} + \tilde{\omega}^{ab}\tilde{\omega}_{ab} - 4\pi G\tilde{\mu} + \Lambda = 0. \tag{45}$$

Notice that Eqs. (43)–(45) are valid to all orders; i.e., these equations are fully nonlinear. A more general equation in a fully covariant form considering the general pressure terms can be found in Eq. (88) of Ref. [26].

We take the comoving gauge condition to all orders, which makes the space part of four-vector with low index vanish, i.e., $\tilde{u}_\alpha \equiv 0$; here we also assume vanishing vector-type perturbation, thus irrotational, which could contribute to $\tilde{u}_\alpha$. As our gauge condition (and the irrotational condition) implies $\tilde{u}_\alpha \equiv 0$, the frame vector is the same as the normal frame; thus, $\tilde{u}_a = \tilde{n}_a$. In such a case we have vanishing rotation of the $\tilde{u}_a$ flow; thus, $\tilde{\omega}_{ab} = 0$. From Eqs. (43)–(45), using

$$\tilde{\mu} \equiv \mu + \delta\mu \tag{46}$$

and the quantities presented in Ref. [10], we can easily derive Eqs. (34), (35), and (27), respectively. A derivation based on the covariant equations is presented in Ref. [27].

After all, the ADM equations (39)–(41) are the same as the covariant equations (43)–(45), expressed in different forms. We can derive the ADM equations by rewriting the covariant equations in the normal-frame vector. Since our comoving gauge condition with irrotational condition makes $\tilde{u}_\alpha \equiv 0$, the frame vector is the same as the normal-frame vector $\tilde{n}_a$. By direct calculations, using the quantities presented in Eqs. (2)–(6) and (14)–(16) of Ref. [10], we can show that

$$\tilde{\mu} = E, \qquad \tilde{\theta} = -K, \qquad \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} = \bar{K}^{\alpha\beta}\bar{K}_{\alpha\beta}, \qquad \tilde{\dot{\tilde{\mu}}} = \hat{E}, \qquad \tilde{\dot{\tilde{\theta}}} = -\hat{K}. \tag{47}$$

Using this, we can show that Eqs. (43)–(45) give Eqs. (39)–(41); these equations are valid considering general background curvature and the tensor-type perturbation (gravitational waves) to all orders.

VI. ANOTHER DERIVATION INCLUDING THE GRAVITATIONAL WAVES

Since Eqs. (34) and (35) are our main results allowing us to conclude about the relativistic-Newtonian correspondence, in the following we will derive these equations in some detail again directly from the fully nonlinear equations in Sec. V. Now we include the gravitational wave contribution. The metric becomes

$$ds^2 = -a^2(1+2\alpha)d\eta^2 - 2a\chi_{,\alpha}d\eta dx^\alpha + a^2[(1+2\varphi)\delta_{\alpha\beta} + 2C^{(t)}_{\alpha\beta}]dx^\alpha dx^\beta, \tag{48}$$

where $C^{(t)}_{\alpha\beta}$ is the transverse and trace-free gravitational waves; its indices are based on $g^{(3)}_{\alpha\beta}$. We work in the temporal comoving gauge. Thus, $C^{(t)}_{\alpha\beta}$ is also evaluated in the comoving gauge and equivalent to a gauge-invariant combination $C^{(t)}_{\alpha\beta v}$.

We introduce

$$E \equiv \mu + \delta\mu, \qquad K \equiv -3\frac{\dot{a}}{a} + \kappa; \tag{49}$$

see Eqs. (45), (72), (178), and (179) of Ref. [10]. We have

$$\hat{E} \equiv \dot{E} - E_{,\alpha}N^\alpha N^{-1} = \dot{\mu} + \delta\dot{\mu} + \frac{1}{a^2}\delta\mu_{,\alpha}\chi^{,\alpha}, \qquad \hat{K} \equiv \dot{K} - K_{,\alpha}N^\alpha N^{-1} = -3\left(\frac{\dot{a}}{a}\right)^{\cdot} + \dot{\kappa} + \frac{1}{a^2}\kappa_{,\alpha}\chi^{,\alpha}. \tag{50}$$

In setting $N = a$, we already have used the comoving gauge condition. Since we take the comoving gauge, we often ignore the subindex v, which indicates the comoving gauge choice (equivalently, the unique corresponding gauge-invariant combination between the variable and v); for example, our δ is the same as a gauge-invariant combination δ_v, which is the same as δ in the comoving gauge setting $v \equiv 0$. Using Eqs. (55), (57), and (175) of Ref. [10], we can show

$$\bar{K}^{\alpha\beta}\bar{K}_{\alpha\beta} = \frac{1}{a^4}\left[\chi^{,\alpha|\beta}\chi_{,\alpha|\beta} - \frac{1}{3}(\Delta\chi)^2\right] + \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a^2}\chi_{,\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right). \tag{51}$$

Equations (39) and (40) become, respectively,

$$\left(\frac{\dot{\mu}}{\mu} + 3\frac{\dot{a}}{a}\right)(1+\delta) + \dot{\delta} - \kappa = \kappa\delta - \frac{1}{a^2}\delta_{,\alpha}\chi^{,\alpha}, \tag{52}$$

$$3\frac{\ddot{a}}{a} + 4\pi G\mu - \Lambda - \dot{\kappa} - 2\frac{\dot{a}}{a}\kappa + 4\pi G\mu\delta = \frac{1}{a^2}\kappa_{,\alpha}\chi^{,\alpha} - \frac{1}{3}\kappa^2 - \frac{1}{a^4}\left[\chi^{,\alpha|\beta}\chi_{,\alpha|\beta} - \frac{1}{3}(\Delta\chi)^2\right] - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a^2}\chi_{,\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right). \tag{53}$$

Now we have to relate $\chi(\equiv \chi_v)$ to our notation. Apparently, we need χ only to the linear order. To the linear order the $\tilde{G}^0_\alpha$ component of the Einstein equation in Eq. (15) gives $(\Delta/a^2)\chi_v = -\kappa_v \equiv \frac{1}{a}\nabla\cdot\mathbf{u}$; we have $\chi_v \equiv \chi - av \equiv -av_\chi$ to the linear order. As our $\mathbf{u}$ is of the potential type, i.e., of the form $\mathbf{u} \equiv u_{,\alpha}$, we have

$$\mathbf{u} = \frac{1}{a}\nabla\chi_v, \tag{54}$$

to the linear order. Thus, we have

$$\left(\frac{\dot{\mu}}{\mu} + 3\frac{\dot{a}}{a}\right)(1+\delta_v) + \dot{\delta}_v + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta_v\mathbf{u}), \tag{55}$$

$$3\frac{\ddot{a}}{a} + 4\pi G\mu - \Lambda + \frac{1}{a}\nabla\cdot\left(\dot{\mathbf{u}} + \frac{\dot{a}}{a}\mathbf{u}\right) + 4\pi G\mu\delta_v = -\frac{1}{a^2}\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha,\beta} + \dot{C}^{(t)}_{\alpha\beta}\right). \tag{56}$$

The perturbed parts give Eqs. (34) and (35) with additional contributions from the gravitational waves in Eq. (35) and, thus, in Eq. (27) as well.

Therefore, in the presence of the tensor-type perturbation we have

$$\dot{\delta}_v + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta_v\mathbf{u}), \tag{57}$$

$$\frac{1}{a}\nabla\cdot\left(\dot{\mathbf{u}} + \frac{\dot{a}}{a}\mathbf{u}\right) + 4\pi G\mu\delta_v = -\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha,\beta} + \dot{C}^{(t)}_{\alpha\beta}\right); \tag{58}$$

thus,

$$\ddot{\delta}_v + 2\frac{\dot{a}}{a}\dot{\delta}_v - 4\pi G\mu\delta_v = -\frac{1}{a^2}\frac{\partial}{\partial t}[a\nabla\cdot(\delta_v\mathbf{u})] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha,\beta} + \dot{C}^{(t)}_{\alpha\beta}\right). \quad (59)$$

The presence of linear-order gravitational waves can generate the second-order scalar-type perturbation by generating the shear terms. Here we note the behavior of the gravitational waves in the linear regime. To the linear order the gravitational waves are described by the well known wave equation [2]

$$\ddot{C}^{(t)}_{\alpha\beta} + 3\frac{\dot{a}}{a}\dot{C}^{(t)}_{\alpha\beta} - \frac{\Delta}{a^2}C^{(t)}_{\alpha\beta} = 0. \quad (60)$$

In the superhorizon scale the nontransient mode of $C^{(t)}_{\alpha\beta}$ remains constant, thus $\dot{C}^{(t)}_{\alpha\beta} = 0$, and in the subhorizon scale, the oscillatory $C^{(t)}_{\alpha\beta}$ redshifts away, thus $C^{(t)}_{\alpha\beta} \propto a^{-1}$. Thus, we anticipate that the contribution of gravitational waves to the scalar-type perturbation is not significant to the second order.

To the second order the equation for tensor-type perturbation (gravitational waves) can be derived from Eqs. (103) and (210) of Ref. [10]. Since we are ignoring the vector-type perturbation from Eqs. (199) and (211) of Ref. [10], we have

$$\ddot{C}^{(t)}_{\alpha\beta} + 3\frac{\dot{a}}{a}\dot{C}^{(t)}_{\alpha\beta} - \frac{\Delta}{a^2}C^{(t)}_{\alpha\beta} = N_{4\alpha\beta} - \frac{3}{2}\left(\nabla_\alpha\nabla_\beta - \frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\right)\Delta^{-2}\nabla^\gamma\nabla^\delta N_{4\gamma\delta}, \quad (61)$$

where we assumed a flat background and set anisotropic stress to be zero. From Eq. (103) of Ref. [10] to the second order we have

$$\begin{aligned} N_{4\alpha\beta} = {}& \frac{1}{a^3}\left\{a^3\left[\frac{2}{a^2}(\varphi\chi_{,\alpha|\beta} + \varphi_{,(\alpha}\chi_{,\beta)}) + 2\varphi\dot{C}^{(t)}_{\alpha\beta} + \frac{2}{a^2}\chi^{,\gamma}{}_{|\beta}C^{(t)}_{\alpha\gamma} + \frac{1}{a^2}\chi^{,\gamma}(2C^{(t)}_{\gamma(\alpha|\beta)} - C^{(t)}_{\alpha\beta|\gamma}) + 2C^{(t)\gamma}_{\alpha}\dot{C}^{(t)}_{\beta\gamma}\right]\right\}^{\cdot} \\ & + \frac{1}{a^4}\chi^{,\gamma}{}_{|\alpha}\chi_{,\gamma|\beta} + \frac{1}{a^2}(\kappa\chi_{,\alpha|\beta} - 4\varphi\varphi_{,\alpha|\beta} - 3\varphi_{,\alpha}\varphi_{,\beta}) + \kappa\dot{C}^{(t)}_{\alpha\beta} + \frac{1}{a^2}[2\varphi^{,\gamma}{}_{|\alpha}C^{(t)}_{\beta\gamma} - 2\Delta\varphi C^{(t)}_{\alpha\beta} - 4\varphi\Delta C^{(t)}_{\alpha\beta} \\ & + \varphi^{,\gamma}(2C^{(t)}_{\gamma(\alpha|\beta)} - 3C^{(t)}_{\alpha\beta|\gamma}) + 2\chi^{,\gamma}{}_{|[\alpha}\dot{C}^{(t)}_{\beta]\gamma} - \chi^{,\gamma}\dot{C}^{(t)}_{\alpha\beta|\gamma} + 2C^{(t)\gamma\delta}(2C^{(t)}_{\gamma(\alpha|\beta)\delta} - C^{(t)}_{\alpha\beta|\gamma\delta} - C^{(t)}_{\gamma\delta|\alpha\beta}) - 2C^{(t)\gamma}_{\alpha}\Delta C^{(t)}_{\beta\gamma} \\ & - C^{(t)\gamma}{}_{\delta|\alpha}C^{(t)\delta}{}_{\gamma|\beta} + 4C^{(t)\gamma|\delta}_{\alpha}C^{(t)}_{\beta[\delta|\gamma]}\Big] - \frac{1}{3}g^{(3)}_{\alpha\beta}\left\{\frac{1}{a^3}\left\{a^3\left[\frac{2}{a^2}(\varphi\Delta\chi + \varphi^{,\gamma}\chi_{,\gamma}) + 2C^{(t)\gamma\delta}\left(\frac{1}{a^2}\chi_{,\gamma|\delta} + \dot{C}^{(t)}_{\gamma\delta}\right)\right]\right\}^{\cdot} \right. \\ & \left. + \frac{1}{a^4}\chi^{,\gamma|\delta}\chi_{,\gamma|\delta} + \frac{1}{a^2}[\kappa\Delta\chi - 4\varphi\Delta\varphi - 3\varphi^{,\gamma}\varphi_{,\gamma} + 2\varphi^{,\gamma|\delta}C^{(t)}_{\gamma\delta} - 4C^{(t)\gamma\delta}\Delta C^{(t)}_{\gamma\delta} + C^{(t)\gamma\delta|\epsilon}(2C^{(t)}_{\gamma\epsilon|\delta} - 3C^{(t)}_{\gamma\delta|\epsilon})]\right\}. \end{aligned} \quad (62)$$

In Eq. (62) we have ignored α and $\dot{\varphi}$ terms which are already quadratic order in the comoving gauge; see Eqs. (12) and (20). Since we are in the comoving gauge, we have $\chi = \chi_v$, $\varphi = \varphi_v$, $\kappa = \kappa_v$, and $C^{(t)}_{\alpha\beta} = C^{(t)}_{\alpha\beta v}$. Apparently, we need χ_v, κ_v, and φ_v to the linear order. We have $\kappa_v = -\frac{1}{a}\nabla\cdot\mathbf{u}$ and $\mathbf{u} = \frac{1}{a}\nabla\chi_v$. For φ_v we have

$$\varphi_v \equiv \varphi - aHv = \varphi_\chi - aHv_\chi, \quad (63)$$

where we have $\varphi_\chi = -\delta\Phi$ and $\mathbf{u} = -\nabla v_\chi$ in Eq. (32). Using these identifications, we can express the scalar-type perturbation variables in Eq. (62) in terms of the Newtonian variables.

Equations (57), (58), and (61) provide a *complete* set describing the scalar- and tensor-type perturbations to the second order in the flat Friedmann background. From these equations we can see that the linear-order scalar-type (tensor-type) perturbation works as a source for the tensor-type (scalar-type) perturbation to the second order. Such effects and the presence of the gravitational waves are purely general relativistic ones.

VII. DISCUSSION

We have shown that to the second order, ignoring the gravitational wave contribution, the zero-pressure relativistic cosmological perturbation equations can be exactly identified with the known equations in a Newtonian system; compare Eqs. (57)–(59) with Eqs. (28)–(31). More precisely, the relativistic equations can be identified with the continuity equation and the divergence of the Euler equation replacing the Newtonian gravitational potential using Poisson's equation. In order to achieve such a correspondence, we need correct identification of gauge-invariant density and velocity perturbation variables as in Eqs. (32) and (33). It is important to notice that we have *avoided* using the potential-like variable in our identification. In fact, we showed that we do *not* have a relativistic variable which corresponds to the Newtonian potential to the second order. This is understandable because the gravitational potential introduced in Poisson's equation reveals the action-at-a-distance nature and the static nature of Newton's gravity theory compared with relativistic gravity.

As a consequence, to the second order, the Newtonian hydrodynamic equations (31), (34), and (35) remain valid in *all* cosmological scales including the superhorizon scale. Although showing the equivalence of the zero-pressure relativistic scalar-type perturbation to the Newtonian ones to the second order may not be entirely surprising, it should not be so obvious, either. It might be as well that our relativistic results give relativistic correction terms appearing to the second order which become important as we approach and go beyond the horizon scale. Our results show that there are *no* such correction terms appearing to the second order, and the correspondence is exact to that order. A complementary result, showing the relativistic-Newtonian correspondence in the Newtonian limit of the post-Newtonian approach, can be found in Ref. [28]. In fact, the Newtonian hydrodynamic equations appear naturally as the zeroth-order post-Newtonian limit [29].

We note that, although we assumed a flat background, our equations are valid with the cosmological constant. Thus, these are compatible with current observations of the large-scale structure and the cosmic microwave background radiation which favor a near flat Friedmann world model with nonvanishing Λ [30]. As we consider a flat background, the ordinary Fourier analysis can be used to study the mode couplings as in the Newtonian case in Ref. [21]. Our result also may have the following important practical cosmological implication. As we have proved that the Newtonian hydrodynamic equations are valid in *all* cosmological scales to the second order, our result has an important cosmological implication that large-scale Newtonian numerical simulation can be used more reliably in the general relativistic context even as the simulation scale approaches near (and goes beyond) the horizon scale.

At this point, it is important to be reminded that we showed the relativistic-Newtonian correspondence for the density and velocity perturbations, but not for the gravitational potential. Therefore, although our result assures that one can trust cold dark matter simulations at *all* scales for the density and velocity fields, it does *not* imply that one can trust the Newtonian simulations for effects involving the gravitational potential, such as the weak gravitational lensing effects. Indeed, in order to handle the lensing effects properly, we often require an extra factor of 2, which comes from the post-Newtonian effects.

Since the Newtonian system is exact to the second order in nonlinearity, besides the gravitational wave contribution to the second and higher order, any nonvanishing third and higher order perturbation terms in the relativistic analysis can be regarded as the pure relativistic corrections. Expanding the fully nonlinear equations in (43)–(45) or (39)–(41) to third and higher order will give the potential correction terms. Our recent investigation of this important open question shows that to the third order there occur pure relativistic correction terms which are of φ_v order higher [31]. Thus, the corrections are independent of the horizon and are small; see the accompanying contribution in Ref. [31].

In this work, we have considered an irrotational single component dust in the flat background. Extending any of these assumptions could lead to situations which deserve further attention. First, it would be interesting to see up to what point the correspondence between the two theories can be extended in the nonflat case. In this way, we can identify possible relativistic effects caused by the nonflat nature of the background. Second, in this work we have ignored the vector-type perturbation because it simply decays in the expanding phase. This has to do with considering only the longitudinal part of $\mathbf{u}$ in Eqs. (35) and (58). It would be interesting to include the rotational mode to see the similarity and difference between the two gravity theories. As the realistic Newtonian simulations include the whole $\mathbf{u}$ vector as the perturbed velocity, it would be practically important to see the role of relativistic vector-type perturbation to the second order and to determine whether the relativistic effect could be important. Third, the usual cosmological simulations include the cold dark matter together with the baryon, thus a system with two components. Thus, the relativistic nonlinear perturbations of the zero-pressure but multicomponent system would be an interesting subject in practice. It is, *a priori*, unclear whether the relativistic-Newtonian correspondence would continue in such a multicomponent case. In the second and the third subjects, the comoving gauge issue should be applied with care. Fourth, the presence of a substantial amount of pressures (both isotropic and anisotropic) would lead to relativistic corrections. Even in the linear perturbation, the pressure terms cause new relativistic correction terms which are not present in the Newtonian system. Thus, including the pressure terms in relativistic second-order perturbation is interesting because most of the terms will be pure relativistic corrections. Such a formulation would be practically interesting because we anticipate the presence of strong pressure in the early Universe. All four subjects are left for future studies.

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