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Langevin analysis of eternal inflation

Steven Gratton and Neil Turok
Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA. United Kingdom
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It has been widely claimed that inflation is generically eternal to the future, even in models where the inflaton potential monotonically increases away from its minimum. The idea is that quantum fluctuations allow the field to jump uphill, thereby continually revitalizing the inflationary process in some regions. In this paper we investigate a simple model of this process, pertaining to $\lambda\phi^4$ inflation, in which analytic progress may be made. We calculate several quantities of interest, such as the expected number of inflationary efolds, first without and then with various selection effects. With no additional weighting, the stochastic noise has little impact on the total number of inflationary efoldings in the model even if the inflaton starts with a Planckian energy density. A ''rolling'' volume factor, i.e. weighting in proportion to the volume at that time, also leads to a monotonically decreasing Hubble constant and hence no eternal inflation. We show how stronger selection effects including a constraint on the initial and final states and weighting with the final volume factor can lead to a picture similar to that usually associated with eternal inflation.

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I. INTRODUCTION

If inflation is to provide a satisfactory explanation of the early universe, it needs both to find a successful microphysical implementation and to answer the question of why the universe started out in a high-energy inflating state. The phenomenon of ''eternal inflation'' has been proposed as a solution of this second problem [1–7]. Even if the inflaton potential monotonically increases away from its minimum, quantum fluctuations allow the inflaton field to jump uphill in some regions which would then expand exponentially. It is argued that this process, once started, can allow inflation to continue indefinitely, and that in all likelihood there was a great deal of inflation in the past of any observers ''like us''. The phrase ''like us'' warns that selection effects are involved, and that these might be important in evaluating the predictions for various inflationary models [8]. At the simplest level an observer ''like us'' could not live in a phase still undergoing high-energy inflation. So, even if such a phase dominates spacetime, no observers ''like us'' are there to see it. Attempts have been made to implement selection effects with the aid of a volume weighting, assigning more weight to larger regions of the universe, which are assumed to contain a greater number of ''typical'' observers.

The anthropic leanings of the discussion can be disguised by attempting to rephrase selection effects in a more physical way, for example, by demanding enough inflation to provide superhorizon correlations on the last scattering surface say. This amounts to the requirement that there be something like 50 or more inflationary efolds in typical models (see e.g. [9]). But the observed superhorizon correlations in the universe (including large scale homogeneity and isotropy) appear to be much stronger than those required for successful galaxy formation. If large amounts of inflation are exponentially less likely than small amounts, one might interpret the existence of superhorizon correlations in our cosmic microwave background as evidence *against* inflation.

In any case, it seems important to try and develop calculations of conditional probabilities within inflationary models, taking into account the backreaction of quantum fluctuations on the process of inflation itself. At first glance, one might think of a slow rolling inflaton field as being similar to an over-damped harmonic oscillator in the presence of weak stationary noise. After a short time such an oscillator ends up at the bottom of its potential and it only rarely fluctuates appreciably upwards. Memory of initial conditions is exponentially suppressed, and small fluctuations away from the minimum are exponentially more common than larger ones. If such a model were correct, one might expect ''our'' universe to only have the minimum possible number of efoldings consistent with the existence of a galaxy say.

As we relate here, taking into account the field dependence of the Hubble damping and noise leads to a qualitatively different picture. As we shall discuss, with a change of variable we see that the system is actually an *upside-down*, over-damped harmonic oscillator, for which there is no stationary state.

Nevertheless, the system can be studied via a simple Langevin equation which, for a particular choice of inflaton potential $V = \lambda\phi^4$ and in the slow-roll approximation, is linear and hence exactly soluble [10]. Even though the dynamical evolution of the system is trivial, the quantities we are interested in, such as the number of inflationary efolds, are highly nonlinear and nonlocal in time and hence

*Electronic address: S.T.Gratton@damtp.cam.ac.uk
†Electronic address: N.G.Turok@damtp.cam.ac.uk

nontrivial to compute. Nevertheless, we are able to make progress and obtain a number of new results, extending earlier works on stochastic inflation [10–17]. The main advantage of the Langevin approach we take over that involving the Fokker-Planck equation (see e.g. [16,18,19]) is that quantities which are nonlocal in time, such as the number of efolds, are more easily treated analytically. We are also able to include various proposed weightings such as volume factors rather straightforwardly.

Before we outline these calculations, however, let us explain the physical setup which we believe is approximately described by the simple stochastic model we employ.

II. CAUSAL INTERPRETATION OF A HUBBLE VOLUME DURING INFLATION

We shall be following a region whose size is the Hubble radius during inflation, and computing the evolution of the spatially-averaged field $\phi(t)$ in this region. The justification for focusing on just one such Hubble volume is that it spans the past light cone of an observer located far to the future. The spacetime volume inside this past light cone can be considered as an isolated physical system: given initial (or final, or mixed initial and final) conditions and a set of dynamical laws, its state should be completely describable without reference to the exterior. As long as causality holds it cannot be influenced by anything outside it.

Consider the past light cone emanating from some point at time t_1 in a universe described by a flat FRW metric with scale factor a. At an earlier time t this light cone encloses a sphere of physical radius

$$r_{\rm phys}(t) = a(t)\int_t^{t_1} dt'/a(t'). \qquad (1)$$

If $a(t)$ is increasing quasiexponentially, i.e. the Hubble parameter $H(t) \equiv \dot{a}/a$ is positive and only slowly varying with time, then the integral is dominated by its lower limit and $r_{\rm phys}(t)$ becomes approximately equal to $1/H(t)$, the Hubble radius at time t.

So we are interested in describing the evolution of the scalar field when averaged on the scale of the Hubble radius at that time. We shall employ a simple stochastic model to describe the scalar field fluctuations which is standard in discussions of eternal inflation. According to the model, the scalar field acquires a fluctuation $\delta\phi \sim H$ on a scale of order the Hubble radius, every Hubble time. We may understand the equations as describing the history of the fluctuating field as seen in the past light cone of a point in the far future.

Usually one hears that the physical size of a region that is known to be inflating increases rapidly. This statement is only accurate in the case that the initial region is so large that causal influences coming from outside the initial region cannot propagate far enough into it so as to shut

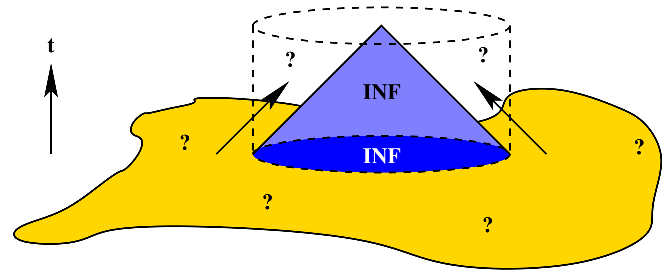

FIG. 1 (color online). Heuristic conformal diagram showing an inflating patch of critical size, indicated by the spacetime within the cone. The physical radius of the patch at time t is $1/H(t)$; the physical volume of the patch does not increase exponentially in time.

inflation down. The critical size turns out to be the inflationary Hubble radius. The stochastic model applies to a region that lies on this knife edge: small enough to lie within the past light cone of a future observer, and large enough to remain in an inflating state well into the future. See Fig. 1.

III. BACKGROUND EVOLUTION

Our starting point for the stochastic approach to inflation will be the Friedmann and scalar field equations for a flat FRW metric:

$$H^2 - \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right) = 0, \qquad (2)$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \qquad (3)$$

One then considers linearized perturbations about a background solution. In each Hubble time, new quantum fluctuations in the scalar field are generated and freeze out on the scale of the Hubble radius with an amplitude of order H [2].

We model the effects of such fluctuations by adding a stochastic noise term onto the right-hand side (RHS) of Eq. (3). We take the noise to be proportional to delta-function-normalized Gaussian white noise, $n(t)$, which obeys

$$\langle n(t)\rangle = 0, \qquad \langle n(t)n(t')\rangle = \delta(t - t'). \qquad (4)$$

Clearly, $n(t)$ has dimensions of mass to the power one half. The coefficient may be determined, up to a numerical coefficient of order unity, by dimensional analysis. The only scale entering into the fluctuations is H, with dimensions of mass. Thus we need $H^{5/2}n(t)$ on the RHS of (3) for the correct dimensions. We shall not be concerned about the numerical coefficient here, but if desired this can be determined by normalizing the model to a calculated quantum correlation function (see e.g. [11]).

There is a long history of attempts to take account of the effects of the fluctuations on the background evolution in inflation, going back to [20] for example. Ref. [2] develops a picture of the field evolution over a flat region of its potential as Brownian motion. It was later realized that this motion could also be important for fields with unbounded potentials and that this effect might be crucial in predicting what an observer might expect to observe (see [3–5]). Numerical simulations of this process have been performed (see e.g. [17–19]).

One can often self-consistently make the slow-roll approximation, even in the presence of noise, which involves neglecting the $\dot{\phi}^2/2$ term in (2) and the $\ddot{\phi}$ term in the noisy version of (3), leaving us with

$$3H\dot{\phi} + V_{,\phi} = H^{5/2}n(t), \tag{5}$$

where $H = H(\phi) = \sqrt{8\pi G V(\phi)/3}$. From now on we shall adopt reduced Planck units, setting $8\pi G = 1$.

IV. EXPLICIT SOLUTION FOR SLOW-ROLL $\lambda\phi^4$

There is a special choice of potential for which (5) simplifies, namely, for $V = \lambda\phi^4$, as noticed by Hodges [10] and Nambu [14]. This is an interesting potential to investigate in its own right and might reasonably be expected to be representative of other models with simple power law potentials such as $m^2\phi^2$. After a change of variable, the Langevin equation becomes linear, with field-independent noise. Remarkably, the new stochastic variable is just the physical Hubble radius.

Dropping the second-derivative term, defining $R = 1/(\sqrt{\lambda/3}\phi^2)$, the physical Hubble radius in the slow-roll approximation, and introducing $\alpha = 8\sqrt{\lambda/3}$ and $\beta = 2\sqrt[4]{\lambda/3}/3$ we have

$$\dot{R} - \alpha R = \beta n(t), \tag{6}$$

where we have changed the sign of the RHS relative to (5). As introduced above, $n(t)$ is delta-normalized Gaussian white noise with zero mean. We use angular brackets $\langle(\ldots)\rangle$ to denote the ensemble average of a quantity $(\ldots)$ over histories of the noise function. We shall use double angle brackets $\langle\langle(\ldots)^q\rangle\rangle$ to denote the qth cumulant of the distribution of a quantity $(\ldots)$, and also for connected correlation functions of products of quantities. Note that powers of β count the number of times that the noise enters into any expression.

Equation (6) describes, as claimed, an over-damped, upside-down harmonic oscillator with linear noise. It immediately reveals a potential problem in that nothing prevents R from crossing zero for some trajectories. Both the stochastic model and the underlying field theory of inflation may be expected to break down there, since the scalar field tends to infinity. A similar pathology occurs in the Fokker-Planck approach [14,16], and we can try to apply analogous workarounds here where necessary.

If we specify an initial condition $R = r_0$ at $t = 0$ say, we have the integral solution

$$R(t) = r_0 e^{\alpha t}\left(1 + \frac{\beta}{r_0}\int_0^t dt_1 e^{-\alpha t_1} n(t_1)\right). \tag{7}$$

Averaging over the noise, we find

$$\langle R(t)\rangle = r_0 e^{\alpha t}, \tag{8}$$

$$\langle R(t)R(t')\rangle = r_0^2 e^{\alpha(t+t')}\left(1 + \frac{\beta^2}{2\alpha r_0^2}(1 - e^{-2\alpha \min(t,t')})\right). \tag{9}$$

For the second expectation value we have multiplied two integral solutions together, taken the ensemble average using the assumed properties of the noise, given in (4), and then performed the integrals over the dummy time variables.

The mean value $\langle R(t)\rangle$ does not involve the noise at all. Changing variables back to ϕ we see this just represents the classical slow-roll solution $\phi = \phi_0 e^{-4\sqrt{\lambda/3}t}$. Because the noise is Gaussian, the R-distribution is also Gaussian, with a mean $\mu = r_o e^{\alpha t}$ from (8) and a variance $\sigma^2 = \beta^2(e^{2\alpha t} - 1)/(2\alpha)$ given by setting $t' = t$ in (9). More generally, one can derive a simple integral expression for the expectation value of any function $f(R(t))$ of R that can be expressed as a Fourier integral, by taking the averaging inside the Fourier integral and writing $\langle e^{ikR}\rangle$ in terms of cumulants as $e^{ik\langle R\rangle - k^2\langle\langle R^2\rangle\rangle/2}$.

Defining X^2 as a dimensionless measure, σ^2/μ^2, of the variance in $R(t)$:

$$X^2(t) \equiv \frac{\langle\langle R^2(t)\rangle\rangle}{\langle R(t)\rangle^2} = \frac{\beta^2}{2\alpha r_0^2}(1 - e^{-2\alpha t}), \tag{10}$$

we see that $R = 0$ lies $1/X$ standard deviations to the left of $\langle R(t)\rangle$. Thus if X is small, one can hope that for reasonable weightings the problems associated with $R = 0$ can be neglected. X is small at small times for all starting values r_0, and remains small for all time if r_0 is large enough that $X_\infty^2 \equiv X^2(\infty) = \beta^2/(2\alpha r_0^2) \ll 1$. X_∞^2 is a quantity with a nice physical meaning: it is, up to a numerical coefficient, the ratio of the initial energy density to the Planck energy density. So if we start well below the Planck scale, we do not expect subtleties at $R = 0$ to be important when following a patch forward in time.

Strictly speaking, one should not trust the theory at all if the density approaches the Planck density. Nevertheless, one can attempt to patch the theory up by imposing reflecting [14] or absorbing [16] boundary conditions at $R = 0$. We shall discuss the latter in Sec. VI below. In our Langevin treatment these two prescriptions correspond, respectively, to averaging or differencing the probability densities over trajectories that start at both r_0 and $-r_0$.

V. ASYMPTOTIC EXPANSION FOR THE NUMBER OF EFOLDINGS

Let us use our integral solution to calculate the expected number of inflationary efolds $N(T) = \int_0^T dt H(t)$. If we were following a comoving region of a homogeneous inflating universe rather than one physical Hubble volume in an inhomogeneous universe, $N(T)$ would simply be the number of inflationary efolds that the region had experienced up to time T. We shall continue to call N the number of efolds, keeping in mind though that it bears no simple relation to the physical size of the volume that we are actually tracking in the calculation.

Since H is just $1/R$ in our approximation,

$$\langle N(T)\rangle = \int_0^T dt\langle H(t)\rangle = \int_0^T dt\left\langle \frac{e^{-\alpha t}}{r_0}(1 + I(t))^{-1}\right\rangle, \tag{11}$$

where

$$I(t) \equiv \frac{\beta}{r_0}\int_0^t dt_1 e^{-\alpha t_1} n(t_1). \tag{12}$$

We now perform a formal Taylor expansion in $I(t)$ of the reciprocal inside (11). The justification for this procedure is that as long as the noise term can be treated as "small", an expansion in the noise makes sense.

We can take the expectation value of powers of I term by term making use of the fact that the noise is assumed to be Gaussian so that all higher-order correlation functions of it can be expressed in terms of products of its two-point function via Wick's theorem. It thus turns out that $\langle I^q(t)\rangle$ vanishes if q is odd, and equals $(2p-1)!!X^{2p}(t)$ if $q = 2p$ is even, with $X(t)$ as defined above and $(2p-1)!! = (2p-1)(2p-3)\ldots 1$. Thus

$$\langle H(t)\rangle = \frac{e^{-\alpha t}}{r_0}(1 + X^2(t) + 3X^4(t) + \ldots + (2p-1)!!X^{2p}(t) + \ldots). \tag{13}$$

Finally, we can perform the integral over t term by term to obtain a series for $\langle N(T)\rangle$.

This procedure works particularly neatly if we take $t \to \infty$, since then the time integrals are straightforward, using $\int_0^\infty dy e^{-y}(1 - e^{-2y})^p = 2^p p!/(2p+1)!!$. With $N_\infty \equiv N(\infty)$, one obtains:

$$\langle N_\infty\rangle = N_{\rm cl}\left(1 + 1\cdot\frac{2}{3}\cdot X_\infty^2 + 3\cdot\frac{16}{30}\cdot X_\infty^4 + \ldots + \frac{2^p p!}{2p+1}X_\infty^{2p} + \ldots\right), \tag{14}$$

where $N_{\rm cl} = \frac{1}{\alpha r_0}$ is the "classical" number of efolds expected in the slow-roll approximation at late times in the absence of noise.

Unfortunately the two series expansions (13) and (14) diverge. We shall investigate the apparent divergence of $\langle H(t)\rangle$ and $\langle N_\infty\rangle$ in more detail in the next section, showing that they are in fact finite if an absorbing boundary condition is imposed at $R = 0$. For now, let us understand these series as asymptotic expansions in terms of X_∞^2, the initial energy density. This is plausible since the Taylor expansion of $(1 + I(t))^{-1}$ inside the integral cannot be valid for all histories of the noise. But if the noise is weak enough we do not expect this inaccuracy to be significant. An analogous example is the calculation of the expectation value of $(1 + \epsilon x^2)^{-1}$ or $(1 - \epsilon x)^{-1}$ say for a Gaussian distribution with zero mean and unit variance in the case that $\epsilon \ll 1$, for which the perturbative expansion, although only asymptotic, is still extremely accurate.

Our second illustration $(1 - \epsilon x)^{-1}$ above is instructive in suggesting a formal procedure to account for effects associated with $R = 0$. Here we have a divergence in the calculation of the expectation value, at $x = -1/\epsilon$, far into the tail of the distribution. If we do not trust the detailed form of the distribution for large negative x, we may feel that such a divergence is unphysical and irrelevant. A natural mathematical way of removing such a divergence, and allowing us nevertheless to continue to use the Gaussian form, is to calculate the expectation value of principal values (denoted P.V.) of linearly-divergent quantities rather than the quantities themselves. In the stochastic model this prescription turns out to be related to imposing absorbing boundary conditions at $R = 0$.

After these comments, let us see what we can glean from Eq. (14). We see that fluctuations increase the expected number of efolds that a patch undergoes, but only by a small multiplicative fraction if the energy density starts low compared to the Planck density. Significant corrections to the classical result only occur at such high energy densities that we can have no confidence in the applicability of the theory.

VI. THE TOTAL NUMBER OF EFOLDINGS

We would like to investigate in greater depth the divergence of our series for $\langle H(t)\rangle$ and $\langle N(T)\rangle$. In the Fokker-Planck approach, a natural way to regulate the problem at $R = 0$ is to impose the "absorbing" boundary condition, that the probability is zero at $R = 0$ [16]. This can be done for a Gaussian probability distribution by reflecting it around $R = 0$ and subtracting the reflected copy off the original one. This new probability distribution must then be renormalized. In the Langevin approach this procedure corresponds to ignoring any paths that reach $R = 0$. One could alternatively impose "reflecting" boundary conditions for which the gradient of the probability distribution at $R = 0$ is zero, or indeed consider a mixture of the two conditions. Only the pure absorbing condition ensures that the probability vanishes at $R = 0$, and for definiteness we shall focus on this case here.

From our Langevin approach, we know that before the boundary condition is imposed R is Gaussian-distributed, with mean μ and variance σ^2 given above. Applying the reflect-and-renormalize procedure outlined above, the expectation value of some function $f(R)$ is given by

$$\langle f(R)\rangle_{\rm abs} = \frac{\int_{0^+}^{\infty} dx f(x)(e^{-(x-\mu)^2/2\sigma^2} - e^{-(x+\mu)^2/2\sigma^2})}{\int_{0^+}^{\infty} dx(e^{-(x-\mu)^2/2\sigma^2} - e^{-(x+\mu)^2/2\sigma^2})}. \tag{15}$$

(The subscript "abs" is for absorbing.) We extend the range of f from the positive reals to the entire real line by demanding that f is odd. Then we let $x \to -x$ in the second term of the upper integral, and divide top and bottom by $\sqrt{2\pi\sigma^2}$. The numerator is then the expectation value of the P.V. of f with respect to the Gaussian we started with over the entire real line. The denominator is the error function $\mathrm{erf}(\mu/(\sqrt{2}\sigma)) = \mathrm{erf}(1/(\sqrt{2}X(t)))$.

Since $H(t) = 1/R(t)$ we can apply this to work out $\langle H(t)\rangle_{\rm abs}$:

$$\langle H(t)\rangle_{\rm abs} = \frac{\left\langle \mathrm{P.V.}\left(\frac{1}{R(t)}\right)\right\rangle}{\mathrm{erf}\left(\frac{1}{\sqrt{2}X(t)}\right)}. \tag{16}$$

To make contact with our asymptotic expansion above, we write $\mathrm{P.V.}(1/R(t))$ as a Fourier integral and then take its Gaussian expectation value as discussed earlier. This gives the numerator as the integral

$$\left\langle \mathrm{P.V.}\left(\frac{1}{R(t)}\right)\right\rangle = \frac{1}{\langle R(t)\rangle}\cdot\frac{1}{X(t)}\int_0^\infty dk \sin\left(\frac{k}{X(t)}\right)e^{-k^2/2}. \tag{17}$$

If we then continually integrate by parts on the sin we generate the asymptotic expansion (13).

We still have to consider the denominator. Now erf(1/(sqrt2X(t))) $\mathrm{erf}(1/(\sqrt{2}X(t))) = 1 - \mathrm{erfc}(1/(\sqrt{2}X(t)))$, and, for small $X(t)$, $\mathrm{erfc}(1/(\sqrt{2}X(t))) \sim \sqrt{2/\pi}X(t)e^{-1/(2X^2(t))}$ (see e.g [21]) which is exponentially small. So the denominator itself is exponentially close to 1, and the entire expression is accurately approximated by the asymptotic expansion (13).

Thus we see that effects coming from imposing boundary conditions at $R = 0$ are negligible to the extent that $X(t)$ is small.

Imposing an absorbing boundary condition at $R = 0$ allows us to go on to investigate the large X regime also. We may rewrite (15), applied to $H(t) = 1/R(t)$, as

$$\langle H(t)\rangle_{\rm abs} = \frac{\int_0^\infty \frac{dx}{x} W(x)}{\int_0^\infty dx W(x)} \tag{18}$$

where

$$W(x) \equiv \sinh\left(\frac{x\mu}{\sigma^2}\right)e^{-x^2/2\sigma^2}. \tag{19}$$

By rescaling the integration variable, one finds that the expected Hubble constant can be expressed in the form

$$\langle H(t)\rangle_{\rm abs} = \frac{\mu}{\sigma^2}F(\sigma^2/\mu^2). \tag{20}$$

One finds the asymptotic behavior:

$$F(z) \sim z, \quad \text{for } z \ll 1; \tag{21}$$

$$F(z) \sim \sqrt{(\pi z)/2}, \quad \text{for } z \gg 1. \tag{22}$$

We thus note that $\langle H(t)\rangle_{\rm abs}$ is finite for all t, as mentioned below Eq. (14). Using these expressions, one estimates the expected total number of efoldings $\langle N_\infty\rangle_{\rm abs} = \int_0^\infty dt \langle H(t)\rangle_{\rm abs}$ as:

$$\langle N_\infty\rangle_{\rm abs} \sim \frac{1}{\alpha r_0} = \frac{\phi_0^2}{8}, \quad \text{for } r_0 \gg 1, \tag{23}$$

which is just the standard classical result, and

$$\langle N_\infty\rangle_{\rm abs} \sim \left(\frac{\beta^2}{2\alpha}\right)^{-1/2}\left(\frac{\pi}{2}\right)^{3/2}\frac{1}{\alpha}, \quad \text{for } r_0 \ll 1, \tag{24}$$

which up to a numerical constant is $\lambda^{-1/2}$, just the standard classical slow-roll result for inflation starting at the Planck density. Hence we conclude that quantum fluctuations, treated as stochastic noise, do not qualitatively alter the expected number of inflationary efoldings beyond the classical result, for initial densities right up to the Planck density. We have numerically confirmed that the RHS of (22) is in fact an upper bound for $F(z)$ for all z, as might be suspected from inspecting the forms of (21) and (22). Thus the RHS of (24) is in fact an upper bound for $\langle N_\infty\rangle_{\rm abs}$ for *all* r_0. Hence we see, as mentioned below Eq. (14), that with absorbing boundary conditions the mean number of efoldings at late times is bounded and does not diverge.

VII. "ETERNAL INFLATION" FROM STRONG SELECTION

Imagine we know that at two times the scalar field was well up the hill within the Hubble volume we are tracking. What can we say about the likely value of the field at intermediate times? Did the scalar field roll downhill and then fluctuate back up, or did it fluctuate up and then roll down to its prescribed final value?

There are two equivalent ways of calculating such an effect. One is to use Bayes' theorem explicitly. The other to exploit the fact that the noise is Gaussian, which means that the full probability distribution for multiple events is given by exponentiating their covariance matrix (this method cannot be used if the boundary conditions at $R = 0$ are important). We shall present the former method here.

Let us take $0 < t < T$. We denote by $p_{1|1}(r_t|r_0)dr$ the probability that, at time t, R lies between r_t and $r_t + dr$ given that $R(0) = r_0$; and by $p_{1|2}(r_t|r_T, r_0)dr$ the probability that, at time t, R lies between r_t and $r_t + dr$ given that $R(T) = r_T$ and $R(0) = r_0$.

Then, by Bayes' theorem,

$$p_{1|2}(r_t|r_T, r_0) = \frac{p_{1|1}(r_t|r_0)p_{1|2}(r_T|r_t, r_0)}{p_{1|1}(r_T|r_0)} \tag{25}$$

and we know what all the terms on the RHS are, since $p_{1|2}(r_T|r_t, r_0) = p_{1|1}(r_T|r_t)$.

If we neglect $R = 0$ effects, the probability distributions are Gaussian and we can use our formulas (8) and (9) to evaluate the means and variances for the RHS. With the constraints on R at 0 and T we find that at the intermediate time t, R is again Gaussian-distributed with a t-dependent mean of

$$\langle R(t)\rangle_{\rm con} = \frac{r_0 \sinh\alpha(T-t) + r_T \sinh\alpha t}{\sinh\alpha T}. \tag{26}$$

(The subscript "con" is for constrained.) In the Appendix we discuss an interesting formal property of $\langle R(t)\rangle_{\rm con}$, which is that it obeys a second order differential equation related to the original first order Eq. (6) in a simple way.

One can also easily find the variance in R by looking at the coefficient of the term quadratic in r_t in the exponent on the RHS of (25). It turns out to be:

$$\langle\langle R^2(t)\rangle\rangle_{\rm con} = \frac{\beta^2}{2\alpha}\frac{2\sinh\alpha(T-t)\sinh\alpha t}{\sinh\alpha T}. \tag{27}$$

Interestingly, this is independent of r_0 and r_T, and attains its maximum value of $(\beta^2/2\alpha)\tanh(\alpha T/2)$ at the midpoint $t = T/2$. This is less than $\beta^2/2\alpha$, so the standard deviation in R here is always bounded by a constant of order unity. As long as $\langle R(t)\rangle_{\rm con}$ is always significantly larger than unity, we expect that we may neglect effects from $R = 0$.

Furthermore, if the minimum of $\langle R(t)\rangle_{\rm con}$ is much less than r_T, then, relatively close to T $\langle R(t)\rangle_{\rm con}$ behaves like $r_T e^{\alpha(t-T)}$, independent of r_0. This is the same as the noise-free slow-roll solution which passes through r_T at time T, and which in the far past would approach arbitrarily small R and thus large ϕ.

For example, if we take $r_T = r_0$, $\langle R(t)\rangle_{\rm con}$ takes a minimum value of $r_0/\cosh(\alpha T/2)$ at $t = T/2$. Demanding that the minimum value is greater than the standard deviation at that time puts a condition on the values of T for which we may reliably neglect effects associated with the $R = 0$ boundary: this condition reads $\sinh\alpha T < 2/X_\infty^2$. Since X_∞^2 is typically small if the field starts low down the potential, we see that as long as $T \lesssim \ln(4/X_\infty^2)/\alpha$ boundary effects from $R = 0$ should be unimportant. As long as this is so, we can now answer the question posed at the beginning of this section. If the field is known to be at the same place on the potential at times 0 and T, the Hubble radius lies at smaller values in the interim, corresponding to the field being further up the potential. See Fig. 2.

So we have shown that if inflation has lasted a reasonably long time (i.e. we know that the field was displaced well up its potential at two widely separated times), the field was likely to have rolled up the hill to higher field values, and then turned round and rolled back down, approaching the standard slow-roll solution. This behavior shares many of the characteristics that are commonly associated with the phrase "eternal inflation".

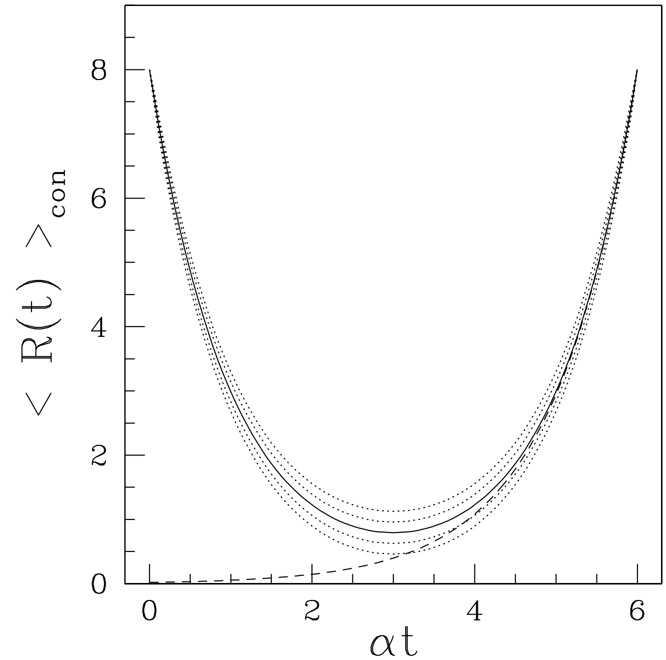

FIG. 2. A plot showing the expectation of $R = 1/(\sqrt{\lambda/3}\phi^2)$ against time for paths constrained to take specified values at two different times. In the example shown, $R = 8$ at $\alpha t = 0$ and $\alpha t = 6$ (solid line). Also indicated are one and two sigma contours for the trajectories (dotted lines), and the noise-free solution for R that passes through $R = 8$ at $\alpha t = 6$ (dashed line). We have set $\beta^2/2\alpha = 1/36$ in making this plot.

As an example, imagine we select paths in which the scalar field takes a value of order the Planck mass at two widely separated times. Recall that this is the value for which the slow-roll approximation fails, near the end of inflation. The above calculation shows that the expected trajectory is one in which the field first runs uphill to very large values before rolling down along a nearly classical slow-roll path to the final value. For such paths, one would expect the usual predictions for the observable density perturbations.

However, by demanding that αT is large we have imposed a very strong selection of paths; the vast majority of paths run downhill. An indication of the strength of this selection is given by considering the fraction of trajectories for $R(t)$ in the unconstrained case that happen to end up at r_0 or less at time T. This fraction may be approximated as $\sqrt{X_\infty^2/2\pi}e^{-1/(2X_\infty^2)} \sim e^{-C/V(\phi_0)}$, with C some number of order unity. Unless one selects trajectories starting and ending at the Planck density, this fraction is an exponentially small number. Of course, in that case one is sensitive to the boundary conditions at $R = 0$. For our numerical example of Fig. 2, one obtains a fraction of order 10^{-500}!

Now let us consider weakening the selection by reducing the time T between our initial and final states. Even if we insist upon 50 efoldings of inflation, the likely trajectory is very different from the classical slow-roll one usually considered. In standard inflation where one ignores the effect of quantum fluctuations on the background, the density perturbation amplitude is approximately $H^2/\dot{\phi}$. However, in the selected ensemble the velocity $\dot{\phi}$ of the scalar field vanishes at an intermediate time. This would lead one to suspect that for the selected ensemble, density perturbations which exited the Hubble radius around the time $T/2$ would be far larger than those in standard inflation. Unless T is chosen to give substantially more than 50 efoldings, the normal predictions of inflationary perturbations will not be obtained.

To conclude this section, we have seen how selection effects can indeed make a scalar field roll uphill before rolling roll back along a near-classical slow-roll path. This is the same type of behavior as that sought from eternal inflation, but we have obtained it through an imposed ad-hoc selection, rather than from volume weighting, the effect to which it is normally attributed. We now turn to an investigation of volume weighting to see whether it can indeed produce such behavior.

VIII. ETERNAL INFLATION AS A VOLUME EFFECT

The usual argument for inflation being eternal to the future with an inflaton field with an unbounded potential is summarized and criticized in [22]. The main idea is that regions of space in which the field fluctuates higher inflate more rapidly, and thus after a long amount of time the majority of the physical volume is dominated by inflating regions at the Planck density. Occasional regions fluctuate downwards and inflation stops, and we are expected to live in one of these regions. The main criticism is that this line of argument is not gauge invariant, i.e. comparisons of volumes depend on the choice of time-slicing.

We argued above that the stochastic equation really only follows the evolution of the field inside one physical Hubble volume. The common supposition is that it rather describes the average behavior of the field in a fixed *comoving* volume. In Fig. 1, this corresponds to assuming that the field on the constant-time slice through the entire conformal cylinder is the same as it is on the piece of the slice inside the cone. In this case, one may then argue that expectation values of various quantities evaluated at time t should be calculated by weighting each trajectory by its volume factor $e^{3N(t)}$. In this section, we shall explore the consequences of this suggestion.

We thus define the "volume-weighted" average of some quantity $(\ldots)$ at time t via:

$$\langle(\ldots)\rangle_{\mu=3,t} \equiv \frac{\langle(\ldots)e^{3N(t)}\rangle}{\langle e^{3N(t)}\rangle}. \tag{28}$$

(The reasoning behind the subscript will become clear shortly.)

Let us attempt to calculate the volume-weighted number of efolds $\langle N(t)\rangle_{\mu=3,t}$. This is an interesting example because the usual claim (see e.g. [6]) is that if ϕ_0 is above a critical value of order $\lambda^{-1/6}$ then the physical volume of space that is inflating should increase without limit, corresponding to eternal inflation. Thus we would expect to see a qualitative change in the behavior of $\langle N(t)\rangle_{\mu=3,t}$ around this critical starting value.

We start by introducing the formal generating function $W_t(\mu) = \ln\langle e^{\mu N(t)}\rangle$, where μ is a number. Differentiating this with respect to μ and then setting $\mu = 3$ gives us the volume-weighted number of efolds, $\langle N(t)\rangle_{\mu=3,t}$ (hence the subscript in (28)). We may think of $W_t(\mu)$ as

$$\sum_{n=1}^{\infty} \frac{\mu^n}{n!}\langle\langle N(t)^n\rangle\rangle. \tag{29}$$

So we can express $\langle N(t)\rangle_{\mu=3,t}$ in terms of the connected moments or cumulants of N. The nth cumulant of N can be obtained as usual from the regular moments of N, which can be calculated order by order in the noise. (Such a procedure is described in more detail in the following section where it is needed for a wider-ranging calculation.)

In the limit $t \to \infty$, it turns out that the nth cumulant goes like $N_{\rm cl}^n$ times a power series in X_∞^2 starting at the $(n-1)$th power of X_∞^2. So to order X_∞^4 we find

$$\langle\langle N_\infty\rangle\rangle = N_{\rm cl}\left(1 + \frac{2}{3}X_\infty^2 + \frac{8}{5}X_\infty^4 + \ldots\right) \tag{30}$$

$$\langle\langle N_\infty^2\rangle\rangle = N_{\rm cl}^2\left(\frac{1}{2}X_\infty^2 + 3X_\infty^4 + \ldots\right) \tag{31}$$

$$\langle\langle N_\infty^3\rangle\rangle = N_{\rm cl}^3\left(\frac{12}{7}X_\infty^4 + \ldots\right) \tag{32}$$

with all higher cumulants being of higher order X_∞^2. Following the aforementioned procedure and collecting terms in X_∞^2, we find

$$\langle N_\infty\rangle_{\mu=3,\infty} = N_{\rm cl}\left\{1 + \left(\frac{3}{2}N_{\rm cl} + \frac{2}{3}\right)X_\infty^2 + \left(\frac{54}{7}N_{\rm cl}^2 + 9N_{\rm cl} + \frac{8}{5}\right)X_\infty^4 + O(X_\infty^6)\right\}. \tag{33}$$

Thus we obtain the fractional correction to the classical number of efolds $N_{\rm cl}$ to fourth order in the noise. One sees that for $N_{\rm cl} \gg 1$ (but still $X_\infty^2 \ll 1$), we in fact seem to have an expansion in $N_{\rm cl}X_\infty^2$ rather than in just X_∞^2. So our formula gives only a small fractional correction to the number of efolds as long as $N_{\rm cl}X_\infty^2 \ll 1$.

But $N_{\rm cl}X_\infty^2 = \beta^2/(2\alpha^2 r_0^3) \sim \lambda\phi_0^6$ and so we see that when $\phi_0 \gtrsim \lambda^{-1/6}$ the volume-weighted number of efolds begins to change substantially from the noise-free result. This is just the criterion mentioned above that is usually given for the onset of eternal inflation.

So our method has established some contact with the usual approach to eternal inflation. However, we caution the reader that the assumption that one should volume-weight is not yet (in our view) well-founded, since the dynamical equations being used are only following the value of the field in one Hubble volume.

IX. A CLOSER INVESTIGATION OF VOLUME-WEIGHTING

In the above section we introduced volume-weighted averages of quantities defined at a given time, where the volume-weighting was performed at that time. We used this to calculate the volume-weighted number of efolds at a given time t, and then considered taking the limit $t \to \infty$.

In this section, we shall consider two sorts of volume weighting. If one believes in volume-weighting as a physical spatial averaging, then to compute observables at time t one might want to weight with the volume at that time. On the other hand, if one thinks of volume-weighting as a selection effect for typical observers, one might want to weight with the "final" volume, at some much later time. To cover both cases, we shall compute the Hubble parameter as a function of time, $H(t_a)$, weighted by the volume at a later time $t > t_a$. Taking the limit $t_a \to t$, we obtain the trajectory of the Hubble parameter with a rolling volume weighting. If instead t is taken to ∞, we obtain the expected history of the Hubble parameter when selected by final volume. This gives what a typical late-time volume-weighted observer should expect to see in his/her past. (Note that our treatment in the previous section took the "spatial averaging" point of view but focused only on the final number of efoldings.)

To calculate such averages involving the Hubble parameter, one needs to generalize the generating function introduced above, promoting μ from a number to a function of time. Thus one defines the formal generating functional $W_t[\mu]$

$$W_t[\mu] = \ln\langle e^{M_t[\mu]}\rangle, \tag{34}$$

where

$$M_t[\mu] \equiv \int_0^t dt_1 \mu(t_1) H(t_1). \tag{35}$$

Now by functionally differentiating with respect to $\mu(t_a)$, and then setting $\mu = 3$ for all time, one obtains volume-weighted cumulants involving $H(t_a)$. Different choices for μ correspond to more general forms of weighting, and we may denote expectation values with respect to them by $\langle\langle \ldots \rangle\rangle_{\mu(t'),t}$ should we wish to use them. Note that $\mu = 0$ corresponds to the natural weighting. It can be useful to take μ to be an arbitrary constant at intermediate stages of the calculation, and only set $\mu = 3$ at the very end. Then one can easily identify any effects coming from the volume weighting.

In order to proceed perturbatively in the noise, we again expand W in terms of the regular (i.e. non volume-weighted) cumulants of $M[\mu]$:

$$W_t[\mu] = \sum_{n=1}^{\infty} \frac{1}{n!} \langle\langle M_t^n[\mu] \rangle\rangle. \tag{36}$$

These cumulants involve multiple integrals $\int dt_1 \ldots \int dt_j$ over products of the Hs. Each H is written as:

$$H(t_i) = \frac{e^{-\alpha t_i}}{r_0}(1 - I_i + I_i^2 - I_i^3 + I_i^4 \ldots) \tag{37}$$

where I_i is a shorthand for $I(t_i)$ as defined in (12). We use Wick's theorem to express higher-order moments of I in terms of the two-point function $\langle I_1 I_2 \rangle$, which is given by

$$\langle I_1 I_2 \rangle = X_\infty^2 (1 - e^{-2\alpha \min(t_1, t_2)}). \tag{38}$$

Each power of X_∞ counts an order of the noise. As in the previous section, it turns out that the nth cumulant is of order $2(n-1)$ in the noise.

Let us now use the above machinery to calculate $\langle H(t_a) \rangle_{\mu=3,t}$ to second order in the noise. We need the first and second cumulants of M, each evaluated to second order in the noise. Functionally differentiating the former leaves us with $\langle H(t_a) \rangle$, or

$$\frac{e^{-\alpha t_i}}{r_0}(1 + X_\infty^2 (1 - e^{-2\alpha t_a})). \tag{39}$$

Functionally differentiating the latter cumulant yields $2 \int_0^t dt_1 \mu(t_1) \langle\langle H(t_1) H(t_a) \rangle\rangle$, or

$$2 \int_0^t dt_1 \mu(t_1) \frac{e^{-\alpha t_1}}{r_0} \frac{e^{-\alpha t_a}}{r_0} X_\infty^2 (1 - e^{-2\alpha \min(t_1, t_a)}). \tag{40}$$

This can be evaluated assuming μ to be constant in time. Utilizing these results and setting $\mu = 3$ leads to

$$\begin{aligned} \langle\langle H(t_a) \rangle\rangle_{\mu=3,t} = \frac{e^{-\alpha t_a}}{r_0} \bigg\{ & 1 + X_\infty^2 (1 - e^{-2\alpha t_a}) \\ & + 3 N_{\rm cl} X_\infty^2 \bigg(1 - e^{-\alpha t} - \frac{1 - e^{-3\alpha t_a}}{3} \\ & - e^{-2\alpha t_a} (e^{-\alpha t_a} - e^{-\alpha t}) \bigg) \bigg\}. \end{aligned} \tag{41}$$

As discussed above, we are interested in two cases. The first, involves taking $t_a \to t$. This choice implements a "rolling volume weighting", where at any time the volume-weighted average is performed using the volume at that time. This choice would correspond to what one might usually consider to be a spatial average over physical volume. We obtain

$$\begin{aligned} \langle\langle H(t) \rangle\rangle_{\mu=3,t} = \frac{e^{-\alpha t}}{r_0} \bigg\{ & 1 + X_\infty^2 (1 - e^{-2\alpha t}) \\ & + 3 N_{\rm cl} X_\infty^2 \bigg(1 - e^{-\alpha t} - \frac{1 - e^{-3\alpha t}}{3} \bigg) \bigg\}. \end{aligned} \tag{42}$$

(This result can also be obtained by calculating $\langle N(t)\rangle_{\mu=3,t}$ as discussed in the previous section to second order in the noise and then differentiating with respect to t.) A Taylor expansion of the correction term coming from the volume-weighting starts at $O((\alpha t)^2)$. This is to be compared with the $O(\alpha t)$ variation coming from the expansion of the $e^{-\alpha t}$ "classical rolling" term. One (initial) Hubble time is $t_H = 1/H_0 = r_0$, so $\alpha t_H = \alpha r_0 = 1/N_{\rm cl}$, which is small if the classical number of efoldings is large. Thus as long as one is interested in times up to of order a few Hubble times from the start, rolling volume-weighting does very little even if $N_{\rm cl}X_\infty^2 \sim \lambda\phi_0^6 \gg 1$. One only starts getting corrections of order $N_{\rm cl}X_\infty^2$ when $\alpha t \sim 1$, that is at times for which, without noise, a substantial fraction of the total number $N_{\rm cl}$ of efolds would have occurred. We have checked that including the fourth order term does not affect this conclusion, and strongly suspect that neither will further higher-order terms. Hence we conclude that a rolling volume weighting leads to a monotonically decreasing Hubble constant, and will not produce the uphill motion required by the eternal inflation picture.

Now let us consider the second limit. This is to take $t \to \infty$. Thus we are weighting trajectories by their final volume. As discussed above, this in some sense corresponds to the history of the field that a physical observer at late times might expect to see in his/her past light cone, assuming that physical observers are evenly distributed over the final three-volume. We obtain:

$$\langle H(t_a)\rangle_{\mu=3,\infty} = \frac{e^{-\alpha t_a}}{r_0}\{1 + X_\infty^2(1 - e^{-2\alpha t_a}) + 2N_{\rm cl}X_\infty^2(1 - e^{-3\alpha t_a})\}. \tag{43}$$

Now the volume-weighting term has a Taylor expansion that starts at $O(\alpha t_a)$, the same order as that from the classical exponential. So now for large enough $N_{\rm cl}$, we can have the expectation value of the Hubble parameter increase with time. The requirement is that $6N_{\rm cl}X_\infty^2 > 1$, or $\phi_0 \gtrsim \lambda^{-1/6}$, again the criterion for eternal inflation. By $\alpha t_a \sim 1$, the fractional correction to the noise-free result is becoming large and we do not trust our result in the details. Of course, if the noise is very important we should not overly trust the $t \to \infty$ limit in the first place.

We can investigate the spread in the final-volume-weighted trajectories by considering the ratio $\langle\langle H^2(t_a)\rangle\rangle_{\mu=3,\infty}/\langle H(t_a)\rangle_{\mu=3,\infty}^2$. The lowest-order term in the numerator is already $O(X_\infty^2)$, so we must calculate the numerator to fourth order to obtain the fractional change in the ratio due to volume-averaging to second order. Thus we need the third cumulant of H for the numerator, but the calculation can be performed. Taking $\alpha t_a \to \infty$ to obtain the late-time behavior and dropping terms down by $N_{\rm cl}$ we find

$$\frac{\langle\langle H^2(\infty)\rangle\rangle_{\mu=3,\infty}}{\langle H(\infty)\rangle_{\mu=3,\infty}^2} \approx X_\infty^2\left(1 + \frac{26}{5}N_{\rm cl}X_\infty^2\right). \tag{44}$$

Thus we see that in the regime of eternal inflation the noise is playing an important role. One might be tempted to conclude that for $\phi_0 \gtrsim \lambda^{-1/6}$ the volume-weighted system becomes "noise-dominated" and a "nonperturbative" way of treating the noise is called for.

X. CONCLUSIONS

We have seen that $\lambda\phi^4$ inflation with quantum jumps may be modeled as an upside-down, over-damped harmonic oscillator with noise. The critical role that selection effects have on expectations for the field's observed history has been clearly demonstrated. In the absence of any selection effects, we have seen that the presence of stochastic noise hardly alters the motion of the inflaton field downhill: the total number of inflationary efolds is essentially the same as the classical result. Through strong selection one can obtain trajectories which travel uphill: if the field is constrained to lie at a particular inflating value at two widely separated times we have shown that it is likely in the interim to be at higher field values. As we saw, the selection required to achieve this must be exponentially strong, with only a minute fraction ($\sim e^{-C/V(\phi_0)}$) of all possible paths being chosen.

We also investigated volume-weighting in our approach. With a rolling weighting, such as might be attributed to averaging over physical volume, we find no significant effect on the Hubble parameter's evolution until late times, for which inflation would classically have completed a substantial fraction of its total number of efolds. If, alternatively, histories are weighted by the *final* volume, qualitative changes in the mean trajectory of the system can occur within a few Hubble times of the start. However in this situation, it is not clear whether the perturbative treatment is valid.

Our techniques might also be applicable for more general forms of averaging. For example, one might consider the possibility of weighting by 4-volume. This would be the right thing to do if observers spring into being from inflating regions in a stochastic manner consistent with Poisson statistics (say by bubble nucleation), and that the appearance of a region supporting physical observers has a negligible effect on the background evolution (unlike bubble nucleation). If the 4-volume were finite, the result would be gauge-invariant and ambiguity-free also. The generating function could then be $\ln\langle \int_0^\infty dt_1 e^{\mu N(t_1)}\rangle$, and differentiating this with respect to μ and then setting $\mu = 3$ would give the 4-volume-weighted number of efolds. The generating function could be calculated order by order in the noise using our methods. As we have explained, however, we are not convinced that any particular form of volume-weighting is physically correct, since the stochastic equation being used only describes the time evolution of a single physical Hubble volume.

There are several possible extensions of the work reported here. Numerical simulations could be used to extend the perturbative treatment we have given and to check the generality of the behavior found here for other potentials. It would be interesting to know whether inflation models with much flatter potentials exhibit the same qualitative behavior. It would also be interesting to extend the model to allow the inflaton field to jump back up from its minimum into an inflating regime, a situation referred to by Garriga and Vilenkin as a "recycling universe" [23] (see also [24]). In this situation one might hope to find a steady state solution, although the typical bout of inflation would involve very small numbers of efolds, and so such a model would be observationally disfavored.

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APPENDIX: A PROPERTY OF $\langle R(t)\rangle_{\mathrm{con}}$

In this Appendix we discuss an interesting fact about the constrained expectation $\langle R(t)\rangle_{\mathrm{con}}$, which is that it satisfies the *second order* deterministic differential equation

$$\left(\frac{d}{dt} - \alpha\right)\left(\frac{d}{dt} + \alpha\right)\langle R(t)\rangle_{\mathrm{con}} = 0, \qquad \text{(A1)}$$

where the operator appearing is the product of the original operator in the equation of motion, (6), and its time reverse. That this should be so is in fact a general consequence of time-translation invariance of the operator appearing in the original equation of motion, (6), and of the noise ensemble. The point is that the constrained expectation $\langle R(t)\rangle_{\mathrm{con}}$ is proportional to the correlator $\langle R(t)R(T)\rangle$, as may be seen by writing the joint (Gaussian) probability distribution for $R(t)$ and $R(T)$. To show the correlator obeys the stated Eq. (A1), note first that

$$\left(\frac{d}{dt} - \alpha\right)\left(\frac{d}{dT} - \alpha\right)\langle R(t)R(T)\rangle = 0 \qquad \text{(A2)}$$

for all $t \neq T$, since from the equation of motion (6) this equals $\beta^2\langle n(t)n(T)\rangle$, which is zero at unequal times. Now, we can write the left hand side as $(\frac{d}{dT} - \alpha)$ acting upon $\beta\langle n(t)R(T)\rangle$. But $R(T)$ is a sum of a particular integral which does not correlate with the noise, and an integral $\beta\int_0^T dt' G(T, t')n(t')$, where $G(T, t')$ is the Green function of the original operator. So $\langle n(t)R(T)\rangle$ is just $\beta G(T, t)$. Because of time-translation invariance, the latter is a function only of the time difference $t - T$. Therefore, we may replace $\frac{d}{dT}$ in (A2) with $-\frac{d}{dt}$, obtaining (A1). The arguments we have used are rather general: it is clear, for example, that constrained expectation values related to a second order stochastic differential equation would obey a fourth order deterministic differential equation.

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