Thermodynamics and stability of higher dimensional rotating (Kerr-)AdS black holes

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We study the thermodynamic and gravitational stability of Kerr anti-de Sitter black holes in five and higher dimensions. We show, in the case of equal rotation parameters, $a_i = a$, that the Kerr-AdS background metrics become stable, both thermodynamically and gravitationally, when the rotation parameters a_i take values comparable to the AdS curvature radius. In turn, a Kerr-AdS black hole can be in thermal equilibrium with the thermal radiation around it only when the rotation parameters become not significantly smaller than the AdS curvature radius. We also find with equal rotation parameters that a Kerr-AdS black hole is thermodynamically favored against the existence of a thermal AdS space, while the opposite behavior is observed in the case of a single nonzero rotation parameter. The five-dimensional case is however different and also special in that there is no high temperature thermal AdS phase regardless of the choice of rotation parameters. We also verify that at fixed entropy, the temperature of a rotating black hole is always bounded above by that of a nonrotating black hole, in four and five dimensions, but not in six and more dimensions (especially, when the entropy approaches zero or the minimum of entropy does not correspond to the minimum of temperature). In this last context, the six-dimensional case is marginal.

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I. INTRODUCTION

Black holes are perhaps the most tantalizing objects in general relativity. Recently, the study of black holes in a background anti-de Sitter spacetime has been well motivated from developments in string/M theory, which naturally incorporate black holes as solitonic D-branes, or simply branes as the higher-dimensional progenitors of black holes. An intriguing example of this is the conjectured duality [1] between string theory on $AdS_5 \times S^5$ background and $\mathcal{N} = 2$ super Yang-Mills theory in four dimensions, and, in particular, Witten's interpretation [2] of the Hawking-Page phase transition between thermal AdS and AdS black hole [3] as the confinement-deconfinement phases of the dual gauge theory defined on the asymptotic boundaries of the AdS space.

Much effort has been put into the weak AdS gravity regime, analyzing the implications of AdS black holes on dual (gauge) theories at nonzero temperature, using the conjectured AdS/CFT (conformal field theory) correspondence. In this context, the most interesting black hole solutions are presumably the five-dimensional Kerr-AdS solutions for a stationary black hole [4]. The thermodynamics of AdS quantum gravity has been extensively used to infer the thermodynamics of quantum field theory in the large N (or weak field) limits, with an AdS gravity dual, such as Schwarzschild-AdS [2], Kerr-Newman-AdS [5–8] and hyperbolic-AdS [9–11] black holes.

The Kerr metric [12] is a simple explicit exact solution of the Einstein vacuum equations describing a rotating black hole in a four-dimensional flat spacetime. Shortly after Kerr's discovery, Carter [13] provided an elegant generalization of the Kerr solution in four-dimensional de Sitter and anti-de Sitter backgrounds. A higher-dimensional generalization of the Kerr metric in a flat background was given by Myers and Perry [14]. But its generalization to five and higher dimensions with a nonzero cosmological constant was given, only recently, by Hawking *et al.* [4] and Gibbons *et al.* [15,16]. There has been recent interest in constructing the analogous charged rotating solutions in gauged supergravity in four, five and seven dimensions [17], and also on the nonuniqueness [18] of those solutions in five and higher dimensions.

The study of noncharged rotating (Kerr) black holes is interesting at least for two reasons. First, the thermodynamics of Kerr black holes, in a background AdS space, can give rise to interesting descriptions in terms of CFTs defined on the (conformal) boundary of AdS space, leading to a better understanding of the AdS/CFT correspondence [4]. Second, astronomically relevant black hole spacetimes are, to a very good approximation, described by the Kerr metric.

As not much is known about the stability of Kerr-AdS black holes in higher dimensions, in this paper we study the thermodynamic stability of these black holes in five and higher dimensions. We also investigate the gravitational stability of a background Kerr-(A)dS spacetime under metric perturbations.

The layout of the paper is as follows. We begin in Sec. II by outlining the (anti)-de Sitter background metrics in d dimensions and their generalizations to Kerr-AdS solutions. In Sec. III we pay special attention to the thermodynamic stability of Kerr-AdS black holes by studying the behavior of Hawking temperature, free energy and specific heat in various dimensions. In Sec. IV we study the gravitational stability of background Kerr-(A)dS metrics under linear tensor perturbations. The stability of a rotating anti-de Sitter background spacetime in dimensions higher than four was not previously studied. Our linearized perturbation equations have other interesting applications. In particular, they allow us to study the stability of background AdS metrics with nontrivial rotation parameters.

Separability of Hamilton-Jacobi and Klein-Gordon equations in the Kerr (anti)-de Sitter backgrounds was discussed in [19], especially in the limit when all rotation parameters take the same value, see [20] for a discussion in five dimensions. An earlier work on separability of the Hamilton-Jacobi equation and quantum radiation from a five-dimensional Kerr black hole with two rotation parameters, but in an asymptotically flat background, can be found in [21]. However, our analysis is different. It corresponds not to a separability of the wave equations for a particle but rather to a separability of radial and angular wave equations under linear tensor perturbations.

II. ADS AND KERR-ADS METRICS

One of the interesting features of the Kerr metric in (anti)-de Sitter spaces is that it can be written in the so-called Kerr-Schild form, where the metric g_{ab} is given exactly by its linear approximation around the (anti)-de Sitter metric $\tilde{g}_{ab}$ as follows [15,19]:

$$ds^2 = g_{ab}dx^a dx^b = \tilde{g}_{ab}dx^a dx^b + \frac{2M}{U}(k_a dx^a)^2, \quad (1)$$

where k_a is a null geodesic with respect to both the full metric g_{ab} and the (A)dS metric $\tilde{g}_{ab}$. Moreover, the Ricci tensor of g_{ab} is related to that of $\tilde{g}_{ab}$ by

$$R_a{}^b = \tilde{R}_a{}^b - \tilde{R}_a{}^c h_c{}^b + \frac{1}{2}(\tilde{\nabla}_c\tilde{\nabla}_a h^{bc} + \tilde{\nabla}^c\tilde{\nabla}^b h_{ac} - \tilde{\nabla}^c\tilde{\nabla}_c h_a{}^b), \quad (2)$$

where $h_{ab} = \frac{2M}{U}k_a k_b$, with M and U being the parameters proportional to the mass and gravitational potential of a Kerr black hole, respectively. Thus, the stability of a Kerr metric under metric perturbation is specific to the stability of the background metric, which is given by $M = 0$.

Let us begin with a five-dimensional (anti)-de Sitter metric in the standard form:

$$\widetilde{ds}^2 = -(1 + cy^2)dt^2 + \frac{dy^2}{1 + cy^2} + y^2\left(\frac{dx^2}{1 - x^2} + (1 - x^2)d\hat{\phi}_1^2 + x^2 d\hat{\phi}_2^2\right), \quad (3)$$

which satisfies $R_{\mu\nu} = -4cg_{\mu\nu}$, with $c > 0$ in AdS space. The apparent singularities at $x = \pm 1$ are merely coordinate singularities. By defining $x = \cos\hat{\theta}$, one sees that the coordinate x has a range $-1 \le x \le 1$ while $(\hat{\phi}_1, \hat{\phi}_2)$ have a period 2π, so $(x, \hat{\phi}_1)$ parametrizes (topologically) a 2-sphere, while $\hat{\phi}_2$ parametrizes an S^1 fiber.

The metric (3) is easily generalized to six and higher dimensions. In six dimensions, one has

$$\widetilde{ds}^2 = -(1 + cy^2)dt^2 + \frac{dy^2}{1 + cy^2} + y^2 d\Sigma_4^2, \quad (4)$$

where

$$d\Sigma_4^2 = \frac{d\Omega^2}{1 - x^2} + (1 - x^2)d\hat{\phi}_3^2 + \hat{\phi}_1^2 d\psi^2, \quad (5)$$

$$d\Omega^2 = (1 - \hat{\phi}_2^2)d\hat{\phi}_1^2 + (1 - \hat{\phi}_1^2)d\hat{\phi}_2^2 + 2\hat{\phi}_1\hat{\phi}_2 d\hat{\phi}_1 d\hat{\phi}_2, \quad (6)$$

and $x^2 = \hat{\phi}_1^2 + \hat{\phi}_2^2$. The apparent singularities at $x = \pm 1$ are again merely coordinate singularities. By defining

$$\hat{\phi}_1 = \sin\theta \sin\varphi, \qquad \hat{\phi}_2 = \sin\theta\cos\varphi,$$

one easily sees that $(\hat{\phi}_1, \hat{\phi}_2, \psi)$ parametrize (topologically) a 3-sphere. In fact, the generalized (anti)-de Sitter metric can be written in a more compact form:

$$\widetilde{ds}^2 = -(1 + cy^2)dt^2 + \frac{dy^2}{1 + cy^2} + y^2 \sum_{k=1}^{N+\varepsilon}(d\hat{\mu}_k^2 + \hat{\mu}_k^2 d\hat{\phi}_k^2) \quad (7)$$

satisfying

$$\sum_{i=1}^{N+\varepsilon} \hat{\mu}_i^2 = 1, \quad (8)$$

where $N = (d - 1)/2$, $\varepsilon = 0$ (if d is odd), or $N = (d - 2)/2$, $\varepsilon = +1$ (if d is even). Both in odd and even dimensions, there are N azimuthal coordinates $\hat{\phi}_i$, each with period 2π. But when d is even there is an extra coordinate $\hat{\mu}_{N+1}$, which lies in the interval $-1 \le \hat{\mu}_{N+1} \le 1$.

In AdS_d spaces the rotation group is $SO(d - 1)$ and the number of independent rotation parameters for a localized object is equal to the number of Casimir operators, which is the integer part of $(d - 1)/2$. Thus in four dimensions the metric of a Kerr black hole can have only one Casimir invariant of the rotation group $SO(3)$, which is uniquely defined by an axis of rotation, while in five dimensions it can have two independent rotation parameters associated with two possible planes of rotation.

One may introduce to (7) N rotation parameters, for example, using the following coordinate transformation:

$$y^2 = \sum_{i=1}^{N} \frac{(r^2 + a_i^2)\mu_i^2}{1 - ca_i^2}, \quad (9)$$

where $\sum_{i=1}^{N+\varepsilon} \mu_i^2 = 1$. The constants a_i which are introduced in (9) merely as parameters in a coordinate transformation may be interpreted as genuine rotation parameters after one adds to (7) the square of an appropriate null vector, as in (1). Using the following coordinate transformations [4]:

$$
\begin{aligned}
dt &= d\tau + \frac{2M}{V-2M}\frac{dr}{(1+cr^2)}, \\
d\hat{\phi}_i &= d\phi_i + ca_i d\tau + \frac{2M}{V-2M}\frac{a_i dr}{(r^2+a_i^2)},
\end{aligned} \tag{10}
$$

and combining the expressions (1), (7), and (9), one would obtain the Kerr (A)dS metrics in Boyer-Lindquist coordinates. We are not going into details of this construction but refer to Ref. [15] for an elegant discussion. In five dimensions, the metric of the Kerr-AdS solution is

$$
\begin{aligned}
ds^2 = {}& -W(1+cr^2)d\tau^2 + \frac{\rho^2 dr^2}{V-2M} + \frac{\rho^2}{\Delta_\theta}d\theta^2 \\
& + \sum_{i=1}^{2}\frac{r^2+a_i^2}{1-ca_i^2}\mu_i^2(d\phi_i + ca_i d\tau)^2 \\
& + \frac{2M}{\rho^2}\left(d\tau - \sum_{i=1}^{2}\frac{a_i\mu_i^2 d\phi_i}{1-ca_i^2}\right)^2,
\end{aligned} \tag{11}
$$

where $\mu_1 = \cos\theta$, $\mu_2 = \sin\theta$,

$$
\begin{aligned}
\rho^2 &= r^2 + a_1^2\cos^2\theta + a_2^2\sin^2\theta, \\
\Delta_\theta &= 1 - ca_1^2\cos^2\theta - ca_2^2\sin\theta^2, \\
V &= \frac{1}{r^2}(1+cr^2)(r^2+a_1^2)(r^2+a_2^2), \\
W &= \frac{\Delta_\theta}{\Xi_1\Xi_2}, \\
\Xi_i &= 1 - ca_i^2.
\end{aligned} \tag{12}
$$

In the limit $a_i \to 0$, one recovers the standard Schwarzschild-AdS metric. As we see shortly, black holes with nonzero rotation parameters, or, in general, Kerr-AdS black holes, enjoy many interesting properties distinct from Schwarzschild-AdS black holes.

III. THERMODYNAMICS OF KERR-ADS SOLUTIONS

Using the standard technique of background subtraction, Gibbons *et al.* [22] have recently calculated the regularized (Euclidean) actions for the Kerr-AdS black holes in arbitrary d (≥ 4) dimensions. The results are

$$
\hat{I} = -\frac{\mathcal{A}_{d-2}}{8\pi G\prod_j \Xi_j}\frac{\beta}{l^2}\left(l^{2N}\prod_{i=1}^{N}(R^2+\alpha_i^2) - ml^2\right) \tag{13}
$$

for odd $d(= 2N+1)$, and

$$
\hat{I} = -\frac{\mathcal{A}_{d-2}}{8\pi G\prod_j \Xi_j}\frac{\beta}{l}\left(Rl^{2N}\prod_{i=1}^{N}(R^2+\alpha_i^2) - ml\right) \tag{14}
$$

for even $d(= 2N+2)$, where

$$
\mathcal{A}_{d-2} = \frac{2\pi^{(d-1)/2}}{\Gamma[(d-1)/2]} \tag{15}
$$

is the volume of the unit $(d-2)$ sphere. In the above we have defined $c \equiv 1/l^2$, with l being the curvature radius of the (bulk) AdS space. The dimensionless parameters are $\Xi_j \equiv 1-\alpha_j^2$, $R \equiv r_+/l$ and $\alpha_i \equiv a_i/l$, where, as usual, r_+ is the radius of the horizon, which occurs at a root of $V - 2M = 0$, and $m \equiv M(r = r_+)$. The Hawking temperature, which is the inverse of the Euclidean period, $T \equiv 1/\beta$, is given by

$$
T = \frac{R}{2\pi l}\left[(1+R^2)\left(\sum_{i=1}^{N}\frac{1}{R^2+\alpha_i^2} + \frac{\varepsilon}{2R^2}\right) - \frac{1}{R^2}\right], \tag{16}
$$

where $\varepsilon = 0$ for odd d and $+1$ for even d.

The calculation of total energy in an asymptotically (A)dS background is somewhat trickier (see e.g. [22]), mainly because the analogous Komar integral for the relevant timelike Killing vector diverges, which then requires a regularization, see also Ref. [23] which presents a general analysis for the conserved charges and the first law of thermodynamics for the four-dimensional Kerr-Newman-AdS and the five-dimensional Kerr-AdS black holes. In this context, the conserved charges (energies) E and E' associated with different Killing vectors, respectively, ∂_t and $\partial_t + l^{-1}\alpha_i\partial_{\phi_i}$ are different. However, the calculation of free energy itself is unambiguous. In fact, one can always identify the free energy of a Kerr-AdS black hole as $F = \hat{I} \times 1/\beta$, and hence

$$
F = \frac{\mathcal{A}_{d-2}}{16\pi G}\frac{(lR)^{d-3}(1-R^2)}{\prod_j \Xi_j}\prod_{i=1}^{N}\left(1+\frac{\alpha_i^2}{R^2}\right). \tag{17}
$$

This result is modified from that of a Schwarzschild-AdS black hole by certain terms in the product which are now functions of R and the rotation parameters α_i.

A. Thermal phase transition

In four spacetime dimensions black holes are stable (see e.g. [24]), but the issue of stability may be raised in five and higher dimensions. The five-dimensional Kerr-AdS solutions are particularly interesting as these could be embedded into IIB supergravity in ten dimensions.

From (17), it is readily seen that a phase transition between the background AdS space and the black hole is set by the scale $R = 1$, so that $R > 1$ corresponds to AdS black hole ($F < 0$) and $R < 1$ to a thermal AdS space ($F > 0$). This behavior may be seen also in terms of the charge or potential if present. In general, when the values of the rotation parameters α_i are decreased, the free energy lowers towards zero at low temperature. For $0 < \alpha \ll 1$, in the small R region, F nearly approaches but never touches the $F = 0$ axis (see Figs. 1 and 2). That is, the free energy curve crosses the $F = 0$ axis only once, namely, when

$R = 1$, which usually corresponds to the Hawking-Page phase transition point. But, in dimensions $d \geq 6$, this alone does not mean that a first order phase transition of a Hawking-Page type is essentially present.

In five dimensions, with $\alpha_1 = \alpha_2 \equiv \alpha > 0$, there is a minimum R below which the temperature appears to be negative and also diverges as $R \to 0$ (see Fig. 1), which is clearly unphysical. In fact, there is a minimum in temperature below which the Kerr black holes simply do not exit. Nevertheless, the plots in Fig. 3 show that the free energy can be a well-defined function of temperature. We also note that the specific heat is a monotonically increasing function of temperature when $\alpha \geq 0.17$.

The Hawking temperature of a Kerr-AdS black hole with a nonvanishing rotation parameter approaches zero as R goes to zero. The free energy is still a smooth function of both the horizon size and the temperature (see Figs. 2 and 4). These all imply a thermodynamic stability of a small Kerr black hole in AdS_5 space.

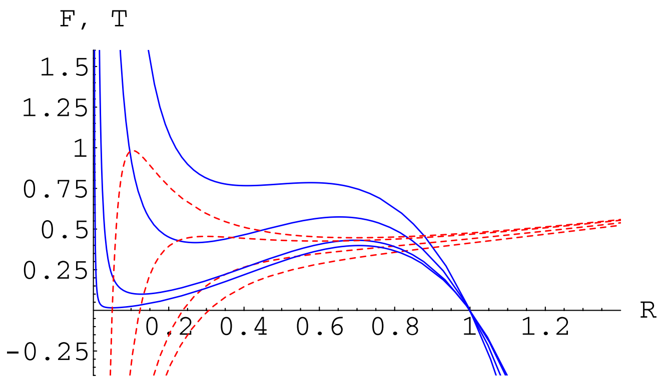

FIG. 1 (color online). ($d = 5$) The free energy (solid lines) and temperature (dashed lines) as a function of horizon position, with $\alpha_1 = \alpha_2 \equiv \alpha$. From top to bottom (free energy) or bottom to top (temperature): $\alpha = 1/3, 0.25, 1/8, 0.05$. In all plots we have set $4G = 1$.

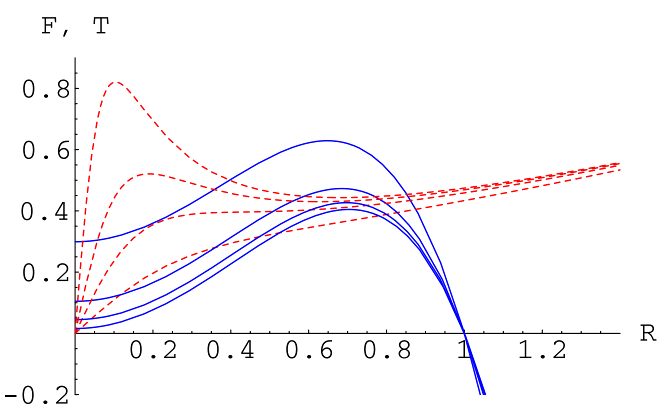

FIG. 2 (color online). ($d = 5$) The free energy (solid lines) and temperature (dashed lines) vs horizon position, with $\alpha_1 \equiv \alpha$, $\alpha_2 = 0$. From top to bottom (free energy) or bottom to top (temperature): $\alpha = 0.4, 0.25, 1/6, 0.1$.

The thermodynamic behavior above is essentially the opposite in six dimensions, where the temperature always diverges at $R = 0$. As the plots in Figs. 5–8 show the thermodynamics of single parameter solutions are quite different from those with equal rotation parameters. We

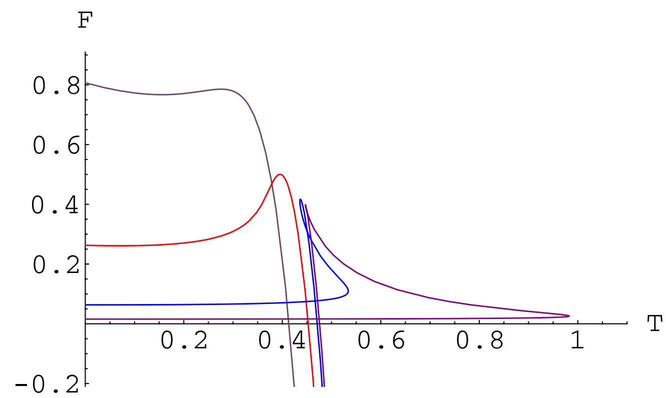

FIG. 3 (color online). ($d = 5$) The free energy vs temperature with $\alpha_1 = \alpha_2 \equiv \alpha$. From top to bottom: $\alpha = 1/3, 0.2, 0.1, 0.05$.

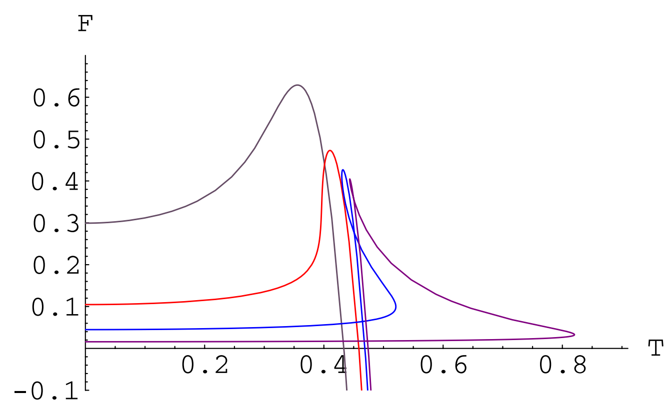

FIG. 4 (color online). ($d = 5$) The free energy vs temperature with $\alpha_1 \equiv \alpha$, $\alpha_2 = 0$. From top to bottom: $\alpha = 0.4, 0.25, 1/6, 0.1$.

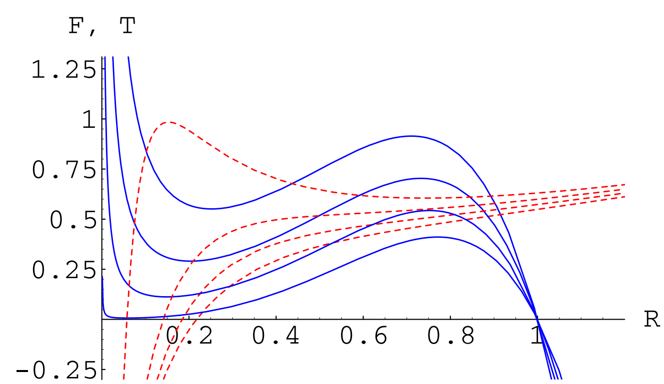

FIG. 5 (color online). ($d = 6$) The free energy and temperature vs horizon position, with $\alpha_1 = \alpha_2 = \alpha$. From top to bottom: $\alpha = 0.4, 1/3, 0.25, 0.1$.

also note that the $F < 0$ region in Figs. 9–12 corresponds to $R > 1$, while the region $F \geq 0$ corresponds to $0 \leq R \leq 1$. When $d \geq 6$, in the case of equal rotation parameters, only small black holes are globally preferred and locally stable, while in the case of a single rotation parameter, a thermal AdS phase is more preferred. The behavior in five dimensions is special in that there is no high temperature

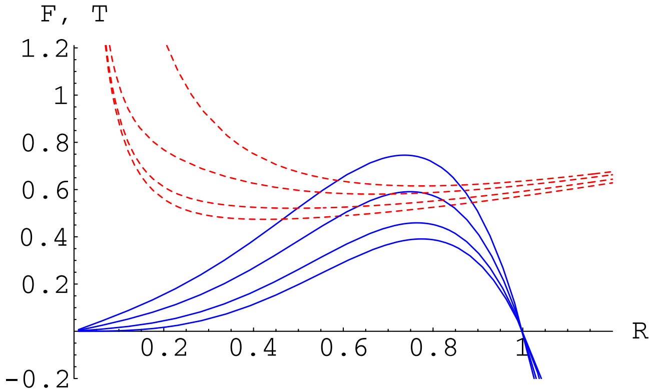

FIG. 6 (color online). ($d = 6$) The free energy and temperature vs horizon position, with $\alpha_1 \equiv \alpha$, $\alpha_2 = 0$. From top to bottom: $\alpha = 0.5, 0.4, 0.25, 0.04$.

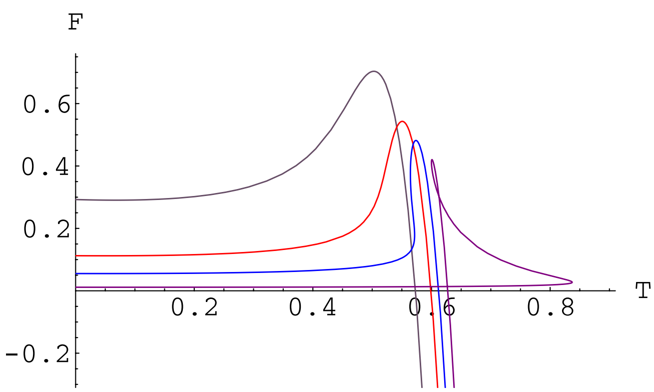

FIG. 7 (color online). ($d = 6$) The free energy vs temperature with $\alpha_1 = \alpha_2 \equiv \alpha$. From top to bottom: $\alpha = 1/3, 0.25, 0.2, 0.12$.

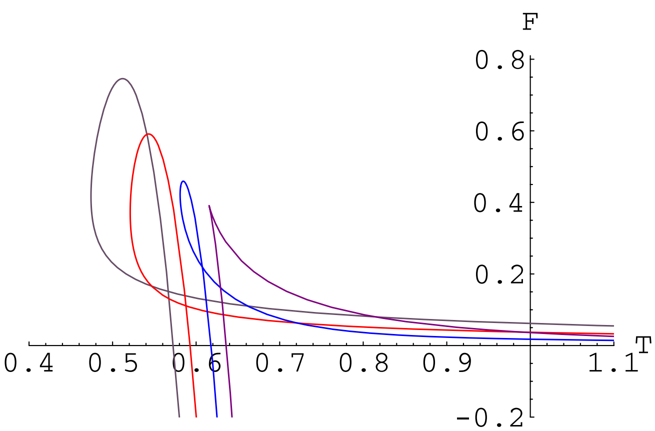

FIG. 8 (color online). ($d = 6$) The free energy vs temperature with $\alpha_1 = \alpha$, $\alpha_2 = 0$. From left to right: $\alpha = 0.5, 0.35, 0.2, 0.05$.

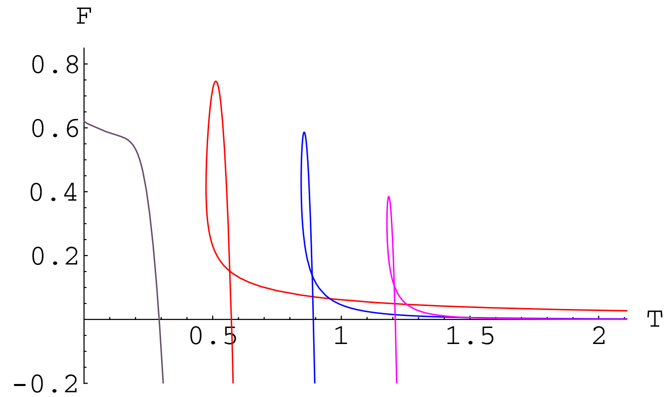

FIG. 9 (color online). The free energy vs temperature with a single nonvanishing rotation parameter α. From left to right $d = 4$ (with $\alpha = 0.3$) and $d = 6, 8, 10$ (with $\alpha = 0.5$).

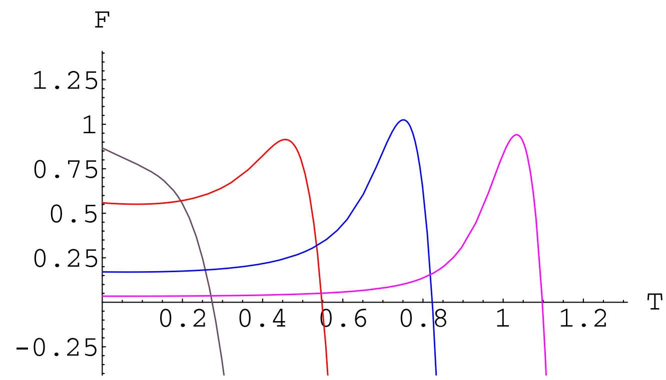

FIG. 10 (color online). The free energy vs temperature with equal rotation parameters, $\alpha_i = \alpha = 0.4$. From left to right $d = 4, 6, 8, 10$.

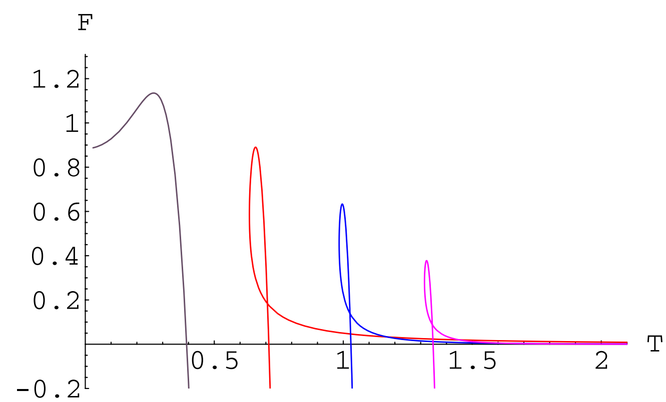

FIG. 11 (color online). The free energy vs temperature with a single nonvanishing rotation parameter α. From left to right: $d = 5, 7, 9, 11$ (with $\alpha = 0.6$).

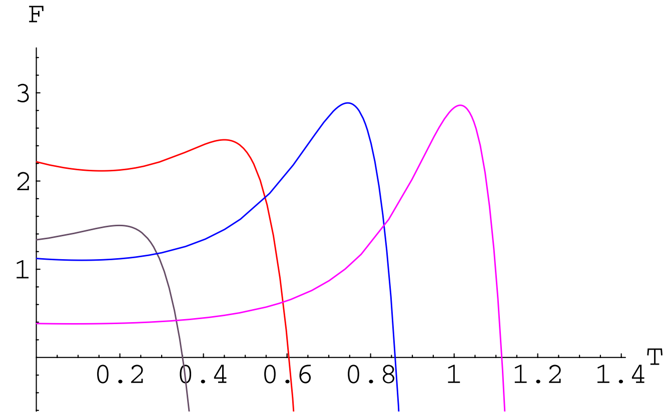

FIG. 12 (color online). The free energy vs temperature with equal rotation parameters, $\alpha_i = \alpha = 0.5$. From left to right $d = 5, 7, 9, 11$.

thermal AdS phase regardless of the choice of rotation parameters.

B. The first law of AdS bulk thermodynamic

One of the simplest ways of calculating the energy in an asymptotically AdS background is to integrate the first law of (bulk) thermodynamics:

$$dE = TdS + \sum_i \Omega_i J_j, \tag{18}$$

where the entropy S and angular momenta (of a rotating black hole) J_i are defined via

$$S = \beta \frac{\partial \hat{I}}{\partial \beta} - \hat{I}, \qquad J_i = -\frac{\partial F}{\partial \Omega_i}, \tag{19}$$

where

$$\Omega_i \equiv \frac{\alpha_i (1 + R^2) l}{R^2 + \alpha_i^2}. \tag{20}$$

In Ref. [22], the mass (energy) of a Kerr-AdS black hole was evaluated, by demanding *as a priori* that entropy of the black hole is one-quarter the area, $S = A/4$, in order to satisfy (18). The results are

$$d = 2N + 1 = \text{odd}:$$
$$E = \frac{m\mathcal{A}_{d-2}}{4\pi \Pi_j \Xi_j}\left(\sum_{i=1}^{N} \frac{1}{\Xi_i} - \frac{1}{2}\right),$$
$$S = \frac{\mathcal{A}_{d-2}}{4} (lR)^{2N-1} \prod_{i=1}^{N}\left(1 + \frac{\alpha_i^2}{R^2}\right)\frac{1}{\Xi_i}, \tag{21}$$

$$d = 2N + 2 = \text{even}:$$
$$E = \frac{m\mathcal{A}_{d-2}}{4\pi \Pi_j \Xi_j} \sum_{i=1}^{N} \frac{1}{\Xi_i},$$
$$S = \frac{\mathcal{A}_{d-2}}{4} (lR)^{2N} \prod_{i=1}^{N}\left(1 + \frac{\alpha_i^2}{R^2}\right)\frac{1}{\Xi_i}. \tag{22}$$

This result differs from the expression of energy suggested by Hawking *et al.* in [4], both in odd and even dimensions,

$$E' = \frac{m\mathcal{A}_{d-2}}{4\pi \prod_{j=1}^{N} \Xi_j} \frac{(d-2)}{2}. \tag{23}$$

The reason for this is that the energy (23) is measured in a frame rotating at infinity with the angular velocities:

$$\Omega_i' = \frac{\alpha_i \Xi_i l}{R^2 + \alpha_i^2}, \tag{24}$$

instead of (20). Since the angular velocities differ by $\Omega_i - \Omega_i' = \alpha_i l$, the two results above, (21) or (22) and (23), agree only in the limit $\alpha_i \to 0$ (i.e. $\Sigma_i \to 1$).

A remark is in order. The energy of background AdS spacetime (i.e. $m = 0$) is expected to be the same as the Casimir energy of a dual field theory in one dimension lower, up to a conformal factor. But from the above result one finds $E = 0$ when $m = 0$. To understand this apparent discrepancy, it should be noted that the Arnowitt-Deser-Misner mass M is only a local definition of black hole energy, while the total energy of a localized object in a curved background normally takes into account the asymptotic value of the background itself (which is nonzero in the AdS space). And, in general, one can write $E = M + E_0$, where E_0 is an integration constant. For example, for a Schwarzschild-AdS black hole with hyperbolic symmetry ($k = -1$), E_0 may be given by $E_0 = -M_e$, where M_e is the black hole mass at the extremal limit, see e.g. [11].

C. The specific heat and thermodynamic stability

A black hole as a thermodynamic system is unstable if it has negative specific heat. As is known, small Schwarzschild-AdS black holes (i.e. with $a_i = 0$) have negative specific heat but large size black holes have positive specific heat. While there also exists a discontinuity of the specific heat as a function of temperature at $R = 1/\sqrt{2}$, and so small and large black holes are found to be somewhat disjoint objects. However, this is essentially not the case when some of a_i are nontrivial, and especially, when $1 \gtrsim \alpha_i \gg 0$, e.g., the small Kerr black holes in AdS_5 space also have positive specific heat.

To this end, we shall study the thermodynamic stability of a Kerr-AdS black hole by evaluating its specific heat, which is given by

$$C_v = \frac{\partial E}{\partial T}. \tag{25}$$

Figures 13 and 14 show the plots of energy and temperature differentials as functions of the horizon size R. In the $d = 5$ case, with equal rotation parameters, there is clearly a minimum R, below which the temperature diverges. There is also a minimum value of rotation parameter below which dT can be negative, which is $\alpha \approx 0.17$ in five dimensions. Above this value, both the temperature differential and the specific heat are positive, see Figs. 15 and 16. When plotted as a function of temperature, the minimum of energy corresponds to the minimum in temperature, and hence the specific heat is a monotonically increasing function of Hawking temperature. This is also the case with a single rotation parameter (see Fig. 17), but now the critical value is $\alpha \approx 0.25$.

It should also be noted that, with $d = 5$ and $\alpha_i = \alpha \leq 1/6$, the thermodynamic behavior of a Kerr-AdS black hole at high temperature can be very different from that at low temperature (see Fig. 17). A similar behavior is found in the case of a single nonvanishing rotation parameter,

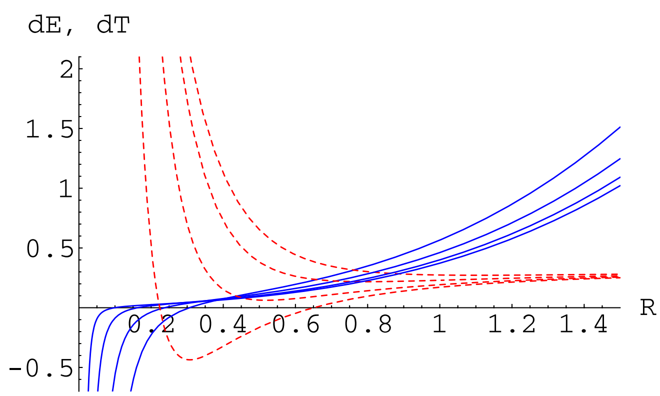

FIG. 13 (color online). ($d = 5$) The energy and temperature differentials vs horizon position, with $\alpha_1 = \alpha_2 \equiv \alpha$. As $R \to 0$, $dE \to +\infty$ and $dT \to -\infty$. From left to right: $\alpha = 0.1, 1/6, 0.25, 1/3$.

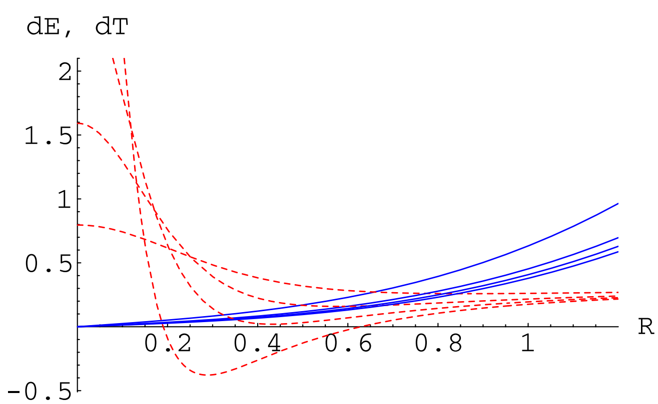

FIG. 14 (color online). ($d = 5$) The energy and temperature differentials vs horizon position, with $\alpha_1 = \alpha$, $\alpha_2 = 0$. As $R \to 0$, $dE \to 0$ and $dT > 0$. From top to bottom (free energy) or bottom to top (temperature) $\alpha = 0.5, 1/3, 0.25, 1/6$.

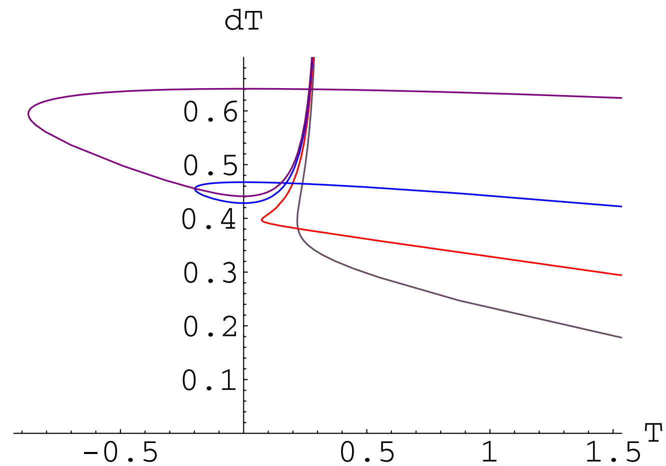

FIG. 15 (color online). ($d = 5$) The temperature differential vs temperature with $\alpha_1 = \alpha_2 = \alpha$; from top to bottom $\alpha = 0.08, 0.12, 0.17, 0.25$.

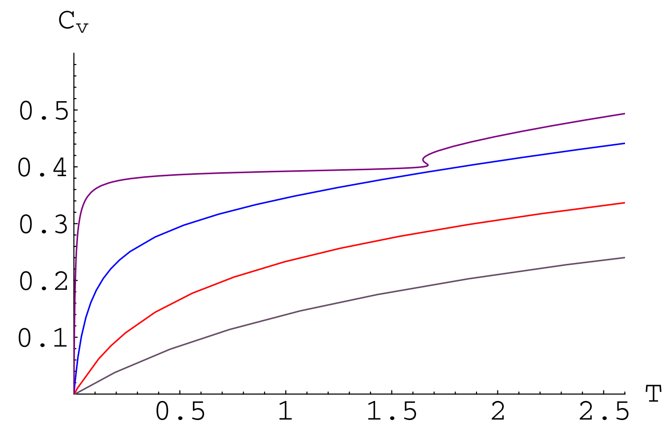

FIG. 16 (color online). ($d = 5$) The specific heat vs temperature with $\alpha_1 = \alpha_2 = \alpha$; from top to bottom $\alpha = 0.17, 1/3, 0.5, 0.6$. When α is $\lesssim 1/6$, then there would appear a new branch with almost constant specific heat at low temperature.

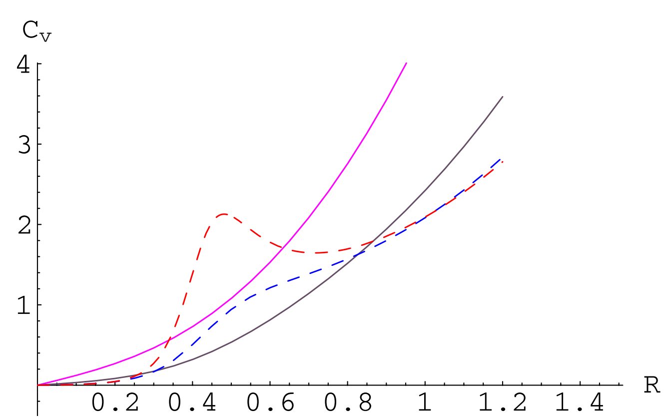

FIG. 17 (color online). ($d = 5$) The specific heat vs horizon position with $\alpha_1 = \alpha$, $\alpha_2 = 0$. From top to bottom: $\alpha = 0.7, 0.5$ (solid lines); and bottom to top: 0.3, 0.26 (dashed lines). With $\alpha \gtrsim 0.245$, the specific heat curve has a single branch and it is a monotonically increasing function of Hawking temperature.

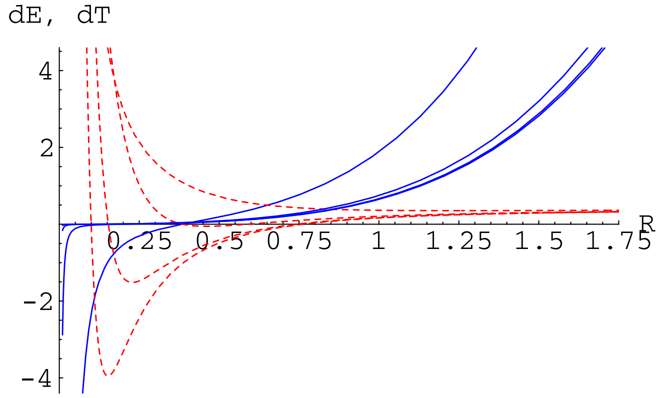

FIG. 18 (color online). ($d = 6$) The energy differential (solid lines) and temperature differential (dashed lines) vs horizon position, with $\alpha_1 = \alpha_2 \equiv \alpha$. From left to right ($dE$ in the $dE > 0$ region) or top to bottom (dT): $\alpha = 0.5, 0.2, 0.1, 1/15$.

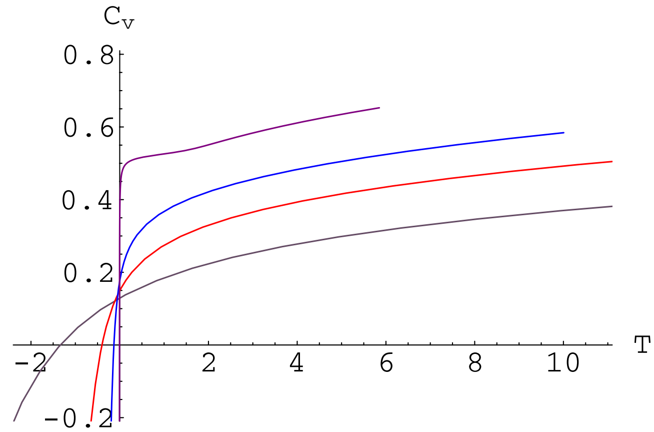

FIG. 20 (color online). ($d = 6$) The specific heat vs temperature with $\alpha_1 = \alpha_2 = \alpha$. From top to bottom $\alpha = 0.25, 0.5, 0.6, 0.7$.

though up to a slightly larger value of α ($\lesssim 1/4$). Put another way, in the AdS_5 background, small rotating black holes are unstable only for rotation parameters of order $0.15l$ or less; the precise limit is dimension dependent and black holes with larger angular velocities are thermodynamically stable.

Similarly, in dimensions $d \geq 6$, the Kerr-AdS black holes become unstable below some critical values of rotation parameters, for which a new branch would appear. When $d = 6$, with equal rotation parameters, the critical value is $\alpha \approx 0.22$ (see Figs. 18–20) and it is slightly higher in the case of a single rotation parameter.

Looking at the behavior of free energy and specific heat as functions of horizon position R, we remark that a five-dimensional Kerr-AdS black hole with a single rotation parameter is thermodynamically more stable over two (equal) rotation parameter solutions. But this is essentially not the case in dimensions six or more.

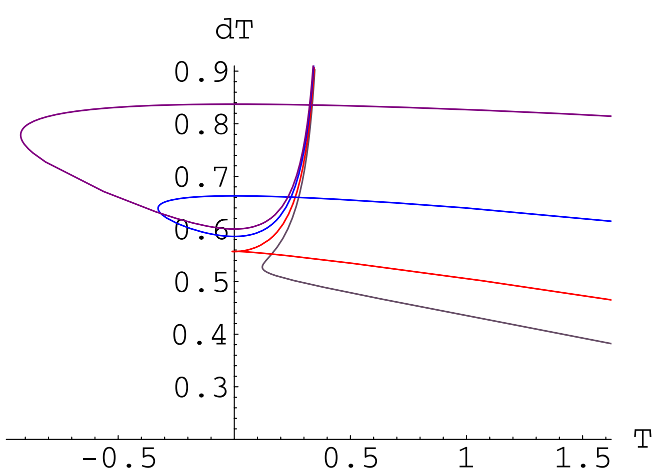

FIG. 19 (color online). ($d = 6$) The temperature differential vs temperature with $\alpha_1 = \alpha_2 \equiv \alpha$. From top to bottom $\alpha = 0.12, 0.16, 0.21, 0.25$.

In all odd dimensions, the specific heat has a single branch at high rotation but two branches at low rotations: the critical value of α which distinguishes these two cases increases with the number dimensions, and also with a number of nontrivial rotation parameters. A similar behavior is observed in all even dimensions $d \geq 6$, but in this case an interesting difference is that the specific heat can never be zero with $T > 0$.

It seems relevant to ask what happens at the critical angular velocity limit, $\alpha_i = 1$. Apparently, the action as well as the entropy is divergent in this limit. Nevertheless, as discussed in [4] (see also [25]), there exists a scaling of the mass parameter $m \rightarrow 0$ which makes the physical charges of the configuration finite. With equal rotation parameters, when $\alpha_i \rightarrow 1$, a Kerr-AdS black hole is more preferred than a thermal AdS phase even at low temperature. In fact, in all dimensions $d \geq 6$, small Kerr-AdS black holes with a single nonvanishing rotation parameter are unstable.

In our plots we have used the energy expressions suggested by Gibbons *et al.* [22], which differ from those suggested by Hawking *et al.* [4] by some overall constant factors. This itself does not introduce any significant difference in the behavior of specific heat and hence the thermodynamic stability of Kerr-AdS solutions. At any rate, the energy measured in a nonrotating frame appeared more suggestive to be used because it can be derived using various other methods [26–30]; the energy (or total mass) expressions given in [28], however, disagree with those in [22] in odd spacetime dimensions.

D. The temperature bound for rotating black holes

It was shown recently in [31] that at fixed entropy, the temperature of a rotating black hole is bounded above by that of a nonrotating black hole, in four and five dimensions, but not in six or more dimensions. We verify this claim by plotting temperature as a function of entropy, in

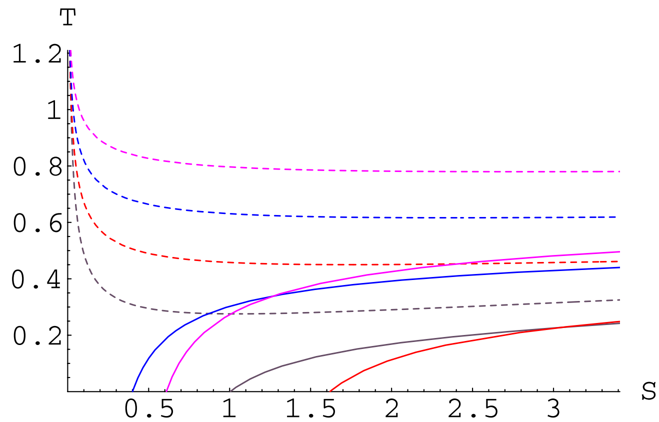

FIG. 21 (color online). The temperature vs entropy, in various dimensions with equal rotation parameters, $\alpha_i \equiv \alpha$. From top to bottom: $d = 7, 6, 5, 4$ with all $\alpha_i = 0$ (dashed lines); from left to right: $d = 6, 7, 4, 5$ (solid lines) each with $\alpha = 0.4$.

various dimensions; some of the plots are depicted in Figs. 21–23. In dimensions six or more, the minimum of entropy is not always the minimum of temperature, it actually depends upon the choice of rotation parameters. This is precisely the case where the inequality $T_{\text{Kerr-AdS}} \geq T_{\text{S-AdS}}$ may be realized with a very small entropy. But in this limit the temperature actually diverges, so the effect like this might be absent in a physical picture. At fixed entropy, but $S \gg 0$, the Hawking temperature of a rotating black hole is always suppressed relative to that of a nonrotating black hole and the inequality $T_{\text{Kerr-AdS}} < T_{\text{S-AdS}}$ holds in all dimensions. This result, presumably, holds with various charges and classical matter fields (such as gauge fields, dilaton, etc.) and is in accord with the earlier observation made by Visser while studying a static spherically symmetric case in four dimensions with no cosmological term [32].

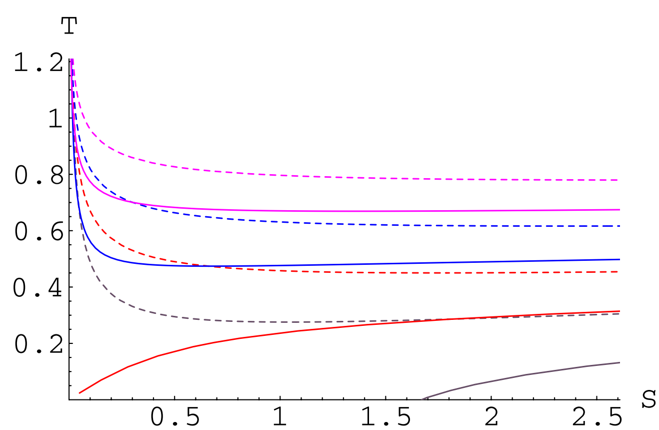

FIG. 22 (color online). The temperature vs entropy with a single rotation parameter α. From top to bottom: $d = 7, 6, 5, 4$ each with $\alpha = 0.4$ (solid lines), and $\alpha_i = 0$ (dashed lines).

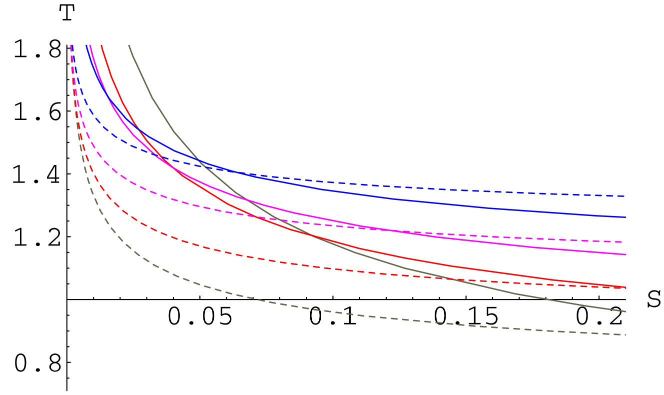

FIG. 23 (color online). The temperature vs entropy, in the small entropy region. From top to bottom (in the region $S > 0.1$): $d = 10, 9, 8, 7$ with $\alpha_1 = \cdots \alpha_{N-1} = 0.01$, $\alpha_N = 0.9$ (solid lines, Kerr-AdS) and $\alpha_i = 0$ (dashed lines, Schwarzschild-AdS).

A five-dimensional Kerr-AdS black hole with a single nonvanishing rotation parameter possesses an interesting (and perhaps desirable) feature; in this case the entropy vanishes when the temperature becomes zero. A similar feature is present in seven dimensions, but with two equal rotation parameters: $\alpha_1 = \alpha_2 \sim 0.33$, $\alpha_3 = 0$.

In a recent work [33], on the evolution of a five-dimensional rotating black hole via the scalar field radiation, Maeda *et al.* observed that, in a flat background ($c = 0$), the asymptotic state of a five-dimensional rotating black hole with a single nonvanishing parameter is described by $a \sim 0.11\sqrt{M}$. It would be interesting to know a similar result in an anti-de Sitter background, $c > 0$.

E. Rotation and the AdS-CFT correspondence

Following [2,4], one would expect the partition function of a Kerr-AdS black hole to be related to the partition function of a CFT in a rotating Einstein universe on the (conformal) boundary of the AdS space.

A curious observation in Ref. [34] is that the Cardy-Verlinde entropy formula works more naturally using the bulk thermodynamic variables defined by Hawking *et al.* [4]. This seems to indicate that the energy expression (23) is still relevant in a dual CFT. The Killing vector is then given by

$$\chi = \frac{\partial}{\partial \tau} + \Omega_i \frac{\partial}{\partial \phi_i}, \tag{26}$$

where ϕ_i are the angular coordinates. This property normally allows the thermal radiation to rotate with black hole's angular velocity all the way to conformal infinity.

One could ask whether or not the bulk thermodynamic variables suggested by Gibbons *et al.* [22], which were measured with respect to a frame that is nonrotating at infinity, can be mapped onto the boundary CFT variables by using the usual scaling argument. This does not seem to be the case as long as the CFT is assumed to be on a surface of large R in Boyer-Linquist coordinates. However, such a mapping might exist when the CFT is assumed to be on a large spherical surface, that is one for which the coordinate $y = \text{const}$ at large y. That is to say, it is possible that the set of bulk variables for Kerr-AdS black holes given by Gibbons *et al.* [22], in some (modified) form, match onto the boundary CFT variables that satisfy the first law of thermodynamics. This was indeed shown to be the case in [31].

Let us briefly discuss the role of nontrivial rotation parameters on the existence of an equilibrium between Kerr-AdS black hole and rotating thermal radiation around it. For this the requirement of a positive specific heat is a necessary condition. In five dimensions, the specific heat is always positive and also a monotonically increasing function of temperature when one (or both) of the rotation parameters takes a value at least one-quarter the AdS length scale l. This means, unlike in Minkowski (infinite) space, the rotating Kerr-AdS black holes can be in equilibrium with rotating thermal radiation around it, when $0 \ll \alpha_i \lesssim 1$; that is, the rotation parameter is not significantly smaller than the AdS curvature radius, so as to attain a stable equilibrium.

IV. STABILITY OF KERR SPACETIME UNDER GRAVITATIONAL PERTURBATIONS

In this section we study the gravitational stability of Kerr-AdS background metrics (with $M = 0$) in dimensions five and higher. For this purpose, it is sufficient to consider the following d-dimensional (time-independent) metric *Ansatz*:

$$g_{ab}(X)dX^a dX^b = g_{\mu\nu}(x)dx^\mu dx^\nu + \gamma(x)^2 d\Sigma_{k,n}^2(\tilde{x}), \quad (27)$$

where the metric $g_{ab}(X)$ is effectively separated into two parts: a diagonal "bulk" line element and $d\Sigma_{k,n}^2$, which is the metric on an n-dimensional base manifold whose curvature has not been specified (so $k = 0$ or ± 1), and hence can be replaced by any Einstein-Kähler metric with the same scalar curvature. However, in the present work we study only the $k = +1$ case, and hence the base $\mathcal{M}^n$ may be viewed as an S^1 fiber over S^{n-1} (for odd n) or as S^n (for even n). For example, in the $d = 5$ (or $n = 3$) case, the event horizon is $S^1 \times S^2$.

Under a small linear metric perturbation

$$g_{ab}(X) \to g_{ab}(X) + h_{ab}(X)$$

with $|h_a^b| \ll 1$, the variation in the Ricci tensor is given by

$$\delta R_{ab} = \frac{1}{2}\Delta_L h_{ab} - \frac{1}{2}\nabla_a \nabla_b h_c^c + \nabla_{(a}\nabla^c h_{b)c}, \quad (28)$$

where the spin-2 Lichnerowicz operator Δ_L is defined by (see, for example, [35])

$$\Delta_L h_{ab} = -\nabla^2 h_{ab} - 2R_{cadb}h^{cd} + 2R_{c(a}h_{b)}^c. \quad (29)$$

The stability of background metrics of the form (27) with $n > 2$, under certain metric perturbations, is specific to tensor perturbations. We therefore would like to restrict our analysis here to the tensor mode fluctuations that satisfy

$$h_{ab}(X) = 0$$

unless $(a, b) = (i, j)$, where the indices $a, b, \ldots$ run from $0 \cdots (d-1)$ and the indices $i, j, \ldots$ will run from $(d-n) \cdots (d-1)$ for the n-dimensional base. The variation of the Ricci tensor on the base $\mathcal{M}^n$ must then satisfy

$$\delta R_{ij} = \frac{1}{2}(\Delta_L h)_{ij} = -c(d-1)h_{ij}, \quad (30)$$

where c is the d-dimensional cosmological constant, with

$$\begin{aligned}\Delta_L h_{ij} = {} & \frac{1}{\gamma^2}\tilde{\Delta}_L h_{ij} + [-g^{\mu\nu}\partial_\mu\partial_\nu]h_{ij} \\ & + \sum_{\nu=1}^{d-n}\left[\partial^\sigma g_{\sigma\nu} - \frac{1}{2}g^{\sigma\rho}\partial_\nu g_{\sigma\rho} + (4-n)\frac{\partial_\nu\gamma}{\gamma}\right]\partial^\nu h_{ij} \\ & - \frac{4}{\gamma^2}[g^{\mu\nu}\partial_\mu\gamma(x)\partial_\nu\gamma(x)]h_{ij}, \end{aligned} \quad (31)$$

where $\tilde{\Delta}_L h_{ij}$ is the spin-2 Lichnerowicz operator acting on the base $\mathcal{M}^n$. The Lichnerowicz operator Δ_L is compatible with the transverse, trace-free (de Donder) gauge for h_{ab}: $h_a^a = 0 = h_{b;a}^a$, see e.g. [24].

A. Dependence on radial coordinate only

Let us first consider a background spacetime where $d = n + 2$, such that we can write the metric as

$$ds^2 = -\alpha(r)^2 dt^2 + \beta(r)^2 dr^2 + \gamma(r)^2 d\Sigma_n^2. \quad (32)$$

We can write the Lichnerowicz operator as

$$\begin{aligned}\Delta_L h_{ij} = {} & \frac{1}{\gamma^2}\tilde{\Delta}_L h_{ij} + \left[\frac{\partial_t^2}{\alpha^2} - \frac{\partial_r^2}{\beta^2}\right]h_{ij} \\ & + \left[\frac{\beta_r}{\beta} - \frac{\alpha_r}{\alpha} + (4-n)\frac{\gamma_r}{\gamma}\right]\frac{\partial_r h_{ij}}{\beta^2} - \frac{4}{\gamma^2}\frac{\gamma_r^2}{\beta^2}h_{ij}, \end{aligned} \quad (33)$$

where the subscripts t, r denote derivatives with respect to t, r respectively. In this case we find it convenient to choose

$$h_{ij} = \Psi(r)e^{\omega t}\tilde{h}_{ij}(\tilde{x}), \quad (34)$$

such that

$$(\tilde{\Delta}_L\tilde{h})_{ij} = \lambda\tilde{h}_{ij}, \quad (35)$$

where $\tilde{x}$ are coordinates on $\mathcal{M}^n$ and λ is the eigenvalue of the Lichnerowicz operator on $\mathcal{M}^n$. We want to write the perturbed Eqs. (30) in the form:

$$(\partial_{r_*}^2 - V(r_*))\Phi(r_*) = \omega^2\Phi(r_*). \quad (36)$$

To facilitate this we introduce two transformations:

$$dr = \frac{\partial r}{\partial r_*} dr_*, \qquad \Psi(r) = \chi(r)\Phi(r) \tag{37}$$

with

$$\chi(r) = C_1 \gamma^{(4-n)/2}. \tag{38}$$

We then find (see the Appendix for details)

$$V(r(r_*)) = \frac{\lambda \alpha^2}{\gamma^2} + \frac{n^2 - 10n + 8}{4}\left(\frac{\gamma_{r_*}}{\gamma}\right)^2 + \frac{(n-4)}{2}\frac{\gamma_{r_* r_*}}{\gamma} + 2(n+1)c\alpha^2, \tag{39}$$

where,

$$\gamma_{r_*} \equiv \frac{\partial r}{\partial r_*}\frac{\partial \gamma}{\partial r} = \frac{\alpha}{\beta}\gamma_r, \qquad \gamma_{r_* r_*} = \frac{\alpha^2}{\beta^2}\left[\gamma_{rr} + \left(\frac{\alpha_r}{\alpha} - \frac{\beta_r}{\beta}\right)\gamma_r\right]. \tag{40}$$

The above potential correctly reproduces the result in [11] [cf. Eq. (41) with $\alpha^2 = f(r)$ and $\gamma^2 = r^2$], see also [36,37]. Apparently, the case $n = 4$ is special.

B. Kerr-AdS backgrounds in odd dimensions

While we believe the stability analysis of Kerr-AdS background metrics can be generalized to nonequal rotation parameters (or angular momenta), we shall focus on the case with equal rotation parameters.

In the case of an odd number of spacetime dimensions $d = 2N + 1 = n + 2$, the zero-mass ($M = 0$) Kerr (anti)-de Sitter background metric may be given by

$$ds^2 = -\frac{(1 + cr^2)dt^2}{(1 - ca^2)} + \frac{r^2 dr^2}{(1 + cr^2)(r^2 + a^2)} + \frac{r^2 + a^2}{1 - ca^2} ds^2(\mathcal{M}^n), \tag{41}$$

where the rotation parameters are set equal (i.e., $a_1 = a_2 = a$). The base space $\mathcal{M}^n$, which is topologically S^{2N-1}, may be parametrized by the metric

$$(d\psi + A)^2 + d\Sigma_{N-1}^2 \tag{42}$$

where $d\Sigma_{N-1}^2$ is the canonically normalized Fubini-Study metric on an $(N-1)$ dimensional complex projective space $\mathbb{CP}^{N-1}$, and A is a local potential for the Kähler form $J = \frac{1}{2}dA$ on $\mathbb{CP}^{N-1}$. For example, in five dimensions, the metric on base $\mathcal{M}^3$ is $ds^2(\mathcal{M}^3) = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2$.

In the above background, the linear tensor perturbations satisfy

$$\Delta_L h_{ij} = -2c(n+1)h_{ij}, \tag{43}$$

where

$$\Delta_L h_{ij} = \left[-\frac{(r^2 + a^2)(1 + cr^2)}{r^2}\left(\frac{\partial^2}{\partial r^2} + \frac{4r^2}{(r^2 + a^2)^2}\right) - \left((n-2)cr + \frac{n-4}{r} - \frac{a^2(1 - cr^2)}{r^3}\right)\frac{\partial}{\partial r} + \frac{1 - ca^2}{1 + cr^2}\frac{\partial^2}{\partial t^2}\right]h_{ij} + \frac{1 - ca^2}{r^2 + a^2}(\tilde{\Delta}_L h)_{ij}. \tag{44}$$

In terms of the Regge-Wheeler type coordinate r_*, which may be defined by

$$dr = \frac{(1 + cr^2)\sqrt{r^2 + a^2}}{r\sqrt{1 - ca^2}} dr_*, \tag{45}$$

and using Eqs. (34) and (37) the differential equation is cast in the standard form:

$$-\frac{d^2\Phi}{dr_*^2} + V(r(r_*))\Phi = -\omega^2\Phi \equiv E^2\Phi, \tag{46}$$

where the potential is

$$V(r(r_*)) = \frac{\lambda(1 + cr^2)}{r^2 + a^2} + \frac{(n^2 - 10n + 8)(1 + cr^2)^2}{4(1 - ca^2)(r^2 + a^2)} + \frac{(3n-2)c(1 + cr^2)}{1 - ca^2}. \tag{47}$$

This potential is well behaved around $r = 0$ unlike for the AdS-Schwarzschild metric (i.e. $a = 0$).

There exists a criterion for stability (e.g. the Schrödinger equation not possessing a bound state with $\omega > 0$), in terms of the minimum Lichnerowicz eigenvalue, $\lambda_{\min}$, on the base manifold $\mathcal{M}^n$. In the case of a vanishing cosmological constant ($c = 0$), this criterion is the same as that for a Schwarzschild-AdS background [36]:

$$\lambda_{\min} \geq \lambda_c = 4 - \frac{(5-n)^2}{4} \Leftrightarrow \text{stability}, \tag{48}$$

a requirement that $\lambda_c \geq 0$ constrains the spacetime dimensions to $n \leq 9$ (or $d \leq 11$). The stability of a potential depends on the eigenvalue λ, ensuring that the potential is positive everywhere and bounded from below. Defining $\mu \equiv ca^2$, with $a > 0$, we require $\mu < 1$ for $c > 0$ and we find

$$\lambda \geq \tilde{\lambda}_c = -\frac{n^2 - 10n + 8 + 4(3n-2)\mu}{4(1-\mu)}. \tag{49}$$

The lower bound on ca^2 required for gravitational stability of the background metric is found to be stronger than that for thermodynamic stability. In the de Sitter case (i.e. $c < 0$), there is a mass gap, so λ starts from a finite value, and ca^2 is unbounded from below.

Instead of solving the Schrödinger equation directly in terms of r_*, one can solve the radial part of Eq. (43) by expressing it as a hypergeometric equation, whose solution is given by linear combinations of

$$\Psi_{\pm}(x,\mu) = \left(\frac{x+\mu}{c}\right)^{(5-n\pm2\nu)/4}(1+\mu)^{i\omega/2\sqrt{c}}{}_2F_1\left(\frac{\pm2\nu-(n-1)}{4}+\frac{i\omega}{2\sqrt{c}}, \frac{\pm2\nu+(n+3)}{4}+\frac{i\omega}{2\sqrt{c}}, \pm\nu+1; -\frac{x+\mu}{1-\mu}\right), \tag{50}$$

where $x \equiv cr^2$, and

$$\nu = \frac{1}{2}\sqrt{4\lambda + (5-n)^2 - 16}. \tag{51}$$

We note that the reality of ν immediately gives the stability condition (48). The reality of the solution also requires $\omega = i\tilde{\omega}$ which implies that there are no exponentially growing (unstable) modes. Requiring the solution to be bounded as $r \to \infty$ fixes one arbitrary constant which leaves Ψ decaying as r^{1-n}. Given that $\Psi = \chi\Phi$ we find that Φ decays as $r^{-(n+2)/2}$. By considering the large r limit of potential (47) we also see that $n \geq 2$ so that Eq. (46) remains bounded as required to make the total energy finite.

C. Kerr-AdS backgrounds in even dimensions

Consider a background spacetime where $d = n + 3$, such that we can write the metric as

$$ds^2 = -\alpha(r,\theta)^2 dt^2 + \beta(r,\theta)^2 dr^2 + \sigma(r,\theta)^2 d\theta^2 + \gamma(r,\theta)^2 d\Sigma_n^2. \tag{52}$$

We can write the Lichnerowicz operator as

$$\Delta_L h_{ij} = \frac{1}{\gamma^2}\tilde{\Delta}_L h_{ij} + \left[\frac{\partial_t^2}{\alpha^2} - \frac{\partial_r^2}{\beta^2} - \frac{\partial_\theta^2}{\sigma^2}\right]h_{ij} + \left[-\frac{\alpha_r}{\alpha} + \frac{\beta_r}{\beta} - \frac{\sigma_r}{\sigma} + (4-n)\frac{\gamma_r}{\gamma}\right]\frac{\partial_r h_{ij}}{\beta^2} + \left[-\frac{\alpha_\theta}{\alpha} - \frac{\beta_\theta}{\beta} + \frac{\sigma_\theta}{\sigma} + (4-n)\frac{\gamma_\theta}{\gamma}\right]\frac{\partial_\theta h_{ij}}{\sigma^2} - \frac{4}{\gamma^2}\left[\frac{\gamma_r^2}{\beta^2} + \frac{\gamma_\theta^2}{\sigma^2}\right]h_{ij}. \tag{53}$$

To this end, we shall consider a Kerr-AdS background metric ($M = 0$) in even dimensions, $n = 2N - 1$, by setting the N rotation parameters equal (i.e. $a_1 = \cdots = a_N = a$). The background metric is [15]

$$ds^2 = -\frac{(1+cr^2)\Delta_\theta}{1-ca^2}dt^2 + \frac{\rho^2}{(1+cr^2)(r^2+a^2)}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{(r^2+a^2)\sin^2\theta}{1-ca^2}\overline{ds}^2(\mathcal{M}^n), \tag{54}$$

where,

$$\rho^2 \equiv r^2 + a^2\cos^2\theta, \qquad \Delta_\theta = 1 - ca^2\cos^2\theta. \tag{55}$$

A calculation gives

$$(\Delta_L h)_{ij} = \left[\frac{1-ca^2}{(1+cr^2)\Delta_\theta}\frac{\partial^2}{\partial t^2} - \frac{(1+cr^2)(r^2+a^2)}{\rho^2}\frac{\partial^2}{\partial r^2} - \frac{\Delta_\theta}{\rho^2}\frac{\partial^2}{\partial\theta^2} - \frac{4}{\rho^2}\left(\frac{r^2(1+cr^2)}{r^2+a^2} + \frac{\Delta_\theta}{\tan^2\theta}\right)\right]h_{ij} + \frac{(1-ca^2)}{(r^2+a^2)\sin^2\theta}\tilde{\Delta}_L h_{ij} + \frac{r}{\rho^2}(2(1-ca^2) - (n-1)(1+cr^2))\partial_r h_{ij} + \frac{1}{\rho^2\tan\theta}(2(1-ca^2) - (n-2)\Delta_\theta)\partial_\theta h_{ij}. \tag{56}$$

Equation (56) may be separated by writing,

$$h_{ij} = \Psi(r)e^{\omega t}S(\theta)\tilde{h}_{ij}(\tilde{x}), \tag{57}$$

and taking the large r limit. Hence,

$$\left(r^2\frac{\partial^2}{\partial r^2} + (n-1)r\frac{\partial}{\partial r} - 2n + \frac{p}{cr^2}\right)\Psi = 0, \tag{58}$$

$$\Delta_\theta\frac{\partial^2 S}{\partial\theta^2} - \frac{1}{\tan\theta}(2(1-ca^2) - (n-2)\Delta_\theta)\frac{\partial S}{\partial\theta} + \left(\frac{4\Delta_\theta}{\tan^2\theta} - \frac{\lambda(1-ca^2)}{\sin^2\theta} - \frac{\omega^2(1-ca^2)}{c\Delta_\theta} - p\right)S = 0, \tag{59}$$

where we have defined $(\tilde{\Delta}_L \tilde{h})_{ij} = \lambda \tilde{h}_{ij}$, so that λ is the eigenvalue of the Lichnerowicz operator on $\mathcal{M}^n$, and p is the separation constant.

The radial equation is easily solved to yield

$$\Psi = c_1 r^{(2-n)/2} J_1\left(\frac{n+2}{2}, \sqrt{\frac{p}{cr^2}}\right) + c_2 r^{(2-n)/2} Y_1\left(\frac{n+2}{2}, \sqrt{\frac{p}{cr^2}}\right). \tag{60}$$

However, regularity of the radial solution at $r = \infty$ requires $c_2 = 0$ and hence as $r \rightarrow \infty$ the radial solution behaves as

$$\Psi(r) \sim \frac{c_1}{r^n}. \tag{61}$$

Equation (59), together with boundary conditions of regularity at $\theta = 0$ and π, constitute an eigenvalue problem for the separation constant p. For $\sin\theta \approx \theta$, $\cos\theta \approx 1$, the solution is

$$S = \theta^{(5-n)/2}[c_1 J_m(z) + c_2 Y_m(z)], \tag{62}$$

where

$$m = \sqrt{\lambda - 4 + \frac{(5-n)^2}{4}}, \qquad z = \sqrt{-\frac{pc + \omega^2}{c(1 - ca^2)}}\theta. \tag{63}$$

The criterion for gravitational stability, in terms of the minimum Lichnerowicz eigenvalue $\lambda_{\min}$ on the base manifold $\mathcal{M}^n$, namely $\lambda_{\min} \geq \lambda_c = 4 - (5-n)^2/4$, now translates into the requirement that $m \in \mathbf{R}$. However we note that $c \neq 0$ in this case. In AdS space, since $c > 0$, for reality of the solution we also require,

$$0 < 1 - ca^2 < 1, \qquad p < -\frac{\omega^2}{c}. \tag{64}$$

For real ω, $p < 0$ and hence $\sqrt{p/cr^2}$ is imaginary, but this is not allowed by the radial wave equation. Therefore there are no normalizable solutions with $\omega \in \mathbf{R}$. For $\omega \rightarrow i\tilde{\omega}$, one requires $p < \tilde{\omega}^2/c$. A useful inequality for stability of the background AdS metric (54) is therefore,

$$0 < p < \frac{\tilde{\omega}^2}{c}. \tag{65}$$

Instead of considering the large r limit in (56), let us now consider the special case where the angular velocity approaches the critical limit, $ca^2 = 1$ (or $a = l$). The eigenfunctions are then the associated Legendre polynomials $P_{\tilde{n}}^m(\cos\theta)$, $Q_{\tilde{n}}^m(\cos\theta)$, where,

$$m = \frac{1}{2}\sqrt{4p - (7-n)(n+1)}, \qquad \tilde{n} = \frac{1}{2}(\sqrt{(n-6)(n+2)} - 1). \tag{66}$$

An interesting case is $n = 7$, which allows one to study supergravity solutions in $d = 10$. It would be interesting to know what the limit $ca^2 \rightarrow 1$ corresponds to in a dual field theory. We leave this issue to future work.

V. CONCLUSION

In this paper we have studied the thermodynamics and stability of higher-dimensional ($d \geq 5$) rotating black holes in a background (anti)-de Sitter spacetime. The thermodynamic quantities for Kerr-AdS black hole solutions suggested by Gibbons *et al.* [22] have been used to study the behavior of the free energy and specific heat (which are defined unambiguously in all spacetime dimensions $d \geq 4$) as functions of temperature and horizon positions. The two apparently different expressions of energy in the Kerr-AdS background suggested by Hawking *et al.* [4] and Gibbons *et al.* [22] do not introduce any significant difference in the behavior of bulk thermodynamic quantities (such as entropy, free energy, specific heat, etc.) and therefore the stability of Kerr-AdS solutions. Nevertheless, the Gibbons *et al.* bulk variables are more suggestive to be used as they map onto the boundary variables with the natural definition of conformal boundary metric, that is the one for which the coordinate $y =$ const for large y, and satisfy the first law of thermodynamics.

As for thermodynamic stability, rotating black holes are found to be stable down to a critical value of the rotation parameter, below which the specific heat becomes negative. For example, a five-dimensional Kerr-AdS black hole is thermodynamically stable when the rotation parameters take values $a_i \approx 0.17l$; larger angular velocities usually stabilize the black hole.

Similarly, a zero-mass Kerr-AdS background is gravitationally stable down to a critical eigenvalue, below which the Schrödinger equation may involve growing tensor mode perturbations. Again larger angular velocities stabilize the background Kerr-AdS spacetimes, although the bound on the rotation parameters required for the gravitational stability of rotating black holes is not directly related to that of thermodynamic stability.

An obvious extension to study, for completeness, would be the inclusion of nonzero charges and all of the possible rotation parameters in dimensions five and higher. A particularly interesting problem would be a study of the gravitational stability of *massive* Kerr-AdS black hole spacetimes in AdS_5 and AdS_7 spaces. Some of the problems will be discussed in a follow-up paper [38].

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APPENDIX: SCHRÖDINGER EQUATION

Consider a second order differential equation of the form

$$(A\partial_r^2 + B\partial_r + C + D\partial_t^2 + E\tilde{\Delta}_L)h = 0, \qquad \text{(A1)}$$

where A, B, C are functions of r only. We find it is convenient to choose $h \equiv \Psi(r)e^{\omega t}\tilde{h}$, such that $\tilde{\Delta}_L\tilde{h} = \lambda\tilde{h}$. We then have

$$(A\partial_r^2 + B\partial_r + C + D\omega^2 + E\lambda)\Psi(r)e^{\omega t}\tilde{h} = 0. \qquad \text{(A2)}$$

For nonzero fluctuations, $e^{\omega t}\tilde{h} \neq 0$, this implies that

$$(A\partial_r^2 + B\partial_r + \tilde{C})\Psi(r) = 0, \qquad \text{(A3)}$$

where $\tilde{C} \equiv C + D\omega^2 + E\lambda$. We would like to write this in the form

$$(\partial_x^2 - V(x(r)))\varphi = \omega^2\varphi. \qquad \text{(A4)}$$

To facilitate this we introduce two transformations:

$$dr = \frac{\partial r}{\partial x}dx, \qquad \Psi = \chi\varphi.$$

The differential equation then takes the form

$$\frac{A}{r_x^2}\varphi'' + \left[\frac{2A}{r_x^2}\frac{\chi'}{\chi} + \frac{B}{r_x} - \frac{Ar_{xx}}{r_x^3}\right]\varphi' + \left[\frac{A}{r_x^2}\frac{\chi''}{\chi} + \left(\frac{B}{r_x} - \frac{Ar_{xx}}{r_x^3}\right)\frac{\chi'}{\chi} + \tilde{C}\right]\varphi = 0, \qquad \text{(A5)}$$

where $r_x \equiv r' = (\partial r/\partial x)$. Let us define

$$r_x^2 = -\frac{A}{D}, \qquad \frac{\chi'}{\chi} = \frac{B}{2Dr_x} + \frac{r_{xx}}{2r_x}.$$

This implies

$$\frac{r_{xx}}{r_x} = \frac{1}{2}\left(\frac{A_x}{A} + \frac{D_x}{D}\right). \qquad \text{(A6)}$$

The differential equation then takes the standard form:

$$\partial_x^2\varphi - V\varphi = \omega^2\varphi, \qquad \text{(A7)}$$

where

$$V = -\left(\frac{\chi'}{\chi}\right)' + \left(\frac{\chi'}{\chi}\right)^2 + \frac{\bar{C}}{D}, \qquad \text{(A8)}$$

where $\bar{C} \equiv C + E\lambda$.

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