Aspects of the tetrahedral neutrino mass matrix

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The four-parameter tetrahedral neutrino mass matrix introduced earlier by the author is studied in two specific limits, both having only two parameters and resulting in $\theta_{13} = 0$, $\theta_{23} = \pi/4$, and $\tan^2\theta_{12} = 1/2$. One limit corresponds to a recent proposal which predicts a normal ordering of neutrino masses; the other is new and allows both inverted and normal ordering.

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The non-Abelian discrete symmetry group of the even permutation of 4 objects has 12 elements and is called A_4. It is also the symmetry group of the tetrahedron and has been shown to be useful for understanding the family structure of charged-lepton and neutrino mass matrices [1,2]. A modified version [3] of this model was proposed a year ago by the author and has 4 parameters which allow arbitrary neutrino masses while predicting both θ_{23} and θ_{12} as functions of θ_{13}. For $0 < |U_{e3}| < 0.16$, this implies $1 > \sin^2 2\theta_{23} > 0.94$ and $0.5 < \tan^2\theta_{12} < 0.52$ respectively. At that time, the central value of $\tan^2\theta_{12}$ in a global fit of all neutrino data [4] was near 0.39, but now (with the most recent SNO analysis [5]) it is given rather by [6]

$$\tan^2\theta_{12} = 0.45 \pm 0.05. \tag{1}$$

This means that the prediction of Ref. [3] is in much better shape and deserves a second look.

In the basis where the charged-lepton mass matrix is diagonal, it was shown in Ref. [3] that a particular application of A_4 results in

$$\mathcal{M}_\nu = \begin{pmatrix} a + (2d/3) & b - (d/3) & c - (d/3) \\ b - (d/3) & c + (2d/3) & a - (d/3) \\ c - (d/3) & a - (d/3) & b + (2d/3) \end{pmatrix}. \tag{2}$$

If $b = c$, then this has the solution

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} \sqrt{2/3} & -\sqrt{1/6} & -\sqrt{1/6} \\ \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \\ 0 & -\sqrt{1/2} & \sqrt{1/2} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}, \tag{3}$$

with

$$m_1 = a - b + d, \qquad m_2 = a + 2b, \qquad m_3 = -a + b + d. \tag{4}$$

Hence $\theta_{13} = 0$, $\theta_{23} = \pi/4$, and $\tan^2\theta_{12} = 1/2$. This pattern is reminiscent of the $\pi - \eta - \eta'$ system in hadronic physics and was conjectured [7] to be applicable in the neutrino sector as well. It is consistent with all present neutrino-oscillation data.

Consider now the 3 neutrino mass eigenvalues of Eq. (4). With the 3 parameters (a, b, d), it is clear that $m_{1,2,3}$ may be chosen to fit whatever experimental values of $|m_2|^2 - |m_1|^2$ and $|m_3|^2 - |m_2|^2$ are necessary. In other words, this model has no prediction on neutrino masses. However, there are 2 interesting special cases of Eq. (4) which have definite predictions: (I) $b = 0$, and (II) $a = 0$.

Case (I) has

$$m_1 = a + d, \qquad m_2 = a, \qquad m_3 = -a + d, \tag{5}$$

and was obtained recently in a detailed model [8] which solves the vacuum alignment problem inherent in the Higgs sector used to obtain Eq. (2) by appealing to extra dimensions. It also avoids the problem of setting $b = c$ which is impossible to maintain by a symmetry if either parameter is nonzero. Since $\Delta m^2_{\rm sol}$ is much smaller than $\Delta m^2_{\rm atm}$ experimentally, Case (I) implies that

$$|d| \simeq -2|a|\cos\phi, \tag{6}$$

where ϕ is the relative phase between a and d. Hence

$$|m_{1,2}|^2 \simeq |a|^2, |m_3|^2 \simeq |a|^2(1 + 8\cos^2\phi), \tag{7}$$

requiring thus a normal ordering of neutrino masses. The ν_e kinematical mass is then given by

$$|m_{\nu_e}|^2 \simeq |m_{1,2}|^2 \simeq \frac{\Delta m^2_{\rm atm}}{8\cos^2\phi}, \tag{8}$$

and the effective ν_e mass in neutrinoless double beta decay is

$$|m_{ee}| = |a + (2d/3)| = \frac{1}{3}|\Delta m^2_{\rm atm}|^{1/2}[(9/8\cos^2\phi) - 1]^{1/2}, \tag{9}$$

resulting in the interesting relationship

$$|m_{\nu_e}|^2 \simeq |m_{ee}|^2 + \Delta m^2_{\rm atm}/9. \tag{10}$$

Case (II) has

$$m_1 = -b + d, \qquad m_2 = 2b, \qquad m_3 = b + d, \tag{11}$$

and has not been considered before. It requires

$$|d| \simeq |b|(\cos\phi + \sqrt{3 + \cos^2\phi}), \tag{12}$$

where ϕ is the relative phase between b and d. Hence

$$|m_{1,2}|^2 \simeq 4|b|^2,$$
$$|m_3|^2 \simeq 4|b|^2 + 4|b||d|\cos\phi \simeq 4|b|^2[1 + \cos\phi(\cos\phi + \sqrt{3 + \cos^2\phi})]. \quad (13)$$

For $\cos\phi = 1$, $|m_3|^2 - |m_{1,2}|^2 \simeq 12|b|^2$ (normal ordering) and $|m_{ee}| = 2|b|$. For $\cos\phi = -1$, $|m_3|^2 - |m_{1,2}|^2 \simeq -4|b|^2$ (inverted ordering) and $|m_{ee}| = (2/3)|b|$. In general,

$$|m_{ee}| = \frac{1}{3}|\Delta m^2_{\rm atm}|^{1/2}[(1 + 3/\cos^2\phi)^{1/2} \pm 1]^{1/2}, \quad (14)$$

where ± 1 refers to $\cos\phi > 0$ or <0, and

$$|m_{\nu_e}|^2 \simeq |m_{1,2}|^2 \simeq \frac{\Delta m^2_{\rm atm}}{\cos\phi(\cos\phi + \sqrt{3 + \cos^2\phi})}, \quad (15)$$

resulting in the relationship

$$|m_{\nu_e}|^2 \simeq 3|m_{ee}|^2 - (2/3)\Delta m^2_{\rm atm}. \quad (16)$$

By choosing $\cos\phi$ near zero, it is clear that the present experimental upper bound of about 0.3 eV for $|m_{ee}|$ may be reached, in which case the 3 neutrino masses are nearly degenerate. This is also possible in Case (I).

In Case (I), $b = c = 0$ is a natural limit of the symmetry. In Case (II), $a = 0$ is a natural limit but $b = c \neq 0$ is not. However for $b \neq c$, it was shown in Ref. [3] that the experimental bound $|U_{e3}| < 0.16$ limits the deviation of $\tan^2\theta_{12}$ from 0.5 to only 0.52. In other words, if extended to allow $b \neq c$, Case (II) does not predict all three angles, but given one, it does predict the other two.

In conclusion, two interesting two-parameter descriptions of the neutrino mass matrix have been discussed, each with the mixing matrix of Eq. (3). One admits only a normal ordering of neutrino masses and predicts Eq. (10); the other allows inverted as well as normal ordering and predicts Eq. (16).

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[1] E. Ma and G. Rajasekaran, Phys. Rev. D **64**, 113012 (2001).
[2] K. S. Babu, E. Ma, and J. W. F. Valle, Phys. Lett. B **552**, 207 (2003).
[3] E. Ma, Phys. Rev. D **70**, 031901R (2004).
[4] See, for example, M. C. Gonzalez-Garcia, hep-ph/0410030.
[5] B. Aharmim *et al.* (SNO Collaboration, nucl-ex/0502021.
[6] See for example A. Strumia and F. Vissani, hep-ph/0503246.
[7] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. B **530**, 167 (2002).
[8] G. Altarelli and F. Feruglio, hep-ph/0504165.