Remarks on a five-dimensional Kaluza-Klein theory of the massive Dirac monopole

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The Gross-Perry-Sorkin spacetime, formed by the Euclidean Taub-Newman-Unti-Tamburino space with the time trivially added, is the appropriate background of the Dirac magnetic monopole without an explicit mass term. We show that there exists a very simple five-dimensional metric of spacetimes carrying massive magnetic monopoles that is an exact solution of the vacuum Einstein equations. Moreover, the same isometry properties as the original Euclidean Taub-Newman-Unti-Tamburino space are preserved. This leads to an Abelian Kaluza-Klein theory whose metric appears as a combination between the Gross-Perry-Sorkin and Schwarzschild ones. The asymptotic motion of the scalar charged test particles is discussed, now by accounting for the mixing between the gravitational and magnetic effects.

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A special class of solutions of the Maxwell or Yang-Mills equations are the instantons and monopoles defined on appropriate flat or curved backgrounds [1–3]. A natural framework is offered by the Kaluza-Klein theories where the gauge degrees of freedom deal with specific extra-coordinates exceeding the physical spacetime. In these theories the basic problem is to find the solutions of the field equations in geometries whose global metrics should be exact solutions of the Einstein equations.

A typical example is the four-dimensional Euclidean Taub-Newman-Unti-Tamburino (NUT) space which involves the potentials of the Dirac magnetic monopole [4] and satisfies the vacuum Einstein equations [5]. Moreover, this space is hyper-Kähler having many interesting properties related to a specific hidden symmetry [6,7]. For this reason the Kählerian geometries were considered for generalizing the Dirac monopole to many dimensions [8]. Other successful generalizations were obtained by integrating the field equations in Kaluza-Klein theories with five or more dimensions [9,10]

In this way a large collection of metrics was found, including some metrics corresponding to massive Dirac monopoles with explicit mass terms. Thus the whole set of the monopole metrics can be divided in metrics of the Schwarzschild or Gross-Perry-Sorkin types [10]. In this paper we show that there exists a hybrid metric giving rise to a simple Abelian five-dimensional Kaluza-Klein theory of a monopole with gravitational mass. This metric is an asymptotic flat solution of the time-dependent vacuum Einstein equations combining Schwarzschild terms with Gross-Perry-Sorkin ones. Our purpose is to evaluate the mixing between the magnetic and gravitational effects produced by this metric in the asymptotic domain.

The Euclidean Taub-NUT manifold is the space of the Abelian Kaluza-Klein theory of the Dirac magnetic monopole that provides a nontrivial generalization of the Kepler problem. This space is a static four-dimensional manifold, $M_4 \sim \mathbb{R}^4$, equipped with the isometry group SO(3) ⊗ U(1) and carrying the Dirac magnetic monopole related to strings along the third axis. This symmetry recommends the use of local charts with spherical coordinates $(r, \theta, \varphi, \alpha)$, where the first three are the usual spherical coordinates of the vector $\vec{x} = (x^1, x^2, x^3)$, with $|\vec{x}| = r$, while α is the angular Kaluza-Klein extra-coordinate. The spherical coordinates can be associated with the Cartesian ones $(\vec{x}, y) = (x^1, x^2, x^3, y)$ where the extra-coordinate y may depend linearly on α. In Cartesian coordinates one has the opportunity to use the vector notation and the scalar products $\vec{x} \cdot \vec{x}'$, which are invariant under the SO(3) rotations.

The Euclidean Taub-NUT space has the virtue to be Ricci flat, its metric being an exact solution of the vacuum Einstein equations [5]. In the Cartesian charts $(\vec{x}, y^{\pm})$ the line elements read

$$ds_{\pm}^2 = G(r)d\vec{x} \cdot d\vec{x} + G(r)^{-1}(dy^{\pm} + \vec{A}^{\pm} \cdot d\vec{x})^2, \tag{1}$$

where

$$G(r) = 1 + \frac{\mu}{r}. \tag{2}$$

The vector potential of the Dirac magnetic monopole produced by a string along the negative third semiaxis is denoted by $\vec{A}^+$ while $\vec{A}^-$ is due to a string along the positive one. These potentials have the components

$$A_1^{\pm} = \mp \frac{\mu x^2}{r(r \pm x^3)}, \qquad A_2^{\pm} = \pm \frac{\mu x^1}{r(r \pm x^3)}, \qquad A_3^{\pm} = 0. \tag{3}$$

The potential $\vec{A}^-$ differs from $\vec{A}^+$ only within a gauge, giving rise to the same magnetic field with central symmetry,

$$\vec{B} = \mathrm{rot}\vec{A}^{\pm} = \mu \frac{\vec{x}}{r^3}. \tag{4}$$

In this manner the original string singularity is reduced to a

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pointlike one interpreted as a magnetic monopole with the magnetic charge μ.

The crucial point of this construction is the correct definition of the transition function between the Cartesian charts $(\vec{x}, y^{\pm})$ or the corresponding spherical ones $(r, \theta, \varphi, \alpha^{\pm})$ [2]. It is obvious that the transition $y^- = y^+ + 2\mu\varphi$ is in accordance with the gauge transformation $\vec{A}^- \cdot d\vec{x} = \vec{A}^+ \cdot d\vec{x} - 2\mu d\varphi$. This transition defines a suitable nontrivial fibration which is a version of the Hopf one, $S^3 \rightarrow S^2$ [2]. In these conditions it is convenient to take $y^{\pm} = -\mu(\alpha^{\pm} \pm \varphi)$ which leads to $\alpha^- = \alpha^+ = \alpha$. Then both the line elements (1) get the same form,

$$ds_o^2 = G(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) + \mu^2 G(r)^{-1}(d\alpha + \cos\theta d\varphi)^2, \tag{5}$$

in the spherical charts where the vector potentials have the components

$$A_r^{\pm} = A_\theta^{\pm} = 0, \qquad A_\varphi^{\pm} = \mu(\pm 1 - \cos\theta). \tag{6}$$

In order to implement the Euclidean Taub-NUT space in physical Kaluza-Klein theories, one has to include the time t. The Gross-Perry-Sorkin metric,

$$d\hat{s}^2 = -dt^2 + ds_o^2, \tag{7}$$

has the remarkable property to remain Ricci flat [5] since it is a solution of the vacuum Einstein equations in five dimensions. Moreover, this leads to a geometry with $SO(3) \otimes U(1) \otimes T(1)_t$ isometries where $T(1)_t$ is the group of the time translations. The metric (7) incorporates the effects of the magnetic charge μ while the gravitational interaction seems to be rather a consequence of the magnetic one. The reason is that there are no terms of the Schwarzschild type with at least one parameter which could be interpreted as the gravitational mass of the magnetic monopole.

Under such circumstances we assume that the metric (7) describes massless monopoles, but we have to look for another simple five-dimensional Kaluza-Klein metric suitable for massive Dirac monopoles. In these geometries we maintain the $SO(3) \otimes U(1) \otimes T(1)_t$ isometries as well as the form of the potentials (3), up to the factor representing the magnetic charge. Requiring the whole spacetime to be Ricci flat, we find an interesting solution of the vacuum Einstein equations with the line element

$$ds^2 = -F(r)dt^2 + G(r)\left(\frac{dr^2}{F(r)} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right) + \frac{\mu_{ef}^2}{G(r)}(d\alpha + \cos\theta d\varphi)^2, \tag{8}$$

where both $\mu_{ef} = \sqrt{\mu(\mu + 2M)}$ and

$$F(r) = 1 - \frac{2M}{r} \tag{9}$$

depend on the monopole mass M. This metric is asymptotically flat embedding gravitational and magnetic terms. We observe that for $M \rightarrow 0$ we recover the line element (7), while for $\mu \rightarrow 0$ the magnetic properties disappear. In the latter case the metric reduces to the usual Schwarzschild one of a particle with the mass M. It is remarkable that the massive monopole has the *effective* magnetic charge μ_{ef}, which depends on the genuine magnetic charge μ and the monopole mass M. Obviously, the nonvanishing components of the potentials (6) now become $A_\varphi^{\pm} = \mu_{ef}(\pm 1 - \cos\theta)$.

Next let us consider the motion of a test particle in the gravitational and magnetic fields of a massive Dirac monopole at large distances. We suppose that the test particle is a quantum scalar particle of the bare mass m and charge $e = q/\mu_{ef}$ where q is the eigenvalue of the operator $Q = -i\partial_\alpha$ [7]. The scalar field Φ of the test particle obeys the five-dimensional Klein-Gordon equation $(\Box - m^2)\Phi = 0$ [11]. Because of the central symmetry, there are particular solutions with separated spherical variables like

$$\Phi(r, \theta, \phi, \alpha) = R_{E,q,l}(r) Y_{lm}^q(\theta, \phi, \alpha), \tag{10}$$

where Y_{lm}^q are the $SO(3) \otimes U(1)$ harmonics we introduced before in Ref. [12]. The radial functions $R_{E,q,l}$ depend on the energy E, the angular quantum number l, and q. Such functions satisfy the radial equation

$$\left[-\frac{1}{r^2}\frac{d}{dr}\left(r^2 F \frac{d}{dr}\right) + \frac{l(l+1)}{r^2} + e^2\left(G^2 - \frac{\mu_{ef}^2}{r^2}\right) + m^2 G - E^2 \frac{G}{F}\right] R_{E,q,l}(r) = 0, \tag{11}$$

but this can not be analytically solved.

However, for large distances, like $r \gg 2M$, the above equation can be approximated by the asymptotic radial equation,

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{\nu}{r}\right]\hat{R}_{E,q,l}(r) = (E^2 - m_{ef}^2)\hat{R}_{E,q,l}(r), \tag{12}$$

where $\hat{R}_{E,q,l}(r) \propto r R_{E,q,l}^{\text{asympt.}}(r)$. The new parameter

$$\nu = 2Mm_{ef}^2 + (E^2 - m_{ef}^2)(\mu + 2M) - \mu e^2 \tag{13}$$

depends on the effective mass of the test particles $m_{ef} = \sqrt{m^2 + e^2}$, which includes the standard Kaluza-Klein contribution. This time the radial Eq. (12) is analytically solvable. Indeed, one deals with a Keplerian motion under the relativistic potential $\phi = -\nu/r$ corresponding to the nonrelativistic one $\phi/2m_{ef}$. Moreover, one finds that for low energies, like $E \sim m_{ef}$, the nonrelativistic potential appears as the Newtonian potential produced by the effective monopole mass

$$M_{ef} = M - \frac{\mu}{2}\frac{e^2}{m^2 + e^2}. \tag{14}$$

The conclusion is that the metric (8) of the background of a massive Dirac monopole mixes the gravitational and the magnetic effects in such a manner that for large distances and low energies the gravity may screen the magnetism. This could explain why it is so difficult to find experimental evidence about possible cosmic objects with magnetic charges.

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