$SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ models for β arbitrary and families with mirror fermions

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A detailed and general study of the fermionic structure of the 331 models with β arbitrary is carried out based on the criterion of cancellation of anomalies. We consider models with an arbitrary number of lepton and quark generations, but which require associating only one lepton and one quark $SU(3)_L$ multiplet for each generation, and at most one right-handed singlet per each left-handed fermion. We see that the number of quark left-handed multiplets must be 3 times the number of leptonic left-handed multiplets. Furthermore, we consider a model with four families and $\beta = -1/\sqrt{3}$ where the additional family corresponds to a mirror fermion of the third generation of the standard model. We also show how to generate ansatz about the mass matrices of the fermions according to the phenomenology. In particular, it is possible to get a natural fit for the neutrino hierarchical masses and mixing angles. Moreover, by means of the mixing between the third quark family and its mirror fermion, a possible solution for the A_{FB}^{b} discrepancy is obtained.

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I. INTRODUCTION

A very common alternative to solve some of the problems of the standard model (SM) consists of enlarging the group of gauge symmetry, where the larger group embeds the SM properly. For instance, the $SU(5)$ grand unification model of Georgi and Glashow [1] can unify the interactions and predicts the electric charge quantization, while the group E_6 can also unify the interactions and might explain the masses of the neutrinos [2]. Nevertheless, such models cannot explain the origin of the fermion families. Some models with larger symmetries address this problem [3]. A very interesting alternative to explain the origin of generations comes from the cancellation of chiral anomalies [4]. In particular, the models with gauge symmetry $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$, also called 331 models, arise as a possible solution to this puzzle, since some of such models require the three families in order to cancel chiral anomalies completely. An additional motivation to study these kinds of models comes from the fact that they can also predict the charge quantization for a three-family model even when neutrino masses are added [5]. Finally, supersymmetric versions of this gauge theory have also been studied [6].

Despite the fact that the 331 models could formally provide an explanation for the number of families, they cannot explain many aspects that the SM cannot explain either; it suggests the presence of new physics. In the current versions of the model it is not possible to explain the mass hierarchy and mixing of the fermions. On the other hand, the model is purely left handed, so that it cannot account for parity breaking. Another point of interest to study in the models is the CP violation, particularly the strong CP violation which might allow help us to understand the values for the electric dipole moment of the neutron and electron.

Although cancellation of anomalies leads to some conditions [7], such criterion alone still permits an infinite number of 331 models. In these models, the electric charge is defined, in general, as a linear combination of the diagonal generators of the group

$$Q = T_3 + \beta T_8 + XI. \tag{1.1}$$

As it has been extensively studied in the literature [7–9], the value of the β parameter determines the fermion assignment and, more specifically, the electric charges of the exotic spectrum. Hence, it is customary to use this quantum number to classify the different 331 models. If we want to avoid exotic charges we are led to only two different models i.e. $\beta = \pm 1/\sqrt{3}$ [7,10].

In the analysis for β arbitrary based on the cancellation of anomalies, we find many possible structures that contain the SM at low energies. In the model with two leptonic left-handed multiplets ($N = 2$), we get a one-family model in which one of the multiplets corresponds to the mirror fermions (MF) of the other, i.e., the quarks and leptons form vector representations with respect to $SU(3)_L$ for each family. Two additional copies are necessary in order to obtain the SM at low energies.

The structure for $N = 4$ families and $\beta = -1/\sqrt{3}$, where three of them refer to the generations at low energies and the other is a mirror family, is a vectorlike model that has two multiplets in the **3** representation and two multiplets in the **3*** representation in both the quark and lepton sectors. This extension of the 331 model is not reduced to the known models with $\beta = -\sqrt{3}, -1/\sqrt{3}$ [8,10], because in such models the leptons are in three 3-dimensional multiplets. From the phenomenological point of view at low energies, the difference would be in generating ansatz

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for the mass matrices in the lepton and quark sectors. Models with vectorlike multiplets are necessary to explain the family hierarchy. Moreover, it is observed that the neutrinos do not exhibit a strong family hierarchy pattern as it happens with the other fermions. The mixing angles for the neutrinos $\theta_{\rm atm}$ and $\theta_{\rm sun}$ are not small. Besides, the quotient $(\delta m^2_{\rm sun}/\delta m^2_{\rm atm})$ is of the order of 0.02–0.03; these facts suggest to modify the seesaw mechanism in order to cancel the hierarchy in the mass generation for the neutrinos. Such modifications are usually implemented by introducing vectorlike fermion multiplets [11].

On the other hand, the deviation of the b-quark asymmetry A_b from the value predicted by the SM (of the order of 3σ) suggests a modification in the right-handed couplings of Z_μ with the b quark, by means of particles that are not completely decoupled at low energies. An alternative is the inclusion of MF because they acquire masses slightly greater than the electroweak scale since their masses are generated when $SU(2)_L \otimes U(1)_Y$ is broken [12]. Further, a model with MF couples with right-handed chirality to the electroweak gauge fields. Hence, these couplings might solve the deviations for A_b and $A^b_{\rm FB}$ [13]. Since the traditional 331 models are left handed and the $Z - Z'$ mixing is so weak ($\sim 10^{-3}$) they do not yield a contribution for these asymmetries [14]. Another interesting possibility to explain the discrepancy would be to modify the right-handed couplings of the top quark, which enter in the correction of the $Zb\bar{b}$ vertex. They could also generate deviations for $|V_{tb}|$, which in turn may give us a hint about the mass generation mechanism for the ordinary fermions. The 331 models with $N \neq 3$ might in principle be able to explain such discrepancy and generate right-handed couplings for the bottom and top quarks.

Furthermore, the introduction of mirror fermions permits one, in a certain sense, to restore the chiral symmetry lost in the standard model, and in principle could serve to solve the problem of strong CP violation [15]. The implementation of these models with more fermions for $N \neq 3$ requires a more complex scalar sector that permits one to generate CP violation in a natural way.

There are some other features that neither SM nor their ordinary 331 extensions can explain at a cosmological level, such as the large scale structure in the Universe [16], galactic halo [17], and gamma ray bursts [18]. They suggest the existence of physics beyond the ordinary 331 models. In many cases mirror fermions will be useful to find solutions to these cosmological problems.

Finally, some additional motivations come from grand unified theories (GUTs). GUTs introduce some non-natural features such as the hierarchy problem with the Higgs boson mass, because of the introduction of a new scale (grand unification scale) much higher than the weak scale; this is in turn related to the "grand desert" that apparently exists between the GUT and electroweak scale. This fact motivates the possibility of considering intermediate steps in the route from GUT to electroweak scales. Some versions of the 331 models permit the chain of breaking GUT → 331 → SM, while protecting the phenomenology from fast proton decay [19].

The study of β arbitrary is interesting because it permits a general phenomenological analysis that could be reduced to the known cases when $\beta = -\sqrt{3}$, and $\beta = 1/\sqrt{3}$ [20], but can also permit the study of other scenarios that could be the source for solving some of the problems cited here.

Recently we have gotten constraints on 331 models by examining the scalar sector [21]. In summary, these constraints are obtained by requiring gauge invariance in the Yukawa sector and finding the possible vacuum alignment structures that respect the symmetry breaking pattern and provides the fermions and gauge bosons of the SM with the appropriate masses. By applying gauge invariance to the Yukawa Lagrangian it is found that the Higgs bosons should lie in either a triplet, antitriplet, singlet, or sextet representation of $SU(3)_L$. On the other hand, cancellation of chiral anomalies demands that the number of fermionic triplets and antitriplets must be equal [22]. Moreover, assuming the symmetry breaking pattern

$$\begin{aligned} SU(3)_c \otimes SU(3)_L \otimes U(1)_X &\rightarrow SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \\ &\rightarrow SU(3)_c \otimes U(1)_Q, \\ 331 \rightarrow 321 &\rightarrow 1 \end{aligned} \tag{1.2}$$

we see that one scalar triplet is necessary for the first symmetry breaking and two scalar triplets for the second to give mass to the up and down sectors of the SM. The possible vacuum alignments that obey this breaking pattern, as well as giving the appropriate masses in the second transition, provide the value of the quantum number X in terms of β. Finally, in some cases it is necessary to introduce a scalar sextet to give masses to all leptons.

In this paper we intend to make a general analysis of the fermionic spectrum for β arbitrary, by using the criteria of economy of the exotic spectrum and the cancellation of anomalies. The scalar and vector sectors of the model will be considered as well.

This paper is organized as follows. In Sec. II we describe the fermion representations and find the restrictions over the general fermionic structure based on the cancellation of anomalies. In Sec. III we show the scalar potential and the scalar spectrum for three Higgs triplets with β arbitrary. Section IV develops the vector spectrum for β arbitrary, and Sec. V shows the corresponding Yang-Mills Lagrangian. In Sec. VI we write down the neutral and charged currents for the three-family version of the model with β arbitrary. Section VII describes a new model with four families where one of them corresponds to a mirror family; from the vectorlike structure of the model, we try to solve the problem of the b-quark asymmetries, and generate ansatz for the fermionic mass matrices. Finally, Sec. VIII is regarded for our conclusions.

II. FERMIONIC SPECTRUM AND ANOMALIES WITH β ARBITRARY

A. Fermion representations

The fermion representations under $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ read

$$\hat{\psi}_L = \begin{cases} \hat{q}_L:(\mathbf{3},\mathbf{3},X_q^L) = (\mathbf{3},\mathbf{2},X_q^L) \oplus (\mathbf{3},\mathbf{1},X_q^L), \\ \hat{\ell}_L:(\mathbf{1},\mathbf{3},X_\ell^L) = (\mathbf{1},\mathbf{2},X_\ell^L) \oplus (\mathbf{1},\mathbf{1},X_\ell^L), \end{cases}$$
$$\hat{\psi}_L^* = \begin{cases} \hat{q}_L^*:(\mathbf{3},\mathbf{3}^*,-X_q^L) = (\mathbf{3},\mathbf{2}^*,-X_q^L) \oplus (\mathbf{3},\mathbf{1},-X_q^L), \\ \hat{\ell}_L^*:(\mathbf{1},\mathbf{3}^*,-X_\ell^L) = (\mathbf{1},\mathbf{2}^*,-X_\ell^L) \oplus (\mathbf{1},\mathbf{1},-X_\ell^L), \end{cases}$$
$$\hat{\psi}_R = \begin{cases} \hat{q}_R:(\mathbf{3},\mathbf{1},X_q^R), \\ \hat{\ell}_R:(\mathbf{1},\mathbf{1},X_\ell^R). \end{cases} \tag{2.1}$$

The second equality comes from the branching rules $SU(2)_L \subset SU(3)_L$. The X_p refers to the quantum number associated with $U(1)_X$. The generator of $U(1)_X$ commutes with the matrices of $SU(3)_L$; hence, it should take the form $X_p \mathbf{I}_{3\times3}$. The value of X_p is related to the representations of $SU(3)_L$ and the cancellation of anomalies. On the other hand, this fermionic content shows that the left-handed multiplets lie in either the **3** or $\mathbf{3}^*$ representations.

B. Chiral anomalies with β arbitrary

The fermion spectrum in the SM consists of a set of three generations with the same quantum numbers; the origin of these three generations is one of the greatest puzzles of the model. On the other hand, the fermionic spectrum of the 331 models must contain such generations, which can be fitted in subdoublets $SU(2)_L \subset SU(3)_L$ according to the structure given by Eq. (2.1). Nevertheless, in such models the number of fermion multiplets and their properties are related by the condition of cancellation of anomalies. As a general starting point, we could introduce sets of multiplets with different quantum numbers; this means that each generation can be represented as a set of triplets with particles of the SM plus exotic particles. Even in models of only one generation the structure of the spectrum could be complex, appearing more than one triplet with different quantum numbers [7]. These kinds of models exhibit a large quantity of free parameters and of exotic charges. Such free parameters increase rapidly when more than one generation is introduced; this leads to a loss of predictability in the sense that we have to resort to phenomenological arguments to reduce the arbitrariness of the infinite possible spectra. In the present work, we intend to study the 331 models keeping certain generality but demanding a fermionic spectrum with a minimal number of exotic particles. So we shall take all those models with N leptonic generations and M quark generations, by requiring to associate only one lepton and one quark $SU(3)_L$ multiplet for each generation, and at most one right-handed singlet associated with each left-handed fermion. Based on these criteria we obtain the fermionic spectrum (containing the SM spectrum) displayed in Table I for the quarks and leptons, where the definition of the electric charge, Eq. (1.1), has been used demanding charges of 2/3 and $-1/3$ to the up- and down-type quarks, respectively, and charges of $-1, 0$ for the charged and neutral leptons, in order to ensure a realistic scenario. In general, it is possible to have in a single model any of the representations described by Eq. (2.1), where each multiplet can transform

TABLE I. Fermionic content of $SU(3)_L \otimes U(1)_X$ obtained by requiring only one lepton and one quark $SU(3)_L$ multiplet for each generation, and no more than one right-handed singlet for each right-handed field. The structure of left-handed multiplets is the one shown in Eqs. (2.1) and (2.2). m and n label the quark and lepton left-handed triplets, respectively, while m^*, n^* label the antitriplets; see Eq. (2.2).

Quarks	Q_ψ	X_ψ
$q_L^{(m)} = \begin{pmatrix} U^{(m)} \\ D^{(m)} \\ J^{(m)} \end{pmatrix}_L : 3 \quad \begin{matrix} U_R^{(m)}:1 \\ D_R^{(m)}:1 \\ J_R^{(m)}:1 \end{matrix}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{1}{6}-\frac{\sqrt{3}\beta}{2} \end{pmatrix} \quad \begin{matrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{1}{6}-\frac{\sqrt{3}\beta}{2} \end{matrix}$	$X_{q^{(m)}}^L = \frac{1}{6}-\frac{\beta}{2\sqrt{3}} \quad \begin{matrix} X_{U^{(m)}}^R = \frac{2}{3} \\ X_{D^{(m)}}^R = -\frac{1}{3} \\ X_{J^{(m)}}^R = \frac{1}{6}-\frac{\sqrt{3}\beta}{2} \end{matrix}$
$q_L^{(m^*)} = \begin{pmatrix} D^{(m^*)} \\ -U^{(m^*)} \\ J^{(m^*)} \end{pmatrix}_L : 3^* \quad \begin{matrix} D_R^{(m^*)}:1 \\ U_R^{(m^*)}:1 \\ J_R^{(m^*)}:1 \end{matrix}$	$\begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{6}+\frac{\sqrt{3}\beta}{2} \end{pmatrix} \quad \begin{matrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{6}+\frac{\sqrt{3}\beta}{2} \end{matrix}$	$X_{q^{(m^*)}}^L = -\frac{1}{6}-\frac{\beta}{2\sqrt{3}} \quad \begin{matrix} X_{D^{(m^*)}}^R = -\frac{1}{3} \\ X_{U^{(m^*)}}^R = \frac{2}{3} \\ X_{J^{(m^*)}}^R = \frac{1}{6}+\frac{\sqrt{3}\beta}{2} \end{matrix}$
Leptons	Q_ψ	X_ψ
$\ell_L^{(n)} = \begin{pmatrix} \nu^{(n)} \\ e^{(n)} \\ E^{(n)} \end{pmatrix}_L : 3 \quad \begin{matrix} \nu_R^{(n)}:1 \\ e_R^{(n)}:1 \\ E_R^{(n)}:1 \end{matrix}$	$\begin{pmatrix} 0 \\ -1 \\ -\frac{1}{2}-\frac{\sqrt{3}\beta}{2} \end{pmatrix} \quad \begin{matrix} 0 \\ -1 \\ -\frac{1}{2}-\frac{\sqrt{3}\beta}{2} \end{matrix}$	$X_{\ell^{(n)}}^L = -\frac{1}{2}-\frac{\beta}{2\sqrt{3}} \quad \begin{matrix} X_{\nu^{(n)}}^R = 0 \\ X_{e^{(n)}}^R = -1 \\ X_{E^{(n)}}^R = -\frac{1}{2}-\frac{\sqrt{3}\beta}{2} \end{matrix}$
$\ell_L^{(n^*)} = \begin{pmatrix} e^{(n^*)} \\ -\nu^{(n^*)} \\ E^{(n^*)} \end{pmatrix}_L : 3^* \quad \begin{matrix} e_R^{(n^*)}:1 \\ \nu_R^{(n^*)}:1 \\ E_R^{(n^*)}:1 \end{matrix}$	$\begin{pmatrix} -1 \\ 0 \\ -\frac{1}{2}+\frac{\sqrt{3}\beta}{2} \end{pmatrix} \quad \begin{matrix} -1 \\ 0 \\ -\frac{1}{2}+\frac{\sqrt{3}\beta}{2} \end{matrix}$	$X_{\ell^{(n^*)}}^L = \frac{1}{2}-\frac{\beta}{2\sqrt{3}} \quad \begin{matrix} X_{e^{(n^*)}}^R = -1 \\ X_{\nu^{(n^*)}}^R = 0 \\ X_{E^{(n^*)}}^R = -\frac{1}{2}+\frac{\sqrt{3}\beta}{2} \end{matrix}$

differently. Indeed, in the most general case, each multiplet can transform as

$$\begin{cases} q_L^{(m)}, q_L^{(m^*)}: m = \underbrace{1, 2, \ldots, k}_{3k \text{ triplets}}; \quad m^* = \underbrace{k+1, k+2, \ldots, M}_{3(M-k) \text{ antitriplets}} \\ \ell_L^{(n)}, \ell_L^{(n^*)}: n = \underbrace{1, 2, \ldots, j}_{j \text{ triplets}}; \quad n^* = \underbrace{j+1, j+2, \ldots, N}_{N-j \text{ antitriplets}} \end{cases} \tag{2.2}$$

where the first $3k$th multiplets of quarks lie in the **3** representation while the latter $3(M-k)$ lie in the $\mathbf{3}^*$ representation for a total of $3M$ quark left-handed multiplets. The factor 3 in the number of quark left-handed multiplets owes to the existence of three colors. Similarly the first j left-handed multiplets of leptons are taken in the representation **3** and the latter $(N-j)$ are taken in the $\mathbf{3}^*$ representation, for a total of N leptonic left-handed multiplets.

Now we proceed to analyze the restrictions over the fermionic structure of Eq. (2.2) from the criterion of cancellation of anomalies. When we demand for the fermionic $SU(3)_c$ representations to be vectorlike, we are left with the following nontrivial triangular anomalies

$$\begin{aligned} [SU(3)_c]^2 \otimes U(1)_X \to A_1 &= \pm 3X_q^L - \sum_{\text{singlet}} X_q^R, \\ [SU(3)_L]^3 \to A_2 &= \frac{1}{2} A_{\alpha\beta\gamma}, \\ [SU(3)_L]^2 \otimes U(1)_X \to A_3 &= \sum_r (\pm X_{\ell^{(r)}}^L) + 3\sum_s (\pm X_{q^{(s)}}^L), \\ [\text{Grav}]^2 \otimes U(1)_X \to A_4 &= 3\sum_r (\pm X_{\ell^{(r)}}^L) + 9\sum_s (\pm X_{q^{(s)}}^L) - 3\sum_{\text{singlet}} (X_q^R) - \sum_{\text{singlet}} (X_\ell^R), \\ [U(1)_X]^3 \to A_5 &= 3\sum_r (\pm X_{\ell^{(r)}}^L)^3 + 9\sum_s (\pm X_{q^{(s)}}^L)^3 - 3\sum_{\text{singlet}} (X_q^R)^3 - \sum_{\text{singlet}} (X_\ell^R)^3 \end{aligned} \tag{2.3}$$

where the sign $+$ or $-$ is chosen according to the representation **3** or $\mathbf{3}^*$. The condition of cancellation of these anomalies imposes, under some circumstances, relations between the values of N, M, j, k and the β parameter. Furthermore, the requirement for the model to be $SU(3)_c$ vectorlike demands the presence of right-handed quark singlets, while right-handed neutral lepton singlets are optional.

1. The $[SU(3)_c]^2 \otimes U(1)_X$ *anomaly*

When we take into account that the fermionic triplets in Eq. (2.1) must contain the SM generations, i.e. they contain subdoublets $SU(2)_L \subset SU(3)_L$, we obtain relations among the X and β numbers that cancel this anomaly. In Table I, we write down these relations in the third column, by assuming that the $SU(2)_L$ subdoublets lie in the two upper components of the triplets.

2. The $[SU(3)_L]^3$ *anomaly*

The cancellation of the $[SU(3)_L]^3$ anomaly demands for the number of $\hat{\psi}_L$ multiplets to be the same as the number of $\hat{\psi}_L^*$ ones. Taking into account the number of quark and lepton multiplets defined in Eq. (2.2), we arrive at the condition

$$3k + j = 3(M-k) + (N-j)$$

or writing it properly

$$N - 2j = -3(M-2k); \quad 0 \le j \le N; \quad 0 \le k \le M. \tag{2.4}$$

The first inequality expresses the fact that the models are limited from representations in which all the left-handed multiplets of leptons transform under $\mathbf{3}^*$ (when $j=0$), to representations in which all left-handed lepton multiplets transform under **3** (when $j=N$). An analogous situation appears for the quarks representations that leads to the second inequality.

3. The $[SU(3)_L]^2 \otimes U(1)_X$ *anomaly*

Applying the definition in Eq. (2.2), we make an explicit separation between **3** and $\mathbf{3}^*$ representations, from which this anomaly reads

$$A_3 = \sum_{n=1}^{j} (X_{\ell^{(n)}}^L) + \sum_{n^*=j+1}^{N} (-X_{\ell^{(n^*)}}^L) + 3\sum_{m=1}^{k} (X_{q^{(m)}}^L) + 3\sum_{m^*=k+1}^{M} (-X_{q^{(m^*)}}^L) = 0.$$

On the other hand, using the particle content of Table I, the equation takes the form

$$-\frac{3}{2}M - \frac{3\sqrt{3}\beta}{2}(M-2k) = -\frac{3}{2}N + \frac{\sqrt{3}\beta}{2}(N-2j). \tag{2.5}$$

4. The $[\mathrm{Grav}]^2 \otimes U(1)_X$ *anomaly*

Taking into account Eq. (2.2), this anomaly takes the form

$$A_4 = 3\sum_{n=1}^{j}(X^L_{\ell^{(n)}}) + 3\sum_{n^*=j+1}^{N}(-X^L_{\ell^{(n^*)}}) + 9\sum_{m=1}^{k}(X^L_{q^{(m)}}) + 9\sum_{m^*=k+1}^{M}(-X^L_{q^{(m^*)}}) - \sum_{n=1}^{j}(Q^R_{\nu^{(n)}} + Q^R_{e^{(n)}} + Q^R_{E^{(n)}})$$
$$- \sum_{n^*=j+1}^{N}(Q^R_{\nu^{(n^*)}} + Q^R_{e^{(n^*)}} + Q^R_{E^{(n^*)}}) - 3\sum_{m=1}^{k}(Q_{U^{(m)}} + Q_{D^{(m)}} + Q_{J^{(m)}}) - 3\sum_{m^*=k+1}^{M}(Q_{U^{(m^*)}} + Q_{D^{(m^*)}} + Q_{J^{(m^*)}}) = 0,$$

where the leptonic right-handed charges can be present or absent. The neutrino has null charge so that the presence (or absence) of right-handed neutrinos does not affect the anomalies, but they are important when choosing Yukawa terms for the masses. On the other hand, $e^{(n)}$ possesses a charge (-1), while $E^{(n)}$ and $E^{(n^*)}$ can, in general, possess charges different from zero. We shall call them generically charged leptons. Since charged singlets affect the anomalies, we should set up a notation to specify whether we choose charged right-handed leptonic singlets or not. Taking into account that we permit at most one right-handed singlet per each left-handed fermion we define

$$\Theta_\ell \equiv \begin{cases} 1 & \text{for models with charged } \ell_R, \\ 0 & \text{for models without charged } \ell_R. \end{cases} \tag{2.6}$$

It is applied to each right-handed leptonic charge, in such a way that the cancellation of this anomaly leads to the condition

$$-\frac{3}{2}N + \frac{\sqrt{3}\beta}{2}(N-2j) = -j\Theta_{e^{(1)}} - j\left(\frac{1}{2} + \frac{\sqrt{3}\beta}{2}\right)\Theta_{E^{(1)}} - (N-j)\Theta_{e^{(j+1)}} - (N-j) \times \left(\frac{1}{2} - \frac{\sqrt{3}\beta}{2}\right)\Theta_{E^{(j+1)}}, \tag{2.7}$$

where we have replaced the values of Q_ψ, X_ψ given in Table I. We should note that Eq. (2.7) is a relation about Θ_ℓ; therefore, it imposes restrictions over the possible choices of right-handed charged leptonic singlets. Finally, from Eq. (2.7) and Table I, we see that when the $E^{(n)}$ or $E^{(n^*)}$ fields are neutral (i.e. $\beta = \pm 1/\sqrt{3}$ for $E^{(j+1)}$ and $E^{(1)}$, respectively), the corresponding singlets do not contribute to the equation of anomalies like in the case of the neutrinos.

5. $[U(1)_X]^3$ *anomaly*

In this case we have

$$A_5 = 3\sum_{n=1}^{j}(X^L_{\ell^{(n)}})^3 + 3\sum_{n^*=j+1}^{N}(-X^L_{\ell^{(n^*)}})^3 + 9\sum_{m=1}^{k}(X^L_{q^{(m)}})^3 + 9\sum_{m^*=k+1}^{M}(-X^L_{q^{(m^*)}})^3 - 3\sum_{m=1}^{k}[(Q_{U^{(m)}})^3 + (Q_{D^{(m)}})^3 + (Q_{J^{(m)}})^3]$$
$$- 3\sum_{m^*=k+1}^{M}[(Q_{U^{(m^*)}})^3 + (Q_{D^{(m^*)}})^3 + (Q_{J^{(m^*)}})^3] - \sum_{n=1}^{j}[(Q^R_{\nu^{(n)}})^3 + (Q^R_{e^{(n)}})^3 + (Q^R_{E^{(n)}})^3]$$
$$- \sum_{n^*=j+1}^{N}[(Q^R_{\nu^{(n^*)}})^3 + (Q^R_{e^{(n^*)}})^3 + (Q^R_{E^{(n^*)}})^3] = 0.$$

Using Eq. (2.6) and Table I, we get

$$-\frac{3}{4}\left(\frac{1}{2}N + M\right) + \frac{3\sqrt{3}\beta}{8}(N-2j) - \frac{3\beta^2}{4}\left(\frac{1}{2}N + M\right) + 9\left(\frac{\beta}{\sqrt{3}}\right)^3\left(\frac{N-2j}{24} - M + 2k\right)$$
$$= -j\Theta_{e^{(1)}} - j\left(\frac{1}{2} + \frac{\sqrt{3}\beta}{2}\right)^3\Theta_{E^{(1)}} - (N-j)\Theta_{e^{(j+1)}} - (N-j)\left(\frac{1}{2} - \frac{\sqrt{3}\beta}{2}\right)^3\Theta_{E^{(j+1)}} \tag{2.8}$$

which arises as an additional condition for the presence of right-handed charged leptonic singlets.

C. General fermionic structure

Equations (2.4), (2.5), (2.7), and (2.8) appear as conditions that guarantee the vanishing of all the anomalies, obtaining a set of four equations [plus the two inequalities of Eq. (2.4)] whose variables to solve for are N, M, j, k, and β. Taking Eqs. (2.4) and (2.5), we find the following solutions:

$$N = M; \qquad j + 3k = 2N. \tag{2.9}$$

This means that the number of left-handed quark multiplets ($3M$) must be 3 times the number of left-handed leptonic multiplets (N). Moreover, the number of leptonic triplets in the representation **3** (j) plus the number of quark triplets in the representation **3** ($3k$) must be twice the number of left-handed leptonic multiplets ($2N$) i.e. an even number. In addition, we can find, by combining the two equations in (2.9), that the number of lepton and quark left-handed multiplets in the $\mathbf{3}^*$ representation must also be equal to $2N$. The solutions in Eqs. (2.9) are represented as restrictions over the integer values of j and k according to the number of left-handed multiplets ($4N$). Table II illustrates some particular cases.

It is important to note that there are only some possible ways to choose the number of triplets and antitriplets for a given number of multiplets. Additionally, there is no solution for models with $N = 1$ under the scheme of using one multiplet per generation; so we have the extra condition $N \geq 2$. In this manner, the possible representations according to Table II depend on the number of multiplets $4N$, as it is shown in Table III. We can see that models with $N = 2$ are possible if the multiplets of quarks and leptons transform in a different way. For $N = 3$, we have two possible solutions. In one of them all the lepton multiplets transform in the same way; two of the quark multiplets transform the same and the other transforms as the conjugate. The second solution corresponds to the conjugate of the first solution. For $N = 4$ the quark and leptonic representations are vectorlike with respect to $SU(3)_L$ as Table III displays. In this case we will have one exotic fermion family, $q^{(i)}$ and $l^{(i)}$, which might be a replication of the heavy or light families of the SM. Such choice could be useful to generate new ansatz about mass matrices for the fermions of the SM. In this way, it is possible to add new exotic generations, though not arbitrarily, but respecting the conditions of Table III.

As for the solution (2.8) with $N = M$, it can be rewritten as

$$\frac{3}{4}(1+\beta^2)\left[-\frac{3}{2}N+\frac{\sqrt{3}\beta}{2}(N-2j)\right] = -j\Theta_{e^{(1)}} - j\left(\frac{1}{2}+\frac{\sqrt{3}\beta}{2}\right)^3\Theta_{E^{(1)}} - (N-j)\Theta_{e^{(j+1)}} - (N-j)\left(\frac{1}{2}-\frac{\sqrt{3}\beta}{2}\right)^3\Theta_{E^{(j+1)}}, \tag{2.10}$$

and using (2.7), we find

$$j(\Theta_{e^{(1)}} - \Theta_{E^{(1)}}) = (j-N)(\Theta_{e^{(j+1)}} - \Theta_{E^{(j+1)}}). \tag{2.11}$$

TABLE III. Possible representations according to Table II. Each value of $q^{(i)}$ represents three left-handed quark multiplets because of the color factor.

N	Allowed representations		
2		$\ell^{(1)}:3$ $\ell^{(2)}:3^*$ $q^{(1)}:3$ $q^{(2)}:3^*$	
3	$\ell^{(1)}, \ell^{(2)}, \ell^{(3)}:3^*$ $q^{(1)}, q^{(2)}:3$ $q^{(3)}:3^*$		$\ell^{(1)}, \ell^{(2)}, \ell^{(3)}:3$ $q^{(3)}:3$ $q^{(1)}, q^{(2)}:3^*$
4		$\ell^{(1)}, \ell^{(2)}:3$ $\ell^{(3)}, \ell^{(4)}:3^*$ $q^{(1)}, q^{(2)}:3$ $q^{(3)}, q^{(4)}:3^*$	
5	$\ell^{(5)}:3$ $\ell^{(1)}, \ell^{(2)}, \ell^{(3)}, \ell^{(4)}:3^*$ $q^{(3)}, q^{(4)}, q^{(5)}:3$ $q^{(1)}, q^{(2)}:3^*$		$\ell^{(1)}, \ell^{(2)}, \ell^{(3)}, \ell^{(4)}:3$ $\ell^{(5)}:3^*$ $q^{(1)}, q^{(2)}:3$ $q^{(3)}, q^{(4)}, q^{(5)}:3^*$
6	$\ell^{(1)}, \ell^{(2)}, \ell^{(3)}, \ell^{(4)}, \ell^{(5)}, \ell^{(6)}:3^*$ $q^{(1)}, q^{(2)}, q^{(5)}, q^{(6)}:3$ $q^{(3)}, q^{(4)}:3^*$		$\ell^{(1)}, \ell^{(2)}, \ell^{(3)}, \ell^{(4)}, \ell^{(5)}, \ell^{(6)}:3$ $q^{(3)}, q^{(4)}:3$ $q^{(1)}, q^{(2)}, q^{(5)}, q^{(6)}:3^*$
		$\ell^{(1)}, \ell^{(2)}, \ell^{(3)}:3$ $\ell^{(4)}, \ell^{(5)}, \ell^{(6)}:3^*$ $q^{(1)}, q^{(2)}, q^{(5)}:3$ $q^{(3)}, q^{(4)}, q^{(6)}:3^*$	

TABLE II. Solutions of Eqs. (2.9) represented as restrictions on the number of lepton triplets (j) and of quark triplets ($3k$) according to the number of left-handed multiplets ($4N$).

N	$0 \leq j \leq N$	$0 \leq 3k \leq 3N$	Solution for $j + 3k = 2N$
1	0, 1	0, 3	no solution
2	0, 1, 2	0, 3, 6	$j = 1; k = 1$
3	0, 1, 2, 3	0, 3, 6, 9	$j = 0; k = 2$
			$j = 3; k = 1$
4	0, 1, 2, 3, 4	0, 3, 6, 9, 12	$j = 2; k = 2$
5	0, 1, 2, 3, 4, 5	0, 3, 6, 9, 12, 15	$j = 1; k = 3$
			$j = 4; k = 2$
6	0, 1, 2, 3, 4, 5, 6	0, 3, 6, 9, 12, 15, 18	$j = 0; k = 4$
			$j = 3; k = 3$
			$j = 6; k = 2$

TABLE IV. Solutions for Eqs. (2.7) and (2.11) that arise when all possible combinations of Θ_ℓ defined by Eq. (2.6) are taken. In the last column, we mark with $\times$ the cases that are ruled out by the criterion of conjugation, while for the cases marked with $\surd$, such criterion does not give additional restrictions.

3 $\Theta_{e^{(1)}}\Theta_{E^{(1)}}$	**3*** $\Theta_{e^{(j+1)}}\Theta_{E^{(j+1)}}$	Solution for Eq. (2.7)	Solution for Eq. (2.11)	Combined solutions	With conjugation criterion
0 0	0 0	$\beta = (\frac{N}{N-2j})\sqrt{3}$	$\forall\ N, j$	$\beta = (\frac{N}{N-2j})\sqrt{3}$	$\beta = \sqrt{3}; j = 0$ $\beta = -\sqrt{3}; j = N$
0 0	0 1	$\beta = (\frac{-2N-j}{j})\frac{1}{\sqrt{3}}$	$N = j$	$\beta = -\sqrt{3}$	$\times$
0 0	1 0	$\beta = (\frac{N+2j}{N-2j})\frac{1}{\sqrt{3}}$	$N = j$	$\beta = -\sqrt{3}$	$\times$
0 0	1 1	$\beta = -\sqrt{3}; \forall\ j \neq 0$ $\forall\ \beta; j = 0$	$\forall\ N, j$	$\beta = -\sqrt{3}; \forall\ j \neq 0$ $\forall\ \beta; j = 0$	$\surd$
0 1	0 0	$\beta = (\frac{3N-j}{N-j})\frac{1}{\sqrt{3}}$	$j = 0; \forall\ N$	$\beta = \sqrt{3}$	$\times$
0 1	0 1	$N = 0; \forall\ j, \beta$	$N = 0; \forall\ j\times$	$\forall\ \beta$	$\times$
0 1	1 0	$\beta = (\frac{N+j}{N-j})\frac{1}{\sqrt{3}}$	$N = 2j$	$\beta = \sqrt{3}$	$\surd$
0 1	1 1	$j = 0; \forall\ N, \beta$	$j = 0; \forall\ N$	$\forall\ \beta$	$\beta = -1/\sqrt{3}$
1 0	0 0	$\beta = (\frac{3N-2j}{N-2j})\frac{1}{\sqrt{3}}$	$j = 0; \forall\ N$	$\beta = \sqrt{3}$	$\times$
1 0	0 1	$\beta = (\frac{-2N+j}{j})\frac{1}{\sqrt{3}}$	$N = 2j$	$\beta = -\sqrt{3}$	$\surd$
1 0	1 0	$\beta = (\frac{N}{N-2j})\frac{1}{\sqrt{3}}$	$N = 0; \forall\ j\times$	$\beta = 0$	$\times$
1 0	1 1	$\beta = \frac{-1}{\sqrt{3}}; \forall\ j \neq 0$ $\forall\ \beta; j = 0$	$j = 0; \forall\ N$	$\forall\ \beta$	$\times$
1 1	0 0	$\beta = \sqrt{3}; \forall\ j \neq N$ $\forall\ \beta; j = N$	$\forall\ N, j$	$\beta = \sqrt{3}; \forall\ j \neq N$ $\forall\ \beta; j = N$	$\surd$
1 1	0 1	$N = j; \forall\ \beta$	$N = j$	$\forall\ \beta$	$\beta = 1/\sqrt{3}$
1 1	1 0	$\beta = \frac{1}{\sqrt{3}}; \forall\ j \neq N$ $\forall\ \beta; j = N$	$N = j$	$\forall\ \beta$	$\times$
1 1	1 1	$\forall\ N, j, \beta$	$\forall\ N, j$	$\forall\ \beta$	if $j = 0 \Rightarrow \beta = -\sqrt{3}$, if $j = N \Rightarrow \beta = \sqrt{3}$

In this way, the solutions (2.7) and (2.11) represent restrictions over the singlet sector that are related with the values of N, j, and β. All the possible combinations of Θ_ℓ that arise when the definition (2.6) is applied lead to the solutions summarized in Table IV. Nevertheless, not all the 16 cases obtained correspond to physical solutions. First of all $\beta = 0$ is not permitted. Additionally, $N \geq 2$, from which the two solutions marked with $\times$ on the fourth column of Table IV are forbidden. On the other hand, there is another important criterion to select possible physical models, which we shall call the criterion of conjugation. The charged leptons are necessarily described by Dirac's spinors; thus we should ensure for each charged lepton to include its corresponding conjugate in the spectrum in order to build up the corresponding Dirac Lagrangian. In the case of the exotic charged leptons the conjugation criterion fixes their electric charges and so the possible values of β, from which additional restrictions for the models are obtained. As an example, for $(\Theta_{e^{(1)}}, \Theta_{E^{(1)}}, \Theta_{e^{(j+1)}}, \Theta_{E^{(j+1)}}) = (0, 1, 1, 1)$ the cancellation of anomalies leads to $j = 0$ (see ninth row of Table IV); then, according to Table I, the structure of charged leptons is shown in Table V. Since the number of leptons having nonzero charge must be even, one of the exotic leptons must be neutral. Therefore, we have the following possibilities: (1) Demanding $E_R^{(n)}$ to be neutral we are led to $\beta = -1/\sqrt{3}$; now if we assume the scheme of conjugation $e_L^{(n^*)} \sim e_R^{(n^*)}; E_L^{(n^*)} \sim E_R^{(n^*)}$ no further restrictions are obtained. (2) Assuming $E_L^{(n^*)}$ neutral yields $\beta = 1/\sqrt{3}$, but all possible combinations of conjugation between the remaining charged fields are forbidden. For instance, the

TABLE V. Structure of leptons for the structure of singlets given by $(\Theta_{e^{(1)}}, \Theta_{E^{(1)}}, \Theta_{e^{(j+1)}}, \Theta_{E^{(j+1)}}) = (0, 1, 1, 1)$.

Leptons	Q_ψ
no triplets 3	no charge
$\nu_R^{(n)}$:1	0
$E_R^{(n)}$:1	$-\frac{1}{2} - \frac{\sqrt{3}\beta}{2}$
$\ell_L^{(n^*)} = \begin{pmatrix} e^{(n^*)} \\ -\nu^{(n^*)} \\ E^{(n^*)} \end{pmatrix}_L$:3*	$\begin{pmatrix} -1 \\ 0 \\ -\frac{1}{2} + \frac{\sqrt{3}\beta}{2} \end{pmatrix}$
$e_R^{(n^*)}$:1	-1
$\nu_R^{(n^*)}$:1	0
$E_R^{(n^*)}$:1	$-\frac{1}{2} + \frac{\sqrt{3}\beta}{2}$

scheme $E_R^{(n)} \sim e_L^{(n^*)}; E_R^{(n^*)} \sim e_R^{(n^*)}$ yields $\beta = 1/\sqrt{3}$ and $\beta = \sqrt{3}$, respectively, leading to a contradiction. (3) Finally, for $E_R^{(n^*)}$ neutral we find $\beta = 1/\sqrt{3}$ and no consistent conjugation structures are possible. In summary, for this singlet structure the only value of β consistent with the conjugation criterion is $\beta = 1/\sqrt{3}$. This restriction should be added to the ones obtained with cancellation of anomalies and yields the solutions shown in the ninth row, last column of Table IV. A similar procedure is done to obtain the restrictions written in the last column of Table IV. The cases marked with $\times$ in the last column are forbidden, while for the cases marked with $\surd$ the conjugation criterion provides no further restrictions with respect to the ones obtained from cancellation of anomalies.

The solutions that survive in Table IV are combined with the ones obtained in Tables II and III [or more generally with Eqs. (2.9)]. The solutions that cancel anomalies and fulfill the conjugation criterion are summarized in Tables VI and VII.

These solutions determine the fermionic structure of the model according to the number of leptonic charged right-handed singlets. However, cancellation of anomalies does not impose any restriction about the right-handed neutral leptonic singlets. Table VI only admits an even number of left-handed leptonic multiplets (N), while Table VII permits, in principle, any number of them as long as $N \geq 2$. It is observed that there are models that fix the values of β, so that they are possible only for certain values of the quantum numbers. However, in three of the cases described in Table VII, there are solutions for β arbitrary.

TABLE VI. Solutions for $N = 2j = 2k \geq 2$.

$\Theta_{e^{(1)}}$	$\Theta_{E^{(1)}}$	$\Theta_{e^{(j+1)}}$	$\Theta_{E^{(j+1)}}$	Solution
1	0	0	1	$\beta = -\sqrt{3}$
0	1	1	0	$\beta = \sqrt{3}$

For the sake of completeness, we shall elaborate about the complications of making an analysis of the most general case where we allow various left-handed multiplets that transform in an identical way with respect to $SU(3)_L$, but with different quantum numbers with respect to $U(1)_X$. In Eq. (2.2), the number of left-handed multiplets is enlarged to include the fact that each representation of $SU(3)_L$ is formed by a subset of several left-handed multiplets:

$$\begin{cases} q^{(m)}: m = \underbrace{1, \ldots, m_1}_{\substack{3m_1 \text{ triplets} \\ 1^{\text{st}} \text{ generation}}}; \underbrace{m_1 + 1, \ldots, 2m_1}_{\substack{3m_1 \text{ triplets} \\ 2^{\text{nd}} \text{ generation}}}; \ldots, \ldots; \underbrace{(k-1)m_1 + 1, \ldots, km_1}_{\substack{3m_1 \text{ triplets} \\ k\text{th generation}}} \\ q^{(m^*)}: m^* = \underbrace{km_1 + 1, \ldots, km_1 + m_1^*}_{\substack{3m_1^* \text{ antitriplets} \\ (k+1)\text{th generation}}}; \underbrace{km_1 + m_1^* + 1, \ldots, km_1 + 2m_1^*}_{\substack{3m_1^* \text{ antitriplets} \\ (k+2)\text{th generation} \cdots}}; \ldots, \\ \ldots; \underbrace{(M-1)m_1^* + 1, \ldots, Mm_1^*}_{\substack{3m_1^* \text{ antitriplets} \\ \cdots\cdots M\text{th generation}}} \\ \ell^{(n)}: n = \underbrace{1, \ldots, n_1}_{\substack{n_1 \text{ triplets} \\ 1^{\text{st}} \text{ generation}}}; \underbrace{n_1 + 1, \ldots, 2n_1}_{\substack{n_1 \text{ triplets} \\ 2^{\text{nd}} \text{ generation}}}; \ldots; \underbrace{(j-1)n_1 + 1, \ldots, jn_1}_{\substack{n_1 \text{ triplets} \\ j\text{th generation}}} \\ \ell^{(m^*)}: m^* = \underbrace{jn_1 + 1, \ldots, jn_1 + n_1^*}_{\substack{n_1^* \text{ antitriplets} \\ (j+1)\text{th generation}}}; \underbrace{jn_1 + n_1^* + 1, \ldots, jn_1 + 2n_1^*}_{\substack{n_1^* \text{ antitriplets} \\ (j+2) \text{ generation}}}; \ldots, \\ \ldots; \underbrace{(N-1)n_1^* + 1, \ldots, Nn_1^*}_{\substack{n_1^* \text{antitriplets} \\ \cdots N\text{th generation}}} \end{cases} \tag{2.12}$$

where $3m_1$, $3m_1^*$, n_1, and n_1^* are the total number of triplets and antitriplets for each generation of quarks (including the color) and the total number of triplets and antitriplets for each generation of leptons, respectively. k and j are the number of generations of quarks and leptons that transform according to **3**, and $(M - k)$, $(N - j)$ are the number of generations under $\mathbf{3}^*$. In this way, the number of parameters is increased, having N, M, n_1, n_1^*, m_1, m_1^*, j, k, and β as free parameters, restricted by only four equations of cancellation of anomalies. Since the number of triplets per generation (characterized by the indices n_1, n_1^*, m_1, m_1^*) has no upper limit, it is always possible to choose a convenient number of them to cancel anomalies, allowing the entrance of an arbitrary number of exotic particles with no reasons but purely phenomenological ones. Therefore, such models lose certain naturalness which is precisely what we look for

TABLE VII. Solutions for $N = \frac{j+3k}{2} \geq 2$, $0 \leq k \leq N$.

$\Theta_{e^{(1)}}$	$\Theta_{E^{(1)}}$	$\Theta_{e^{(j+1)}}$	$\Theta_{E^{(j+1)}}$	Solution
0	0	0	0	$\beta = \sqrt{3}; j = 0$ $\beta = -\sqrt{3}; j = N$
0	0	1	1	$\beta = -\sqrt{3}; \forall\ j \neq 0$ $\forall\ \beta; j = 0$
1	1	0	0	$\beta = \sqrt{3}; \forall\ j \neq N$ $\forall\ \beta; j = N$
0	1	1	1	$j = 0, \forall\ N, \beta = -1/\sqrt{3}$
1	1	0	1	$j = N, \beta = 1/\sqrt{3}$
1	1	1	1	$\forall\ \beta, \forall\ N, j \neq 0, N$ if $j = 0 \Rightarrow \beta = -\sqrt{3}$ if $j = N \Rightarrow \beta = \sqrt{3}$

TABLE VIII. Solutions for β and the fermionic structure with $N = 3$.

$\Theta_{e^{(1)}}\Theta_{E^{(1)}}$	$\Theta_{e^{(j+1)}}\Theta_{E^{(j+1)}}$	Solution
0 0	0 0	$\beta = \sqrt{3}; j = 0, k = 2$ ● $\beta = -\sqrt{3}; j = 3, k = 1$ ●
0 0	1 1	$\beta = -\sqrt{3}; j = 3, k = 1$ ◆ $\forall\ \beta; j = 0, k = 2$ □
0 1	1 1	$\beta = -1/\sqrt{3}; j = 0, k = 2$ ★
1 1	0 0	$\beta = \sqrt{3}; j = 0, k = 2$ ◇ $\forall\ \beta; j = 3, k = 1$ □
1 1	0 1	$\beta = 1/\sqrt{3}; j = 3, k = 1$ ★
1 1	1 1	$\beta = -\sqrt{3}; j = 0, k = 2$ ★ $\beta = \sqrt{3}; j = 3, k = 1$ ★

when we build up a model from basic principles with a minimum of free parameters.

For the case $N = 3$ in Table VII, solutions exist only for $j = 0$ or 3 (see Table II). These solutions are displayed in Table VIII. It should be emphasized that the models without leptonic right-handed singlets (marked with ●) are divided into two according to the value of j to be 0 or 3, which are precisely the models discussed by Pleitez and Frampton [8,9], where $\beta = \pm\sqrt{3}$. The solutions marked with ★ are not discarded by anomalies nor conjugation, but lead to more than one right-handed singlet for each left-handed field. On the other hand, the solutions marked with ◆ and ◇ give no restriction on the number of right-handed leptonic singlets associated with $\mathbf{3}^*$ and $\mathbf{3}$ representations, respectively. Finally, the solutions marked with □ are the only ones that permit arbitrary values of β.

As for the two models with β arbitrary, they exist only if leptonic singlets associated with all the particles in either representation are introduced. In the framework of these two solutions, the particular cases of $\beta = \mp 1/\sqrt{3}$ are discussed by Long in Refs. [10,23], respectively.

It is interesting to note that, as well as the models of Pleitez, Frampton, and Long (with $\beta = \pm\sqrt{3}, \pm 1/\sqrt{3}$), models with other different values of β arise. On the other hand, additional models with $\beta = \pm\sqrt{3}, \pm 1/\sqrt{3}$ but with different structures of right-handed lepton singlets appear as well.

III. HIGGS POTENTIAL AND SPECTRUM FOR β ARBITRARY

A. Potential

The scalar sector of the 331 models has also been studied in the literature [21,24]. The most important features of the scalar potential are [21]

(i) The scalars should lie in either the singlet, triplet, antitriplet, or sextet representation of $SU(3)_L$.

(ii) For the first transition 331 → 321 we could have triplet, antitriplet, or sextet representations. The vacuum alignments for triplet and antitriplet representations are indicated in Table IX for $\beta \neq \pm 1/\sqrt{3}$. While for the sextet representation, the vacuum alignment reads

$$\langle S^{ij}\rangle_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \nu_6 \end{bmatrix}.$$

These vacuum expectation values (VEVs) induce the masses of the exotic fermions.

(iii) In the second transition 321 → 31, triplets, antitriplets, and sextets are also allowed. For the particular case of triplet (or antitriplet) representations we get that pairs of solutions are obtained according to the value of β. Both multiplets are necessary to give masses to the quarks of type up and down, respec-

TABLE IX. Vacuum alignments for the Higgs triplets necessary to get the SSB scheme: $31 \to 21 \to 1$ for $\beta \neq \pm 1/\sqrt{3}$. In the case of Higgs antitriplets, we find the same structure but replacing $\chi, \rho, \eta \to \chi^*, \rho^*, \eta^*$.

		$\beta \neq \pm\frac{1}{\sqrt{3}}$
1st SSB	$\langle\chi\rangle_0$	$\begin{pmatrix} 0 \\ 0 \\ \nu_{\chi_3} \end{pmatrix}$
	X_χ	$\frac{\beta}{\sqrt{3}}$
2nd SSB	$\langle\rho\rangle_0$	$\begin{pmatrix} 0 \\ \nu_{\rho_2} \\ 0 \end{pmatrix}$
	X_ρ	$\frac{1}{2} - \frac{\beta}{2\sqrt{3}}$
	$\langle\eta\rangle_0$	$\begin{pmatrix} \nu_{\eta_1} \\ 0 \\ 0 \end{pmatrix}$
	X_η	$-\frac{1}{2} - \frac{\beta}{2\sqrt{3}}$

tively. So in the second transition, we have to introduce two triplets (or antitriplets) ρ and η associated with each pair of solutions. We show in Table IX the vacuum structure of this pair of triplets for $\beta \neq \pm 1/\sqrt{3}$. On the other hand, the possible vacuum structures for the second transition with Higgs sextets for $\beta \neq \pm 1/\sqrt{3}, \pm\sqrt{3}$ are shown in Table X.

(iv) In some scenarios the Higgs sextet is necessary to give masses to all leptons [4,22].

In the case of β arbitrary (different from $\pm\sqrt{3}, \pm 1/\sqrt{3}$), and taking a scalar content of three Higgs triplets, the most general Higgs potential, renormalizable and $SU(3)_L \otimes U(1)_X$ invariant, is [21]

$$\begin{aligned} V_{\text{Higgs}} = {}& \mu_1^2 \chi^i \chi_i + \mu_2^2 \rho^i \rho_i + \mu_3^2 \eta^i \eta_i + f(\chi_i \rho_j \eta_k \varepsilon^{ijk} + \text{h.c.}) \\ & + \lambda_1 (\chi^i \chi_i)^2 + \lambda_2 (\rho^i \rho_i)^2 + \lambda_3 (\eta^i \eta_i)^2 \\ & + \lambda_4 \chi^i \chi_i \rho^j \rho_j + \lambda_5 \chi^i \chi_i \eta^j \eta_j + \lambda_6 \rho^i \rho_i \eta^j \eta_j \\ & + \lambda_7 \chi^i \eta_i \eta^j \chi_j + \lambda_8 \chi^i \rho_i \rho^j \chi_j + \lambda_9 \eta^i \rho_i \rho^j \eta_j. \end{aligned} \tag{3.1}$$

As it was mentioned above, in some models the choice of three triplets is not enough to provide all leptons with masses [4,22]. Hence, an additional sextet is introduced. The choice of one of these solutions depends on the fermionic sector to which we want to give masses. The introduction of a sextet S leads us to additional terms that

TABLE X. Vacuum alignments for the second SSB with Higgs sextets, and for $\beta \neq \pm 1/\sqrt{3}, \pm\sqrt{3}$.

	$\beta \neq \pm\frac{1}{\sqrt{3}}; \pm\sqrt{3}$			
$\langle\rho^{ij}\rangle_0$	$\begin{pmatrix} \nu_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \nu_3 \\ 0 & 0 & 0 \\ \nu_3 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu_4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \nu_5 \\ 0 & \nu_5 & 0 \end{pmatrix}$
$X_{\rho^{ij}}$	$-\frac{1}{2} - \frac{\beta}{2\sqrt{3}}$	$-\frac{1}{4} + \frac{\beta}{4\sqrt{3}}$	$\frac{1}{2} - \frac{\beta}{2\sqrt{3}}$	$\frac{1}{4} + \frac{\beta}{4\sqrt{3}}$

should be added to the Higgs potential of Eq. (3.1),

$$\begin{aligned} V(S) = {}& \mu_5^2 S^{ij} S_{ij} + S^{ij} S_{ij} (\lambda_{15} \chi^k \chi_k + \lambda_{16} \rho^k \rho_k \\ & + \lambda_{17} \eta^k \eta_k) + \lambda_{18} \chi^i S_{ij} S^{jk} \chi_k + \lambda_{19} \rho^i S_{ij} S^{jk} \rho_k \\ & + \lambda_{20} \eta^i S_{ij} S^{jk} \eta_k + \lambda_{21} (S^{ij} S_{ij})^2 + \lambda_{22} S^{ij} S_{jk} S^{kl} S_{li}. \end{aligned} \tag{3.2}$$

B. Mass spectrum for β arbitrary

In this section we analyze the general case for β arbitrary ($\beta \neq \pm\sqrt{3}, \pm 1/\sqrt{3}$). With three Higgs triplets, it is obtained the potential given by Eq. (3.1), which correspond to the solution shown in Table IX for $\beta \neq \pm 1/\sqrt{3}$. In Table XI we show the fields explicitly with their corresponding charges, where $Q_1 = \frac{1}{2} + \frac{\sqrt{3}\beta}{2}$ and $Q_2 = -\frac{1}{2} + \frac{\sqrt{3}\beta}{2}$ refer to the electric charge of the fields, which satisfy the property $Q_1 - Q_2 = 1$. When we apply the minimum conditions, the following relations are obtained:

$$\begin{aligned} \mu_1^2 &= -2\lambda_1 \nu_\chi^2 - \lambda_4 \nu_\rho^2 - \lambda_5 \nu_\eta^2 - f \frac{\nu_\eta \nu_\rho}{\nu_\chi}, \\ \mu_2^2 &= -2\lambda_2 \nu_\rho^2 - \lambda_4 \nu_\chi^2 - \lambda_6 \nu_\eta^2 - f \frac{\nu_\eta \nu_\chi}{\nu_\rho}, \\ \mu_3^2 &= -2\lambda_3 \nu_\eta^2 - \lambda_5 \nu_\chi^2 - \lambda_6 \nu_\rho^2 - f \frac{\nu_\rho \nu_\chi}{\nu_\eta}, \end{aligned}$$

and we replace them again in the scalar potential to find the physical spectrum of the fields and their masses. From the second derivatives with respect to the fields, we obtain the mass matrices $M_{\zeta\zeta}^2$ for the imaginary sector, $M_{\xi\xi}^2$ for the scalar real sector, and three decoupled matrices M_ϕ^2 for the scalar charged sector.

In order to obtain the eigenvalues and eigenvectors we shall suppose that there is a strong hierarchy between the scales of the first and the second transition, from which it is natural to assume

$$\langle\chi\rangle_0 \gg \langle\rho\rangle_0, \langle\eta\rangle_0 \Rightarrow |\nu_\chi| >> |\nu_\rho|, |\nu_\eta|. \tag{3.3}$$

In addition, since some of the Higgs bosons of the first transition are proportional to $f\nu_\chi$ we shall make the assumption

$$|f| \approx |\nu_\chi| \tag{3.4}$$

where f is the trilinear coupling constant defined in the scalar potential equation (3.1). This assumption prevents the introduction of another scale different from the ones defined by the two transitions. In our approach we shall keep only the matrix elements that are quadratic in ν_χ, i.e. the terms proportional to $\nu_\chi^2, f\nu_\chi$, unless otherwise indicated. Under these approximations, the mass matrices and eigenvalues are written in explicit form in Eqs. (A1)–(A16) in Appendix A. Summarizing we get all the scalar bosons described in Table XII.

TABLE XI. Quantum numbers of three scalar triplets for any $\beta \neq \pm 1/\sqrt{3}$.

	Q_Φ	Y_Φ	X_Φ	$\langle\Phi\rangle_0$
$\chi = \begin{pmatrix} \chi_1^{\pm Q_1} \\ \chi_2^{\pm Q_2} \\ \xi_\chi \pm i\zeta_\chi \end{pmatrix}$	$\begin{pmatrix} \pm(\frac{1}{2}+\frac{\sqrt{3}\beta}{2}) \\ \pm(-\frac{1}{2}+\frac{\sqrt{3}\beta}{2}) \\ 0 \end{pmatrix}$	$\begin{pmatrix} \pm\frac{\sqrt{3}\beta}{2} \\ \pm\frac{\sqrt{3}\beta}{2} \\ 0 \end{pmatrix}$	$\frac{\beta}{\sqrt{3}}$	$\begin{pmatrix} 0 \\ 0 \\ \nu_\chi \end{pmatrix}$
$\rho = \begin{pmatrix} \rho_1^{\pm} \\ \xi_\rho \pm i\zeta_\rho \\ \rho_3^{\mp Q_2} \end{pmatrix}$	$\begin{pmatrix} \pm 1 \\ 0 \\ \mp(-\frac{1}{2}+\frac{\sqrt{3}\beta}{2}) \end{pmatrix}$	$\begin{pmatrix} \pm\frac{1}{2} \\ \pm\frac{1}{2} \\ \mp(-\frac{1}{2}+\frac{\sqrt{3}\beta}{2}) \end{pmatrix}$	$\frac{1}{2}-\frac{\beta}{2\sqrt{3}}$	$\begin{pmatrix} 0 \\ \nu_\rho \\ 0 \end{pmatrix}$
$\eta = \begin{pmatrix} \xi_\eta \pm i\zeta_\eta \\ \eta_2^{\mp} \\ \eta_3^{\mp Q_1} \end{pmatrix}$	$\begin{pmatrix} 0 \\ \mp 1 \\ \mp(\frac{1}{2}+\frac{\sqrt{3}\beta}{2}) \end{pmatrix}$	$\begin{pmatrix} \mp\frac{1}{2} \\ \mp\frac{1}{2} \\ \mp(\frac{1}{2}+\frac{\sqrt{3}\beta}{2}) \end{pmatrix}$	$-\frac{1}{2}-\frac{\beta}{2\sqrt{3}}$	$\begin{pmatrix} \nu_\eta \\ 0 \\ 0 \end{pmatrix}$

TABLE XII. Spectrum of scalars for $\beta \neq \pm 1/\sqrt{3}, \pm\sqrt{3}$.

Scalar fields	Square masses	Feature
$\phi_2^0 \simeq -\zeta_\chi$	$M^2_{\phi_2^0} = 0$	Goldstone associated with Z'_μ
$\phi_3^0 \simeq S_\beta\zeta_\rho - C_\beta\zeta_\eta$	$M^2_{\phi_3^0} = 0$	Goldstone associated with Z_μ
$\phi_1^{\pm} = S_\beta\rho_1^{\pm} - C_\beta\eta_2^{\pm}$	$M^2_{\phi_1^{\pm}} = 0$	Goldstone associated with $W^{\pm}_\mu$
$\phi_2^{\pm Q_1} \simeq \chi_1^{\pm Q_1}$	$M^2_{\phi_2^{\pm}} = 0$	Goldstone associated with $K_\mu^{\pm Q_1}$
$\phi_3^{\pm Q_2} \simeq -\chi_2^{\pm Q_2}$	$M^2_{\phi_3^{\pm}} = 0$	Goldstone associated with $K_\mu^{\pm Q_2}$
$h_1^0 \simeq C_\beta\zeta_\rho + S_\beta\zeta_\eta$	$M^2_{h_1^0} \simeq -2f\nu_\chi(\frac{\nu_\eta}{\nu_\rho}+\frac{\nu_\rho}{\nu_\eta})$	Higgs
$h_3^0 \simeq S_\beta\xi_\rho + C_\beta\xi_\eta$	$M^2_{h_3^0} \simeq \frac{8}{\nu_\eta^2+\nu_\rho^2} \times [\lambda_2\nu_\rho^4 + 2\lambda_6\nu_\rho^2\nu_\eta^2 + \lambda_3\nu_\eta^4]$	Higgs
$h_4^0 \simeq -C_\beta\xi_\rho + S_\beta\xi_\eta$	$M^2_{h_4^0} \simeq -2f\nu_\chi(\frac{\nu_\eta}{\nu_\rho}+\frac{\nu_\rho}{\nu_\eta})$	Higgs
$h_5^0 \simeq \xi_\chi$	$M^2_{h_5^0} \simeq 8\lambda_1\nu_\chi^2$	Higgs
$h_1^{\pm Q_1} = \eta_3^{\pm Q_1}$	$M^2_{h_1^{\pm}} \simeq \lambda_7\nu_\chi^2 - f\nu_\chi\frac{\nu_\rho}{\nu_\eta}$	Higgs
$h_2^{\pm} = C_\beta\rho_1^{\pm} + S_\beta\eta_2^{\pm}$	$M^2_{h_2^{\pm}} \simeq -f\nu_\chi(\frac{\nu_\eta}{\nu_\rho}+\frac{\nu_\rho}{\nu_\eta})$	Higgs
$h_3^{\pm Q_2} = \rho_3^{\pm Q_2}$	$M^2_{h_3^{\pm}} \simeq \lambda_8\nu_\chi^2 - f\nu_\chi\frac{\nu_\eta}{\nu_\rho}$	Higgs

IV. VECTOR SPECTRUM WITH β ARBITRARY

The gauge bosons associated with the $SU(3)_L$ group transform according to the adjoint representation and are written in the form

$$W_\mu = W_\mu^\alpha G_\alpha = \frac{1}{2}\begin{bmatrix} W_\mu^3 + \frac{1}{\sqrt{3}}W_\mu^8 & \sqrt{2}W_\mu^+ & \sqrt{2}K_\mu^{Q_1} \\ \sqrt{2}W_\mu^- & -W_\mu^3 + \frac{1}{\sqrt{3}}W_\mu^8 & \sqrt{2}K_\mu^{Q_2} \\ \sqrt{2}K_\mu^{-Q_1} & \sqrt{2}K_\mu^{-Q_2} & -\frac{2}{\sqrt{3}}W_\mu^8 \end{bmatrix}. \quad (4.1)$$

Therefore, the electric charge takes the general form

$$Q_W \to \begin{bmatrix} 0 & 1 & \frac{1}{2}+\frac{\sqrt{3}\beta}{2} \\ -1 & 0 & -\frac{1}{2}+\frac{\sqrt{3}\beta}{2} \\ -\frac{1}{2}-\frac{\sqrt{3}\beta}{2} & \frac{1}{2}-\frac{\sqrt{3}\beta}{2} & 0 \end{bmatrix}. \quad (4.2)$$

As for the gauge field associated with $U(1)_X$, it is represented as $\mathbf{B}_\mu = B_\mu \mathbf{I}_{3\times 3}$ which is a singlet under $SU(3)_L$ and has no electric charge. From the previous expressions we see that three gauge fields with charges equal to zero are obtained, and in the basis of mass eigenstates they correspond to the photon, Z and Z'. Moreover, there are two fields with charges ± 1 associated with $W^{\pm}$, as well as four fields with charges that depend on the choice of β (denoted by $K^{\pm Q_1}$ and $K^{\pm Q_2}$). Demanding that the model contains no exotic charges in this sector is equivalent to setting up $\beta = -1/\sqrt{3}$ [10], and $\beta = 1/\sqrt{3}$ [21]. It is important to take into account the scalar sector and the symmetry breakings to fix this quantum number, which in turn determine the would-be Goldstone bosons associated with the gauge fields, with the same electric charge of the gauge fields that are acquiring mass in the different scales of breakdown.

A. Charged sector

The masses for $W^{\pm}, K^{\pm Q_1}, K^{\pm Q_2}$ charged gauge fields read

$$M^2_{W^{\pm}} = \frac{g^2}{2}(\nu_\rho^2 + \nu_\eta^2), \qquad M^2_{K^{\pm Q_1}} = \frac{g^2}{2}(\nu_\chi^2 + \nu_\eta^2),$$
$$M^2_{K^{\pm Q_2}} = \frac{g^2}{2}(\nu_\chi^2 + \nu_\rho^2)$$

where the terms proportional to ν_χ^2 acquire heavy masses of the order of the first symmetry breaking. The other fields acquire a mass proportional to the electroweak scale and correspond to the gauge fields $W^\pm$. The mass eigenstates are given by

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2); \qquad K_\mu^{\pm Q_1} = \frac{1}{\sqrt{2}}(W_\mu^4 \mp iW_\mu^5);$$
$$K_\mu^{\pm Q_2} = \frac{1}{\sqrt{2}}(W_\mu^6 \mp iW_\mu^7). \tag{4.3}$$

B. Neutral sector

The mass matrix is given in Appendix B. This matrix has null determinant corresponding to the mass of the photon. After the proper rotation the mass eigenstates become

$$\begin{aligned} A_\mu &= S_W W_\mu^3 + C_W\left(\beta T_W W_\mu^8 + \sqrt{1-\beta^2 T_W^2} B_\mu\right), \\ Z'_\mu &= -\sqrt{1-\beta^2 (T_W)^2} W_\mu^8 + \beta T_W B_\mu, \\ Z_\mu &= C_W W_\mu^3 - S_W\left(\beta T_W W_\mu^8 + \sqrt{1-\beta^2 T_W^2} B_\mu\right). \end{aligned} \tag{4.4}$$

The corresponding eigenvalues are

$$\begin{aligned} M_{A_\mu}^2 &= 0; \qquad M_{Z'_\mu}^2 \simeq \frac{2[g^2+\beta^2 g'^2]}{3}\nu_\chi^2; \\ M_{Z_\mu}^2 &\simeq \frac{g^2}{2}\left[\frac{g^2+(1+\beta^2)g'^2}{g^2+\beta^2 g'^2}\right](\nu_\rho^2+\nu_\eta^2) \end{aligned} \tag{4.5}$$

where the Weinberg angle is defined (in terms of β) as

$$S_W \equiv \sin\theta_W = \frac{g'}{\sqrt{g^2+(1+\beta^2)g'^2}}, \tag{4.6}$$

and g, g' correspond to the coupling constants of the groups $SU(3)_L$ and $U(1)_X$, respectively. Further, a small mixing between the Z_μ and Z'_μ could occur getting

$$\begin{aligned} Z_{1\mu} &= Z_\mu \cos\theta + Z'_\mu \sin\theta; \\ Z_{2\mu} &= -Z_\mu \sin\theta + Z'_\mu \cos\theta, \qquad \tan\theta = \frac{1}{\Lambda+\sqrt{\Lambda^2+1}}; \\ \Lambda &= \frac{-2S_W C_W^2 g'^2 \nu_\chi^2 + \frac{3}{2} S_W T_W^2 g^2(\nu_\eta^2+\nu_\rho^2)}{gg' T_W^2[3S_W^2\beta(\nu_\eta^2+\nu_\rho^2) + C_W^2(\nu_\eta^2-\nu_\rho^2)]}. \end{aligned} \tag{4.7}$$

It is interesting to note that from the definition of the charge in Eq. (1.1), we obtain a matching condition among the coupling constants, that in turn leads to the following expression:

$$\frac{g'^2}{g^2} = \frac{S_W^2}{1-S_W^2(1+\beta^2)}.$$

By running the Weinberg angle through renormalization group equations, we can find a scale to which a singularity of this quotient appears. In some models and for certain values of β, this pole could appear at the TeV scale [25].

We point out that when $\beta = -\sqrt{3}$, we get the same definitions and diagonalizations of the model of Pleitez and Frampton [8,9]. The β parameter can be written explicitly in terms of the exotic charges as $\beta = (2Q_1 - 1)/\sqrt{3} = (2Q_2+1)/\sqrt{3}$, from which it is obtained that, in general, $Q_1 - Q_2 = 1$, so independently of the model the difference in charges between the charged gauge fields will be equal to the unity.

V. YANG-MILLS COUPLINGS

In general, the Yang-Mills Lagrangian for $SU(3)_L \times U(1)_Y$ is given by

$$\begin{aligned} \mathfrak{L}_{\rm YM} = &-\frac{1}{4}W_{\mu\nu}^i W_i^{\mu\nu} + \frac{1}{2} g f_{ijk} W_{\mu\nu}^i W^{\mu j} W^{\nu k} \\ &-\frac{g^2}{4} f^{ijk} f_{ilm} W_\mu^j W^{\mu k} W_\nu^l W^{\nu m} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \end{aligned} \tag{5.1}$$

where $W_i^{\mu\nu} = \partial^\mu W_i^\nu - \partial^\nu W_i^\mu$. After writing this Lagrangian in terms of the mass eigenstates, the cubic couplings read

$$\begin{aligned} \mathfrak{L}_{\rm cubic} = {}& e\{[r-p]^\mu g^{\alpha\nu} + [p-q]^\nu g^{\alpha\mu} + [q-r]^\alpha g^{\nu\mu}\} A_\nu W_\alpha^+ W_\mu^- + Q_1 e\{[r-p]^\mu g^{\alpha\nu} + [p-q]^\nu g^{\alpha\mu} \\ & + [q-r]^\alpha g^{\nu\mu}\} A_\nu K_\alpha^{+Q_1} K_\mu^{-Q_1} + Q_2 e\{[r-p]^\mu g^{\alpha\nu} + [p-q]^\nu g^{\alpha\mu} + [q-r]^\alpha g^{\nu\mu}\} A_\nu K_\alpha^{+Q_2} K_\mu^{-Q_2} \\ & + gC_W\{[p-q]^\mu g^{\alpha\nu} + [q-r]^\alpha g^{\nu\mu} + [r-p]^\nu g^{\alpha\mu}\} Z_\mu W_\alpha^+ W_\nu^- + \left[\frac{gC_W}{2} + \frac{(2Q_1-1)eT_W}{2}\right]\{[p-q]^\mu g^{\alpha\nu} \\ & + [q-r]^\alpha g^{\nu\mu} + [r-p]^\nu g^{\alpha\mu}\} Z_\mu K_\alpha^{+Q_1} K_\nu^{-Q_1} + \left[\frac{-gC_W}{2} + \frac{(2Q_2+1)eT_W}{2}\right]\{[p-q]^\mu g^{\alpha\nu} + [q-r]^\alpha g^{\nu\mu} \\ & + [r-p]^\nu g^{\alpha\mu}\} Z_\mu K_\alpha^{+Q_2} K_\nu^{-Q_2} + \frac{\sqrt{3}g}{2}\sqrt{1-\beta^2 T_W^2}\{[q-p]^\mu g^{\alpha\nu} + [r-q]^\alpha g^{\nu\mu} + [p-r]^\nu g^{\alpha\mu}\} Z'_\mu K_\alpha^{+Q_1} K_\nu^{-Q_1} \\ & + \frac{\sqrt{3}g}{2}\sqrt{1-\beta^2 T_W^2}\{[q-p]^\mu g^{\alpha\nu} + [r-q]^\alpha g^{\nu\mu} + [p-r]^\nu g^{\alpha\mu}\} Z'_\mu K_\alpha^{+Q_2} K_\nu^{-Q_2}. \end{aligned}$$

In passing to the space of momenta we associate $\partial_\mu = -ip_\mu$ and the following assignments of momenta: p_μ for the positively charged fields W_ν^+, $K_\nu^{+Q_1}$, and $K_\nu^{+Q_2}$; q_μ for the negatively charged fields, i.e. for W_ν^-, $K_\nu^{-Q_1}$, $K_\nu^{-Q_2}$; finally, r_μ

for the neutral fields A_ν, Z_ν, Z'_ν. It is assumed that all the momenta enter the vertex of interaction and that the sum of them vanishes. Note that the coupling $Z'_\mu W^\pm W^\mp$ does not appear at tree level, because of the form of the f_{ijk} structure constant.

Further, the quartic Hermitian couplings are

$$\begin{aligned}\mathfrak{L}_{\text{quartic}} &= g^2 W_\alpha^- W_\beta^+ \Big\{ -\mathfrak{g}_1^{\alpha\delta\gamma\beta}[W_\gamma^+ W_\delta^- + K_\gamma^{-Q_2} K_\delta^{+Q_2}] + \mathfrak{g}_2^{\alpha\beta\gamma\delta}[S_W^2 A_\gamma A_\delta + C_W^2 Z_\gamma Z_\delta + S_W C_W A_\gamma Z_\delta] \\ &\quad - \frac{1}{2}\mathfrak{g}_3^{\beta\delta\alpha\gamma} K_\gamma^{-Q_1} K_\delta^{+Q_1} \Big\} + g^2 K_\alpha^{-Q_1} K_\beta^{+Q_1} \Big\{ -\mathfrak{g}_1^{\alpha\delta\gamma\beta} K_\gamma^{+Q_1} K_\delta^{-Q_1} + \mathfrak{g}_2^{\alpha\beta\gamma\delta}\Big[S_W^2 Q_1^2 A_\gamma A_\delta + \frac{C_W^2}{4}(\sqrt{3}\beta T_W^2 - 1)^2 Z_\gamma Z_\delta \\ &\quad - \frac{S_W C_W Q_1}{2}(\sqrt{3}\beta T_W^2 - 1) A_\gamma Z_\delta - \frac{\sqrt{3} S_W Q_1}{2}\sqrt{1 - \beta^2 T_W^2} A_\gamma Z'_\delta + \frac{3}{4}(1 - \beta^2 T_W^2) Z'_\gamma Z'_\delta \\ &\quad + \frac{\sqrt{3} C_W}{4}\sqrt{1 - \beta^2 T_W^2}(\sqrt{3}\beta T_W^2 - 1) Z_\gamma Z'_\delta \Big]\Big\} + g^2 K_\alpha^{-Q_2} K_\beta^{+Q_2} \Big\{ -\mathfrak{g}_1^{\alpha\delta\gamma\beta} K_\gamma^{+Q_2} K_\delta^{-Q_2} + \mathfrak{g}_2^{\alpha\beta\gamma\delta}\Big[S_W^2 Q_2^2 A_\gamma A_\delta \\ &\quad + \frac{C_W^2}{4}(\sqrt{3}\beta T_W^2 + 1)^2 Z_\gamma Z_\delta - \frac{S_W C_W Q_2}{2}(\sqrt{3}\beta T_W^2 + 1) A_\gamma Z_\delta - \frac{\sqrt{3} S_W Q_2}{2}\sqrt{1 - \beta^2 T_W^2} A_\gamma Z'_\delta \\ &\quad + \frac{\sqrt{3} C_W}{4}\sqrt{1 - \beta^2 T_W^2}(\sqrt{3}\beta T_W^2 + 1) Z_\gamma Z'_\delta + \frac{3}{4}(1 - \beta^2 T_W^2) Z'_\gamma Z'_\delta \Big] - \frac{1}{2}\mathfrak{g}_3^{\beta\delta\alpha\gamma} K_\gamma^{-Q_2} K_\delta^{+Q_2} \Big\} \\ &\quad - \frac{\sqrt{3} g^2}{2\sqrt{2}}\sqrt{1 - \beta^2 T_W^2}\,\mathfrak{g}_2^{\alpha\beta\gamma\delta} K_\alpha^{+Q_2} K_\beta^{-Q_1} W_\gamma^+ Z'_\delta + \text{h.c.} + \frac{g^2 S_W}{2\sqrt{2}}[(Q_1 + Q_2)\mathfrak{g}_2^{\alpha\beta\gamma\delta} + \mathfrak{g}_3^{\beta\delta\alpha\gamma} - \mathfrak{g}_1^{\alpha\delta\gamma\beta}] \\ &\quad \times K_\alpha^{+Q_2} K_\beta^{-Q_1} W_\gamma^+ A_\delta + \text{h.c.} - \frac{g^2 C_W}{\sqrt{2}}\Big[\frac{(\sqrt{3}\beta T_W^2 + 1)}{2}\mathfrak{g}_2^{\alpha\beta\gamma\delta} + \mathfrak{g}_1^{\alpha\delta\gamma\beta}\Big] K_\alpha^{+Q_2} K_\beta^{-Q_1} W_\gamma^+ Z_\delta + \text{h.c.},\end{aligned} \tag{5.2}$$

where $\mathfrak{g}_1^{\alpha\delta\gamma\beta} = -2g^{\alpha\delta}g^{\gamma\beta} + g^{\alpha\beta}g^{\gamma\delta} + g^{\alpha\gamma}g^{\beta\delta}$, $\mathfrak{g}_2^{\alpha\beta\gamma\delta} = -2g^{\alpha\beta}g^{\gamma\delta} + g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}$, and $\mathfrak{g}_3^{\beta\delta\alpha\gamma} = -2g^{\beta\delta}g^{\alpha\gamma} + g^{\alpha\beta}g^{\gamma\delta} + g^{\alpha\delta}g^{\beta\gamma}$. These results are in agreement with Ref. [26] for $\beta = \sqrt{3}$ and Ref. [27] for $\beta = \frac{-1}{\sqrt{3}}$.

VI. MODEL FOR $N = 3$ WITH β ARBITRARY

In Sec. II C, we found that if $\beta \neq \pm\sqrt{3}, \pm 1/\sqrt{3}$, there are only two solutions for the fermionic structure when $N = 3$ (the ones marked with □ in Table VIII), where the solutions are the complex conjugate of each other. Then, we take the option with $j = 3$ (three lepton triplets), $k = 1$ (one quark triplet and two antitriplets) valid for all β. Applying this solution to the fermionic content given in Table I, we obtain the fermionic spectrum given in Table XIII. We should note that for this solution, we must introduce right-handed leptonic singlets associated with each left-handed lepton ($\Theta_{e^{(1)}} = \Theta_{E^{(1)}} = 1$) for Eqs. (2.7) and (2.11) to be accomplished ensuring the vanishing of the anomalies.

A. Neutral and charged currents

The Dirac Lagrangian contains the couplings between gauge bosons and fermions, given by

$$\begin{aligned}\mathcal{L}_{\mathcal{F}} &= i\overline{\psi_L}\not{\partial}\psi_L \mp g\overline{\psi_L}\not{W}\psi_L \mp g'\overline{\psi_L}\not{B}X_p^L\psi_L + i\overline{\psi_R}\not{\partial}\psi_R \\ &\quad - g'\overline{\psi_R}\not{B}X_p^R\psi_R\end{aligned}$$

where the sign is chosen according to the representation **3** or **3***, respectively. Since the mass matrices mix the quarks among each other, the mass basis is different from the gauge basis. So when we write the Lagrangian in terms of mass eigenstates we get

$$\begin{aligned}\mathcal{L}_q &= eQ_{q_j}\overline{\mathbf{Q}}_j\gamma_\mu A^\mu \mathbf{Q}_j + \frac{g}{C_W}\overline{\mathbf{Q}_j}\gamma_\mu[T_3 P_L - Q_{q_j}S_W^2]Z^\mu \mathbf{Q}_j + \frac{g}{\sqrt{2}}\overline{d_{iL}}\gamma_\mu (U_{\theta j}^i)^* W^{\mu -} u_{jL} + \frac{g}{\sqrt{2}}\overline{u_{jL}}\gamma_\mu W^{\mu +} U_{\theta j}^i d_{iL} \\ &\quad + \frac{g'}{2T_W}\overline{q_{m^*}}\gamma_\mu[(2T_8 + \beta Q_{q_{m^*}} T_W^2 \Lambda_1)P_L + 2\beta Q_{q_{m^*}} T_W^2 P_R]Z^{\mu\prime} q_{m^*} + \frac{g'}{2T_W}\overline{q_3}\gamma_\mu[(-2T_8 + \beta Q_{q_3} T_W^2 \Lambda_2)P_L \\ &\quad + 2\beta Q_{q_3} T_W^2 P_R]Z^{\mu\prime} q_3 - \frac{g}{\sqrt{2}}\overline{d_{iL}}\gamma_\mu (U_{\theta j}^i)^* \delta_j^{n^*} K^{\mu -Q_1} U_{\phi n^*}^{m^*} J_{m^* L} - \frac{g}{\sqrt{2}}\overline{J_{m^* L}}\gamma_\mu (U_{\phi n^*}^{m^*})^* K^{\mu +Q_1}\delta_{n^*}^j U_{\theta j}^i d_{iL} \\ &\quad + \frac{g}{\sqrt{2}}\overline{J_{3L}}\gamma_\mu K^{\mu -Q_1} t_L + \frac{g}{\sqrt{2}}\overline{t_L}\gamma_\mu K^{\mu +Q_1} J_{3L} + \frac{g}{\sqrt{2}}\overline{u_{n^* L}}\gamma_\mu K^{\mu -Q_2} U_{\phi n^*}^{m^*} J_{m^* L} + \frac{g}{\sqrt{2}}\overline{J_{m^* L}}\gamma_\mu (U_{\phi n^*}^{m^*})^* K^{\mu +Q_2} u_{n^* L} \\ &\quad + \frac{g}{\sqrt{2}}\overline{J_{3L}}\gamma_\mu K^{\mu -Q_2}\delta_3^j U_{\theta j}^i d_{iL} + \frac{g}{\sqrt{2}}\overline{d_{iL}}\gamma_\mu (U_{\theta j}^i)^* \delta_j^3 K^{\mu +Q_2} J_{3L}.\end{aligned} \tag{6.1}$$

The couplings associated with A^μ and Z^μ have been written in a SM-like notation, i.e. $\mathbf{Q}_j$ with $j = 1, 2, 3$ refers to triplets in the **3** representation associated with the three generations of quarks.

TABLE XIII. Fermionic content for $N = 3$ with β arbitrary. $m^* = 1, 2$ and $j = 1, 2, 3$.

Representation	Q_ψ	X_ψ
$q_{m^*L} = \begin{pmatrix} d, s \\ -u, -c \\ J_1, J_2 \end{pmatrix}_L \mathbf{3}^*$	$\begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{6} + \frac{\sqrt{3}\beta}{2} \end{pmatrix}$	$X^L_{q^{(m^*)}} = -\frac{1}{6} - \frac{\beta}{2\sqrt{3}}$ $X^R_{u^{(m^*)}} = \frac{2}{3}$ $X^R_{d^{(m^*)}} = -\frac{1}{3}$ $X^R_{J^{(m^*)}} = \frac{1}{6} + \frac{\sqrt{3}}{2}\beta$
$d_{m^*R} = d_R, s_R{:}1$	$-\frac{1}{3}$	
$u_{m^*R} = u_R, c_R{:}1$	$\frac{2}{3}$	
$J_{m^*R} = J_{1R}, J_{2R}{:}1$	$\frac{1}{6} + \frac{\sqrt{3}}{2}\beta$	
$q_{3L} = \begin{pmatrix} t \\ b \\ J_3 \end{pmatrix}_L {:}\mathbf{3}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{1}{6} - \frac{\sqrt{3}\beta}{2} \end{pmatrix}$	$X^L_{q^{(3)}} = \frac{1}{6} - \frac{\beta}{2\sqrt{3}}$
$u_{3R} = b_R{:}1$	$-\frac{1}{3}$	$X^R_b = -\frac{1}{3}$
$d_{3R} = t_R{:}1$	$\frac{2}{3}$	$X^R_t = \frac{2}{3}$
$J_{3R} = J_{3R}{:}1$	$\frac{1}{6} - \frac{\sqrt{3}\beta}{2}$	$X^R_{J_3} = \frac{1}{6} - \frac{\sqrt{3}\beta}{2}$
$\ell_{jL} = \begin{pmatrix} \nu_e, \nu_\mu, \nu_\tau \\ e^-, \mu^-, \tau^- \\ E_1^{-Q_1}, E_2^{-Q_1}, E_3^{-Q_1} \end{pmatrix}_L {:}\mathbf{3}$	$\begin{pmatrix} 0 \\ -1 \\ -\frac{1}{2} - \frac{\sqrt{3}\beta}{2} \end{pmatrix}$	$X^L_{\ell^{(m)}} = -\frac{1}{2} - \frac{\beta}{2\sqrt{3}}$
$(e_j^-)_R = e^-, \mu^-, \tau^-_R{:}1$	-1	$X^R_{e^{(m)}} = -1$
$E_j^{-Q_1} = E_1^{-Q_1}, E_2^{-Q_1}, E_3^{-Q_1}{:}1$	$-\frac{1}{2} - \frac{\sqrt{3}\beta}{2}$	$X^R_{E_m} = -\frac{1}{2} - \frac{\sqrt{3}\beta}{2}$

On the other hand, the couplings of the exotic gauge bosons with the two former families are different from the ones involving the third family. It is because the third family transforms differently (see Table XIII). Consequently, there are terms where only the components $m^*, n^* = 1, 2$ are summed, leaving the third one in a term apart. q_{m^*} refers to the two triplets of quarks with $q_{1,2}$ in the $\mathbf{3}^*$ representation and q_3 in the $\mathbf{3}$ representation. $Q_{q_{m^*}}$ are their electric charges shown in Table XIII. We define $\Lambda_1 \equiv \mathrm{diag}(-1, 1/2, 2)$ and $\Lambda_2 \equiv \mathrm{diag}(\frac{1}{2}, -1, 2)$. We also have used the projectors $P_{R,L} = (1 \pm \gamma_5)/2$. Flavor mixings appear owing to the charged gauge bosons W^μ, $K^{\mu^{\pm Q_1}}$, and $K^{\mu^{\pm Q_2}}$, where the Cabibbo Kobayashi Maskawa (CKM) matrix U_θ has been defined with the usual mixing angles θ_i of the SM and a matrix U_ϕ with a mixing angle ϕ_c associated with the exotic quarks J_1 and J_2 (the quark J_3 is decoupled in the mass matrices because of its different electric charge). Equation (6.1) includes the SM couplings properly.

As for the leptons, we have for the three families

$$\begin{aligned} \mathcal{L}_{\ell_j} &= eQ_{\ell_j}\overline{\ell_j}\gamma_\mu A^\mu \ell_j + \frac{g}{C_W}\overline{\ell_j}\gamma_\mu[T_3P_L - Q_{\ell_j}S_W^2]Z^\mu \ell_j + \frac{g}{\sqrt{2}}\overline{\nu_{jL}}\gamma_\mu W^{\mu+}e^-_{jL} + \frac{g}{\sqrt{2}}\overline{e^-_{jL}}\gamma_\mu W^{\mu-}\nu_{jL} \\ &+ \frac{g'}{2T_W}\overline{\ell_j}\gamma_\mu[(-2T_8 - \beta T_W^2\Lambda_3)P_L + 2Q_{\ell_j}\beta T_W^2 P_R]Z^{\mu\prime}\ell_j + \frac{g}{\sqrt{2}}\overline{\nu_{jL}}\gamma_\mu K^{\mu^{+Q_1}}E^{-Q_1}_{jL} + \frac{g}{\sqrt{2}}\overline{E^{-Q_1}_{jL}}\gamma_\mu K^{\mu^{-Q_1}}\nu_{jL} \\ &+ \frac{g}{\sqrt{2}}\overline{e^-_{jL}}\gamma_\mu K^{\mu^{+Q_2}}E^{-Q_1}_{jL} + \frac{g}{\sqrt{2}}\overline{E^{-Q_1}_{jL}}\gamma_\mu K^{\mu^{-Q_2}}e^-_{jL}, \end{aligned} \tag{6.2}$$

with ℓ_j denoting the leptonic triplets shown in Table XIII, and with Q_{ℓ_j} denoting their electric charges; finally $\Lambda_3 = \mathrm{diag}(1, 1, 2Q_1)$.

VII. MODEL WITH $N = 4$ AND $\beta = -\frac{1}{\sqrt{3}}$

We consider a model with $\beta = -1/\sqrt{3}$ which is similar to the model described in Ref. [10] at low energies due to the electromagnetic charges assigned to different multiplets. However, this model is not the same as the one in Ref. [10] because the multiplets' structure for the quark sector is $SU(3)_C \otimes SU(3)_L$ vectorlike, and the leptonic part is not necessary to cancel the quark anomalies. The leptonic multiplets are also vectorlike and anomaly free (see Table XIV). In the models described in the literature, the quarks' anomalies are canceled out with the leptonic anomalies. In the model with $N = 4$ and $\beta = -1/\sqrt{3}$ there are two 3-multiplets for leptons and two 3-multiplets for quarks and they generate the two heavy families of the SM. Two 3^*-multiplets for quarks and leptons correspond to the first SM family; and the other two 3^*, q_L^{4*} and l_L^{4*}, correspond to a mirror fermion family of the third SM family. So with this assignment, it is possible to get mixing between the bottom quark and its mirror quark d^c in order to modify the right-handed coupling of the bottom quark with the Z gauge boson which in turn might explain the

TABLE XIV. Fermionic content of $SU(3)_L \otimes U(1)_X$, with $N = 4$, and $m, n = 1, 2$. The fourth families which are in the $\mathbf{3}^*$ representation are the mirror fermions of one of the families in the $\mathbf{3}$ representation.

Quarks	Q_ψ	X_ψ
$q_L^{(m)} = \begin{pmatrix} u^{(m)} \\ d^{(m)} \\ J^{(m)} \end{pmatrix}_L :3$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$	$X^L_{q^{(m)}} = \frac{1}{3}$
$u_R^{(m)}, d_R^{(m)}, J_R^{(m)}$:1	$\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$	$X^R_{q^{(m)}} = Q_{q^{(m)}}$
$q_L^{(3^*)} = \begin{pmatrix} d^{3^*} \\ -u^{3^*} \\ J^{3^*} \end{pmatrix}_L :3^*$	$\begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$	$X^L_{q^{3^*}} = 0$
$d_R^{3^*}, u_R^{3^*}, J_R^{3^*}$:1	$-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}$	$X^R_{q^{3^*}} = Q_{q^{3^*}}$
$q_L^{4^*} = \begin{pmatrix} \tilde{u}^c \\ \tilde{d}^c \\ \tilde{J}^c \end{pmatrix}_L :3^*$	$\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$	$X^L_{q^{4^*}} = \frac{1}{3}$
$\tilde{u}_R^c, \tilde{d}_R^c, \tilde{J}_R^c$:1	$-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}$	$X^R_{q^{4^*}} = Q_{q^{4^*}}$

Leptons	Q_ψ	X_ψ
$\ell_L^{(n)} = \begin{pmatrix} \nu^{(n)} \\ e^{(n)} \\ N^{0(n)} \end{pmatrix}_L :3$	$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$	$X^L_{\ell^{(n)}} = -\frac{1}{3}$
$\nu_R^{(n)}, e_R^{(n)}$:1	$0, -1, 0$	$X^R_{\ell^{(n)}} = Q_{\ell^{(n)}}$
$\ell_L^{3^*} = \begin{pmatrix} e^{3^*} \\ -\nu^{3^*} \\ E^{3^*-} \end{pmatrix}_L :3^*$	$\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$	$X^L_{\ell^{3^*}} = \frac{2}{3}$
$e_R^{3^*}, \nu_R^{3^*}, E_R^{3^*-}$:1	$-1, 0, -1$	$X^R_{\ell^{3^*}} = Q_{\ell^{3^*}}$
$\ell_L^{4^*} = \begin{pmatrix} \tilde{\nu}^c \\ \tilde{e}^c \\ \tilde{N}^{0c} \end{pmatrix}_L :3^*$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$X^L_{\ell^{4^*}} = -\frac{1}{3}$
$\tilde{\nu}_R^c, \tilde{e}_R^c$:1	$0, 1, 0$	$X^R_{\ell^{4^*}} = Q_{\ell^{4^*}}$

asymmetry deviations A^b and A^b_{FB} [13]. Such discrepancy cannot be explained by a model with only left-handed multiplets such as the SM [28] or the traditional 331 models [14]. The mixing in the mass matrix between the b quark and its mirror fermion permits a solution because the mirror couples with right-handed chirality to the Z_μ gauge field of the SM. On the other hand, the mirror fermions in the leptonic sector are useful to build up ansatz about mass matrices in the neutrino and charged sectors. For the neutrinos corresponding to $SU(2)_L$ doublets, right-handed neutrino singlets are introduced to generate masses of Dirac type.

TABLE XV. Scalar sector with $N = 4$ and its VEVs. χ, ρ, η are triplets in the $\mathbf{3}$ representation, ϕ is a multiplet in the adjoint representation, and S lies in the sextet representation. ν_χ is of the order of the first symmetry breaking. ν_ρ, ν_η are of the order of the electroweak scale. V is much lower than the electroweak VEV.

$\langle\chi\rangle_0$	$(0 \quad 0 \quad \nu_\chi)^T$	$X_\chi = -1/3$
$\langle\rho\rangle_0$	$(0 \quad \nu_\rho \quad 0)^T$	$X_\rho = 2/3$
$\langle\eta\rangle_0$	$(\nu_\eta \quad 0 \quad 0)^T$	$X_\eta = 2/3$
$\langle\phi\rangle_0$	$\nu_\chi \mathrm{diag}(1 \quad 1 \quad -2)$	$X_\chi = 0$
$\langle S^{ij}\rangle_0$	$V\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$X_S = -1/3$

As for the scalar spectrum, three types of representations are considered. First, we have the three minimal triplets (whose VEVs are shown in Table XV) that assure the spontaneous symmetry breaking (SSB) 331 → 321 → 31, and that provide the masses for the gauge fields. Further, an additional scalar in the adjoint representation is included. Such multiplet permits a mixing of the mirror fermions with the ordinary fermions of the SM in order to generate different ansatz for masses. The adjoint representation acquires the VEVs displayed in Table XV. Finally, a sextet representation can also be introduced as shown in Table XV; it acquires very small VEVs compared with the VEVs of the electroweak scale ν_χ, ν_ρ, and ν_η since they belong to triplet components of $SU(2)_L$ and would not break the relation for $\Delta\rho$. They also permit one to generate Majorana masses for neutrinos.

A. Mass matrix for quarks

The Yukawa Lagrangian for quarks has the form

$$\begin{aligned}\mathcal{L}_Y^q = {} & \sum_{\Phi}\sum_{\text{sing.}}\sum_{m,m'=1}^{2} h^{m\varphi}_{q_R}\overline{q_L^{(m)}}q_R\Phi + \frac{1}{2}\overline{q_L^{i(m)}}(q_L^{j(m')})^c[h_\Phi^{mm'}\varepsilon^{ijk}\Phi_k + h_S^{mm'}S^{ij}] + h^{3\Phi}_{q_R}\overline{q_L^{(3^*)}}q_R\Phi^* + h^{4\Phi}_{q_R}\overline{q_L^{(4^*)}}q_R\Phi^* \\ & + \frac{1}{2}\overline{q_{iL}^{(3^*)}}(q_{jL}^{(3^*)})^c[Y_\Phi^{33}\varepsilon_{ijk}\Phi^k + Y_S^{33}S_{ij}] + \frac{1}{2}\overline{q_{iL}^{(4^*)}}(q_{jL}^{(4^*)})^c[Y_\Phi^{44}\varepsilon_{ijk}\Phi^k + h_S^{44}S_{ij}] + \frac{1}{2}\overline{q_{iL}^{(3^*)}}(q_{jL}^{(4^*)})^c[Y_\Phi^{34}\varepsilon_{ijk}\Phi^k + Y_S^{34}S_{ij}] \\ & + \frac{1}{2}\overline{q_{iL}^{(4^*)}}(q_{jL}^{(3^*)})^c[Y_\Phi^{43}\varepsilon_{ijk}\Phi^k + Y_S^{43}S_{ij}] + \frac{1}{2}h_\phi^{n3}\overline{q_L^{i(n)}}(q_{jL}^{(3^*)})^c\phi^i_j + \frac{1}{2}h_\phi^{3n}\overline{q_{iL}^{(3^*)}}(q_L^{j(n)})^c\phi^j_i + \frac{1}{2}h_\phi^{n4}\overline{q_L^{i(n)}}(q_{jL}^{(4^*)})^c\phi^i_j \\ & + \frac{1}{2}h_\phi^{4n}\overline{q_{iL}^{(4^*)}}(q_L^{j(n)})^c\phi^j_i + \text{h.c.,}\end{aligned} \tag{7.1}$$

with Φ being any of the η, ρ, χ multiplets, while ϕ and S correspond to the scalar adjoint and the sextet representation of $SU(3)_L$, respectively. The third and fourth families are written explicitly, since the fourth one corresponds to a mirror fermion. The constants $h_{\Phi}^{mm'}$ and Y_{Φ}^{34} are antisymmetric. It should be noted that all possible terms with scalar triplets, adjoints, and sextets are involved. When we take the VEVs from Table XV, the mass matrices are obtained.

For the mixing among up-type quarks in the basis $(u_{3^*}, u_1, u_2, \tilde{u}, J_1, J_2, \tilde{J})$ we get

$$M^{\rm up} = \begin{pmatrix} \mathcal{M}_U & \mathcal{M}_{JU} \\ \mathcal{M}_{UJ} & \mathcal{M}_J \end{pmatrix}, \tag{7.2}$$

where

$$\mathcal{M}_U = \begin{pmatrix} \nu_\rho h_{u_3}^{3\rho} & \nu_\rho h_{u_1}^{3\rho} & \nu_\rho h_{u_2}^{3\rho} & h_\chi^{43}\nu_\chi \\ \nu_\eta h_{u_3}^{1\eta} & \nu_\eta h_{u_1}^{1\eta} & \nu_\eta h_{u_2}^{1\eta} & h_\phi^{14}\nu_\chi \\ \nu_\eta h_{u_3}^{2\eta} & \nu_\eta h_{u_1}^{2\eta} & \nu_\eta h_{u_2}^{2\eta} & h_\phi^{24}\nu_\chi \\ 0 & 0 & 0 & \nu_\eta h_{\tilde{u}}^{4\eta} \end{pmatrix},$$

$$\mathcal{M}_J = \begin{pmatrix} \nu_\chi h_{J_1}^{1\varkappa} & \nu_\chi h_{J_2}^{1\varkappa} & -2h_\phi^{14}\nu_\chi \\ \nu_\chi h_{J_1}^{2\varkappa} & \nu_\chi h_{J_2}^{2\varkappa} & -2h_\phi^{24}\nu_\chi \\ 0 & 0 & \nu_\chi h_{\tilde{J}}^{4\varkappa} \end{pmatrix},$$

$$\mathcal{M}_{UJ} = \begin{pmatrix} \nu_\chi h_{u_3}^{1\varkappa} & \nu_\chi h_{u_1}^{1\varkappa} & \nu_\chi h_{u_2}^{1\varkappa} & 0 \\ \nu_\chi h_{u_3}^{2\varkappa} & \nu_\chi h_{u_1}^{2\varkappa} & \nu_\chi h_{u_2}^{2\varkappa} & 0 \\ 0 & 0 & 0 & \nu_\eta h_{\tilde{J}}^{4\eta} \end{pmatrix},$$

$$\mathcal{M}_{JU} = \begin{pmatrix} \nu_\rho h_{J_1}^{3\rho} & \nu_\rho h_{J_2}^{3\rho} & h_\eta^{34}\nu_{\eta_1} \\ \nu_{\eta_1} h_{J_1}^{1\eta} & \nu_{\eta_1} h_{J_2}^{1\eta} & 0 \\ \nu_{\eta_1} h_{J_1}^{2\eta} & \nu_{\eta_1} h_{J_2}^{2\eta} & 0 \\ 0 & 0 & \nu_\chi h_{\tilde{u}}^{4\chi} \end{pmatrix}.$$

and (u_{3^*}, u_1, u_2) correspond to the three families of the SM, $\tilde{u}$ refers to the mirror fermion of either u_1 or u_2, and $J_1, J_2, \tilde{J}$ are the exotic quarks with 2/3 electromagnetic charge.

For down-type quarks in the basis $(d_{3^*}, d_1, d_2, \tilde{d}, J_{3^*})$, the mass matrix yields

$$M^{\rm down} = \begin{pmatrix} \nu_\eta h_{d_3}^{3\eta} & \nu_\eta h_{d_1}^{3\eta} & \nu_\eta h_{d_2}^{3\eta} & Y_\chi^{34}\nu_\chi & \nu_\eta h_{J_3}^{3\eta} \\ \nu_\rho h_{d_3}^{1\rho} & \nu_\rho h_{d_1}^{1\rho} & \nu_\rho h_{d_2}^{1\rho} & h_\phi^{14}\nu_\chi & \nu_\rho h_{J_3}^{1\rho} \\ \nu_\rho h_{d_3}^{2\rho} & \nu_\rho h_{d_1}^{2\rho} & \nu_\rho h_{d_2}^{2\rho} & h_\phi^{24}\nu_\chi & \nu_\rho h_{J_3}^{2\rho} \\ 0 & 0 & 0 & \nu_\rho Y_{\tilde{d}}^{4\rho} & 0 \\ \nu_\chi h_{d_3}^{3\varkappa} & \nu_\chi h_{d_1}^{3\varkappa} & \nu_\chi h_{d_2}^{3\varkappa} & Y_\eta^{43}\nu_\eta & \nu_\chi h_{J_3}^{3\varkappa} \end{pmatrix}. \tag{7.3}$$

(d_{3^*}, d_1, d_2) are associated with the three SM families, $\tilde{d}$ is a down-type mirror quark of either d_1 or d_2, and J_{3^*} is an exotic down-type quark. When the adjoint representation of the scalar fields is not taken into account, the mixing between $q^{(m)}$ and the quark mirrors $q^{(4^*)}$ does not appear. Such mixing is important to change the right-handed coupling of the b quark with the Z_μ gauge field, and to look for a possible solution for the deviation of the asymmetries A_b and $A_{\rm FB}^b$ of the SM with respect to the experimental data. If the mixing with the mirror quarks were withdrawn and the exotic particles were decoupled, the mirror quarks would acquire masses of the order of the electroweak scale $\nu_\rho h_{\tilde{d}}^{4\rho}$, $\nu_\eta h_{\tilde{u}}^{4\rho}$ for the up and down sectors, respectively.

B. Mass matrix for leptons

The Yukawa Lagrangian for leptons keeps the general form shown in Eq. (7.1) for the quarks. However, Majorana terms could arise because of the existence of neutral fields. By taking the whole spectrum including right-handed neutrino singlets, Dirac terms are obtained for the charged sector while Dirac and Majorana terms appear in the neutral sector.

By including all the possible structures of VEVs, the charged sector in the basis $(e_{3^*}, e_1, e_2, \tilde{e}, E_{3^*}^-)$ has the following form:

$$M^{\ell\pm} = \begin{pmatrix} \nu_\eta h_{e_3}^{3\eta} & \nu_\eta h_{e_1}^{3\eta} & \nu_\eta h_{e_2}^{3\eta} & h_\chi^{34}\nu_\chi & \nu_\eta h_{J_3}^{3\eta} \\ \nu_\rho h_{e_3}^{1\rho} & \nu_\rho h_{e_1}^{1\rho} & \nu_\rho h_{e_2}^{1\rho} & h_\phi^{14}\nu_\chi & \nu_\rho h_{J_3}^{1\rho} \\ \nu_\rho h_{e_3}^{2\rho} & \nu_\rho h_{e_1}^{2\rho} & \nu_\rho h_{e_2}^{2\rho} & h_\phi^{24}\nu_\chi & \nu_\rho h_{J_3}^{2\rho} \\ 0 & 0 & 0 & \nu_\rho h_{\tilde{e}}^{4\rho} & 0 \\ \nu_\chi h_{e_3}^{3\varkappa} & \nu_\chi h_{e_1}^{3\varkappa} & \nu_\chi h_{e_2}^{3\varkappa} & h_\eta^{43}\nu_{\eta_1} & \nu_\chi h_{E_3}^{3\varkappa} \end{pmatrix};$$

the first three components e_i correspond to the ordinary leptons of the SM, $\tilde{e}$ is a mirror lepton of e_1 or e_2, and E_{3^*} is an exotic lepton. Like in the case of the quark sector, direct mixings are obtained between all the fields $\ell^{(n)}$, ℓ^{3^*} and the mirrors ℓ^{4^*} by means of the scalars χ, ρ, η and the adjoint ϕ. The mass matrix of charged leptons is similar to the mass matrix of the down-type quarks.

For the neutral lepton sector, we take the following basis of fields:

$$\begin{aligned} \psi_L^0 &= (\nu_{3L}, \nu_{1L}, \nu_{2L}, (\tilde{\nu}_R)^c, N_{1L}^0, N_{2L}^0, (\tilde{N}_R^0)^c)^T, \\ \psi_R^0 &= (\nu_{3R}, \nu_{1R}, \nu_{2R}, (\tilde{\nu}_L)^c)^T, \end{aligned} \tag{7.4}$$

where ν_{iL} are the SM fields, and ν_{iR} are sterile neutrinos and the right-handed components of SM neutrinos. With these components the Dirac mass matrix is constructed like the up quarks' mass matrix; $(\tilde{\nu}_{L,R})^c$ are mirror fermions, and N_{iL}^0 are exotic neutral fermions. The mass terms are written as

$$\mathfrak{L}_Y^0 = (\overline{\psi_L^0}; \overline{(\psi_R^0)^c})\begin{pmatrix} M_L & m_D \\ m_D^T & M_R \end{pmatrix}\begin{pmatrix} (\psi_L^0)^c \\ \psi_R^0 \end{pmatrix} + \text{h.c.}, \tag{7.5}$$

where very massive Majorana terms M_R have been introduced between the singlets ψ_R^0, corresponding to sterile neutrinos with right-handed chirality. We shall suppose that in this basis the mass matrix M_R is diagonal. Such terms can be introduced without a SSB because they are $SU(3)_L \otimes U(1)_X$ invariant. Besides, they correspond to heavy Majorana mass terms for the sterile heavy neutrinos. The Majorana contribution M_L takes the form

$$M_L = \frac{1}{2}\begin{pmatrix} \mathcal{M}_\nu & \mathcal{M}_{N\nu} \\ \mathcal{M}_{\nu N} & \mathcal{M}_N \end{pmatrix}, \tag{7.6}$$

where

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & 0 & 0 & -h_\chi^{34}\nu_\chi \\ 0 & Vh_S^{11} & Vh_S^{12} & h_\phi^{14}\nu_\chi \\ 0 & Vh_S^{21} & Vh_S^{22} & h_\phi^{24}\nu_\chi \\ h_\chi^{43}\nu_\chi & h_\phi^{14}\nu_\chi & h_\phi^{24}\nu_\chi & Vh_S^{44} \end{pmatrix}.$$

The entries of the upper 3×3 submatrix correspond to Majorana masses for the ordinary neutrinos of the three SM families, which are generated with the six-dimensional representation of the scalar sector. If such VEVs were taken as null, or if we chose discrete symmetries to forbid these terms, they can be generated through the seesaw mechanism of the form $m_D^\dagger M_R^{-1} m_D$. The other mass matrices are given by

$$\mathcal{M}_N = \begin{pmatrix} Vh_S^{11} & Vh_S^{12} & -2h_\phi^{14}\nu_\chi \\ Vh_S^{21} & Vh_S^{22} & -2h_\phi^{24}\nu_\chi \\ -2h_\phi^{14}\nu_\chi & -2h_\phi^{24}\nu_\chi & Vh_S^{44} \end{pmatrix},$$

$$\mathcal{M}_{\nu N} = \begin{pmatrix} 0 & h_\rho^{11}\nu_\rho & h_\rho^{12}\nu_\rho & 0 \\ 0 & h_\rho^{21}\nu_\rho & h_\rho^{22}\nu_\rho & 0 \\ -\nu_{\eta_1}h_\eta^{43} & 0 & 0 & h_\rho^{44}\nu_\rho \end{pmatrix},$$

$$\mathcal{M}_{N\nu} = \begin{pmatrix} 0 & 0 & h_\eta^{34}\nu_{\eta_1} \\ -h_\rho^{11}\nu_\rho & -h_\rho^{12}\nu_\rho & 0 \\ -h_\rho^{21}\nu_\rho & -h_\rho^{22}\nu_\rho & 0 \\ 0 & 0 & -h_\rho^{44}\nu_\rho \end{pmatrix},$$

where we have taken into account the VEVs of the scalar triplets χ, ρ, η, the adjoint ϕ, and the sextet S. The adjoint VEVs ensure the direct mixings between $\ell^{(n)}$ and the mirrors $\ell^{(4^*)}$. The Dirac terms of (7.5) are

$$m_D = \frac{1}{2}\begin{pmatrix} \nu_\rho h_{\nu_3}^{3\rho} & \nu_\rho h_{\nu_1}^{3\rho} & \nu_\rho h_{\nu_2}^{3\rho} & \nu_\rho h_{\tilde\nu}^{3\rho} \\ \nu_\eta h_{\nu_3}^{1\eta} & \nu_\eta h_{\nu_1}^{1\eta} & \nu_\eta h_{\nu_2}^{1\eta} & \nu_\eta h_{\tilde\nu}^{1\eta} \\ \nu_\eta h_{\nu_3}^{2\eta} & \nu_\eta h_{\nu_1}^{2\eta} & \nu_\eta h_{\nu_2}^{2\eta} & \nu_\eta h_{\tilde\nu}^{2\eta} \\ \nu_\eta h_{\nu_3}^{4\eta} & \nu_\eta h_{\nu_1}^{4\eta} & \nu_\eta h_{\nu_2}^{4\eta} & \nu_\eta h_{\tilde\nu}^{4\eta} \\ \nu_\chi h_{\nu_3}^{1\chi} & \nu_\chi h_{\nu_1}^{1\chi} & \nu_\chi h_{\nu_2}^{1\chi} & \nu_\chi h_{\tilde\nu}^{1\chi} \\ \nu_\chi h_{\nu_3}^{2\chi} & \nu_\chi h_{\nu_1}^{2\chi} & \nu_\chi h_{\nu_2}^{2\chi} & \nu_\chi h_{\tilde\nu}^{2\chi} \\ \nu_\chi h_{\nu_3}^{4\chi} & \nu_\chi h_{\nu_1}^{4\chi} & \nu_\chi h_{\nu_2}^{4\chi} & \nu_\chi h_{\tilde\nu}^{4\chi} \end{pmatrix}. \tag{7.7}$$

When the quarks and leptons spectra are compared (see Table XIV), it is observed that they are equivalent in the sense that both introduce the same quantity of particles in the form of left-handed triplets and right-handed singlets (singlet components of neutrinos are taken). Nevertheless, the Yukawa Lagrangian (and hence the mass matrices) of quarks and leptons are not equivalent because the quarks have different values of the X quantum number with respect to the leptons; this fact puts different restrictions on the terms of both Yukawa Lagrangians.

In the limit ν_ρ, $\nu_\eta \ll \nu_\chi$ and $V = 0$, the physics beyond the SM could be decoupled at low energies leaving an effective theory at low energies similar to a two Higgs doublet model (2HDM) with the fermionic fields of the SM and the right-handed neutrinos that we introduced in the particle content ν_{1R}, ν_{2R}, ν_{3R} to generate Dirac type masses and could be able to relate the neutrino sector with the up quark sector. It allows one to give a large mass to the up quark sector and the mass pattern for the neutrinos. In this limit, the mass matrices that are generated would be similar to the ansatz proposed in Ref. [29]. Considering the upper 3×3 submatrix of m_D in Eq. (7.7) and imposing discrete symmetries, it can be written in the form

$$m_D = \frac{1}{2}\begin{pmatrix} \nu_\rho h_{\nu_3}^{3\rho} & \nu_\rho h_{\nu_1}^{3\rho} & 0 \\ \nu_\eta h_{\nu_3}^{1\eta} & \nu_\eta h_{\nu_1}^{1\eta} & \nu_\eta h_{\nu_2}^{1\eta} \\ 0 & \nu_\eta h_{\nu_1}^{2\eta} & \nu_\eta h_{\nu_2}^{2\eta} \end{pmatrix}. \tag{7.8}$$

Considering the same Yukawa couplings within each generation (i.e. the same $h_{\nu_m}^{n\Phi}$ for each pair of $n\Phi$), we can write the matrix (7.8) as

$$m_D = \frac{\nu_\eta}{\sqrt{2}}\begin{pmatrix} ct_\beta & ct_\beta & 0 \\ \delta b & b & b \\ 0 & a & a \end{pmatrix}, \tag{7.9}$$

where $t_\beta = \nu_\rho/\nu_\eta$ is the scalar mixing angle given by (A3), and δ is a real parameter that is fitted in agreement with the neutrino oscillation data. If the third generation is ν_3, the second is ν_1 and the first is ν_2, and taking $M_R = M_{\text{diag}}(\epsilon_{M3}, \epsilon_{M2}, \epsilon_{M1})$, we obtain the same mass ansatz and mixing as in Ref. [29]. Thus, from the seesaw mechanism we get

$$m_\nu = -m_D^\dagger M_R^{-1} m_D = m_\nu^0\begin{pmatrix} \delta^2\bar\epsilon + \omega & \delta\bar\epsilon + \omega & \delta\bar\epsilon \\ \delta\bar\epsilon + \omega & \epsilon + \omega & \epsilon \\ \delta\bar\epsilon & \epsilon & \epsilon \end{pmatrix}, \tag{7.10}$$

with $m_\nu^0 = \nu_\eta^2/2M$, $\epsilon = (a^2/\epsilon_{M_1}) + (b^2/\epsilon_{M_2})$, $\bar\epsilon = b^2/\epsilon_{M_2}$, $\omega = c^2 t_\beta^2/\epsilon_{M_3}$, $\tan 2\theta_{23} \sim 2r\omega/[\epsilon(\delta^2 - r)]$, $\tan 2\theta_{12} \sim \frac{2g}{f}$, $\theta_{13} \sim [\epsilon(\delta + r)]/(2^{3/2} r\omega)$, $m_1 \sim \epsilon m_\nu^0\{1 - g\sin 2\theta_{12} + f\sin^2\theta_{12}\}$, $m_2 \sim \epsilon m_\nu^0\{1 + g\sin 2\theta_{12} + f\cos^2\theta_{12}\}$, $m_3 \sim 2\omega m_\nu^0$, $r = \frac{\bar\epsilon}{\epsilon}$, $g = \frac{|r-\delta|}{\sqrt{2}r}$, and $f = (\delta^2 - 2\delta - r)/2r$. As it is discussed in Ref. [29], if $m_3 \sim \sqrt{\Delta m_{\text{atm}}^2}$, $m_2 \sim \sqrt{\Delta m_{\text{sol}}^2}$, and taking $t_\beta = \nu_\rho/\nu_\eta \gg O(1)$, it is possible to obtain a natural fit for the observed neutrino hierarchical masses and mixing angles. This result shows the good behavior of the model.

C. The mixing between the bottom quark and its mirror

In order to look for a solution to the deviation from the b asymmetries, let us assume that the exotic quarks with charge 1/3 acquire their mass in the first SSB and that they are basically decoupled at electroweak energies. On the other hand, let us suppose that the mass matrix of the three generations of down quarks is approximately diagonal. In this way the mixing between the down quark of the third generation (b quark) and its corresponding mirror can be written as [see Eq. (7.3)]

$$\begin{pmatrix} \bar{d}_2 & \bar{\tilde{d}} \end{pmatrix}_L M \begin{pmatrix} d_2 \\ \tilde{d} \end{pmatrix}_R, \qquad M \equiv \begin{pmatrix} h_{d_2}^{2\rho} \nu_\rho & h_\phi^{24} \nu_\chi \\ 0 & Y_{\tilde{d}}^{4\rho} \nu_\rho \end{pmatrix}. \tag{7.11}$$

The eigenvalues of this mass matrix M that correspond to the masses of the b-quark and the mirror fermion are $h_{d_2}^{2\rho} \nu_\rho$ and $Y_{\tilde{d}}^{4\rho} \nu_\rho$, respectively. To diagonalize the mass matrix the following rotation is proposed:

$$\begin{pmatrix} b \\ \tilde{b} \end{pmatrix}_{L(R)} = V_{L(R)}^\dagger \begin{pmatrix} d_2 \\ \tilde{d} \end{pmatrix}_{L(R)} \tag{7.12}$$

where b and $\tilde{b}$ are the mass eigenstates for the bottom quark and its mirror fermion, respectively. V_L and V_R are 2×2 matrices of rotation obtained from the matrices $MM^\dagger$ and $M^\dagger M$, respectively [see Eq. (7.11)]. We shall assume that the rotation angle of the left-handed quarks (θ_L) is small enough, since it would be tightly restricted by the electroweak processes. For the right-handed angle we get

$$\tan 2\theta_R = \frac{2 h_\phi^{24} \nu_\chi Y_{\tilde{d}}^{4\rho} \nu_\rho}{(Y_{\tilde{d}}^{4\rho} \nu_\rho)^2 - (h_{d_2}^{2\rho} \nu_\rho)^2 - (h_\phi^{24} \nu_\chi)^2} \approx \frac{2 M_{Z'} M_F}{M_F^2 - M_{Z'}^2}. \tag{7.13}$$

In the last line the b-quark mass was neglected and the VEV ν_χ was approximated to $M_{Z'}$.

When writing the neutral currents for the d_2 and its mirror $\tilde{d}$ we get

$$\begin{aligned} \mathcal{L}_b^{\rm NC} &= \frac{g}{2C_W} \overline{d_2} \gamma_\mu \left[\left(1 - \frac{2}{3} S_W^2 \right) P_L - \frac{2}{3} S_W^2 P_R \right] Z^\mu d_2 \\ &\quad + \frac{g}{2C_W} \bar{\tilde{d}} \gamma_\mu \left[\left(1 - \frac{2}{3} S_W^2 \right) P_R - \frac{2}{3} S_W^2 P_L \right] Z^\mu \tilde{d}. \end{aligned} \tag{7.14}$$

After making the rotations for left- and right-handed components of d_2, $\tilde{d}$ quarks, and taking $\theta_L = 0$, we can write the right-handed current of the quark bottom mass eigenvalues as

$$\frac{g}{2C_W} \bar{b} \gamma_\mu \left(\sin^2\theta_R - \frac{2}{3} S_W^2 \right) P_R Z^\mu b \tag{7.15}$$

and the electroweak right-handed coupling is modified by a factor

$$\delta g_R = \sin^2\theta_R. \tag{7.16}$$

By making a combined fit for the CERN LEP and Slac Linear Collider measurements in terms of the left and right currents of the b quark, and subtracting the central value of the SM it is obtained that [30]

$$\delta g_R = 0.02. \tag{7.17}$$

This means that in order to solve the problem of the deviation of the anomaly A_b, it is necessary for the right-handed mixing angle to be of the order of $\sin\theta_R \approx 0.1$. Replacing this value in Eq. (7.13) we find that $M_{Z'} \approx 10 M_F$. This is a reasonable value if the mirror fermions lie at the electroweak scale and the first breaking of the 331 model is of the order of the TeV scale.

VIII. CONCLUSIONS

We have studied the fermionic spectrum of the 331 models with β arbitrary by the criterion of cancellation of anomalies. In order to minimize the exotic spectrum we assume that only one lepton and only one quark $SU(3)_L$ multiplet is associated with each generation, and that there is no more than one right-handed singlet associated with each left-handed fermion field. By considering models with an arbitrary number of lepton and quark generations we find the constraints that cancellation of anomalies provides for the possible fermionic structures. After assuming that the fermionic $SU(3)_c$ representations are vectorlike, and that the SM fermion representations must be embedded in the triplet 331 representations, we obtain five conditions from the vanishing of anomalies. The first condition becomes trivial when the SM is embedded in the 331 model. Two of them restrict the structure of the left-handed fermionic multiplets, while the other two restrict the structure of right-handed charged leptonic singlets. The right-handed neutral leptonic singlets are left unconstrained by the equations of anomalies. Under the assumptions made above, the number of left-handed quark multiplets must be 3 times the number of left-handed leptonic multiplets because of the color factor. Besides, models with only one lepton multiplet are forbidden. In addition, the Higgs and vector spectra, as well as the Yang-Mills Lagrangian, are calculated for β arbitrary.

The interest for studying the case of β arbitrary is twofold. On one hand, it permits a general phenomenological analysis that could lead to the cases studied in the literature. On the other hand, it also permits the study of other scenarios that could be the source for solving some of the problems of the SM.

In particular, we studied models with three and four lepton multiplets ($N = 3, 4$). Models with $N = 3$ are allowed even if no right-handed charged leptonic singlets are introduced (models of Pleitez and Frampton i.e. $\beta = \pm\sqrt{3}$) as it is indicated in Table VIII. However, for arbitrary values of β, the three-family versions require the introduction of right-handed charged leptonic singlets in order to cancel anomalies, and only two types of solutions are possible (see Table VIII).

The version with $N = 4$ and $\beta = -1/\sqrt{3}$ is a vectorlike model consisting of three triplets containing the SM fermions plus one triplet containing mirror fermions of one of the SM families. We choose the mirror fermions to be associated with the third family of the SM. This $N = 4$ model is different from similar 331 versions considered in the literature, and possesses strong phenomenological motivations: the right-handed coupling of the b quark with the Z_μ gauge boson could be modified and may in turn explain the deviation of the b asymmetries with respect to the SM prediction. In order to solve the A_b puzzle, the right-handed mixing angle should be of the order of $\sin\theta_R \approx 0.1$, which in turn leads to $M_{Z'} \approx 10 M_F$ with $M_{Z'}$ and M_F denoting the masses for the exotic neutral gauge boson and the mirror fermion, respectively; this relation is reasonable if M_F lies in the electroweak scale and the breaking of the 331 model lies at the TeV scale. On the other hand, vectorlike models are necessary to explain the family hierarchy. From the phenomenological point of view, the model provides the possibility of generating ansatz for masses at low energies in the quark and lepton sector. It is worth saying that the physics beyond the SM could be decoupled at low energies leaving an effective theory of two Higgs doublets with right-handed neutrinos, and that the mass matrices generated are similar to the ansatz proposed by Ref. [29]. From such ansatz, a natural fit for the neutrino hierarchical masses and mixing angles can be obtained.

Finally, this general approach opens a window to analyze other possible 331 versions. For instance, we can analyze the model with $N = 4$ but with the mirror fermion associated with another SM family. Moreover, several models with $N \geq 4$, with more mirror fermions, could be studied from phenomenological grounds (see Table III). In particular, we observe from Table III that $N = 6$ contains models that are vectorlike with respect to $SU(3)_L$ in the quark and lepton sectors.

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APPENDIX A: SCALAR MASSES WITH β ARBITRARY

1. Imaginary sector

The mass matrix is built up in the basis $\zeta_\chi, \zeta_\rho, \zeta_\eta$:

$$M^2_{\zeta\zeta} = -2f \begin{bmatrix} \frac{\nu_\eta \nu_\rho}{\nu_\chi} & \nu_\eta & \nu_\rho \\ \nu_\eta & \frac{\nu_\eta \nu_\chi}{\nu_\rho} & \nu_\chi \\ \nu_\rho & \nu_\chi & \frac{\nu_\chi \nu_\rho}{\nu_\eta} \end{bmatrix}. \tag{A1}$$

The eigenvalues and eigenvectors are given by

$$\begin{aligned} &P_1 = P_2 = 0; \quad P_3 = -2f\nu_\chi\left(\frac{\nu_\eta}{\nu_\rho} + \frac{\nu_\rho}{\nu_\eta} + \frac{\nu_\eta \nu_\rho}{\nu_\chi^2}\right), \\ &\phi_2^0 = N^0_{\phi_2}(-\nu_\chi \zeta_\chi + \nu_\eta \zeta_\eta) \approx -\zeta_\chi, \\ &\phi_3^0 = N^0_{\phi_3}[-\nu_\chi \nu_\eta^2 \zeta_\chi + \nu_\rho(\nu_\chi^2 + \nu_\eta^2)\zeta_\rho - \nu_\chi^2 \nu_\eta \zeta_\eta] \\ &\quad \approx S_\beta \zeta_\rho - C_\beta \zeta_\eta, \\ &h_1^0 = C_\beta \zeta_\rho + S_\beta \zeta_\eta, \end{aligned} \tag{A2}$$

obtained by using the approximations in Eqs. (3.3) and (3.4). The scalars ϕ_2^0 and ϕ_3^0 are the would-be Goldstone bosons corresponding to the gauge fields Z'_μ and Z_μ, respectively. N denotes normalization factors. The mixing angle is defined by

$$t_\beta \equiv \tan\beta = \frac{\nu_\rho}{\nu_\eta}. \tag{A3}$$

2. Real sector

The basis is $\xi_\chi, \xi_\rho, \xi_\eta$:

$$M^2_{\xi\xi} = \begin{bmatrix} 8\lambda_1 \nu_\chi^2 - 2f\frac{\nu_\eta \nu_\rho}{\nu_\chi} & 4\lambda_4 \nu_\chi \nu_\rho + 2f\nu_\eta & 4\lambda_5 \nu_\chi \nu_\eta + 2f\nu_\rho \\ 4\lambda_4 \nu_\chi \nu_\rho + 2f\nu_\eta & 8\lambda_2 \nu_\rho^2 - 2f\frac{\nu_\eta \nu_\chi}{\nu_\rho} & 4\lambda_6 \nu_\eta \nu_\rho + 2f\nu_\chi \\ 4\lambda_5 \nu_\chi \nu_\eta + 2f\nu_\rho & 4\lambda_6 \nu_\eta \nu_\rho + 2f\nu_\chi & 8\lambda_3 \nu_\eta^2 - 2f\frac{\nu_\chi \nu_\rho}{\nu_\eta} \end{bmatrix}. \tag{A4}$$

Keeping only quadratic terms in ν_χ in the matrix (A4), we obtain

$$M^2_{\xi\xi} \simeq \begin{bmatrix} 8\lambda_1 \nu_\chi^2 & 0 & 0 \\ 0 & -2f\frac{\nu_\eta \nu_\chi}{\nu_\rho} & 2f\nu_\chi \\ 0 & 2f\nu_\chi & -2f\frac{\nu_\chi \nu_\rho}{\nu_\eta} \end{bmatrix}, \tag{A5}$$

where we get the following decoupled matrices:

$$M^2_{\xi_\rho \xi_\eta} \simeq \begin{bmatrix} -2f\frac{\nu_\eta \nu_\chi}{\nu_\rho} & 2f\nu_\chi \\ 2f\nu_\chi & -2f\frac{\nu_\chi \nu_\rho}{\nu_\eta} \end{bmatrix}, \qquad M^2_{\xi_\chi \xi_\chi} \to 8\lambda_1 \nu_\chi^2. \tag{A6}$$

The submatrix $M^2_{\xi_\rho \xi_\eta}$ written in Eq. (A6) has the following eigenvalues:

$$P_2 = 0; \qquad P_3 = -2f\nu_\chi\left(\frac{\nu_\eta}{\nu_\rho} + \frac{\nu_\rho}{\nu_\eta}\right).$$

The first eigenvalue is zero because of the approximation made in (A4). If the approximation is not considered, the matrix in (A6) takes the form

$$M^2_{\xi_{\rho^2}\xi_\eta} = \begin{bmatrix} 8\lambda_2\nu_\rho^2 - 2f\frac{\nu_\eta\nu_\chi}{\nu_\rho} & 4\lambda_6\nu_\eta\nu_\rho + 2f\nu_\chi \\ 4\lambda_6\nu_\eta\nu_\rho + 2f\nu_\chi & 8\lambda_3\nu_\eta^2 - 2f\frac{\nu_\chi\nu_\rho}{\nu_\eta} \end{bmatrix}, \tag{A7}$$

and the corresponding eigenvalues are different; they are

$$P_2 = \frac{8(\lambda_2\nu_\rho^4 + \lambda_3\nu_\eta^4 + \lambda_6\nu_\rho^2\nu_\eta^2)}{\nu_\eta^2 + \nu_\rho^2}; \qquad P_3 = -2f\nu_\chi\left(\frac{\nu_\rho}{\nu_\eta} + \frac{\nu_\eta}{\nu_\rho}\right), \tag{A8}$$

and the eigenvectors read

$$h_5^0 = \xi_\chi; \qquad h_3^0 = S_\beta\xi_\rho + C_\beta\xi_\eta; \qquad h_4^0 = -C_\beta\xi_\rho + S_\beta\xi_\eta. \tag{A9}$$

3. Charged sector

The basis is $\chi_1^{\pm Q_1}$, $\eta_3^{\pm Q_1}$:

$$M^2_{\phi^{\pm Q_1}} = \begin{bmatrix} \lambda_7\nu_\eta^2 - f\frac{\nu_\eta\nu_\rho}{\nu_\chi} & \lambda_7\nu_\chi\nu_\eta - f\nu_\rho \\ \lambda_7\nu_\chi\nu_\eta - f\nu_\rho & \lambda_7\nu_\chi^2 - f\frac{\nu_\chi\nu_\rho}{\nu_\eta} \end{bmatrix}. \tag{A10}$$

The mass matrix in the basis $\chi_2^{\pm Q_2}$, $\rho_3^{\pm Q_2}$ is

$$M^2_{\phi^{\pm Q_2}} = \begin{bmatrix} \lambda_8\nu_\rho^2 - f\frac{\nu_\eta\nu_\rho}{\nu_\chi} & \lambda_8\nu_\chi\nu_\rho - f\nu_\eta \\ \lambda_8\nu_\chi\nu_\rho - f\nu_\eta & \lambda_8\nu_\chi^2 - f\frac{\nu_\chi\nu_\eta}{\nu_\rho} \end{bmatrix}. \tag{A11}$$

The mass matrix in the basis $\rho_1^\pm$, $\eta_2^\pm$ reads

$$M^2_{\phi^\pm} = \begin{bmatrix} \lambda_9\nu_\eta^2 - f\frac{\nu_\chi\nu_\eta}{\nu_\rho} & \lambda_9\nu_\eta\nu_\rho - f\nu_\chi \\ \lambda_9\nu_\eta\nu_\rho - f\nu_\chi & \lambda_9\nu_\rho^2 - f\frac{\nu_\rho\nu_\chi}{\nu_\eta} \end{bmatrix}. \tag{A12}$$

It is found that the matrices are singular; it is that

$$\det(M^2_{\phi 1\pm}) = \det(M^2_{\phi 2\pm}) = \det(M^2_{\phi 3\pm}) = 0, \tag{A13}$$

giving a total of six would-be Goldstone bosons. For the matrix $M^2_{\phi^{\pm Q_1}}$ of Eq. (A10) the corresponding eigenvalues and eigenvectors are found to be

$$\begin{aligned} &P_1 = 0, \qquad P_2 = \lambda_7(\nu_\eta^2 + \nu_\chi^2) - f\nu_\rho\left(\frac{\nu_\chi}{\nu_\eta} + \frac{\nu_\eta}{\nu_\chi}\right), \\ &\phi_2^{\pm Q_1} = N_{\phi_2}^{Q_1}(-\nu_\chi\chi_1^{\pm Q_1} + \nu_\eta\eta_3^{\pm Q_1}) \approx -\chi_1^{\pm Q_1}, \\ &h_1^{\pm Q_1} = N_{h_1}^{\pm Q_1}(\nu_\eta\chi_1^{\pm Q_1} + \nu_\chi\eta_3^{\pm Q_1}) \approx \eta_3^{\pm Q_1}, \end{aligned} \tag{A14}$$

where the approximations of Eqs. (3.3) and (3.4) have been taken into account, getting two would-be Goldstone bosons $\phi_2^{\pm Q_1}$ associated with the gauge fields $K_\mu^{\pm Q_1}$ and two massive Higgs bosons $h_1^{\pm Q_1}$.

For $M^2_{\phi^{\pm Q_2}}$, from Eq. (A11), we find

$$\begin{aligned} &P_3 = 0, \qquad P_4 = \lambda_8(\nu_\rho^2 + \nu_\chi^2) - f\nu_\eta\left(\frac{\nu_\chi}{\nu_\rho} + \frac{\nu_\rho}{\nu_\chi}\right), \\ &\phi_3^{\pm Q_2} = N_{\phi_3}^{Q_2}(-\nu_\chi\chi_2^{\pm Q_2} + \nu_\rho\rho_3^{\pm Q_2}) \approx -\chi_2^{\pm Q_2}, \\ &h_3^{\pm Q_2} = N_{h_3}^{Q_2}(\nu_\rho\chi_2^{\pm Q_2} + \nu_\chi\rho_3^{\pm Q_2}) \approx \rho_3^{\pm Q_2}. \end{aligned} \tag{A15}$$

Again we have used the approximations in Eqs. (3.3) and (3.4), obtaining two would-be Goldstone bosons $\phi_3^{\pm Q_2}$ associated with the gauge fields $K_\mu^{\pm Q_2}$ and two massive Higgs bosons $h_3^{\pm Q_2}$.

Finally, for $M^2_{\phi^\pm}$ and from Eq. (A12) we have

$$\begin{aligned} &P_5 = 0, \qquad P_6 = \lambda_9(\nu_\rho^2 + \nu_\eta^2) - f\nu_\chi\left(\frac{\nu_\eta}{\nu_\rho} + \frac{\nu_\rho}{\nu_\eta}\right), \\ &\phi_1^\pm = S_\beta\rho_1^\pm - C_\beta\eta_2^\pm, \qquad h_2^\pm = C_\beta\rho_1^\pm + S_\beta\eta_2^\pm, \end{aligned} \tag{A16}$$

where $\phi_1^\pm$ give mass to $W_\mu^\pm$.

APPENDIX B: THE MASS MATRIX FOR THE NEUTRAL GAUGE SECTOR

The basis for the mass matrix for the neutral gauge sector is W^3, W^8, B:

$$\begin{bmatrix} \frac{g^2}{2}(\nu_\eta^2 + \nu_\rho^2) & \frac{g^2}{2\sqrt{3}}(\nu_\eta^2 - \nu_\rho^2) & \frac{gg'}{2}\left[-\nu_\eta^2\left(1 + \frac{\beta}{\sqrt{3}}\right) - \nu_\rho^2\left(1 - \frac{\beta}{\sqrt{3}}\right)\right] \\ \frac{g^2}{2\sqrt{3}}(\nu_\eta^2 - \nu_\rho^2) & \frac{g^2}{6}(\nu_\eta^2 + \nu_\rho^2 + 4\nu_\chi^2) & \frac{gg'}{6}[-\nu_\eta^2(\sqrt{3} + \beta) + \nu_\rho^2(\sqrt{3} - \beta) - 4\nu_\chi^2\beta] \\ \frac{gg'}{2}\left[-\nu_\eta^2\left(1 + \frac{\beta}{\sqrt{3}}\right) - \nu_\rho^2\left(1 - \frac{\beta}{\sqrt{3}}\right)\right] & \frac{gg'}{6}[-\nu_\eta^2(\sqrt{3} + \beta) + \nu_\rho^2(\sqrt{3} - \beta) - 4\nu_\chi^2\beta] & \frac{g'^2}{6}[\nu_\eta^2(\sqrt{3} + \beta)^2 + \nu_\rho^2(\sqrt{3} - \beta)^2 + 4\nu_\chi^2\beta^2] \end{bmatrix}. \tag{B1}$$

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