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Cosmic growth history and expansion history

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The cosmic expansion history tests the dynamics of the global evolution of the universe and its energy density contents, while the cosmic growth history tests the evolution of the inhomogeneous part of the energy density. Precision comparison of the two histories can distinguish the nature of the physics responsible for the accelerating cosmic expansion: an additional smooth component—dark energy—or a modification of the gravitational field equations. With the aid of a new fitting formula for linear perturbation growth accurate to 0.05%–0.2%, we separate out the growth dependence on the expansion history and introduce a new growth index parameter γ that quantifies the gravitational modification.

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I. INTRODUCTION

Acceleration of the cosmic expansion reveals fundamentally new physics missing from our picture of the universe, yet key for the understanding of the recent and present history and the fate of the universe. Furthermore, this new physics tells us that our standard models of gravitation and particle physics may be woefully incomplete. The acceleration may lead us to insights about new high energy physics and the nature of the quantum vacuum, or about gravitation beyond Einstein's general relativity. Perhaps most exciting would be a signal that both are involved, providing clues to a theory of quantum gravity.

The first scenario includes physical components such as the cosmological constant, dynamical scalar field models, or effective potentials from string theory. The second scenario includes extensions of the Einstein-Hilbert action, e.g. to higher derivative theories, scalar-tensor theories, generalized functions of the Ricci scalar, theories of supergravity or quantum gravity, and infrared modifications of gravity such as from hidden spacetime dimensions. We can say that searching for the nature of the accelerating expansion is seeking to answer one or the other question: "Does nothing weigh something?" or "Is nowhere somewhere?"

To distinguish the many different theoretical possibilities requires accurate observations of the cosmic expansion history, but even this will leave some degeneracies between explanations. Models with different physical origins but the same global expansion properties could not be separated. Fortunately, the overall smooth characteristics of the universe are not the only observables. The energy density contents have evolved from the hot, dense, smooth state of the early universe to a relatively cool, diffuse, and in the case of matter, clustered state. While the first two properties are purely due to the expansion of the zeroth order, homogeneous universe, being qualitatively kinematical, the last property arises from the perturbed, inhomogeneous universe, being intrinsically dynamical [1]. The growth of large scale structure in the universe provides an important companion test, and the cosmic expansion history and growth history together provide discernment of the nature of the new accelerating physics.

In Sec. II we discuss the expansion history and the effective equation of state entering the acceleration. The growth of linear perturbations in the matter component in a generalized cosmological model is reviewed in Sec. III. The growth equation is extended in Sec. IV to allow other theories of gravitation besides general relativity, and formal solutions given. For practical use in constraining models by observational data we introduce a highly accurate fitting formula in Sec. V and apply it to a braneworld gravity model and models with coupling between the matter and dark energy density. We present the conclusions in Sec. VI.

II. EXPANSION HISTORY

The expansion history of the universe is a key quantity in cosmology, appearing directly in the metric in the form $a(t)$. Kinematically, this is all that is needed to define distances and volumes (together with the spatial curvature constant k, which we take to be zero, though this does not affect the form of the following discussion). To evaluate the distances for a specific cosmology, dynamics or equations of motion from the gravity theory are required, together with information on the energy density contents. The expansion history, together with the amount of clustering matter and any interactions of the matter with other components, is the central ingredient as well as in the growth of matter perturbations.

The Friedmann expansion equation in terms of the Hubble parameter $H = \dot{a}/a$ is

$$H^2 = (8\pi G/3)\sum_i \rho_i, \tag{1}$$

where we sum over all components of the energy density. Since we are especially interested in the matter component, e.g. since we are positive it exists and since we will later examine its growth into large scale structure, it is convenient to separate it out from the sum. Then in terms of dimensionless energy density we can write

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$$H^2/H_0^2 = \Omega_m a^{-3} + \sum_{i'} \Omega_{i'} e^{3\int_a^1 [1+w_{i'}(a')]da'/a'} \quad (2)$$

$$= \Omega_m a^{-3} + \delta H^2/H_0^2, \quad (3)$$

where the set i' does not include matter, $\Omega_m + \sum_{i'} \Omega_{i'} = 1$, and $w(a)$ is the equation of state of each component.

Without imposing any physical interpretation on δH^2 as actually being due to an energy density component as opposed to a modification of the Friedmann expansion equation, we can define an effective "acceleration physics" or "dark energy" equation of state [2] (cf. [3])

$$w(a) = -1 - \frac{1}{3}\frac{d\ln\delta H^2}{d\ln a} \quad (4)$$

$$= -\frac{1}{3}\frac{d\ln[\Omega_m(a)^{-1} - 1]}{d\ln a}, \quad (5)$$

writing $\Omega_m(a) = \Omega_m a^{-3}/(H/H_0)^2$.

Volumes and distances are built up out of the conformal distance

$$\eta(a) = \int_a^1 (da'/a')(a'H)^{-1} = \int_0^z dz'/H. \quad (6)$$

Models with the same expansion history will have the same distances and volumes. Note that formally we could obtain the same expansion history for two models by keeping their Ω_m the same and matching their $w(a)$, or by allowing different Ω_m and compensating for this in $w(a)$. Since the latter case corresponds to misestimating the matter density rather than any new physics, we do not consider it further.

While the definition of an effective equation of state in terms of the expansion history is powerful, allowing different models to be talked about with a common language and treated in a model independent parameter space, this feature is also a bug. Measurements of the expansion history, through distances and volumes to arbitrary precision, will not be able to distinguish different physical origins for the same expansion behavior. This is where the growth history comes in.

III. GROWTH OF MATTER DENSITY PERTURBATIONS

The universe has not remained homogeneous on all scales since its early, essentially smooth state. While the largest volumes can still be treated as homogeneous and isotropic Robertson-Walker universes, smaller scale evolution must take into account perturbations to the metric in the form of gravitational potentials.

Note that recent speculation [4,5] about the interaction of these potentials to affect significantly even the global expansion seems misplaced; investigation of a realistic inhomogeneous universe metric by Jacobs, Linder, and Wagoner [6,7] derived a Green function solution for the potential. This "post-Newtonian" solution corrects the Newton-Poisson equation and shows that no infrared divergences exist in the potential, rather a suppression as the Hubble scale is approached. The Appendix summarizes the effects.

Using the perturbed equations of motion for the gravity theory, one can derive the growth of density perturbations. Concentrating on perturbations in the matter density $\delta = \delta\rho_m/\rho_m$, assuming all other components are smooth, within general relativity the time evolution is

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_m\delta = 0. \quad (7)$$

The physical interpretation is very simple: the perturbations grow according to a source term involving the amount of matter able to cluster and are restricted by a friction term, or Hubble drag, arising from the expansion of the universe. General discussion of the physics dependence on the expansion rate is in [2].

It is convenient to study the growth evolution in terms of the expansion scale a or characteristic (e fold) scale $\ln a$, rather than time t. Since the pure matter universe solution has $\delta \sim a$, it is also useful in studying the dark energy to divide out this behavior and switch to the growth variable $g = \delta/a$. Finally, since we will be interested in modifications of gravity, we hereafter normalize G by Newton's constant, i.e. where G appears in equations it stands for $G/G_{\rm Newton}$.

Denoting derivatives with respect to $\ln a$ as primes, we have

$$g'' + \left[4 + \frac{1}{2}(\ln H^2)'\right]g' + \left[3 + \frac{1}{2}(\ln H^2)' - \frac{3}{2}G\Omega_m(a)\right]g = 0, \quad (8)$$

$$g'' + [3 - q]g' + \left[2 - q - \frac{3}{2}G\Omega_m(a)\right]g = 0, \quad (9)$$

$$g'' + \left[\frac{5}{2} - \frac{3}{2}w(a)\Omega_w(a)\right]g' + \frac{3}{2}[1 - w(a)]G\Omega_w(a)g = 0, \quad (10)$$

$$g'' + \left[\frac{5}{2} - \frac{1}{2}(\ln\Omega_m(a))'\right]g' + \left[\frac{3}{2} - \frac{1}{2}(\ln\Omega_m(a))' - \frac{3}{2}G\Omega_m(a)\right]g = 0. \quad (11)$$

All these forms are equivalent. They clearly show that the Hubble drag is increased, and hence growth is suppressed, for an accelerating universe, as the deceleration parameter $q = -a\ddot{a}/\dot{a}^2$ or w become more negative. We emphasize that they also demonstrate that within general relativity the linear theory growth factor depends purely on the expansion history, e.g. $H(a)$ or $w(a)$ or $\Omega_m(a)$ or $\Omega_w(a) = 1 - \Omega_m(a)$. So a discrepancy between the growth observed and that predicted based on an observed expansion history tests the theoretical framework and can point up modifications to the theory of gravity.

IV. GENERALIZATION TO GRAVITATIONAL MODIFICATIONS

To study other theories of gravity we can consider a change to the effective Newton's constant G entering the above equations (remember the G in the equations really means $G/G_{\rm Newton}$), or more generally some nonzero right-hand side. First we examine this as a generic change, and later treat a specific example within braneworld gravity.

The deviation in G from its Newtonian value caused by some time variation in G can be viewed as a subset of a nonzero right-hand side, since we may write a left-hand side source term $XG(a)$ as the usual $XG_{\rm Newton}$ and add a term $X[G_{\rm Newton} - G(a)]$ to the right-hand side. Indeed, any difference between two cosmological models that only changes the source term, and keeps it linear in g, can be viewed as transforming the solution of the homogeneous differential equation for model 1 into a solution for model 2 of the effective inhomogeneous differential equation.

Using a Green function method one can obtain a formal solution

$$g(a_i, a) = \bar{g}(a_i, a) + \int_{a_i}^{a} du[\bar{Q}(u) - Q(u)]g(a_i, u)u^5 H(u)\bar{g}(u)\bar{g}(a) \times \int_u^a dv \bar{g}^{-2}(v) v^{-5} H^{-1}(v), \qquad (12)$$

where Q is the source term.[1] The barred quantities represent some model 1 for which the solution is known (e.g. general relativity), and the integral gives the particular solution in the second model, for growth between any two scale factors (we can set $a_i = 0$ to get the total growth up to some a).

The solution can also be written as a recursion relation

$$g(a_i, a) = \bar{g}(a_i, a) + \int_{a_i}^{a} du \bar{g}(a_i, u) \sum_{i=1}^{\infty} K_i(u, a), \qquad (13)$$

$$K_{i+1}(u, a) = \int_u^a dx K_1(u, x) K_i(x, a), \qquad (14)$$

$$K_1(u, a) = [\bar{Q}(u) - Q(u)]u^5 H(u)\bar{g}(u)\bar{g}(a) \times \int_u^a dv \bar{g}^{-2}(v) v^{-5} H^{-1}(v). \qquad (15)$$

This is particularly useful when considering small perturbations between models, e.g. when the gravitational coupling is slowly changing, as in the case of some scalar-tensor theories. (Retaining only the first order term, K_1, is basically a Born approximation.)

Another virtue of the Green function solution is the ability to see broad physical trends as the models change. This follows the approach of [8,9] who considered the relation between distances as the cosmological model changed, including an analogous change in the theoretical framework (there in terms of allowing a clumpy universe). Here we consider the relation between growth factors. If $\bar{Q} > Q$ then $K_i > 0$ and so $g(a_i, a) > \bar{g}(a_i, a)$. Thus we have a criterion for when the growth will be stronger, or when it will be more suppressed. With the expansion history fixed, the criterion $\bar{Q} > Q$ simply becomes $\bar{G} < G$; i.e. if the effective gravitational coupling is stronger than Newton's constant then the growth is enhanced. For more elaborate modifications of gravity, a nonzero right-hand side to the growth equation can contribute to Q as well, but the prescription above still applies.

V. A NEW FITTING FORMULA FOR GROWTH

The general growth solutions of the previous section are formal, and while we saw that they can present generic physical insights they are somewhat cumbersome for application to cosmological models. One might draw an analogy to trying to map the expansion history. While one can calculate the expansion history in a specific model, say from a high energy physics scalar field potential, this is inefficient for comparison of the observations to many possible models. Instead a useful approach is a model independent one, using a parametrization of the expansion history, for example, in terms of the equation of state $w(z)$ value and variation: $w_0 = w(z=0)$ and $w_a = (-dw/da)|_{z=1}$ (this is also similar to the inflationary power spectrum index and tilt parameters). In this section we derive an analogous model independent parametrization of the growth history, putting it on equal footing with the cosmic expansion history.

Rather than attempting to fit observations of growth history with an effective equation of state $w_{\rm grow}(z)$, it is better to render the physics appropriately: the expansion effects on the growth are described in terms of the standard expansion $w(z)$, and the gravitational modifications giving deviations from the expected growth history are treated as additional inputs. Again, in the standard framework the expansion history completely determines the growth history. Thus, we would like to write the growth history $g(a)$ as a function of an expansion history quantity plus a new, framework testing characteristic.

Since the growth concerns matter density perturbations we take the expansion history in terms of the matter density history $\Omega_m(a)$. Two models with the same $\Omega_m(a) = \Omega_m a^{-3}/[H/H_0]^2$ for all redshifts will have the same expansion history. So we look for a functional expression $g(\Omega_m(a))$. One that works superbly well, as both a highly

[1] Technically, Q is the source term divided by the growth variable g, and also multiplied by a^{-2} since Eq. (12) uses a dependent variable of a rather than $\ln a$. For example $Q = [(3/2) - (1/2)(\ln\Omega_m(a))' - (3/2)G\Omega_m(a)]a^{-2}$ in Eq. (11).

accurate approximation to the exact solution and as a simple characterization stimulating physical intuition, is

$$g(a) = e^{\int_0^a d\ln a[\Omega_m(a)^\gamma - 1]}. \tag{16}$$

Here γ is our new parameter for the growth history, called the growth index, encompassing deviations in the theoretical framework. Models with identical expansion histories but different gravitational theories will possess different γ parameters.

A. Accuracy tests

First we establish the accuracy of the fitting formula, Eq. (16), over a wide range of dark energy cosmologies. Note that [10] (also see [11,12]) has found that a similar formula provides estimations of the normalized growth factor at the accuracy level of about 1%. However that approach normalized to the growth factor today [so it could not predict its value, and [2] showed that $g(z)/g(0)$ varies by only a few percent innately between models] and fixed the growth index. Furthering the pioneering work of [10–12] we can remove both those restrictions and obtain an order of magnitude better accuracy.

In terms of the expansion history dark energy equation of state, within general relativity, we find excellent fits, to better than 0.2%, using

$$\gamma = 0.55 + 0.05[1 + w(z = 1)]. \tag{17}$$

Employing the value of the equation of state evaluated at $z = 1$ allows simple treatment of dynamical models where the equation of state varies with redshift, as it generically does.

For the cosmological constant case, the fitting formulas Eqs. (16) and (17) reproduce the exact growth history for any redshift (including the total growth to the present) to better than 0.05% over the range $\Omega_m \in [0.22, 1]$. It remains accurate to better than 1% all the way down to $\Omega_m = 0.01$.

Models with equation of state $w = -0.8$ (-0.5) have accurately fit growth histories to within 0.2% (0.4%) for $\Omega_m \in [0.2, 1]$. A dynamical model such as SUGRA with $w_0 = -0.82$, $w_a = 0.58$ is fit to within 0.25%. Models with equations of state $w < -1$ are similarly well approximated. When $w = -1.2$ (-1.5), the fit is good to 0.3% (0.5%). If we are willing to slightly modify the simplest fit of Eq. (17) to

$$\gamma = 0.55 + 0.02[1 + w(z = 1)] \quad \text{for } w < -1, \tag{18}$$

for the phantom models $w < -1$, then we achieve an astonishing 0.05% accuracy for these fits. (Note that the fitting function of [13,14], also containing a single integral, is accurate to only 5% for models with $w = -0.8$ or -1.2 and $\Omega_m = 0.3$.)

While impressive in accuracy, the growth function fitting formula's primary purpose is not a fit as such (the exact solution requires only solving a second order differential equation), but rather its usefulness in physical intuition and in parametrizing modifications of the Einstein growth equation beyond the expansion behavior [just as $w(z)$ parametrizes modifications of the Friedmann expansion equation]. The fitting function provides us access to the acceleration physics that exists beyond what the expansion history sees.

B. Example: braneworld gravity

The growth of matter perturbations in gravitational theories beyond general relativity is not well developed. Here we consider one theory that has been shown to be self-consistent [15–19], the Dvali-Gabadadze-Porrati (DGP) [15,16] braneworld theory of gravity. In this theory gravity has infrared modifications due to spacetime possessing a large extra dimension (making our view a 4D brane within a 5D bulk), causing a weakening of gravity on large scales approaching the Hubble scale.

The expansion history for this braneworld theory follows from the modified Friedmann equation,

$$H^2 - H/r_c = (8\pi/3)\rho, \tag{19}$$

where $r_c = H_0^{-1}/(1 - \Omega_m)$ is the crossover distance, related to the 5D Planck mass. Equivalently the expansion history has an effective equation of state

$$w(a) = -[1 + \Omega_m(a)]^{-1}, \tag{20}$$

as noted elegantly by [17]. The braneworld expansion history can be well approximated by a simple scalar field model with $w_0 = -0.78$, $w_a = 0.32$. Indeed these two very different physical origins for the acceleration agree in distance measurements to within 0.5% (0.01 mag) out to $z = 2$.

Information from the growth history, however, as stated before can break this degeneracy in the nature of the acceleration physics. Comparing the braneworld model with a scalar field model with an identical expansion history shows deviations in the present growth factor of 7%. Figure 1 illustrates how the growth history depends on both expansion history and the gravitational framework. Taking into account only the expansion history in the growth equation, the braneworld and scalar field models appear to have the same growth history. However, proper treatment of the gravitational modifications inherent in the braneworld scenario separates these models. This has been pointed out as well in [17,20,21].

In the linear power spectrum the deviation ranges from 4% at $z = 2$ to 15% today. While a scalar field model that matched the modified growth of the braneworld model is possible, it in turn can be distinguished through the expansion history. We see that expansion measurements and growth measurements work in important complementarity to reveal the nature of the new physics.

Figure 2 demonstrates this synergy explicitly. Supposing the universe was described by a braneworld model with

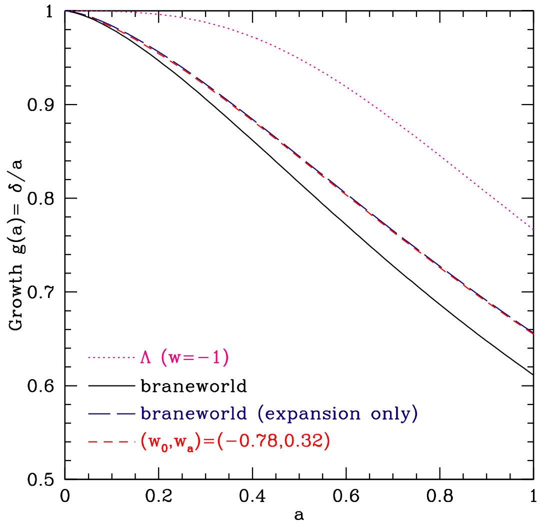

FIG. 1 (color online). The growth history is shown for an extra dimensional braneworld model (long dashed, blue curve) and a quintessence model with $w_0 = -0.78$, $w_a = 0.32$ (short dashed, red), having nearly identical expansion histories. When proper account is taken of the effects of altered gravity on the braneworld growth history (solid, black curve) this allows distinction of these models. The expansion history can in turn rule out a quintessence model degenerate with the solid curve.

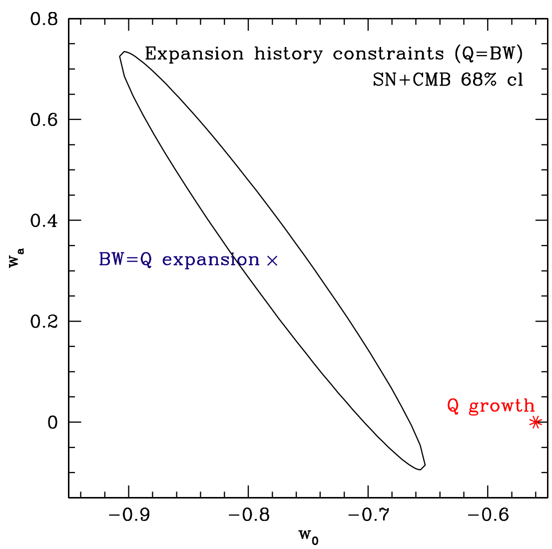

FIG. 2 (color online). Expansion history and growth history constraints on the dark energy equation of state parameters can test the theoretical framework by looking for inconsistent results. The blue cross gives the best fit for the expansion history of a quintessence (Q) universe matching the braneworld (BW) scenario, but the red star gives the best fit for the growth history to a quintessence model, assuming general relativity. The black ellipse shows the constraints at 68% confidence level from next generation data composed of SNAP supernovae data and Planck CMB last scattering distance measurement.

$\Omega_m = 0.28$, distance measurements of the quality of the proposed Supernova/Acceleration Probe (SNAP; [22]) supernova data set, together with a 0.7% measurement of the distance to the CMB last scattering surface from Planck [23], would provide the constraints (at 68% confidence level, marginalizing over other parameters such as Ω_m) in the dark energy equation of state parameter space shown by the ellipse. The best fit for growth measurements would concur with the solution using braneworld gravity equations, showing the consistency of the data with this model. However, if the growth equation employed general relativity, the best fit would lie at the red star, $w_0 = -0.56$, $w_a = 0$, clearly inconsistent. The comparison of expansion and growth histories reveal a breakdown of the theoretical framework, this discrepancy alerting us to a possible modification of gravity (or experimental systematic errors).

Of course if the measurements were too coarse and imprecise, we would not necessarily have noticed a statistically significant discrepancy. The braneworld model expansion is compatible with an expansion history of a constant $w = -0.71$ model, to 1% in distance out to $z = 1.7$. So the expansion history measurements find a "distance" in equation of state space of $\Delta w_0 = 0.15$ between the effective scalar field model from the expansion history and that from the growth history. Conversely, the expansion history of $(w_0, w_a) = (-0.56, 0)$ can be fit by $(-0.63, 0.32)$, so the distance from the expansion fit to the braneworld model of $(-0.78, 0.32)$ is again $\Delta w_0 = 0.15$. This suggests that for a 3σ detection of framework inconsistency we should strive for experiments that provide an uncertainty of $\sigma(w_0) < 0.05$.

Likewise one can estimate from the different orientations of the expansion history and growth history constraints in the $w_0 - w_a$ plane that the precision of measurements on w_a should be $\sigma(w_a) < 0.2$. This comparison of growth to expansion provides one of the only ways of putting an absolute scale on the measurement precision that should be striven for in experiments to reveal the nature of the accelerating physics—a significant breakthrough (see also [24]). This is somewhat dampened by the realization that this scale is particular to the braneworld scenario. Note that the relative precisions between w_0 and w_a obey the relation

$$\sigma(w') \equiv \sigma(w_a)/2 \approx 2\sigma(w_0), \tag{21}$$

found in [24,25], though that analysis was within the scalar field context. This relation signifies that a precise measurement of w_0 is of limited use without concomitant constraint on w_a, since a sufficiently different w_a can spoof w_0. That is, the uncertainty in seeing a discrepancy will be dominated by the largest error among the two equation of state parameters.

To move beyond a mere alarm that there is an inconsistency, we need to employ the growth parametrization of Eq. (16) to obtain a quantitative measure of the deviation from the growth behavior predicted by the expansion history measurements. We find that the fitting formula works for the braneworld scenario including gravitational modifications, using a growth index $\gamma = 0.68$ (note that the pioneering paper of [17] indicated the equivalent of $\gamma = 2/3$). In fact, the growth history using the fitting function Eq. (16) and $\gamma = 0.68$ matches the exact solution to within 0.2% (for $\Omega_m \geq 0.2$).

The approximation of a single growth parameter beyond the expansion history effects on the growth can be validated by asking what values of γ as a function of redshift reproduce the exact solution. For the case $\Omega_m = 0.28$, the (now) function $\gamma(z)$ ranges between 0.665 at $z = 0$ to 0.683 at $z = 1$ to 0.687 at high redshift. (The constancy of γ with redshift holds even better for quintessence models.) This, as well as the excellent fit to the growth function, justifies the use of a single parameter γ, the growth index.

With this model independent parametrization in hand, we can obtain quantitative measures of the deviations between models, even those that involve gravitational modifications. To the cosmological parameters Ω_m, w_0, and w_a, we add the growth index γ and can plot the resulting parameter estimation uncertainties, marginalizing over subsets of parameters. Figure 3 illustrates an example.

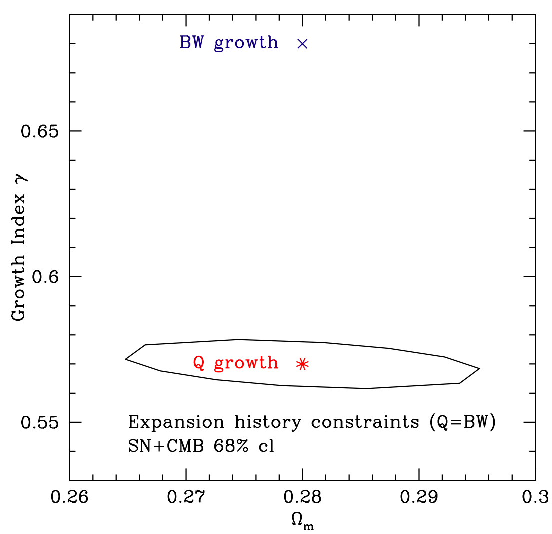

FIG. 3 (color online). While Fig. 2 showed that expansion history and growth history constraints on the dark energy equation of state parameters could test the theoretical framework by looking for inconsistent results, here we see quantitative measures of framework breaking by gravitational modification of the growth index γ. The red star gives the best fit for a quintessence (Q) model matching the expansion history of the braneworld (BW) scenario, but the blue cross gives the true result for the braneworld growth history. The black ellipse shows the constraints at 68% confidence level from next generation data composed of SNAP supernovae data and Planck CMB last scattering distance measurement, marginalized over the equation of state parameters w_0, w_a.

The growth index γ that would fit the braneworld growth history is clearly distinct from the values allowed by a scalar field model that matches the expansion history. The distance in γ is 0.11; to attain the value $\gamma = 0.68$ would require, by extrapolation of Eq. (17), $w = +1.6$. A 3σ distinction of the framework breaking would need a measurement with precision $\sigma(\gamma) = 0.037$ (marginalized over the other parameters). This corresponds roughly to a 2% measurement of the growth history. Indeed, this is in good agreement with the results of Fig. 1, which showed growth deviations between the two models with identical expansion histories at up to the 7% level, so the same 3σ criterion leads to $\sim$2% precision.

C. Coupling of matter and dark energy

While the primary purpose of the formalism here is to test the gravitational framework, in the case of a physical dark energy there can enter microphysical effects. These can include spatial perturbations to the dark energy or coupling to the matter component. We leave the first of these to future work, but note that growth probes involving correlations of large scale structure with the CMB might play a role (e.g. [26–28]). Here we consider whether the fitting formula and growth index approach remain valid in the presence of coupling. Without a microphysical theory these are necessarily toy models, and we only consider the effects on matter growth, neglecting early universe or fifth force constraints.

Interaction between matter and a dark energy component is treated through a coupling of the evolution equations:

$$\dot{\rho}_i = -3H(1 + w_i)\rho_i + \Gamma_i(a, \rho_m, \rho_{de}). \tag{22}$$

We have considered the cases $\Gamma_m = -\Gamma_{de} = \alpha\rho_m$, $\alpha\rho_{de}$, and $\alpha a^n H$, the slinky inflation model [29], and the undulant universe model [30]. Note the undulant universe is a special case of the slinky model, without coupling, and is ruled out by having a very low growth factor ($g_0 = 0.03$).

All these models follow the growth index formalism if the coupling is not too strong. As the coupling increases [e.g. if the dimensionless coupling $\alpha' = \alpha/\rho_m(0) \gtrsim 0.5$ in the $\Gamma_m = \alpha H$ case], this will start to break down because the equation of state of matter begins to deviate significantly from zero (plus in some cases the high redshift universe is not matter dominated). As a simple example, consider the decaying matter scenario where $\Gamma_m = -\alpha\rho_m$. This was treated in detail in [31], and the matter equation of state is $\alpha/(3H)$ [32]. Generically, a coupling of the form $\Gamma_i = \alpha\rho_i$ will change the equation of state of component i, defined by the effective conservation equation $\dot{\rho}_i = -3H(1 + w_{\rm eff})\rho_i$, from w_i to $w_{\rm eff} = w_i - \alpha/(3H)$ (providing a way for a $w > -1$ component to look phantom, $w < -1$).

Further research into the effects of coupling on growth of perturbations is underway [33] (also see [34]). Lack of a consistent microphysical theory is the major obstacle. For example, is the "new" energy density in a component distributed uniformly or in the same spatial distribution as the component it came from? Issues of evolution from that point by clumping or free streaming make rigorous calculation complicated.

VI. CONCLUSIONS

To reveal the physical origin of the acceleration of the universe, both probes of the expansion history (such as the distance-redshift relation) and of the growth history (such as weak gravitational lensing measurements involving the mass power spectrum) are required. While the two types of probes in synergy give enhanced constraints on the effective dark energy equation of state, in comparison they can test the theoretical framework of cosmology and general relativity.

The growth history of mass in the universe follows the source and friction term behaviors governed by the expansion. Deviations from this reveal a breakdown of the framework such as from modification of gravity. By rendering the growth function in a physically appropriate manner, separating the expansion effects from framework extensions, we presented here a new, physically intuitive and highly accurate (0.05%–0.2%) fitting function, Eq. (16), for the linear growth of perturbations in generalized cosmologies. This allows model independent quantification of gravitational modifications in terms of a new parameter, the growth index γ.

This research suggests a new paradigm for understanding the nature of the acceleration physics: accurate measurement of expansion and growth separately, for example, through type Ia supernovae and weak gravitational lensing. A useful, model independent, quantitative parameter set was shown to be the equation of state value w_0 and variation w_a and the growth index γ. In the specific worked case of comparing an extra dimensional braneworld scenario with scalar field physics in general relativity, the desired measurement precisions should be of order $\sigma(w_0) \leq 0.05$, $\sigma(w_a) \leq 0.2$, $\sigma(\gamma) \leq 0.04$. These should be technically feasible and should be within the reach of next generation experiments such as the Joint Dark Energy Mission.

The formalism presented here has further applications for future investigation, such as seeing the effect of perturbations in a physical dark energy component, couplings between dark energy and matter, and scalar-tensor gravity. To reveal the nature of the new physics responsible for the universe-shaking acceleration, we will require a comprehensive suite of cosmological probes. The significance of the discoveries is so great that every robust method is needed to strengthen the accuracy, and the confidence in our understanding. With clear measurements of the cosmic expansion history and the cosmic growth history together we can learn if nothing weighs something, if nowhere is somewhere, or even more unexpected insights.

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APPENDIX: INHOMOGENEITIES AND COSMIC EXPANSION

We have taken the cosmic growth history to not "backreact" on the cosmic expansion history. That is, the global homogeneous expansion independent of the growth of matter structure is a valid treatment. This is a topic of great interest and comment; here we simply present a brief summary of the dependence of the metric on gravitational potentials and the lack of large contributions by gravitational potentials (in particular no infrared divergence) to the cosmic expansion.

The approach taken by [6,7] is a straightforward calculation to obtain the metric of a realistically inhomogeneous universe. In particular, it did not rely on any averaging procedure, rather a harmonic decomposition of the perturbations. The second key aspect was no *a priori* assumption on the size of matter density fluctuations; rather it used a post-Newtonian parametrization, essentially a weak field, slow motion expansion. This followed work of Futamase [35–37] and can be traced back to the mean field theory, or two length scale, approach of Chandrasekhar [38].

For potentials parametrized by an amplitude $\epsilon^2 \ll 1$, and characteristic length scale κ, the slow motion or, more physically, the shear condition $\epsilon^2/\kappa \ll 1$ applies. Violation of this condition leads to ray crossing in light propagation (see [7,39]) and eventually relativistically moving matter structures, contrary to observations of our universe. Landau and Lifshitz [40] pointed out that the dominant first order effect on the cosmic expansion entered at what they called pseudotensor order: ϵ^4/κ^2. Thus the shear condition ensures that the expansion is insignificantly affected, and conversely a significant backreaction of inhomogeneities on the expansion would generically lead to visible anisotropies.

However, here we concentrate on the post-Newtonian gravitational potentials, and modification of the Newton-Poisson equation relating the potentials to the matter density distribution. The general solution obtained by [6,7] was

$$\phi(\eta, \vec{x}) = -\frac{4\pi}{3}\int_{\eta_0}^{\eta}\frac{du}{a'(u)} \times \int d^3\vec{y}a^3(u)\delta\rho(u, \vec{y})\mathcal{G}(u, \eta, \vec{x}, \vec{y}), \qquad \text{(A1)}$$

plus an initial condition term. The Green function is

$$\mathcal{G}(u, \eta, \vec{x}, \vec{y}) = [a(u)/a(\eta)] \times [4\pi C(u, \eta)]^{-3/2} e^{-|\vec{y}-\vec{x}|^2/[4C(u,\eta)]}, \tag{A2}$$

$$C(u, \eta) = (1/3) \int_u^\eta dv (a/a'), \tag{A3}$$

where a prime denotes a derivative with respect to the conformal time η.

These expressions show that there is no divergence of the potential or its derivatives in the presence of inhomogeneities. In contrast, the post-Newtonian Green function solution, while matching the Newton-Poisson equation on small scales, shows an exponential suppression of the potential as one approaches horizon scales. These limits are treated in detail in [7], and the physical problem is shown to be closely analogous to the displacement probability distribution for isotropic random walks, and for diffusion in a uniform medium.

[1] Tegmark has written particularly clearly about this view of observational tests in terms of zeroth order and perturbed universe properties. See, e.g., M. Tegmark, Science **296**, 1427 (2002).
[2] E. V. Linder and A. Jenkins, Mon. Not. R. Astron. Soc. **346**, 573 (2003).
[3] U. Alam, V. Sahni, T. D. Saini, and A. Starobinsky, Mon. Not. R. Astron. Soc. **344**, 1057 (2003).
[4] E. W. Kolb, S. Matarrese, A. Notari, and A. Riotto, Phys. Rev. D **71**, 023524 (2005).
[5] E. W. Kolb, S. Matarrese, A. Notari, and A. Riotto, hep-th/0503117.
[6] M. W. Jacobs, E. V. Linder, and R. V. Wagoner, Phys. Rev. D **45**, R3292 (1992).
[7] M. W. Jacobs, E. V. Linder, and R. V. Wagoner, Phys. Rev. D **48**, 4623 (1993).
[8] E. V. Linder, Astron. Astrophys. **206**, 190 (1988).
[9] P. Schneider and A. Weiss, Astrophys. J. **327**, 526 (1988).
[10] L. Wang and P. J. Steinhardt, Astrophys. J. **508**, 483 (1998).
[11] D. H. Weinberg, New Astron. Rev. (to be published).
[12] L. Amendola, C. Quercellini, and E. Giallongo, Mon. Not. R. Astron. Soc. **357**, 429 (2005).
[13] D. J. Heath, Mon. Not. R. Astron. Soc. **179**, 351 (1977).
[14] P. J. E. Peebles, *Large-Scale Structure in the Universe* (Princeton University Press, Princeton, 1980).
[15] G. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. B **485**, 208 (2000).
[16] C. Deffayet, G. Dvali, and G. Gabadadze, Phys. Rev. D **65**, 044023 (2002).
[17] A. Lue, R. Scoccimarro, and G. D. Starkman, Phys. Rev. D **69**, 124015 (2004).
[18] G. Dvali, A. Gruzinov, and M. Zaldarriaga, Phys. Rev. D **68**, 024012 (2003).
[19] G. Gabadadze, in *From Fields to Strings: Circumnavigating Theoretical Physics*, edited by M. Shifman, A. Vainshtein, and J. Wheater (World Scientific, Singapore, 2004).
[20] Y.-S. Song, Phys. Rev. D **71**, 024026 (2005).
[21] L. Knox, Y.-S. Song, and J. A. Tyson, astro-ph/0503644.
[22] SNAP, http://snap.lbl.gov; G. Aldering *et al.*, astro-ph/0405232.
[23] Planck, http://sci.esa.int/planck
[24] R. R. Caldwell and E. V. Linder, astro-ph/0505494 [Phys. Rev. Lett. (to be published)].
[25] R. R. Caldwell and E. V. Linder (unpublished).
[26] L. Pogosian *et al.*, astro-ph/0506396.
[27] F. Giovi, C. Baccigalupi, and F. Perrotta, Phys. Rev. D **71**, 103009 (2005).
[28] B. Gold, Phys. Rev. D **71**, 063522 (2005).
[29] G. Barenboim and J. Lykken, astro-ph/0504090.
[30] G. Barenboim, O. Mena, and C. Quigg, Phys. Rev. D **71**, 063533 (2005).
[31] M. S. Turner, Phys. Rev. D **31**, 1212 (1985).
[32] E. V. Linder, Astron. Astrophys. **206**, 175 (1988).
[33] L. Barnes, M. Francis, G. F. Lewis, and E. V. Linder (unpublished).
[34] L. Amendola and C. Quercellini, Phys. Rev. Lett. **92**, 181102 (2004).
[35] T. Futamase, Phys. Rev. Lett. **61**, 2175 (1988).
[36] T. Futamase, Mon. Not. R. Astron. Soc. **237**, 187 (1989).
[37] T. Futamase and M. Sasaki, Phys. Rev. D **40**, 2502 (1989).
[38] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).
[39] E. V. Linder, Mon. Not. R. Astron. Soc. **243**, 362 (1990).
[40] L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields* (Pergamon, Oxford, 1975).