Ghosts, strong coupling, and accidental symmetries in massive gravity

C. Deffayet[1,2,*] and J.-W. Rombouts[3,†]

[1]*APC*[‡], *11 place Marcelin Berthelot, 75005 Paris Cedex 05, France*
[2]*GReCO/IAP*[§], *98 bis boulevard Arago, 75014 Paris, France*
[3]*Department of Physics, New York University, 4 Washington Place, New York, New York 10003, USA*
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We show that the strong self-interaction of the scalar polarization of a massive graviton can be understood in terms of the propagation of an extra ghostlike degree of freedom, thus relating strong coupling to the sixth degree of freedom discussed by Boulware and Deser in their Hamiltonian analysis of massive gravity. This enables one to understand the Vainshtein recovery of solutions of massless gravity as being due to the effect of the exchange of this ghost, which gets frozen at distances larger than the Vainshtein radius. Inside this region, we can trust the two-field Lagrangian perturbatively, while at larger distances one can use the higher derivative formulation. We also compare massive gravity with other models, namely, deconstructed theories of gravity, as well as the Dvali-Gabadadze-Porrati model. In the latter case, we argue that the Vainshtein recovery process is of a different nature, not involving a ghost degree of freedom.

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I. INTRODUCTION

The construction of a consistent theory of massive gravity has proven to be an extremely difficult task. The usual Pauli-Fierz approach is to add a quadratic mass term to the quadratic Einstein-Hilbert action for a perturbation (the would-be massive graviton) over a flat background [1]. At this level, the action for the massive graviton is consistent in the sense that it does not propagate ghosts or tachyons. Nevertheless, it is incompatible with experiment because it propagates an unwanted extra scalar degree of freedom (DOF) that couples to the trace of the energy-momentum tensor [2] (further referred to as the vDVZ scalar). It leads to physical predictions different from the massless theory, even in the limit where the graviton mass goes to zero. This is true, for instance, for the light bending around the Sun and is known as the van Dam-Veltman-Zakharov (vDVZ) discontinuity [2]. Another question raised by the quadratic Pauli-Fierz theory is the link between the background flat metric and the graviton. This should be answered by a proper nonlinear theory of massive gravity. However, attempts to go beyond the quadratic Pauli-Fierz action are raising even more questions. For example, Boulware and Deser [3] introduced an extraneous metric, so that the theory they considered can be thought of as some kind of bimetric theory [4], the dynamics of one of the two metrics being frozen, and the coupling between the two metrics being such that it reproduces the quadratic Pauli-Fierz action for small perturbations. In the following, we will refer to this type of theory, quite loosely speaking, as massive gravity; note, however, that such a nonlinear completion is not unique. In a particular example of such a theory, simply obtained by adding the quadratic Pauli-Fierz mass term to the full Einstein-Hilbert action, a standard Hamiltonian treatment shows that the number of propagating DOFs is six (instead of the five DOFs described by the quadratic action), of which the sixth DOF has a ghost character [3]. This increment in the number of the propagating DOFs with respect to the quadratic theory, as well as the unboundedness from below of the Hamiltonian, was shown to persist in a more general class of bimetric theories [5]. This could have consequences at the classical level already (see, e.g., [6]). On the other hand, it was argued in Refs. [7,8] that the vDVZ scalar has a strong self-interaction on a flat background at an extremely low scale $\Lambda = (M_P m^4)^{1/5}$. This, at the classical level already, has dramatic consequences. Indeed, Vainshtein [9] first noticed that nonlinearities in massive gravity dominate the linear terms at distance scales smaller than $r_V = (GM/m^4)^{1/5}$ (the Vainshtein region) for a classical source of mass M. He then proposed that these nonlinearities cure the vDVZ discontinuity within this region, due to a nonperturbative resummation of the source expansion [9] (see also [7]). This ‘‘Vainshtein resummation’’ seems to be problematic for massive gravity [10], if one follows the resummation beyond the first terms, but there remains the possibility that it does work for more sophisticated models (see, e.g., [7]). Nonlinearities could also cure some of the above mentioned pathologies by selecting a nonasymptotically flat vacuum [5].

In this paper, we will consider only expansions of massive gravity over a flat background. Namely, we want to discuss the relation between the strong coupling and the, seemingly unrelated, ghost appearing at nonlinear level in massive gravity. That there could exist a link between these two issues is suggested in part by the following reasoning. Consider, e.g., a classical nonrelativistic source. There it

*Electronic address: deffayet@iap.fr
†Electronic address: jwr218@nyu.edu
‡UMR 7164 (CNRS, Université Paris 7, CEA, Observatoire de Paris).
§UMR 7095 (CNRS, Université Paris 6).

seems, if one follows Vainshtein's reasoning, that within the Vainshtein region, one somehow compensates for the extra attraction exerted by the vDVZ scalar that is responsible (in a perturbative sense) for the discontinuity (this compensation being at worst valid only far from the singularities discussed in Ref. [10]). One may suspect that there is a ghost degree of freedom responsible for this cancellation.[1] A natural candidate for this ghost seems to be the "sixth degree of freedom" mentioned above, but it is not clear why this ghost should work only within a certain distance scale. We will discuss in this paper how one can reformulate the strong coupling as a ghost problem in the theory, and our analysis will clarify the above features.

Although we will focus here on massive gravity (as defined above), we would like to motivate this work by more recent developments. Indeed, massive Einstein gravity is just one example of a theory that modifies gravity at large distance scales. In recent years, a number of different models have been proposed to modify gravity in the infrared. In particular, the Dvali-Gabadadze-Porrati (DGP) [13] braneworld model (also known as *brane induced gravity*) modifies gravity at a large distance while it can give an alternative way of producing the observed late-time cosmic acceleration [14]. DGP gravity shows certain similarities with massive gravity. More specifically, it exhibits a vDVZ discontinuity at linearized level and also has a related strong coupling [7,15,16], the exact consequences of which have been subject to a debate [7,15–18]. With the link between strong coupling and ghosts in massive gravity, one would hence suspect the DGP model also to contain ghosts when properly analyzed (i.e. at the nonlinear level). As we will argue, the situation in DGP is quantitatively different from massive gravity, in that the leading operator that grows strong in the UV has dimension seven, as opposed to massive gravity, where this operator has dimension nine. This fact and the particular tensor structure of the operator describing the scalar content of the graviton, as found in Refs. [15,17], suggest that the issue is more subtle and differs from the case of massive gravity.

This paper is organized as follows. In the next section we will discuss how, in massive gravity, the strong coupling can be reformulated as a ghost problem. Namely, we will argue that there is a propagating ghost DOF, which appears only at the cubic order in the perturbation theory over a flat background and which is responsible for the cancellation, à la Vainshtein, of the attraction exerted by the vDVZ scalar around heavy sources. We will show how the ghost formulation allows a perturbative treatment of massive gravity *within* the Vainshtein region (so that in our reformulation, the theory stays weakly coupled in this region). This perturbative ghost generates the same leading order corrections as found in the Vainshtein resummation as discussed in Refs. [7,9], while it freezes out at a large distance, where one is left with only the vDVZ scalar. This relates the old Hamiltonian formulation to the effective field theoretical formulation. In Sec. III, we want to understand how the appearance of ghosts in massive gravity is related to the breaking of accidental symmetries, present in the linearized theory, at nonlinear level. There we will discuss the link between the ghost discussed in Sec. II, and the "canonical" sixth ghost DOF as discussed by Boulware and Deser. This section will further give a "geometrical" meaning to the ghost problem, which we further explore by considering discretized gravity in Sec. IV. There we also compare massive gravity with the DGP model. We suggest that the situation in the latter is very different as far as the relation between strong coupling and ghosts is concerned, namely, because the operator that grows large *at the lowest energy scale* in the DGP model is not associated with propagating ghost modes. We conclude with some final remarks on our work and other IR modifications of gravity. Note that our discussion will be purely classical, and we will not be concerned with the issue of the quantum consistency of the theories considered.

II. GHOST OR STRONG COUPLING?

As recalled above, the quadratic mass term for a spin-two excitation is uniquely defined by demanding the absence of ghost and tachyonic modes and takes the Pauli-Fierz form [1]. Beyond quadratic level, completion is not uniquely defined. Two possible choices are, for example, given by the following actions:

$$S_{\rm BD} = M_P^2 \int d^4x\sqrt{-g}R(g) + M_P^2 m^2 \int d^4x\sqrt{-g^{(0)}}h_{\mu\nu}h_{\alpha\beta}(g_{(0)}^{\mu\nu}g_{(0)}^{\alpha\beta} - g_{(0)}^{\mu\alpha}g_{(0)}^{\nu\beta}), \quad (1)$$

$$S_{\rm AGS} = M_P^2 \int d^4x\sqrt{-g}R(g) + M_P^2 m^2 \int d^4x\sqrt{-g}h_{\mu\nu}h_{\alpha\beta}(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}), \quad (2)$$

where $h_{\mu\nu} \equiv g_{\mu\nu} - g_{\mu\nu}^{(0)}$, $g_{\mu\nu}^{(0)}$ being some extra metric field.[2] The above actions define bigravity theories, where the dynamics of $g_{\mu\nu}^{(0)}$ has been frozen.[3] Note that the above theory is invariant under 4D diffeomorphisms (in the same sense as bigravity theories considered in Ref. [4] are) and

[1]Such a mechanism was also found in the Gregory-Rubakov-Sibiryakov model [11], responsible for the instability of the model [12].

[2]$g_{(0)}^{\mu\nu}$ is the inverse of $g_{\mu\nu}^{(0)}$.

[3]This theory is in the "Pauli-Fierz universality class," to use the phrasing of Ref. [5].

arises naturally in the process of "deconstructing" gravity [8,19]. The action (1) was used by Boulware and Deser in their Hamiltonian analysis of massive gravity [3], while action (2) was used by Arkani-Hamed *et al.* in Ref. [8]. In this last paper, an action for the scalar polarization of the massive graviton was obtained by introducing the "hopping" fields $Y^\alpha(x)$ and replacing $h_{\mu\nu}$ by $H_{\mu\nu}$, given by

$$H_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\alpha\beta}(Y)\partial_\mu Y^\alpha \partial_\nu Y^\beta. \tag{3}$$

Taking $g^{(0)}_{\mu\nu}$ and $g_{\mu\nu}$ to be flat and expanding Y^α as $Y^\alpha = x^\alpha + \pi^\alpha$, one obtains

$$H_{\mu\nu} = h_{\mu\nu} + \pi_{\mu,\nu} + \pi_{\nu,\mu} + \pi_{\alpha,\mu}\pi^\alpha_{,\nu}. \tag{4}$$

Focusing on the scalar mode of the Goldstone vector π^μ, $\pi_\mu = \partial_\mu \phi$, one gets (after an integration by part) the cubic action for ϕ

$$2M_P^2 m^2 \int d^4x((\Box\phi)^3 - (\Box\phi)(\partial^\mu\partial^\nu\phi)(\partial_\mu\partial_\nu\phi)). \tag{5}$$

Note that, due to general covariance of the Einstein-Hilbert action and to the particular tensorial structure of the Pauli-Fierz mass term, there is no kinetic term arising directly for ϕ from the above procedure. Rather, the Goldstone scalar obtains a kinetic term only through mixing with $h_{\mu\nu}$. This gives a m^2 dependence (and possibly very small coefficient) to the scalar kinetic term $\sim M_P^2 m^4 \phi\Box\phi$. After a proper diagonalization [8], one obtains the following action for the canonically normalized scalar (we will still denote ϕ):

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi + \frac{1}{\Lambda^5}(\Box\phi)^3 - \frac{1}{\Lambda^5}(\Box\phi)(\partial^\mu\partial^\nu\phi)(\partial_\mu\partial_\nu\phi) - \frac{1}{M_P}\phi T, \tag{6}$$

where T is the trace of the energy-momentum tensor, and the energy scale Λ is given by

$$\Lambda \equiv (m^4 M_P)^{1/5}. \tag{7}$$

This Lagrangian contains strong cubic interaction terms for the Goldstone field, with dimension nine operators growing strong at the scale Λ. One can check that the cubic terms are the dominant terms in the effective Lagrangian, in the sense that they grow strong at a much smaller energy scale than any other interaction terms [8], which makes this Lagrangian a useful description of the ϕ low-energy dynamics. This amounts to taking the limit $M_P \to \infty$, $T \to \infty$, and $m \to 0$ with $\Lambda = cst$ and $T/M_p = cst$ in the original Lagrangian (2). The limiting procedure eliminates all other self-interaction terms of ϕ as well as mixing terms of the Goldstone scalar with other spin components. A similar limit has been considered in Ref. [17] in the case of DGP gravity. Note also that, had we applied the same procedure from action (1), we would have obtained the same action as in (6) with a global minus sign in front of the cubic derivative interaction terms. This means that the following discussion will apply for both cases and, indeed, as we will argue, much more generally.

To investigate the properties of the ϕ sector and with no fundamental differences in the conclusion, we simplify our discussion here by omitting the third term in the Lagrangian (6) (in the appendix we explain how to extend the treatment below for the full Lagrangian). So the starting point in our discussion will be the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi + \frac{1}{\Lambda^5}(\Box\phi)^3 - \frac{1}{M_P}\phi T. \tag{8}$$

This Lagrangian obviously implies an equation of motion of fourth order, reading

$$\Box\phi + \frac{3}{\Lambda^5}\Box((\Box\phi)^2) - \frac{T}{M_P} = 0. \tag{9}$$

This means, if one thinks in terms of the Cauchy problem, that one can expect this action to describe two, rather than one, scalar DOFs. As is well known, similar conclusions are reached in higher derivative scalar field theories (see, e.g., [20]) or higher derivative theories of gravity (see, e.g., [21]). One can typically reformulate a higher derivative theory of some fundamental fields as a standard two-derivative Lagrangian of the fundamental fields plus extra degrees of freedom, with certain interaction terms, and which encode the extra derivatives appearing in the initial action. It is also known that some of these extra DOFs, quite generically, have a ghost character (see, e.g., [22]), so that one should be able to reinterpret, classically, the strong coupling discussed in Refs. [7,8] in terms of an extra propagating ghost DOF. This is what we do in the following. Note that one could question the consequence of this because we consider here a truncated theory. For example, if one looks at a truncated derivative expansion of some perfectly safe underlying theory, one could conclude erroneously that the latter is sick. Here, however, the situation is quite different because of the fact that the Lagrangian started from has some well defined range of applicability, as was recalled previously.

We thus introduce a new field λ, modifying the Lagrangian (8) into

$$\mathcal{L}_{\rm eq} = \frac{1}{2}\phi\Box\phi + \frac{1}{\Lambda^5}(\Box\phi)^3 + F(\lambda, \Box\phi) - \frac{1}{M_P}\phi T. \tag{10}$$

The equations of motion for λ and ϕ are given by

$$\Box\phi + \frac{3}{\Lambda^5}\Box((\Box\phi)^2) + \Box F^{(0,1)} - \frac{T}{M_P} = 0 \tag{11}$$

and

$$F^{(1,0)} = 0, \tag{12}$$

where $F^{(i,j)}$ means the derivative of F, i times with respect to its first variable and j times with respect to its last one. We then ask those equations of motion not to contain derivatives of order higher than two and to be equivalent to Eq. (9). A suitable ansatz for F is given by

$$F(\lambda, \Box\phi) = \frac{2}{3\sqrt{3}}\Lambda^{5/2}\lambda^3 + \lambda^2\Box\phi - \frac{1}{\Lambda^5}(\Box\phi)^3, \quad (13)$$

leading to the equation of motion

$$\Box\phi + \Box\lambda^2 - \frac{1}{M_P}T = 0, \quad (14)$$

$$\lambda\left(\frac{\Lambda^{5/2}}{\sqrt{3}}\lambda + \Box\phi\right) = 0. \quad (15)$$

Those for a nonvanishing λ are equivalent to the equation of motion (9). Note that the phase space of solutions to Eqs. (14) and (15) is, in fact, larger than that of solutions to (9), because the former also includes solutions to the equation (obtained for a vanishing λ) $\Box\phi + T/M_P = 0$. It means that the equivalence we are talking about here has to be understood as an equivalence restricted to a suitable subset of solutions. Similar restrictions also occur, e.g., when showing the equivalence between a $f(R)$ theory of gravity (f being some arbitrary function) and a scalar tensor theory, where the equivalence holds only between solutions defined away from the critical points where the second derivative of f vanishes [21] (see also [23]). The choice (13) leads to a Lagrangian where the field λ^2 does get a kinetic term through a mixing with ϕ. Defining then φ as $\phi = \varphi - \lambda^2$, one obtains the Lagrangian

$$\mathcal{L}_{\rm eq} = \frac{1}{2}\varphi\Box\varphi - \frac{1}{2}\lambda^2\Box\lambda^2 + \frac{2}{3\sqrt{3}}\lambda^3\Lambda^{5/2} - \frac{1}{M_P}\varphi T + \frac{1}{M_P}\lambda^2 T. \quad (16)$$

This can be then rewritten as a quadratic action defining ψ as $\psi = \lambda^2$, so that $\mathcal{L}_{\rm eq}$ becomes

$$\mathcal{L}_{\rm eq} = \frac{1}{2}\varphi\Box\varphi - \frac{1}{2}\psi\Box\psi - \epsilon\frac{2}{3\sqrt{3}}\psi^{3/2}\Lambda^{5/2} - \frac{1}{M_P}\varphi T + \frac{1}{M_P}\psi T, \quad (17)$$

where $\epsilon = \pm 1$ is the sign of $\Box(\varphi - \psi)$. Notice above the nonanalytic form of the potential for ψ, as well as the fact that ψ is a positive definite field variable. The two different possible choices for ϵ in the above equation indicate that the equivalence between the dynamics obtained for ϕ from the Lagrangian (8) and that obtained from the Lagrangian (17) with a specified value of ϵ is restricted to a set of solutions to the equations of motion having the corresponding sign of $\Box\phi \equiv \Box(\varphi - \psi)$. If one considers, e.g., the Cauchy problem associated with the equation of motion (9), and if one wants to derive the solution to this Cauchy problem with the equivalent Lagrangian (17), one should choose the sign ϵ corresponding to the initial Cauchy data provided (of course, the sign of $\Box\phi$ can change along the initial surface so that the equivalence we are talking about here is really a local property). In any case, one can also use the Lagrangian (16) where this sign issue does not arise, but as we saw, this also leads to solutions of the equations of motion not contained in those of (8).

Thus, we rewrote a Lagrangian of a field with nonrenormalizable interactions as a Lagrangian of 2 degrees of freedom; one free field φ, corresponding to the *vDVZ scalar* at large distance from a source, and one ghost ψ, with a *relevant* interaction term.

Let us for now consider a pointlike source. Then, within a certain distance scale to the source, the ghost exactly cancels the vDVZ field, up until when the ghost freezes out due to its self-interaction, leaving one propagating DOF. For our discussion, it is convenient to consider the dimensionless fields $\tilde{\varphi} \equiv \varphi/M_P$ and $\tilde{\psi} \equiv \psi/M_P$, which correspond with no further normalization factor to the dimensionless massive graviton $h_{\mu\nu}$. The equations of motion for those dimensionless fields read

$$\Box\tilde{\varphi} = GT, \quad (18)$$

$$\Box\tilde{\psi} + \epsilon\frac{m^2}{\sqrt{3}}\tilde{\psi}^{1/2} = GT, \quad (19)$$

where $G = 1/M_P^2$. Considering a classical source with $T = -M\delta^3(x)$, one finds that φ is given by the usual Newtonian potential (with an appropriate sign)

$$\tilde{\varphi}^{(0)} = \frac{GM}{r}, \quad (20)$$

where r is the distance to the source. If one then considers the expansion of the solution for $\tilde{\psi}$ around $\tilde{\psi}^{(0)} \equiv \tilde{\varphi}^{(0)}$ as $\tilde{\psi} = \tilde{\psi}^{(0)} + \tilde{\psi}^{(1)} + \ldots$, one finds that $\tilde{\psi}^{(1)}$ obeys to the equation

$$\Box\tilde{\psi}^{(1)} + \epsilon\frac{m^2}{\sqrt{3}}(\tilde{\psi}^{(0)})^{1/2} = 0, \quad (21)$$

which is solved by

$$\tilde{\psi}^{(1)} = -\epsilon\frac{4}{15\sqrt{3}}m^2\sqrt{GM}r^{3/2}. \quad (22)$$

This matches the correction[4] obtained in Ref. [9] (see [7]). Here we traced the origin of these corrections back to the self-interaction of the perturbative ghost $\tilde{\psi}$, which cancels the contribution from the vDVZ scalar φ at small distances

[4]One can verify that $\Box\phi$ obtained from this solution has indeed the same sign as ϵ. One should choose the sign of ϵ so that the solution has the right asymptotic behavior.

from the source. Notice that $\tilde{\psi}^{(0)}$ becomes of the same order as $\tilde{\psi}^{(1)}$ at the distance $r_V = (GM/m^4)^{1/5}$, which is the Vainshtein radius [9]. Going back to the original field $\phi = \varphi - \psi$, we thus see that, inside the Vainshtein region, $\phi^{(0)} = 0$ and

$$\phi^{(1)} = \epsilon \frac{4}{15\sqrt{3}} m^2 \sqrt{GM} r^{3/2}, \qquad r \ll r_V. \tag{23}$$

Outside the Vainshtein region, one cannot trust perturbation theory for the ghost, but in this region, one can simply use the original Lagrangian (6) for the scalar field perturbatively, since the self-interaction of the scalar is small in this region. The equations of motion for this Lagrangian read:

$$\begin{aligned} &\Box\phi + \frac{3}{\Lambda^5}\Box(\Box\phi)^2 - \frac{1}{\Lambda^5}\Box(\partial^\mu\partial^\nu\phi\partial_\mu\partial_\nu\phi) \\ &\quad - \frac{2}{\Lambda^5}\partial^\mu\partial^\nu((\Box\phi)\partial_\mu\partial_\nu\phi) - T/M_P \\ &\quad = 0. \end{aligned} \tag{24}$$

If we now look for an expansion of the form $\phi = \phi^{(0)} + \phi^{(1)} + \ldots$ with $\phi^{(1)} \ll \phi^{(0)}$, we see that $\phi^{(0)}$ obeys

$$\Box\phi^{(0)} = T/M_P, \tag{25}$$

so that $\phi^{(0)}$ is given by

$$\phi^{(0)} = \frac{M}{rM_P} \tag{26}$$

for a pointlike source. $\phi^{(1)}$ is now given by

$$\Box\phi^{(1)} = \frac{1}{\Lambda^5}\Box(\partial^\mu\partial^\nu\phi^{(0)}\partial_\mu\partial_\nu\phi^{(0)}) \tag{27}$$

(the terms in $\Box\phi^{(0)}$ vanish outside of the source). This is solved by

$$\phi^{(1)} \propto \frac{M^2}{M_P^3 m^4}\frac{1}{r^6}, \qquad r \gg r_V, \tag{28}$$

which is the form of the correction expected (recall that the dimensionless metric fluctuation is given by ϕ/M_P).

To summarize, we rewrote the Lagrangian for massive gravity in which the vDVZ field gets strongly coupled at a scale Λ as a system of two fields that do not have strong coupling but of whom one is a ghost. In this ghost formulation, we can trust the Lagrangian perturbatively within the Vainshtein (or strong coupling) region, while outside of the Vainshtein region, we can use the original higher derivative interaction perturbatively.

III. SYMMETRIES AND HAMILTONIAN ANALYSIS OF MASSIVE GRAVITY

The "Goldstone" formalism that we used as a starting point in the previous section is especially useful to focus on the sick scalar sector of massive gravity. Here we relate our discussion to the original analysis of Boulware and Deser, who first discussed, in the "unitary gauge," the unboundedness from below of the Hamiltonian of massive gravity [3]. For that discussion, our starting point will be the action (1), even if, as we already stressed, the Boulware and Deser findings have been shown to be valid for generic bimetric gravity theories (and thus massive gravities as considered here) by Damour and Kogan.

Let us first recall what is going on, from the Hamiltonian point of view, if one retains only the quadratic part of the action (1) for the "graviton" $h_{\mu\nu}$. That is to say, one sets in (1) the background metric $g^{(0)}_{\mu\nu}$ to be the Minkowski metric and expands the Einstein-Hilbert action [first line of (1)] to quadratic order in $h_{\mu\nu}$. One obtains then the Pauli-Fierz action, which is ghost-free and propagates 5 degrees of freedom. The counting of DOFs goes as follows. One start from a symmetric tensor, the graviton, which has 10 components. However, it turns out that neither h_{00} nor h_{0i} (with $i = 1, 2, 3$ being spatial indices) are dynamical degrees of freedom. This is a consequence of the same properties of massless gravity which shares the same kinetic terms with the theory considered here.[5] Namely, h_{00} and h_{0i} appear as Lagrange multipliers in the kinetic term obtained from expanding the Einstein-Hilbert action. This means that out of the 10 initial DOFs one is left with $10 - 4 = 6$ DOFs at this stage. Let us then consider the mass term; here h_{00} and h_{0i} play very different roles. Indeed, the mass term reads

$$M_P^2 m^2 \int d^4x (h_{ij}h_{ij} - 2h_{0i}h_{0i} - h_{ii}h_{jj} + 2h_{ii}h_{00}), \tag{29}$$

where it appears that h_{00} is a true Lagrange multiplier for the theory considered, since it enters linearly in both the kinetic part and the mass term of the action. As a consequence, the h_{00} equation of motion generates a constraint reading

$$(\nabla^2 - m^2)h_{ii} - h_{ij,ij} = 0, \tag{30}$$

which enables one to eliminate one more DOF, leaving five propagating DOFs in the Pauli-Fierz action. Note that the equations of motion for the h_{0i}, which enter quadratically in the mass term, do not eliminate other degrees of freedom (in contrast to the massless case), but rather determine those variables in terms of the others.

We now turn to fully nonlinear gravity. In the massless case, that is to say, Einstein general relativity, formulated in the Hamiltonian language [Arnowitt-Deser-Misner (ADM) formalism [24]], the lapse and the shift field N and shift N^i fields are Lagrange multipliers associated with the reparametrization symmetry of the Einstein-Hilbert

[5]For the same reason, the A_0 component of a massive Proca field does not propagate as is the case for the massless photon.

action.[6] The latter are defined as

$$N \equiv (-g^{00})^{-1/2} \tag{31}$$

and

$$N_i \equiv g_{0i}, \tag{32}$$

in terms of the component of the metric $g_{\mu\nu}$. They generate constraints related, respectively, to the time and space reparametrization symmetries of the action. Those constraints eliminate 4 degrees of freedom, out of the 6 remaining (10 − 4), leaving the two propagating polarizations of the massless graviton. The addition of a "mass term" such as the one of the last two lines of (1) changes, however, dramatically the character of N and N^i. Indeed, the action (1) reads in the first order formalism

$$M_P^2 \int d^4x\{(\pi^{ij}\dot{g}_{ij} - NR^0 - N_iR^i) - m^2(h_{ij}h_{ij} - 2N_iN_i - h_{ii}h_{jj} + 2h_{ii}(1 - N^2 + N_k g^{kl} N_l))\}, \tag{33}$$

where π^{ij} are the conjugate momentum to g_{ij}, and R^0 and R^i are, respectively, the Hamiltonian and momentum constraints of massless gravity (generated by the lapse and shift fields). The dramatic observation made in Ref. [3] is that, now, neither N_i nor N are a true Lagrange multiplier of the full nonlinear massive gravity. Thus, the number of propagating DOFs is now six rather than five. It is also remarked in Ref. [3] that the full reduced Hamiltonian is unbounded from below as we now recall. For this purpose, we introduce the variable n defined as

$$N \equiv 1 + n \tag{34}$$

and rewrite the action (33) as

$$M_P^2 \int d^4x\{(\pi^{ij}\dot{g}_{ij} - (1+n)R^0 - N_iR^i) - m^2(h_{ij}h_{ij} - 2N_iN_i - h_{ii}h_{jj} - 2h_{ii}(2n + \beta n^2 - \alpha N_k g^{kl} N_l))\}, \tag{35}$$

with β and α equal to one. For future reference, we will, however, keep explicitly the β and α dependence upon the process of reducing the Hamiltonian. Notice, in particular, that upon the substitution $N_i \to h_{0i}$ and $2n \to -h_{00}$, the mass term (29) is obtained from the above expression by taking β and α to vanish. The N_i and n equation of motion read, respectively,

$$R^i = 4m^2(\eta^{ij} - \alpha h_{kk} g^{ij})N_j, \tag{36}$$

$$R^0 = 4m^2 h_{ii}(1 + \beta n). \tag{37}$$

Those can be used to extract n and N^i

$$N_j = \frac{1}{4m^2}(\eta - \alpha h_{kk} g)^{-1}_{ij} R^i, \tag{38}$$

$$n = \frac{1}{4\beta h_{ii} m^2}(R^0 - 4m^2 h_{jj}). \tag{39}$$

Inserting those expressions in action (35), we get the reduced Lagrangian (in first order form)

$$M_P^2 \int d^4x\left\{\pi^{ij}\dot{g}_{ij} - m^2(h_{ij}h_{ij} - h_{ii}h_{jj}) - \frac{1}{8m^2}R^l(\eta - \alpha h_{ii} g)^{-1}_{lm} R^m - \frac{1}{8m^2\beta h_{ii}}(R^0)^2 - \frac{2m^2}{\beta}h_{ii}\right\}. \tag{40}$$

One reads immediately from the above expression the Hamiltonian of the system and discovers that it can have arbitrary sign and absolute value, since it is true, in particular, for the second term of the second line of (40). This expression (40) shows in a very clear way that the number of propagating DOFs is now six. Note, however, that the Hamiltonian is singular when one makes β vanish. This manifests the fact that the sixth DOF arises only from the integration of n, as well as that we have chosen to integrate the constraint (39) by extracting n, which is not possible if β and h_{ii} vanish. So, in this language, the decoupling of the sixth DOF is subtle and the above Lagrangian can be considered analogous to the Lagrangian (17) where the 2 degrees of freedom appear explicitly.

We would like now to relate the unboundedness of the Hamiltonian for the extra propagating DOF to the analysis done in Sec. II of this paper. We will discuss this issue in the covariant formulation, following first the discussion of Ref. [3].

Starting from the action (1), one obtains the following equations of motion for the metric $g_{\mu\nu}$:

$$\mathcal{G}^{\mu\nu}(g = \eta + h) - 2m^2\frac{\sqrt{-g_{(0)}}}{\sqrt{-g}}h_{\alpha\beta}(g^{\mu\nu}_{(0)}g^{\alpha\beta}_{(0)} - g^{\mu\alpha}_{(0)}g^{\nu\beta}_{(0)}) = GT^{\mu\nu}, \tag{41}$$

where $\mathcal{G}^{\mu\nu}$ is the Einstein tensor for the metric g. In this formulation we can again see how at linearized level only five DOFs appear. Indeed, by taking the divergence and trace of the linearized form of the equations of motion, one obtains at the linear level

$$\partial_\mu(h^{\mu\nu} - h\eta^{\mu\nu}) = 0, \tag{42}$$

$$h = -\frac{G}{6m^2}T. \tag{43}$$

The first equation implies that the linearized curvature R^L is zero, R^L being given by

$$R^L = 2\partial_\mu(h^{\mu\nu}{}_{,\nu} - \partial^\mu h^\nu_\nu). \tag{44}$$

[6] That is to say, the Einstein-Hilbert action, in its first order form, is already in the "parametrized" form; see, e.g., [24,25].

The second equation states that the trace is not propagating. The latter constraint, obtained by taking the trace (with respect to $g_{\mu\nu}$) of Eq. (41), generalizes to the full nonlinear case into

$$-R - 2m^2 \frac{\sqrt{-g_{(0)}}}{\sqrt{-g}} g_{\mu\nu} h_{\alpha\beta} (g_{(0)}^{\mu\nu} g_{(0)}^{\alpha\beta} - g_{(0)}^{\mu\alpha} g_{(0)}^{\nu\beta}) = GT, \tag{45}$$

which, involving second derivatives, has to be treated as an equation of motion. This is indeed where the sixth ghost degree of freedom comes in the covariant formulation. Now the first constraint in Eq. (42) *has* a nonlinear generalization which is obtained from the Bianchi identities for the metric g. Indeed, taking the covariant derivative with respect to $g_{\mu\nu}$ of (41), one obtains

$$\left(\frac{\sqrt{-g_{(0)}}}{\sqrt{-g}} h_{\alpha\beta} (g_{(0)}^{\mu\nu} g_{(0)}^{\alpha\beta} - g_{(0)}^{\mu\alpha} g_{(0)}^{\nu\beta})\right)_{;\mu} = 0. \tag{46}$$

Taking the ordinary divergence of this equation allows one to write the linearized curvature in terms of higher order terms in $h_{\mu\nu}$. Schematically the above equation reads

$$\partial_\nu \partial_\mu [h^{\mu\nu} - \eta^{\mu\nu} h] + \Box h^2 + O(h^3) = 0, \tag{47}$$

yielding $R_L(h) = \Box h^2$ up to terms of $O(h^3)$. When plugging back in the trace equation (45), we get, again schematically, at quadratic order in the field equations:

$$\Box h^2 + m^2 h = GT. \tag{48}$$

If we replace $h^2 \to \psi$, we see that this equation has exactly the same form as the equation for the ghost of the previous section Eq. (19).

IV. A COMPARISON WITH OTHER MODELS

A. Deconstructed gravity

In the previous section, we saw that the Pauli-Fierz mass term was such that the equivalent of the $R^0 = 0$ constraint, associated with time reparametrization of Einstein (massless) general relativity, was still present in the massive theory at the level of the quadratic action and is given by Eq. (30). Thus, somehow, the time reparametrization symmetry is still present in the quadratic Pauli-Fierz action and eliminates 2 degrees of freedom. However, it disappears from the fully nonlinear massive gravity, as we just reminded. It is interesting to remember a similar property arising in the process of deconstructing gravity [8,19,26],[7] where geometry along one (in the simplest case considered here) dimension is given up by discretization. The starting point is here the five-dimensional Einstein-Hilbert action reading

$$M_{(5)}^3 \int d^4x dy \sqrt{-g} \mathcal{N} \{R + K_{\mu\nu} K_{\alpha\beta} (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta})\}, \tag{49}$$

where we used an ADM split of the 5D metric along the fifth spacelike dimension to be discretized (see [19]). One then latticizes the fifth dimension, with S lattice sites [labeled with Latin indices $(i), (j), \ldots$] with spacing a (we will assume periodic identification of the end point of the lattice for simplicity). We obtain the discretized action

$$S = M_{(5)}^3 a \sum_{(i)} \int d^4x \sqrt{-g_{(i)}} \mathcal{N}_{(i)} \bigg[R(g_{(i)}) + \frac{1}{4\mathcal{N}_{(i)}^2} \times (\Delta_{\mathcal{L}} g_{(i)})_{\mu\nu} (\Delta_{\mathcal{L}} g_{(i)})_{\alpha\beta} (g_{(i)}^{\mu\nu} g_{(i)}^{\alpha\beta} - g_{(i)}^{\mu\alpha} g_{(i)}^{\nu\beta}) \bigg], \tag{50}$$

where we took the natural discrete definition of the Lie derivative

$$\Delta_{\mathcal{L}} T_{(i)} = \frac{W(i, i+1) T_{(i+1)} - T_{(i)}}{a}. \tag{51}$$

In this expression W is the transport operator from site $i+1$ (located at coordinate $y_{(i+1)}$ along the fifth dimension) to site i (located at coordinate $y_{(i)} = y_{(i+1)} - a$ along the fifth dimension). It is given by the Wilson line for transport along the fifth dimension (in analogy with non-Abelian gauge theory)

$$W(y', y) = P \exp \int_y^{y'} dz \bar{N}^\mu \partial_\mu, \tag{52}$$

with $\bar{N}^\mu$ the shift fields of the ADM split used in this discretization. The hopping field Y^μ [analogous to those of Eq. (3)] considered in Ref. [8] can be explicitly constructed out of W as $Y^\mu(y, y_0; x) = W(y, y_0)(x)$.

It turns out to be convenient to work in the Einstein frame for the metrics on the different sites. Namely, we perform a Weyl rescaling $g_{\mu\nu}^{(i)} = \exp(-\phi_{(i)}/\sqrt{3}) \gamma_{\mu\nu}^{(i)}$, and $\mathcal{N}_i \equiv \exp(\phi_{(i)}/\sqrt{3})$, such as the action (50) now reads

$$S = \sum_i M_p^2 \int d^4x \sqrt{-\gamma_{(i)}} \bigg\{ R(\gamma_{(i)}) - \frac{1}{2} \gamma_{(i)}^{\mu\nu} \nabla_\mu \phi_{(i)} \nabla_\nu \phi_{(i)} + e^{-\sqrt{3}\phi_{(i)}} Q_{\mu\nu}^{(i)} Q_{\alpha\beta}^{(i)} (\gamma_{(i)}^{\mu\nu} \gamma_{(i)}^{\alpha\beta} - \gamma_{(i)}^{\mu\alpha} \gamma_{(i)}^{\nu\beta}) \bigg\}, \tag{53}$$

with

$$Q_{\mu\nu}^{(i)} = \frac{1}{2} \bigg\{ \Delta_{\mathcal{L}} \gamma_{\mu\nu}^{(i)} - \gamma_{\mu\nu}^{(i)} \frac{\Delta_{\mathcal{L}} \phi^{(i)}}{\sqrt{3}} \bigg\}, \tag{54}$$

and $M_P^2 = M_{(5)}^3 a$. Eventually, we want to discuss the symmetries and counting of DOFs of this action at the quadratic level for some fluctuation over a flat background. We thus expand the different fields as follows:

[7] See also Ref. [27].

$$\gamma^{(i)}_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_p} h^{(i)}_{\mu\nu}, \qquad \phi^{(i)} = \frac{\varphi^{(i)}}{M_p}, \qquad X^{\mu}(i, i+1) = x^{\mu} + \frac{a}{M_p} n^{\mu}_{(i)}, \tag{55}$$

and define furthermore the discrete Fourier transform of the discrete fields as

$$\hat{\mathcal{F}}_{(k)} = \sum_j \frac{1}{\sqrt{N}} \mathcal{F}_{(j)} e^{-i2\pi jk/N}. \tag{56}$$

This gives the following quadratic action:

$$\begin{aligned} S = \int d^4x \sum_k \frac{1}{4} \{ &\partial_\rho \hat{h}^{\mu\nu}_{(k)} \partial_\sigma \hat{h}^{*\alpha\beta}_{(k)} (\eta^{\rho\sigma}\eta_{\mu\nu}\eta_{\alpha\beta} - \eta^{\rho\sigma}\eta_{\mu\alpha}\eta_{\nu\beta} + 2\delta^\sigma_{(\nu}\eta_{\mu)\beta}\delta^\rho_\alpha - \eta_{\mu\nu}\delta^\sigma_\beta\delta^\rho_\alpha \\ &- \eta_{\alpha\beta}\delta^\sigma_\nu\delta^\rho_{\mu)}\} - \frac{1}{2}\sum_k \partial_\mu \hat{\varphi}^{(k)} \partial_\nu \hat{\varphi}^{*(k)} \eta^{\mu\nu} - \frac{1}{4}(\partial_\mu \hat{n}^{(0)}_\nu - \partial_\nu \hat{n}^{(0)}_\mu)(\partial^\mu \hat{n}^{\nu(0)} - \partial^\nu \hat{n}^\mu_{(0)}) + \sum_{k\neq 0} \frac{1}{a^2}\sin^2\frac{\pi k}{N}\Bigg\{\Bigg(\bigg(\hat{h}^{(k)}_{\mu\nu} \\ &- \frac{\eta_{\mu\nu}}{\sqrt{3}}\hat{\varphi}^{(k)}\bigg) - \frac{2a\partial_{(\mu}\hat{n}^{(k)}_{\nu)}}{e^{i2\pi k/N}-1}\Bigg)\Bigg(\bigg(\hat{h}^{*(k)}_{\alpha\beta} - \frac{\eta_{\alpha\beta}}{\sqrt{3}}\hat{\varphi}^{*(k)}\bigg) - \frac{2a\partial_{(\alpha}\hat{n}^{*(k)}_{\beta)}}{e^{-i2\pi k/N}-1}\Bigg)(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta})\Bigg\}. \end{aligned} \tag{57}$$

As we will now argue, this action describes, leaving aside zero modes, a tower of massive spin-two fields with a mass spectrum given by

$$m_k^2 = \frac{1}{a^2}\sin^2\frac{\pi k}{N}. \tag{58}$$

Indeed, if one concentrates on the massive modes, the symmetries of this action are as follows. First one has the following Stuckelberg symmetry acting at each mass level, namely:

$$\delta\hat{h}^{(k)}_{\mu\nu} = 2\partial_{(\mu}\xi^{(k)}_{\nu)}, \qquad \delta\hat{n}^{(k)}_\mu = \frac{(e^{i2\pi k/N}-1)}{a}\xi^{(k)}_\mu, \tag{59}$$

so that the vector fields are the Goldstone bosons which get absorbed by the $(S-1)$ gravitons that become massive. This symmetry was fully expected from the symmetry of the original action (50) since it comes from its invariance under the product of all the 4D diffeomorphism invariance on each site. It eliminates $8(S-1)$ DOFs out of the $10S + 4(S-1) + S = 15S - 4$ DOFs present in the action and corresponding, respectively, to the S 4D metrics at each site, the $S-1$ hopping fields Y in between adjacent sites and the S lapse functions on each site. However, there is an extra accidental symmetry acting at the quadratic level (in the action); the latter reads

$$\delta\hat{h}^{(k)}_{\mu\nu} = \eta_{\mu\nu}f^{(k)}, \qquad \delta\hat{\varphi}^{(k)} = \sqrt{3}f^{(k)}, \qquad \delta\hat{n}^{(k)}_\mu = \frac{a}{1-e^{-i2\pi k/N}}\partial_\mu f^{(k)}, \qquad k \neq 0, \tag{60}$$

and eliminates an extra $2(S-1)$ DOFs. We see that this leaves $5(S-1)$ propagating DOFs (not counting zero modes) corresponding to the announced $(S-1)$ massive gravitons. Interestingly, this symmetry is inherited from the reparametrization invariance along the discretized direction, which is, however, broken by the discretization. This is thus analogous to what was shown to happen for the Pauli-Fierz theory with the time reparametrization invariance. In particular, it is not expected that one can extend this symmetry at the nonlinear level so that extra degrees of freedom will start to propagate there.

B. DGP gravity

The DGP model is a four dimensionally covariant braneworld model that is closely related to massive gravity [13]. The DGP setup describes a three brane in a five-dimensional flat bulk, on which a large Einstein-Hilbert term is present. The action defining the model is taken to be

$$S = M_P^2 \int \sqrt{|\bar{g}|}\bar{R}d^4x + M_*^3 \int \sqrt{|g|}Rd^5X, \tag{61}$$

with $M_P \gg M_*$ and where a usual Gibbons-Hawking term taking care of the brane is implicit. What is important for us is that, from the point of view of an effective four-dimensional brane observer, the DGP action produces infrared modified Einstein gravity in a way related to massive gravity. Indeed, in a particular gauge, the fluctuations around Minkowski vacuum are described by a Pauli-Fierz-like effective action for a massive graviton with "running mass" $m_g^2 = p/L_{\rm dgp}$, where p is the graviton momentum and $L_{\rm dgp} = M_p^2/M_*^3$ is the DGP length scale. This scale was found to be the crossover scale between a small distance four-dimensional behavior of the Newtonian potential exchanged by nonrelativistic sources on the brane and a large distance five-dimensional behavior [13]. As in Pauli-Fierz theory, DGP theory propagates gravitons with five polarizations at linearized level. As such, one might expect that the model shares the same difficulties as massive gravity, especially the strong coupling and ghost problem. Indeed, the former property of DGP was made explicit in Refs. [15,16] in confirmation of the work [7] discussing the appearance of a Vainshtein scale in the model. The exact consequences of this strong coupling are subject to a debate [17,18]. Moreover, exact solutions [7,28] as well as approximate ones [29] indicate that a Vainshtein mechanism is at work in the DGP model, allowing one to recover solutions of Einstein general relativity (see, however, [30]). In light of our previous discussion of massive gravity, one might suspect that this Vainshtein "resummation" is, also in the DGP model, due to a dynamical ghost DOF at work. However, with the DGP model being 4D covariant, one should investigate this issue in detail. There is one obvious first qualitative difference between the two theories. Indeed, the operator in the DGP effective action, which grows large at a low scale (the strong coupling scale), is a dimension seven operator [15], as opposed to the dimension nine operator found in the Pauli-Fierz action. The effective theory for the longitudinal mode of the graviton in DGP is obtained by decoupling in a similar way to what was recalled in Sec. II for massive gravity. It reads [17]

$$\mathcal{L}_{\rm dgp} = 3\pi\Box\pi - \frac{1}{\Lambda_{\rm dgp}^3}(\partial_\mu\pi)^2\Box\pi + \frac{\pi T}{2M_p}. \tag{62}$$

The scale at which the π mode gets strongly coupled is given by $\Lambda_{\rm dgp} = (M_p/L_{\rm dgp}^2)^{1/3}$. One might think that the higher derivative operator appearing above also corresponds to extra degrees of freedom in an equivalent formulation. However, the equation of motion for this scalar are easily seen to be

$$6\Box\pi - \frac{1}{\Lambda_{\rm dgp}^3}(\partial_\mu\partial_\nu\pi)^2 + \frac{1}{\Lambda_{\rm dgp}^3}(\Box\pi)^2 = -\frac{T}{4M_p}. \tag{63}$$

Although this is a nonlinear equation of motion, it is a *second order* differential equation for the mode π, unlike the analogous equation for its Pauli-Fierz counterpart. Hence, as seen from a Cauchy problem perspective, Eq. (63) describes only *one* propagating degree of freedom. In Ref. [17], the geometrical meaning of this equation was clarified by noting that in the full theory it descends from a combination of the Gauss-Codazzi equations and 4D Einstein equations on the brane, which, by the relation $K_{\mu\nu} \sim -(1/\Lambda_{\rm dgp}^3)(\partial_\mu\partial_\nu\pi)$, makes Eq. (63) an algebraic equation for the extrinsic curvature of the brane. Considering now a pointlike nonrelativistic source, as done at the end of Sec. II, one finds that, inside the Vainshtein region, the solution for π is given at dominant order by the solution to the quadratic part of Eq. (63), that is to say, the equation obtained by dropping the first term in the left-hand side of (63) [17]. Thus, in order to estimate the contribution of π to the Newtonian potential around the source, one may say that π obeys in this region an equation similar to Eq. (21) obeyed by $\tilde{\psi}^{(1)}$, with the role of $\tilde{\psi}^{(0)}$ being played by T/M_p. This yields indeed corrections to the Schwarzschild solution that goes (correctly) as $(r/r_v^*)^{3/2}$, where $r_v^* \equiv (r_c^2 GM)^{1/3}$ is the Vainshtein radius for the DGP model. However, here there is no need for the cancellation of an extra contribution coming from another DOF.

Note that this does not tell us what the nature is of the small fluctuations around some background solution. In particular, in Ref. [17] it is shown that the kinetic term for this scalar is enhanced significantly in the Vainshtein region, which incidentally means that the π mode interacts more and more weakly as one approaches the source.

V. CONCLUSION

In this work, we compared the Goldstone formalism for massive gravity [8] to the old Hamiltonian approach of Boulware and Deser [3]. In the first approach it was found that the scalar polarization (the vDVZ scalar) of the graviton acquired a strong cubic interaction, while in the second approach it was shown that, at nonlinear level, massive gravity propagates a sixth degree of freedom with negative energy. We showed that one can reinterpret the strong coupling of the vDVZ scalar as the propagation of a ghost, in agreement with the Boulware-Deser finding. The ghost can then be seen as responsible for the cancellation of the attraction exerted by the vDVZ scalar at distances around a nonrelativistic source smaller than the Vainshtein radius. Inside this region, we can trust the two-field Lagrangian perturbatively, while at larger distances one can use the higher derivative formulation. Of course, within this region, the presence of the ghost is expected to signal instabilities. In particular, one is tempted to interpret the failure of the *full* Vainshtein resummation found in Ref. [10] as linked to the presence of this sick DOF, even if one should be careful to draw conclusions from energy arguments in general relativity. Note that the presence of this ghostlike DOF is a generic feature of massive gravity as shown in the work of Damour and Kogan [5].

We also compared massive gravity with other models, namely, deconstructed theories of gravity, as well as the DGP model. In the latter case we argued that the Vainshtein resummation process was of a different nature, not involving a ghost degree of freedom. There are other variations on the DGP model which could free of ghosts, too. For instance, Ref. [31] describes a model in which the five (or higher) dimensional Einstein-Hilbert action has a profile in the extra dimensions (or, alternatively, has a varying Planck mass in the extra space). In this case, the absence of ghosts or strong coupling naturally introduces an incurable vDVZ discontinuity in the theory. We also note that the setup of "soft massive gravity" [32] uses higher dimensional generalizations of the DGP model where the usual induced Einstein-Hilbert action is supplied with additional UV operators, and Ref. [32] argues that it does not suffer from any strong coupling issues. One interesting other way to avoid problems of massive gravity might be to break Lorentz invariance [33–36].

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APPENDIX: DOF REDUCTION IN THE GENERAL CASE

We start from the Lagrangian (6),

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{\Lambda^5}((\Box\phi)^3 - \Box\phi(\partial_\mu\partial_\nu\phi)^2), \tag{A1}$$

and want to analyze the degrees of freedom it propagates. There are several ways to proceed. There is the "constructive approach" (see, e.g., [23], which uses a similar mechanism to reduce higher derivative gravity theories) or the formal Ostogradski method for higher derivative scalar field theories, as in, e.g., Ref. [20]. Both methods yield, in fact, the same result, and we will discuss here only the first one, which is in direct correspondence with what we did in Sec. II.

One can rewrite the Lagrangian (A1) as

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + F(\partial_\mu\partial_\nu\phi), \tag{A2}$$

where $F(X_{\mu\nu}) = (1/\Lambda^5)(X_{\mu\mu}^3 - X_{\mu\mu}X_{\mu\nu}^2)$. Next we consider

$$\tilde{\mathcal{L}} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{dF}{dX_{\mu\nu}}(\partial_\mu\partial_\nu\phi - X_{\mu\nu}) + F(X_{\mu\nu}). \tag{A3}$$

The equations of motion for $X_{\mu\nu}$ and ϕ obtained from this Lagrangian read

$$0 = \Box\phi + \partial_\mu\partial_\nu\frac{dF}{dX_{\mu\nu}}, \tag{A4}$$

$$0 = \frac{d^2F}{dX_{\mu\nu}dX_{\alpha\beta}}[\partial_\alpha\partial_\beta\phi - X_{\alpha\beta}]. \tag{A5}$$

Those equations reduce to those deduced from the Lagrangian (A1) if the 16×16 matrix $\mathcal{M}_{(\mu,\nu),(\alpha,\beta)}$ given by

$$\mathcal{M}_{(\mu,\nu),(\alpha,\beta)} = \frac{d^2F}{dX_{\mu\nu}dX_{\kappa\lambda}} \tag{A6}$$

is invertible. In our case, this matrix is given by

$$\frac{d^2F}{dX_{\mu\nu}dX_{\alpha\beta}} = \frac{2}{\Lambda^5}(3\eta^{\mu\nu}\eta^{\alpha\beta}X - \eta^{\mu\nu}X^{\alpha\beta} - \eta^{\alpha\beta}X^{\mu\nu} - X\eta^{\mu\alpha}\eta^{\nu\beta}), \tag{A7}$$

where X stands for $X_{\mu\nu}\eta^{\mu\nu}$. The determinant of the above defined matrix reads

$$\det\mathcal{M} = -2^{18}X^{14}\Lambda^{-80}(4X^2 + X_{\alpha\beta}X^{\alpha\beta}) \tag{A8}$$

and does not vanish, in general. Now we define

$$\pi^{\mu\nu} = \frac{dF(X)}{dX_{\mu\nu}}, \tag{A9}$$

which we invert to get $X_{\mu\nu}(\pi)$ as a function of $\pi^{\mu\nu}$. This inversion is guaranteed to exist again (at least locally) when $\mathcal{M}$ can be inverted, but the explicit form of the inverse function may not be easy to obtain. Also, one may have to divide the theory into different branches if the function is not globally well defined, in a similar way as what was done in Sec. II. In any case, our new Lagrangian is now

$$\tilde{\mathcal{L}} = -\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \pi^{\mu\nu}(\partial_\mu\partial_\nu\phi - X_{\mu\nu}(\pi)) + G(\pi_{\rho\sigma}), \tag{A10}$$

where $G(\pi_{\rho\sigma}) \equiv F(X_{\mu\nu}(\pi_{\rho\sigma}))$. The function G might now be regarded as a potential for the field $\pi_{\rho\sigma}$ [together with $\pi^{\mu\nu}X_{\mu\nu}(\pi)$]. The investigation of the nature of the degree(s) of freedom in $\pi^{\mu\nu}$ in our case simplifies drastically because (unlike generic higher derivative gravity, for example) we have a special mixing with a scalar: The coupling $\pi^{\mu\nu}\partial_\mu\partial_\mu\phi$ guarantees that there is only one component of $\pi^{\mu\nu}$ that propagates due to mixing with ϕ. Indeed, decomposing the symmetric tensor into its orthogonal components

$$\pi^{\mu\nu} = h_{tt}^{\mu\nu} + \partial^{(\mu}A_t^{\nu)} + \left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right)a + \frac{\partial^\mu\partial^\nu}{\Box}\lambda, \tag{A11}$$

where $h_{tt}^{\mu\nu}$ and A_t^ν are, respectively, transverse-traceless and transverse, we see that only λ gets a kinetic term by mixing with ϕ. The above form of the Lagrangian ensures that λ is always a ghost. Indeed, the Lagrangian is easily diagonalized by the transformation $\phi \to \psi = \phi - \lambda$. The final Lagrangian for the two DOFs is

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\psi\partial^\mu\psi + \tfrac{1}{2}\partial_\mu\lambda\partial^\mu\lambda - V(\pi_{\rho\sigma}(\lambda)), \tag{A12}$$

with $V(\pi_{\rho\sigma}) = \pi^{\mu\nu}X_{\mu\nu}(\pi_{\rho\sigma}) - G(\pi_{\rho\sigma})$. Although the above procedure is a bit formal, we did exactly this for the interaction $(\Box\phi)^3$, where indeed we encountered two branches and found a potential for the ghost λ by inverting the scalar analogue of Eq. (A9). The exact form of the potential is now harder to find, having to solve Eq. (A9), which reads

$$\pi^{\mu\nu} = \frac{1}{\Lambda^5}(3\eta^{\mu\nu}X^2 - \eta^{\mu\nu}X_{\alpha\beta}X^{\alpha\beta} - 2XX^{\mu\nu}) \tag{A13}$$

for $X_{\mu\nu}$ as a function of $\pi^{\rho\sigma}$. The least one can say by dimensional analysis is that the potential will be again IR relevant, as opposed to the UV relevant operator that we started from in (A1).

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