

Effective field theory Lagrangians for baryons with two and three heavy quarks

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By analogy with potential nonrelativistic QCD, we construct effective field theories suitable to describe the heavy-quark sector of baryons made of two and three heavy quarks. A long-standing discrepancy between the hyperfine splitting of doubly heavy baryons obtained in the heavy-quark effective theory and potential models is solved. The one-loop matching of the 4-quark operators of dimension 6 is provided.

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I. INTRODUCTION

The SELEX Collaboration at Fermilab recently reported evidence of five resonances that may possibly be identified with doubly charmed baryon states [1]. Tentatively the states have been interpreted as ccd^+ (3443), ccd^+ (3520), ccu^{++} (3460), ccu^{++} (3541) and ccu^{++} (3780). Subsequently the ccd^+ (3520) state has been confirmed in two different decay modes ($\Xi_{cc}^+ \rightarrow \Lambda_c^+ K^- \pi^+$; $\Xi_{cc}^+ \rightarrow p D^+ K^-$) at a mass of 3518.7 ± 1.7 MeV with an average lifetime less than 33 fs. Although these findings need to be confirmed by other experiments and larger statistical samples, they have triggered a renewed theoretical interest in doubly heavy-baryon systems.

Doubly heavy baryons have been studied with several methods, mostly nonrelativistic potential models (for some reviews see [2,3]), but also relativistic models [4], sum rules [5,6] and in a chiral Lagrangian framework [7]. Masses of the lowest lying resonances have been obtained from lattice calculations [8–12]. Doubly heavy baryons are also suited to be studied in a QCD effective field theory (EFT) framework. Indeed, they are characterized by at least two widely separated scales: the large heavy-quark masses, m , and the low momentum transfer between the heavy and the light quarks, which is of order Λ_{QCD} . If one assumes that the typical momentum transfer between the two heavy quarks is larger than Λ_{QCD} , then a QQq baryon is very similar to a bound state of a heavy antiquark and a light quark. This has first been noted in [13], where at leading order in Λ_{QCD}/m the hyperfine splitting of the doubly heavy-baryon ground state has been related to the ground-state hyperfine splitting of the heavy-light meson. In [14] nonleptonic and semileptonic decays of doubly heavy baryons have been examined in the context of $SU(3)$ flavor symmetry. After this original work no further step has been made in the direction of providing a system-

atic description of doubly heavy baryons in an EFT framework that fully combines the dynamics of the two heavy quarks with that one of the light one. Following some suggestions in [15], with this work we attempt to make that further step. In particular, we identify the degrees of freedom and write the low-energy EFT Lagrangian that describes doubly heavy-baryon systems in the heavy-quark sector, once the heavy-quark momentum-transfer scale has been integrated out. The framework is similar to the one developed in the past years for heavy-quarkonium systems (for a review see [16]).

Baryons made of three heavy quarks QQQ have not been observed yet. Their relevance has been emphasized since long ago [17]. They reveal a pure baryonic spectrum without light-quark complications and provide valuable insight into the quark-confinement mechanism. Indeed, the three-quark static Wilson loop is intensively studied on the lattice [18,19] as a source of information about the baryon heavy-quark potential and the type of confining configurations [20–22]. In this work we will identify the degrees of freedom and write the low-energy EFT Lagrangian that describes heavy baryons made of three heavy quarks, once the heavy-quark momentum-transfer scale has been integrated out. We will express the leading-order and spin-dependent potentials in terms of Wilson loop amplitudes along the lines developed for heavy quarkonia in [23].

A recent review that also discusses the present status of the art, experimental and theoretical, including lattice, for heavy baryons made with two or three heavy quarks is Ref. [24]. We refer to it for a more complete bibliography on the subject. This work is partially based on [25]. We refer to it for details in some of the derivations.

The paper is distributed as follows. In Sec. II we introduce nonrelativistic QCD (NRQCD) for heavy baryons. In Sec. III we write the low-energy EFT for QQq baryons and calculate the hyperfine splitting of the ground state. In Sec. IV we write the low-energy EFT for QQQ baryons and give some exact nonperturbative expressions for the leading-order and spin-dependent potentials. Section V is

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devoted to the conclusions. Some technical details may be found in the Appendixes.

II. NRQCD

NRQCD is the EFT suitable to describe systems made of two or more heavy quarks. It is obtained from QCD by integrating out modes of energy of the order of the heavy-quark masses [26].

We are interested here only in the heavy-quark sector of the NRQCD Lagrangian. The 2-heavy-quark sector coincides with the Lagrangian of the heavy-quark effective theory (HQET). Up to order $1/m^2$ it reads

$$\begin{aligned} \mathcal{L}_Q^{\text{NRQCD}} = \sum_{h=1}^{N_Q} Q_h^\dagger & \left[iD_0 + \frac{\mathbf{D}^2}{2m_h} + c_F^{(h)} \frac{\boldsymbol{\sigma} \cdot \mathbf{gB}}{2m_h} \right. \\ & \left. + c_D^{(h)} \frac{[\mathbf{D} \cdot, \mathbf{gE}]}{8m_h^2} + i c_S^{(h)} \frac{\boldsymbol{\sigma} \cdot [\mathbf{D} \times, \mathbf{gE}]}{8m_h^2} \right] Q_h, \end{aligned} \quad (1)$$

where N_Q is the number of heavy-quark flavors, Q_h the Pauli spinor field that annihilates the quark of flavor h and mass m_h , $iD_0 = i\partial_0 - gA^0$, $i\mathbf{D} = i\nabla + g\mathbf{A}$, $[\mathbf{D} \cdot, \mathbf{E}] = \mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D}$, $[\mathbf{D} \times, \mathbf{E}] = \mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D}$, $\mathbf{E}^i = F^{i0}$, $\mathbf{B}^i = -\epsilon_{ijk} F^{jk}/2$ ($\epsilon_{123} = 1$) and $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3)$ are the Pauli matrices. The matching coefficients $c_F^{(h)}$, $c_D^{(h)}$ and $c_S^{(h)}$ may be found at one loop, for instance, in [27]. The Lagrangian (1) [with $\mathcal{O}(1/m^3)$ terms included, but all matching coefficients set equal to 1] has been used to perform lattice calculations of the spectra of heavy baryons in [11].

At order $1/m^2$ the NRQCD Lagrangian relevant to describe baryons made of two or more heavy quarks exhibits also a 4-heavy-quark sector:

$$\begin{aligned} \mathcal{L}_{QQ}^{\text{NRQCD}} = \sum_{h' \geq h=1}^{N_Q} & \left(\frac{d_{Q_h Q_{h'}}^{ss}}{m_h m_{h'}} Q_h^\dagger Q_h Q_{h'}^\dagger Q_{h'} \right. \\ & + \frac{d_{Q_h Q_{h'}}^{sv}}{m_h m_{h'}} Q_h^\dagger \boldsymbol{\sigma} Q_h \cdot Q_{h'}^\dagger \boldsymbol{\sigma} Q_{h'} \\ & + \frac{d_{Q_h Q_{h'}}^{vs}}{m_h m_{h'}} \sum_{a=1}^8 Q_h^\dagger T^a Q_h Q_{h'}^\dagger T^a Q_{h'} \\ & \left. + \frac{d_{Q_h Q_{h'}}^{vv}}{m_h m_{h'}} \sum_{a=1}^8 Q_h^\dagger T^a \boldsymbol{\sigma} Q_h \cdot Q_{h'}^\dagger T^a \boldsymbol{\sigma} Q_{h'} \right). \end{aligned} \quad (2)$$

The matching coefficients $d_{Q_h Q_{h'}}$ start getting contributions at order α_s^2 . They have been calculated to this order in Appendix A.

Six-quark operators contribute to heavy baryons made of three heavy quarks. They show up at order $1/m^5$. The corresponding matching coefficients start getting contribu-

tions at order α_s^4 . Hence, these operators are highly suppressed and will be neglected in the rest of the paper.

In the following we will make a step further and construct the EFT suitable to describe baryons made of two (Sec. III) and three (Sec. IV) heavy quarks once gluons of energy or momentum of the order of the momentum transfer between the heavy quarks have been integrated out. The procedure and the resulting EFT will be quite similar to the one developed for heavy quarkonium in [28,29]. For this reason we will call the EFT with the same name: potential NRQCD (pNRQCD).

III. pNRQCD FOR QQq BARYONS

In this section we deal with baryons made of two heavy quarks Q_1, Q_2 (bb, bc or cc) with masses m_1 and m_2 respectively and one massless quark q . The dynamics of these systems is expected to mix aspects typical of heavy quarkonium with aspects typical of heavy-light mesons. On the one hand, the interaction of the two heavy quarks is one of a nonrelativistic quark pair close to threshold moving with relative velocity v . It is, therefore, characterized by the energy scales: $m \gg mv \gg mv^2$, where mv is the scale of the typical momentum transfer between the two heavy quarks (or of the inverse of their typical distance) and mv^2 is the typical binding energy. On the other hand, the energy scale that governs the interaction between the heavy quarks and the light one is Λ_{QCD} . Two different situations are possible.

(A).—If $mv \gg \Lambda_{\text{QCD}}$, at a scale μ such that $mv \gg \mu \gg \Lambda_{\text{QCD}}$ the heavy-quark distance cannot be resolved. The $Q_1 Q_2$ pair behaves like a pointlike particle (sometimes also called diquark [30]) in an antitriplet or sextet color configuration. In the antitriplet configuration the two heavy quarks attract each other. The interaction of the antitriplet field with the light quark is similar to one of a heavy antiquark with a light quark in a D or B meson [13]. However, the spectrum is expected to be richer and more complex due to the internal excitations of the heavy-quark system. These include radial and spin excitations, but also color excitations to sextet configurations. Considering that the scale μ is perturbative, in this situation pNRQCD has the following degrees of freedom: light quarks, gluons of energy and momentum lower than mv (also called ultrasoft) and heavy quarks. To ensure that the gluons are of energy and momentum lower than mv , gluons appearing in vertices involving heavy-quark fields are multipole expanded in the relative distance $r \sim 1/(mv)$ between the two heavy quarks. This corresponds to expanding in $r\Lambda_{\text{QCD}} \sim \Lambda_{\text{QCD}}/(mv)$ or $r(mv^2) \sim v$. In the situation $mv \gg \Lambda_{\text{QCD}}$, in principle one may further distinguish between the subcases $mv^2 \gg \Lambda_{\text{QCD}}$, $mv^2 \sim \Lambda_{\text{QCD}}$ and $\Lambda_{\text{QCD}} \gg mv^2$. In the first case, one may expand in $\Lambda_{\text{QCD}}/(mv^2)$ and disentangle the heavy-heavy dynamics, which is completely accessible to perturbation theory, from the heavy-light one. In general, excitations of the heavy-

heavy system will dominate over excitations of the heavy-light system. In the latter case, the potential governing the heavy-heavy system gets nonperturbative contributions.¹ Since the kinetic energy of the heavy quarks is smaller than Λ_{QCD} , the heavy-light dynamics dominates in this situation over the heavy-heavy one. At leading order in the $mv^2/\Lambda_{\text{QCD}}$ expansion the flavor symmetry typical of the HQET is restored.

In the rest of this section, we will deal with the general situation $mv \gg \Lambda_{\text{QCD}}$, without assuming any special hierarchy between the scales mv^2 and Λ_{QCD} . To be definite, one may think that we work in the situation $mv^2 \sim \Lambda_{\text{QCD}}$.

(B).—If $mv \sim \Lambda_{\text{QCD}}$ the distances between the three quarks are of the same magnitude. Hence, we cannot disentangle the heavy-quark pair dynamics from the light-quark one. Moreover, the potential between the two heavy quarks is nonperturbative. At the level of NRQCD, the system may be studied with lattice calculations. In this situation it seems unlikely that a simple diquark–light-quark picture holds. In general, from an EFT point of view it does not seem consistent to have a diquark–light-quark picture for the heavy-quarks–light-quark interaction, which implicitly assumes $mv \gg \Lambda_{\text{QCD}}$, and at the same time a confining potential binding the two heavy quarks, which requires $mv \sim \Lambda_{\text{QCD}}$, as so often done in potential models.

In the following, we will work out pNRQCD in situation (A). This situation is expected to be appropriate for the description of at least doubly heavy baryons in the ground state.

A. Lagrangian

In this section we write the pNRQCD Lagrangian that describes heavy baryons of the type $Q_1 Q_2 q$ in the situation where the typical momentum transfer between the two heavy quarks is much larger than Λ_{QCD} . This corresponds to the case labeled (A) above.

The number of allowed operators is reduced if we choose to have a manifestly gauge invariant Lagrangian. This may be obtained by projecting the Lagrangian on the heavy-heavy sector of the Fock space, by splitting the heavy-heavy fields into an antitriplet and sextet component

$$[\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2, \mathbf{R} = (m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2)/(m_1 + m_2)],$$

$$Q_{1i}(\mathbf{x}_1, t) Q_{2j}(\mathbf{x}_2, t) \sim \sum_{\ell=1}^3 T^\ell(\mathbf{r}, \mathbf{R}, t) \underline{\mathbf{T}}_{ij}^\ell + \sum_{\sigma=1}^6 \Sigma^\sigma(\mathbf{r}, \mathbf{R}, t) \underline{\Sigma}_{ij}^\sigma, \quad (3)$$

$$i, j = 1, 2, 3,$$

and by building the Lagrangian from these operators. The tensors $\underline{\mathbf{T}}_{ij}^\ell$ and $\underline{\Sigma}_{ij}^\sigma$ are defined in Appendix B 1. The Lagrangian is constrained to satisfy all the symmetries of QCD. In particular, it is symmetric under the exchange of the heavy quarks. Such symmetry transformation changes $m_1 \leftrightarrow m_2$ and \mathbf{r} to $-\mathbf{r}$. The gluon fields are even, because multipole expanded around the center of mass of the heavy-heavy system. For what concerns the heavy-quark fields, from Eq. (3) it follows that T^ℓ is even, because $\underline{\mathbf{T}}_{ij}^\ell$ is odd under the exchange $i \leftrightarrow j$, and Σ^σ is odd, because $\underline{\Sigma}_{ij}^\sigma$ is even under the exchange $i \leftrightarrow j$.

The resulting Lagrangian $\mathcal{L}_{\text{pNRQCD}} = \mathcal{L}_{\text{pNRQCD}}(\mathbf{R}, t)$ at $\mathcal{O}(1/m)$ in the $1/m$ expansion and at $\mathcal{O}(r)$ in the multipole expansion is

$$\mathcal{L}_{\text{pNRQCD}} = \mathcal{L}_{\text{gluon}} + \mathcal{L}_{\text{light}} + \mathcal{L}_{\text{pNRQCD}}^{(0,0)} + \mathcal{L}_{\text{pNRQCD}}^{(0,1)} + \mathcal{L}_{\text{pNRQCD}}^{(1,0)}, \quad (4)$$

with

$$\mathcal{L}_{\text{gluon}} = -\frac{1}{4} \sum_{a=1}^8 F_{\mu\nu}^a F^{a\mu\nu}, \quad (5)$$

$$\mathcal{L}_{\text{light}} = \sum_{f=1}^3 \bar{q}_f i \not{D} q_f, \quad (6)$$

$$\mathcal{L}_{\text{pNRQCD}}^{(0,0)} = \int d^3 r T^\dagger [iD_0 - V_T^{(0)}] T + \Sigma^\dagger [iD_0 - V_\Sigma^{(0)}] \Sigma, \quad (7)$$

$$\mathcal{L}_{\text{pNRQCD}}^{(0,1)} = - \int d^3 r V_{Tr \cdot E \Sigma}^{(0,1)} \sum_{a=1}^8 \sum_{\ell=1}^3 \sum_{\sigma=1}^6 \left[\left(\sum_{ijk=1}^3 \underline{\mathbf{T}}_{ij}^\ell T_{jk}^a \underline{\Sigma}_{ki}^\sigma \right) T^{\ell\dagger} \mathbf{r} \cdot g \mathbf{E}^a \Sigma^\sigma - \left(\sum_{ijk=1}^3 \underline{\Sigma}_{ij}^\sigma T_{jk}^a \underline{\mathbf{T}}_{ki}^\ell \right) \Sigma^{\sigma\dagger} \mathbf{r} \cdot g \mathbf{E}^a T^\ell \right] - \frac{m_1 - m_2}{2m_R} V_{Tr \cdot ET}^{(0,1)} \sum_{a=1}^8 T^\dagger \mathbf{r} \cdot g \mathbf{E}^a T_3^a T - \frac{m_1 - m_2}{2m_R} V_{\Sigma r \cdot E \Sigma}^{(0,1)} \sum_{a=1}^8 \Sigma^\dagger \mathbf{r} \cdot g \mathbf{E}^a T_6^a \Sigma, \quad (8)$$

¹An analogous situation for the quarkonium system has been treated in [29].

$$\begin{aligned}
 \mathcal{L}_{\text{pNRQCD}}^{(1,0)} = & \int d^3r T^\dagger \left[\frac{\mathbf{D}_R^2}{2m_R} + \frac{\nabla_r^2}{2m_r} \right] T + \Sigma^\dagger \left[\frac{\mathbf{D}_R^2}{2m_R} + \frac{\nabla_r^2}{2m_r} \right] \Sigma \\
 & + V_{T\sigma\cdot\mathbf{B}\Sigma}^{(1,0)} \sum_{a=1}^8 \sum_{\ell=1}^3 \sum_{\sigma=1}^6 \left[\left(\sum_{ijk=1}^3 \mathbf{T}_{ij}^\ell T_{jk}^a \underline{\Sigma}_{ki}^\sigma \right) T^{\ell\dagger} \left(-\frac{c_F^{(1)} \boldsymbol{\sigma}^{(1)}}{2m_1} + \frac{c_F^{(2)} \boldsymbol{\sigma}^{(2)}}{2m_2} \right) \cdot g \mathbf{B}^a \Sigma^\sigma \right. \\
 & \left. - \left(\sum_{ijk=1}^3 \underline{\Sigma}_{ij}^\sigma T_{jk}^a \mathbf{T}_{ki}^\ell \right) \Sigma^{\sigma\dagger} \left(-\frac{c_F^{(1)} \boldsymbol{\sigma}^{(1)}}{2m_1} + \frac{c_F^{(2)} \boldsymbol{\sigma}^{(2)}}{2m_2} \right) \cdot g \mathbf{B}^a T^\ell \right] \\
 & + \frac{V_{T\sigma\cdot\mathbf{B}T}^{(1,0)}}{2} \sum_{a=1}^8 T^\dagger \left(\frac{c_F^{(1)} \boldsymbol{\sigma}^{(1)}}{2m_1} + \frac{c_F^{(2)} \boldsymbol{\sigma}^{(2)}}{2m_2} \right) \cdot g \mathbf{B}^a T_3^a T + \frac{V_{\Sigma\sigma\cdot\mathbf{B}\Sigma}^{(1,0)}}{2} \sum_{a=1}^8 \Sigma^\dagger \left(\frac{c_F^{(1)} \boldsymbol{\sigma}^{(1)}}{2m_1} + \frac{c_F^{(2)} \boldsymbol{\sigma}^{(2)}}{2m_2} \right) \cdot g \mathbf{B}^a T_6^a \Sigma \\
 & + V_{T\mathbf{L}\cdot\mathbf{B}\Sigma}^{(1,0)} \frac{m_1 - m_2}{2m_R m_r} \sum_{a=1}^8 \sum_{\ell=1}^3 \sum_{\sigma=1}^6 \left[\left(\sum_{ijk=1}^3 \mathbf{T}_{ij}^\ell T_{jk}^a \underline{\Sigma}_{ki}^\sigma \right) T^{\ell\dagger} \mathbf{L}_r \cdot g \mathbf{B}^a \Sigma^\sigma - \left(\sum_{ijk=1}^3 \underline{\Sigma}_{ij}^\sigma T_{jk}^a \mathbf{T}_{ki}^\ell \right) \Sigma^{\sigma\dagger} \mathbf{L}_r \cdot g \mathbf{B}^a T^\ell \right] \\
 & + \frac{V_{T\mathbf{L}\cdot\mathbf{B}T}^{(1,0)}}{4} \left(\frac{1}{m_r} - \frac{2}{m_R} \right) \sum_{a=1}^8 T^\dagger \mathbf{L}_r \cdot g \mathbf{B}^a T_3^a T + \frac{V_{\Sigma\mathbf{L}\cdot\mathbf{B}\Sigma}^{(1,0)}}{4} \left(\frac{1}{m_r} - \frac{2}{m_R} \right) \sum_{a=1}^8 \Sigma^\dagger \mathbf{L}_r \cdot g \mathbf{B}^a T_6^a \Sigma, \tag{9}
 \end{aligned}$$

where $m_R = m_1 + m_2$, $m_r = m_1 m_2 / (m_1 + m_2)$, $\boldsymbol{\sigma}^{(h)}$ is the Pauli matrix acting on the heavy quark h , $i\mathbf{D}_R = i\nabla_R + g\mathbf{A}$, $\mathbf{L}_r = \mathbf{r} \times (-i\nabla_r)$, $T = (T^1, T^2, T^3)$, $\Sigma = (\Sigma^1, \Sigma^2, \dots, \Sigma^6)$, the gauge fields in the covariant derivatives acting on the antitriplet and sextet are understood in the antitriplet and sextet representation, respectively, T_3^a and T_6^a have been defined in Appendix B 3 and all gluon fields are evaluated in (\mathbf{R}, t) . The coefficients $c_F^{(1)}$ and $c_F^{(2)}$ are the Wilson coefficients of NRQCD introduced in Sec. II. The functions V are the Wilson coefficients of pNRQCD for doubly heavy baryons. They encode the contributions coming from gluons of energy or momentum of order mv , which have been integrated out. They are nonanalytic functions of r . As we will discuss in the next section, at tree level they are

$$\begin{aligned}
 V_{T\mathbf{r}\cdot\mathbf{E}\Sigma}^{(0,1)} &= V_{T\mathbf{r}\cdot\mathbf{E}T}^{(0,1)} = V_{\Sigma\mathbf{r}\cdot\mathbf{E}\Sigma}^{(0,1)} = 1, \\
 V_{T\sigma\cdot\mathbf{B}\Sigma}^{(1,0)} &= V_{T\sigma\cdot\mathbf{B}T}^{(1,0)} = V_{\Sigma\sigma\cdot\mathbf{B}\Sigma}^{(1,0)} = 1, \\
 V_{T\mathbf{L}\cdot\mathbf{B}\Sigma}^{(1,0)} &= V_{T\mathbf{L}\cdot\mathbf{B}T}^{(1,0)} = V_{\Sigma\mathbf{L}\cdot\mathbf{B}\Sigma}^{(1,0)} = 1,
 \end{aligned} \tag{10}$$

while $V_T^{(0)}$ and $V_\Sigma^{(0)}$ get the first nonvanishing contribution at order α_s . In Eqs. (8) and (9) we have displayed only the operators that have a nonvanishing tree-level matching coefficient. The coefficients in front of the \mathbf{D}_R^2 and ∇_r^2 operators in (9) are equal to 1, due to Poincaré invariance or dynamical considerations similar to those developed in [31]. We observe that in the case $m_1 \neq m_2$, electric dipole transitions between antitriplet states induced by the term

$$-\frac{m_1 - m_2}{2m_R} \sum_{a=1}^8 T^\dagger \mathbf{r} \cdot g \mathbf{E}^a T_3^a T \tag{11}$$

are allowed [3,32].

The power counting of the Lagrangian (4) in the center-of-mass frame goes as follows: $\nabla_r \sim mv$, $\mathbf{r} \sim 1/(mv)$, $\mathbf{D}_R \sim \Lambda_{\text{QCD}}$, mv^2 , $V_{T,\Sigma}^{(0)} \sim mv^2$ and \mathbf{E} , $\mathbf{B} \sim \Lambda_{\text{QCD}}^2$,

$(mv^2)^2$. The power counting is not unique, because the scales mv^2 and Λ_{QCD} are still entangled in the dynamics. The Lagrangian at leading order reads

$$\begin{aligned}
 \mathcal{L}_{\text{pNRQCD}}^{\text{LO}} = & \int d^3r T^\dagger \left[iD_0 + \frac{\nabla_r^2}{2m_r} - V_T^{(0)} \right] T \\
 & + \Sigma^\dagger \left[iD_0 + \frac{\nabla_r^2}{2m_r} - V_\Sigma^{(0)} \right] \Sigma \\
 & - \frac{1}{4} \sum_{a=1}^8 F_{\mu\nu}^a F^{a\mu\nu} + \sum_{f=1}^3 \bar{q}_f i \not{D} q_f. \tag{12}
 \end{aligned}$$

B. Matching

The matching from NRQCD to pNRQCD is, in general, performed by calculating Green functions in the two theories and imposing that they are equal order by order in the inverse of the mass and in the multipole expansion. Since we are working in the situation where the typical momentum transfer between the heavy quarks is larger than Λ_{QCD} , we can, in addition, perform the matching order by order in α_s .

If we aim at calculating the matching at tree level a convenient approach consists in projecting the NRQCD Hamiltonian on the two-quark Fock space spanned by

$$\int d^3x_1 d^3x_2 \sum_{ij=1}^3 \Phi_{Q_1 Q_2}^{ij}(\mathbf{x}_1, \mathbf{x}_2) Q_1^{i\dagger}(\mathbf{x}_1) Q_2^{j\dagger}(\mathbf{x}_2) |0\rangle, \tag{13}$$

where $|0\rangle$ is the Fock subspace containing no heavy quarks but an arbitrary number of ultrasoft gluons and light quarks and $\Phi_{Q_1 Q_2}(\mathbf{x}_1, \mathbf{x}_2)$ is a $3 \otimes 3$ tensor in color space and a $2 \otimes 2$ tensor in spin space. This is similar to what is done in [33]. After projection, all gluon fields are multipole expanded in \mathbf{r} . In order to make gauge invariance explicit at the Lagrangian level, it is useful to decompose $\Phi_{Q_1 Q_2}(\mathbf{x}_1, \mathbf{x}_2)$ into a field $T(\mathbf{r}, \mathbf{R}, t)$, which transforms like

a color antitriplet, and a field $\Sigma(\mathbf{r}, \mathbf{R}, t)$, which transforms like a color sextet:

$$\begin{aligned} \Phi_{Q_1 Q_2}^{ij}(\mathbf{x}_1, \mathbf{x}_2, t) &= \sum_{i'j'=1}^3 \phi_{i'j'}(\mathbf{x}_1, \mathbf{R}, t) \phi_{jj'}(\mathbf{x}_2, \mathbf{R}, t) \\ &\times \left(\sum_{\ell=1}^3 T^\ell(\mathbf{r}, \mathbf{R}, t) \underline{\mathbf{T}}_{i'j'}^\ell \right. \\ &\left. + \sum_{\sigma=1}^6 \Sigma^\sigma(\mathbf{r}, \mathbf{R}, t) \underline{\Sigma}_{i'j'}^\sigma \right), \end{aligned} \quad (14)$$

where

$$\phi(\mathbf{y}, \mathbf{x}, t) \equiv P \exp \left(ig \int_0^1 ds (\mathbf{y} - \mathbf{x}) \cdot \mathbf{A}(\mathbf{x} + (\mathbf{y} - \mathbf{x})s, t) \right). \quad (15)$$

P stands for path ordering. At leading order in the coupling constant, $\phi_{ij}(\mathbf{x}_1, \mathbf{R}, t) = \delta_{ij}$ and

$$\Phi_{Q_1 Q_2}^{ij}(\mathbf{x}_1, \mathbf{x}_2, t) \approx \sum_{\ell=1}^3 T^\ell(\mathbf{r}, \mathbf{R}, t) \underline{\mathbf{T}}_{ij}^\ell + \sum_{\sigma=1}^6 \Sigma^\sigma(\mathbf{r}, \mathbf{R}, t) \underline{\Sigma}_{ij}^\sigma. \quad (16)$$

After projecting on (13), one obtains the Lagrangian (7)–(9) with the matching conditions (10).

As an example, let us consider the calculation that leads to the term

$$\begin{aligned} \delta \mathcal{L}_{\text{pNRQCD}} &= \int d^3 r \frac{1}{2} \sum_{a=1}^8 T^\dagger \left(\frac{c_F^{(1)} \boldsymbol{\sigma}^{(1)}}{2m_1} + \frac{c_F^{(2)} \boldsymbol{\sigma}^{(2)}}{2m_2} \right) \\ &\cdot g \mathbf{B}^a T_{\frac{3}{2}}^a T \end{aligned} \quad (17)$$

in the pNRQCD Lagrangian [see Eq. (9)]. We start from the NRQCD term

$$\delta \mathcal{L}_{\text{NRQCD}} = Q_1^\dagger c_F^{(1)} \frac{\boldsymbol{\sigma} \cdot g \mathbf{B}}{2m_1} Q_1 + Q_2^\dagger c_F^{(2)} \frac{\boldsymbol{\sigma} \cdot g \mathbf{B}}{2m_2} Q_2. \quad (18)$$

Projecting onto (16), we obtain in the antitriplet-antitriplet sector

$$\begin{aligned} \delta \mathcal{L}_{\text{pNRQCD}} &= \int d^3 r \sum_{a=1}^8 \sum_{\ell \ell' j k=1}^3 \left(T^{\ell \dagger} \underline{\mathbf{T}}_{ij}^\ell \frac{c_F^{(1)} \boldsymbol{\sigma}^{(1)}}{2m_1} \right. \\ &\cdot g \mathbf{B}^a T_{ik}^a T^{\ell'} \underline{\mathbf{T}}_{kj}^{\ell'} + T^{\ell \dagger} \underline{\mathbf{T}}_{ij}^\ell \frac{c_F^{(2)} \boldsymbol{\sigma}^{(2)}}{2m_2} \\ &\left. \cdot g \mathbf{B}^a T_{jk}^a T^{\ell'} \underline{\mathbf{T}}_{ik}^{\ell'} \right). \end{aligned} \quad (19)$$

Using the definition (B1), we have

$$\sum_{ijk=1}^3 \underline{\mathbf{T}}_{ij}^\ell T_{ik}^a \underline{\mathbf{T}}_{kj}^{\ell'} = -\frac{T_{\ell \ell'}^a}{2} = \frac{(T_{\frac{3}{2}}^a)_{\ell \ell'}}{2}, \quad (20)$$

$$\sum_{ijk=1}^3 \underline{\mathbf{T}}_{ij}^\ell T_{jk}^a \underline{\mathbf{T}}_{ik}^{\ell'} = -\frac{T_{\ell \ell'}^a}{2} = \frac{(T_{\frac{3}{2}}^a)_{\ell \ell'}}{2}, \quad (21)$$

and eventually end up with Eq. (17). This fixes $V_{T\boldsymbol{\sigma}\cdot\mathbf{B}T}^{(1,0)} = 1$ at leading order. Note that Eq. (17) differs by a factor 1/2 from Eqs. (9) and (10) in [13], which seem to miss the correct color normalization of the antitriplet states.

One may ask what happens to $V_{T\boldsymbol{\sigma}\cdot\mathbf{B}T}^{(1,0)}$ beyond tree level. Order α_s corrections may only come from one-gluon corrections to the NRQCD vertex of Eq. (18), because all other spin-dependent operators in NRQCD contribute to higher orders in $1/m$. One-loop corrections to the external (transverse) gluon or to a quark line or involving a gluon attached to the external gluon and to the quark line coupled to it vanish in dimensional regularization, once we have expanded in the external energies. Gluons attached to a quark line are longitudinal. It is convenient to use the Coulomb gauge. In the Coulomb gauge, longitudinal gluons exchanged between different quark lines cancel in the matching with equal contributions from the pNRQCD side. Finally, longitudinal gluons attached to the external gluon line and a heavy-quark line not coupled to it contribute to higher-order operators, since the three-gluon vertex is proportional to the external energies. We conclude that $V_{T\boldsymbol{\sigma}\cdot\mathbf{B}T}^{(1,0)}$ does not get contributions at one loop, so that $V_{T\boldsymbol{\sigma}\cdot\mathbf{B}T}^{(1,0)} = 1 + \mathcal{O}(\alpha_s^2)$. Similar considerations hold for $V_{\Sigma\boldsymbol{\sigma}\cdot\mathbf{B}\Sigma}^{(1,0)}$ and $V_{T\boldsymbol{\sigma}\cdot\mathbf{B}\Sigma}^{(1,0)}$.

The perturbative matching of the static potentials $V_T^{(0)}$ and $V_\Sigma^{(0)}$ goes as follows (see [29] for the quarkonium case). In NRQCD we compute static Green functions, whose initial and final states overlap with the antitriplet and sextet fields in pNRQCD. Since we work order by order in α_s , it is not necessary for the Green functions to be gauge invariant. A possible choice is

$$\begin{aligned} I_{\mathcal{M}}^{uv} &\equiv \sum_{ijj'=1}^3 \langle 0 | \underline{\mathcal{M}}_{ij}^u Q_i(\mathbf{R}, \mathbf{x}_1, T/2) Q_j(\mathbf{R}, \mathbf{x}_2, T/2) \\ &\times \underline{\mathcal{M}}_{j'j}^v Q_{j'}^\dagger(\mathbf{R}, \mathbf{y}_1, -T/2) Q_{j'}^\dagger(\mathbf{R}, \mathbf{y}_2, -T/2) | 0 \rangle, \end{aligned} \quad (22)$$

$$(1) \quad \text{if } \mathcal{M} = T, \quad \underline{\mathcal{M}}_{ij}^u = \underline{\mathbf{T}}_{ij}^u, \quad u, v = 1, 2, 3,$$

$$(2) \quad \text{if } \mathcal{M} = \Sigma, \quad \underline{\mathcal{M}}_{ij}^u = \underline{\Sigma}_{ij}^u, \quad u, v = 1, 2, \dots, 6,$$

where

$$Q(\mathbf{R}, \mathbf{x}, t) \equiv \phi(\mathbf{R}, \mathbf{x}, t) Q(\mathbf{x}, t), \quad (23)$$

and $\phi(\mathbf{R}, \mathbf{x}, t)$ has been defined in Eq. (15). Integrating out the static-quark fields from $I_{\mathcal{M}}^{uv}$ we obtain

$$I_{\mathcal{M}}^{uv} = \delta^3(\mathbf{x}_1 - \mathbf{y}_1)\delta^3(\mathbf{x}_2 - \mathbf{y}_2)\langle 0|(W_{QQ}^{\mathcal{M}})^{uv}|0\rangle \quad (24)$$

with $W_{QQ}^{\mathcal{M}}$ diagrammatically represented in Fig. 1 and explicitly given by

$$\begin{aligned} (W_{QQ}^{\mathcal{M}})^{uv} \equiv & P \sum_{ijkn'j'k'n'=1}^3 \underline{\mathcal{M}}_{ij}^u \phi_{i'j'}(\mathbf{R}, \mathbf{x}_1, T/2) \\ & \times \phi_{i'k'}(T/2, -T/2, \mathbf{x}_1)\phi_{k'k}(\mathbf{x}_1, \mathbf{R}, -T/2) \\ & \times \phi_{jj'}(\mathbf{R}, \mathbf{x}_2, T/2)\phi_{j'n'}(T/2, -T/2, \mathbf{x}_2) \\ & \times \phi_{n'n}(\mathbf{x}_2, \mathbf{R}, -T/2)\underline{\mathcal{M}}_{kn}^v. \end{aligned} \quad (25)$$

In the large T limit, the Green functions I_T^{uv} and I_{Σ}^{uv} are reduced to the antitriplet and sextet propagators of pNRQCD, respectively. If we neglect subleading loop corrections to the pNRQCD side of the matching, we obtain

$$\begin{aligned} \lim_{T \rightarrow \infty} \langle 0|(W_{QQ}^{\mathcal{M}})^{uv}|0\rangle &= \lim_{T \rightarrow \infty} Z_{\mathcal{M}}(r) \exp(-iV_{\mathcal{M}}^{(0)}(r)T) \\ & \times \langle 0| \sum_{ijkn=1}^3 \underline{\mathcal{M}}_{ij}^u \phi_{ik}(T/2, -T/2, \mathbf{R}) \\ & \times \phi_{jn}(T/2, -T/2, \mathbf{R})\underline{\mathcal{M}}_{kn}^v |0\rangle, \end{aligned} \quad (26)$$

where $Z_{\mathcal{M}}$ is a normalization factor. At order α_s we end up with the well-known result [34]:

$$V_T^{(0)}(r) = -\frac{2}{3} \frac{\alpha_s}{|\mathbf{r}|}, \quad (27)$$

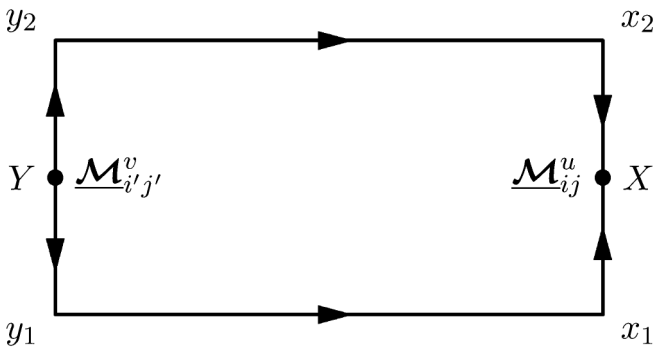


FIG. 1. Static Wilson loop with edges $x_1 = (\mathbf{x}_1, T/2)$, $x_2 = (\mathbf{x}_2, T/2)$, $y_1 = (\mathbf{x}_1, -T/2)$, $y_2 = (\mathbf{x}_2, -T/2)$ and insertions of the tensors $\underline{\mathcal{M}}_{ij}^u$ and $\underline{\mathcal{M}}_{i'j'}^v$ in $X = (\mathbf{R}, T/2)$ and $Y = (\mathbf{R}, -T/2)$ respectively.

$$V_{\Sigma}^{(0)}(r) = \frac{1}{3} \frac{\alpha_s}{|\mathbf{r}|}. \quad (28)$$

The antitriplet channel is attractive, the sextet one repulsive.

C. Hyperfine splitting

In the dynamical situation considered here [case (A) of Sec. III], a doubly heavy baryon is mainly a bound state of a heavy-quark (or antiquark) pair in an antitriplet (or triplet) configuration and a light quark (or antiquark). Sextet field configurations show up in loops with ultrasoft gluons. Their contribution is suppressed either in the multipole expansion or in $1/m$ [see Eqs. (8) and (9)]. The leading-order pNRQCD Lagrangian is shown in Eq. (12). It does not depend on the spin of the heavy quarks. As a consequence, QQq baryons will appear in degenerate multiplets of the total spin $\mathbf{S}_{QQq} = \mathbf{S}_{QQ} + \mathbf{S}_l$, where \mathbf{S}_l is the spin of the light degrees of freedom and \mathbf{S}_{QQ} of the heavy-quark pair. This symmetry is similar to the spin symmetry of the HQET. Differently from the HQET, however, the pNRQCD Lagrangian depends at leading order on the heavy-quark flavor. This is a consequence of the fact that we cannot, in general, neglect the kinetic energy.²

We will consider in this section the S -wave ground state of a doubly heavy baryon made of two identical heavy quarks Q of mass m_Q . In this case, since an (anti)triplet state is antisymmetric in color, due to the Fermi statistics, the two heavy quarks are allowed only in a spin 1 (symmetric) state. In the standard notation, the lowest energy states for QQu or QQd are called Ξ_{QQ} (Ξ_{QQ}^*) for spin 1/2 (3/2), and for QQs , Ω_{QQ} (Ω_{QQ}^*). Since the heavy-quark pair spin is fixed, the hyperfine splitting may only originate from spin-dependent couplings of the heavy quarks with the light one. The leading-order operator (in a Λ_{QCD}/m expansion) is given by Eq. (17). We will derive a simple formula that relates at leading order in the Λ_{QCD}/m expansion the hyperfine splitting of a QQq doubly heavy-baryon ground state with the hyperfine splitting of a Qq heavy-light meson ground state. The framework will be that one of pNRQCD, developed in the previous sections. The calculation will be similar to that one of Ref. [13].

Let us consider, first, the case of a heavy-light meson $\bar{Q}q$. The heavy antiquark may be described by a two-component field $Q_c = i\sigma_2 Q^*$, where Q is the Pauli spinor that annihilates the heavy quark. We rename $Q_c^1 = Q_+$ and $Q_c^2 = Q_-$ since $Q_{\pm}^{\dagger}|0\rangle = |\mathbf{S}_{\bar{Q}}^z = \pm 1/2\rangle$. $\mathbf{S}_{\bar{Q}}$ is the spin of the heavy antiquark and $\mathbf{S}_{\bar{Q}q}$ the total spin of the meson. The leading-order HQET Lagrangian does not contain spin-interaction terms, therefore, states that differ only in the spin quantum numbers are degenerate. In particular,

²An exception may be the special case $\Lambda_{\text{QCD}} \gg mv^2$.

this happens for the three lowest $S_{\bar{Q}q} = 1$ states ($\mathbf{S}_{\bar{Q}q}^z = 1, 0, -1$), which we denote by $|P_Q^*\rangle$, and for the lowest $S_{\bar{Q}q} = 0$ state, which we denote by $|P_Q\rangle$. An expression for these states that makes explicit their heavy (anti)quark field content is given in Appendix C. The Hamiltonian responsible for the leading contribution to the hyperfine separation is

$$\begin{aligned} \delta H_{\text{HQET}} &= -c_F^{(Q)} \int d^3R \sum_{a=1}^8 Q_c^\dagger(\mathbf{R}) \frac{\boldsymbol{\sigma} \cdot g\mathbf{B}^a(\mathbf{R})T_3^a}{2m_Q} Q_c(\mathbf{R}) \\ &= -\frac{c_F^{(Q)}}{2m_Q} \int d^3R \sum_{a=1}^8 [(Q_+^\dagger T_3^a Q_+ - Q_-^\dagger T_3^a Q_-)g\mathbf{B}^{3a} \\ &\quad + i(Q_-^\dagger T_3^a Q_+ - Q_+^\dagger T_3^a Q_-)g\mathbf{B}^{2a} \\ &\quad + (Q_+^\dagger T_3^a Q_- + Q_-^\dagger T_3^a Q_+)g\mathbf{B}^{1a}], \end{aligned} \quad (29)$$

where, after the last equality, we have dropped the explicit coordinate dependence of the fields. From Eqs. (29) and (C1)–(C4) it is straightforward to derive

$$\begin{aligned} &\langle P_Q^* | \delta H_{\text{HQET}} | P_Q^* \rangle - \langle P_Q | \delta H_{\text{HQET}} | P_Q \rangle \\ &= -2 \frac{c_F^{(Q)}}{m_Q} \int d^3R \langle \mathbf{S}_i^z = 1/2 | \sum_{a=1}^8 g\mathbf{B}^{3a} T_3^a | \mathbf{S}_i^z = 1/2 \rangle. \end{aligned} \quad (30)$$

In the case of a doubly heavy baryon QQq we proceed in a similar way. The triplet field T is a $2 \otimes 2$ tensor in spin space, which may be decomposed as $2 \otimes 2 = 1 \oplus 3$, i.e. in

a scalar component, $T^{(S)}$, and a vector one, $T^{(V)}$:

$$\begin{aligned} T_{ij}(\mathbf{r}, \mathbf{R}, t) &= \left(\frac{i\boldsymbol{\sigma}_2}{\sqrt{2}} \right)_{ij} T^{(S)}(\mathbf{r}, \mathbf{R}, t) \\ &\quad + \sum_{k=1}^3 \left(\frac{i\boldsymbol{\sigma}_k \boldsymbol{\sigma}_2}{\sqrt{2}} \right)_{ij} T^{(V)k}(\mathbf{r}, \mathbf{R}, t), \quad i, j = 1, 2. \end{aligned} \quad (31)$$

The indices ij refer to the spin space. Note that the matrices $(i\boldsymbol{\sigma}_2/\sqrt{2})_{ij}$ and $(i\boldsymbol{\sigma}_k \boldsymbol{\sigma}_2/\sqrt{2})_{ij}$ are, respectively, antisymmetric and symmetric in ij . It is convenient to rewrite the fields $T^{(V)k}$ as

$$T_0 = T^{(V)3} \quad \text{and} \quad T_\pm = \frac{\mp T^{(V)1} + iT^{(V)2}}{\sqrt{2}}, \quad (32)$$

since $T_0^\dagger |0\rangle = |\mathbf{S}_{QQ}^z = 0\rangle$ and $T_\pm^\dagger |0\rangle = |\mathbf{S}_{QQ}^z = \pm 1\rangle$. As we argued above, the leading-order pNRQCD Lagrangian describing doubly heavy baryons does not contain spin-interaction terms, therefore, states that differ only in the spin quantum numbers are degenerate. In particular, this happens for the four lowest $S_{QQq} = 3/2$ states ($\mathbf{S}_{QQq}^z = \pm 3/2, \pm 1/2$), which we denote by $|\Xi_{QQ}^*\rangle$, and for the two lowest $S_{QQq} = 1/2$ states ($\mathbf{S}_{QQq}^z = \pm 1/2$), which we denote by $|\Xi_{QQ}\rangle$. An explicit expression of these states in terms of heavy (anti)triplet fields is given in Appendix C. The Hamiltonian responsible for the leading contribution to the hyperfine separation is

$$\begin{aligned} \delta H_{\text{pNRQCD}} &= -\frac{c_F^{(Q)}}{2m_Q} \int d^3R \int d^3r V_{T\boldsymbol{\sigma}\cdot\mathbf{B}T}^{(1,0)}(r) \sum_{a=1}^8 T^\dagger(\mathbf{r}, \mathbf{R}) \frac{\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}}{2} \cdot g\mathbf{B}^a(\mathbf{R})T_3^a T(\mathbf{r}, \mathbf{R}) \\ &= -\frac{c_F^{(Q)}}{2m_Q} \int d^3R \int d^3r V_{T\boldsymbol{\sigma}\cdot\mathbf{B}T}^{(1,0)}(r) \sum_{a=1}^8 \sum_{ikj=1}^3 T^{(V)i\dagger} i\epsilon_{ikj} g\mathbf{B}^{ka} T_3^a T^{(V)j} \\ &= -\frac{c_F^{(Q)}}{2m_Q} \int d^3R \int d^3r V_{T\boldsymbol{\sigma}\cdot\mathbf{B}T}^{(1,0)}(r) \sum_{a=1}^8 \left[(T_+^\dagger T_3^a T_+ - T_-^\dagger T_3^a T_-)g\mathbf{B}^{3a} \right. \\ &\quad + \frac{i}{\sqrt{2}} (-T_+^\dagger T_3^a T_0 + T_0^\dagger T_3^a T_+ + T_-^\dagger T_3^a T_0 - T_0^\dagger T_3^a T_-)g\mathbf{B}^{2a} \\ &\quad \left. + \frac{1}{\sqrt{2}} (T_+^\dagger T_3^a T_0 + T_0^\dagger T_3^a T_+ + T_-^\dagger T_3^a T_0 + T_0^\dagger T_3^a T_-)g\mathbf{B}^{1a} \right], \end{aligned} \quad (33)$$

where the second equality follows from Eq. (31) and the third one from Eq. (32). From Eqs. (33) and (C5)–(C10) it is straightforward to derive

$$\begin{aligned} \langle \Xi_{QQ}^* | \delta H_{\text{pNRQCD}} | \Xi_{QQ}^* \rangle - \langle \Xi_{QQ} | \delta H_{\text{pNRQCD}} | \Xi_{QQ} \rangle &= -3 \frac{c_F^{(Q)}}{2m_Q} \int d^3R \langle \mathbf{S}_i^z = 1/2 | \sum_{a=1}^8 g\mathbf{B}^{3a} T_3^a | \mathbf{S}_i^z = 1/2 \rangle \\ &\quad \times \int d^3r \varphi_{QQ}^*(\mathbf{r}) V_{T\boldsymbol{\sigma}\cdot\mathbf{B}T}^{(1,0)}(r) \varphi_{QQ}(\mathbf{r}), \end{aligned} \quad (34)$$

where φ_{QQ} is the ground-state eigenfunction of $-\nabla_r^2/(2m_r) + V_T^{(0)}$. At next-to-leading order $V_{T\boldsymbol{\sigma}\cdot\mathbf{B}T}^{(1,0)} = 1$, therefore

$\int d^3r \varphi_{QQ}^*(\mathbf{r}) V_{T\sigma\text{-}BT}^{(1,0)}(\mathbf{r}) \varphi_{QQ}(\mathbf{r}) = 1 + \mathcal{O}(\alpha_s^2)$. This result crucially depends on the fact that we have multipole expanded the gluon fields. As a consequence, \mathbf{B} does not depend on \mathbf{r} and the magnetic dipole transition term (differently from the electric one $\propto \mathbf{r} \cdot g\mathbf{E}$) does not exhibit any explicit dependence on \mathbf{r} . Comparing Eq. (30) with Eq. (34) we obtain (M_{Ξ} and M_P are the baryon and meson masses, respectively)

$$M_{\Xi_{QQ}^*} - M_{\Xi_{QQ}} = \frac{3m_{Q'}}{4m_Q} \frac{c_F^{(Q)}}{c_F^{(Q')}} (M_{P_{Q'}^*} - M_{P_{Q'}}) \times \left[1 + \mathcal{O}\left(\alpha_s^2, \frac{\Lambda_{\text{QCD}}}{m_Q}, \frac{\Lambda_{\text{QCD}}}{m_{Q'}}\right) \right]. \quad (35)$$

Up to a factor 1/2, the formula is the one derived in [13]. In the previous section, the origin of the discrepancy has been traced back to a missing color normalization factor in the spin antitriplet interaction term (17) [and (33)]. On the other hand, the relation $M_{\Xi_{QQ}^*} - M_{\Xi_{QQ}} = (3m_{Q'}/4m_Q) \times (M_{P_{Q'}^*} - M_{P_{Q'}})$ has been derived since long time in non-relativistic potential models. Surprisingly the discrepancy between this formula and the formula in [13] has to the best of our knowledge never been noticed before in the literature.³ Even more surprisingly some of the literature has explicitly claimed agreement between the potential model prediction and the formula in [13].

From [35] we read that $M_{D^*} - M_D = 142.12 \pm 0.07$ MeV and $M_{B^*} - M_B = 45.78 \pm 0.35$ MeV. Both data may be used to obtain $M_{\Xi_{cc}^*} - M_{\Xi_{cc}}$ and $M_{\Xi_{bb}^*} - M_{\Xi_{bb}}$ from Eq. (35). If $Q \neq Q'$, we use $c_F^{(Q)}$ at next-to-leading logarithmic accuracy calculated in [36], and $m_b = M_{Y(1S)}/2$ and $m_c = M_{J/\psi}/2$. For $M_{\Xi_{cc}^*} - M_{\Xi_{cc}}$ we obtain about 107 MeV from the D data and about 133 MeV from the B data. Taking the average and estimating $\Lambda_{\text{QCD}}/m_c \sim \alpha_s^2(m_c \alpha_s) \approx 0.3$, our result is

$$M_{\Xi_{cc}^*} - M_{\Xi_{cc}} = 120 \pm 40 \text{ MeV}. \quad (36)$$

Similarly for $M_{\Xi_{bb}^*} - M_{\Xi_{bb}}$ we obtain about 27 MeV from the D data and about 34 MeV from the B data. Taking only the estimate based on the B data, because affected by the smaller uncertainty $\Lambda_{\text{QCD}}/m_b \sim \alpha_s^2(m_b \alpha_s) \approx 0.1$, our result is

$$M_{\Xi_{bb}^*} - M_{\Xi_{bb}} = 34 \pm 4 \text{ MeV}. \quad (37)$$

These results compare well with the quenched QCD lattice simulation of [12], whose result is $M_{\Xi_{cc}^*} - M_{\Xi_{cc}} = 89 \pm 15$ MeV, and of [9], whose result is $M_{\Xi_{cc}^*} - M_{\Xi_{cc}} = 80 \pm 10^{+3}_{-7}$ MeV, and with the quenched NRQCD lattice simulations of [8,11], whose results for bbq baryons are $M_{\Xi_{bb}^*} - M_{\Xi_{bb}} = 20 \pm 6^{+2}_{-3}$ MeV and $M_{\Xi_{bb}^*} - M_{\Xi_{bb}} =$

$20 \pm 6^{+3}_{-4}$ MeV respectively. The figures of [9,11] refer to the lattice calculations at largest β .

IV. PNQCD FOR QQQ BARYONS

In this section we consider baryons formed by three heavy quarks of which at least two with the same mass $m_1 = m_2 \equiv m$. Baryons of this type may be composed by bbb , bbc , bcc or ccc quarks. We define

$$m_R = 2m + m_3, \quad m_\rho = \frac{m}{2}, \quad m_\lambda = \frac{2mm_3}{m_R}, \quad (38)$$

$$\mathbf{R} = \frac{m(\mathbf{x}_1 + \mathbf{x}_2) + m_3\mathbf{x}_3}{m_R}, \quad \boldsymbol{\rho} = \mathbf{x}_1 - \mathbf{x}_2, \quad (39)$$

$$\boldsymbol{\lambda} = \frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{x}_3.$$

There are, in principle, several physical scales that may play an important role in the dynamics: the masses m_R , m_ρ and m_λ , which we assume to be of the same order, the typical relative three momenta of the heavy quarks, the typical kinetic energies and the scale of nonperturbative physics Λ_{QCD} . In the following, we will keep the discussion as simple as possible by not exploiting any possible hierarchy among the relative momenta and the kinetic energies. We will assume that the typical relative momenta of the heavy quarks, generically denoted by mv , are all much smaller than the heavy-quark masses and much larger than the kinetic energies, generically denoted by mv^2 . We may distinguish two situations.

(A).—The typical relative momenta of the heavy quarks are much larger than Λ_{QCD} . We call this situation weakly coupled.

(B).—The typical relative momenta of the heavy quarks are of the order of Λ_{QCD} . We call this situation strongly coupled.

A. pNRQCD for weakly coupled QQQ baryons

1. Lagrangian and degrees of freedom

If we assume that the typical distances ρ and λ in the baryon, which are of order $1/(mv)$, are much smaller than $1/\Lambda_{\text{QCD}}$, then gluons of momentum or energy of order mv may be integrated out from NRQCD order by order in α_s . The resulting EFT has light quarks, gluons of energy and momentum lower than mv (ultrasoft gluons), and heavy quarks as degrees of freedom. Gluons appearing in vertices involving heavy-quark fields are multipole expanded in ρ and λ to ensure that they are ultrasoft. This corresponds to expanding in $\rho\Lambda_{\text{QCD}} \sim \Lambda_{\text{QCD}}/(mv)$ and $\lambda\Lambda_{\text{QCD}} \sim \Lambda_{\text{QCD}}/(mv)$, or $\rho(mv^2) \sim v$ and $\lambda(mv^2) \sim v$. Like in the case of doubly heavy baryons the number of allowed operators is consistently reduced if we choose to have a manifestly gauge-invariant Lagrangian. This may be done by projecting the Lagrangian on the heavy-heavy-heavy

³From private communications we know, however, that at least T. Mehen and the authors of [9] were aware of it.

sector of the Fock space, by splitting the heavy-heavy-heavy fields into a singlet, two octet and a decuplet component,

$$Q_{1i}(\mathbf{x}_1, t)Q_{2,j}(\mathbf{x}_2, t)Q_{3k}(\mathbf{x}_3, t) \sim S(\boldsymbol{\rho}, \boldsymbol{\lambda}, \mathbf{R}, t)\underline{\mathbf{S}}_{ijk} + \sum_{a=1}^8 O^{Aa}(\boldsymbol{\rho}, \boldsymbol{\lambda}, \mathbf{R}, t)\underline{\mathbf{O}}_{ijk}^{Aa} + \sum_{a=1}^8 O^{Sa}(\boldsymbol{\rho}, \boldsymbol{\lambda}, \mathbf{R}, t)\underline{\mathbf{O}}_{ijk}^{Sa} + \sum_{\delta=1}^{10} \Delta^\delta(\boldsymbol{\rho}, \boldsymbol{\lambda}, \mathbf{R}, t)\underline{\mathbf{\Delta}}_{ijk}^\delta, \quad i, j = 1, 2, 3, \quad (40)$$

and by building the Lagrangian from these operators. The tensors $\underline{\mathbf{S}}_{ijk}$, $\underline{\mathbf{O}}_{ijk}^{Aa}$, $\underline{\mathbf{O}}_{ijk}^{Sa}$ and $\underline{\mathbf{\Delta}}_{ijk}^\delta$ are defined in Appendix B 2. $\underline{\mathbf{S}}_{ijk}$ and $\underline{\mathbf{\Delta}}_{ijk}^\delta$ are real and, respectively, totally antisymmetric and symmetric. We chose $\underline{\mathbf{O}}_{ijk}^{Aa}$ and $\underline{\mathbf{O}}_{ijk}^{Sa}$ to be, respectively, antisymmetric and symmetric in the first two indices. The Lagrangian is constrained to satisfy all the symmetries of QCD. In particular, in the case $m_1 = m_2$ that we consider here, it must be invariant under the exchange of the heavy quarks labeled 1 and 2. Under such transformation, $\boldsymbol{\rho}$ goes into $-\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$ goes into $\boldsymbol{\lambda}$. The gluon fields are even, because multipole expanded around the center of mass of the heavy-heavy-heavy system. For what concerns the heavy-quark fields, from Eq. (40) it follows that S and O^{Aa} are even, because $\underline{\mathbf{S}}_{ijk}$ and $\underline{\mathbf{O}}_{ijk}^{Aa}$ are odd under exchange $i \leftrightarrow j$, and O^{Sa} and $\underline{\mathbf{\Delta}}_{ijk}^\delta$ are odd, because $\underline{\mathbf{O}}_{ijk}^{Sa}$ and $\underline{\mathbf{\Delta}}_{ijk}^\delta$ are even under exchange $i \leftrightarrow j$. It is also useful to consider the combination of the above transformation with parity, $(1 \leftrightarrow 2) \times P$, which is also a symmetry of the Lagrangian. Under this transformation $\boldsymbol{\rho}$ goes into $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$ goes into $-\boldsymbol{\lambda}$. The gluon fields transform like $A_\mu(t, \mathbf{R}) \rightarrow A^\mu(t, -\mathbf{R})$, which means that, up to reflection of the internal spatial coordinates, chromo-

electric fields are odd and chromomagnetic fields are even. Up to reflection of the internal spatial coordinates, the heavy-quark fields transform like in the case of the $(1 \leftrightarrow 2)$ exchange.

The resulting Lagrangian $\mathcal{L}_{\text{pNRQCD}} = \mathcal{L}_{\text{pNRQCD}}(\mathbf{R}, t)$ at $\mathcal{O}(\lambda, \rho)$ in the multipole expansion [we also display at $\mathcal{O}(1/m)$ the kinetic energy terms] is

$$\mathcal{L}_{\text{pNRQCD}} = \mathcal{L}_{\text{gluon}} + \mathcal{L}_{\text{light}} + \mathcal{L}_{\text{pNRQCD}}^{(0,0)} + \mathcal{L}_{\text{pNRQCD}}^{(0,1)} + \mathcal{L}_{\text{pNRQCD}}^{(1,0)}, \quad (41)$$

with $\mathcal{L}_{\text{gluon}}$ and $\mathcal{L}_{\text{light}}$ defined in Eqs. (5) and (6) respectively and

$$\mathcal{L}_{\text{pNRQCD}}^{(0,0)} = \int d^3\rho d^3\lambda S^\dagger [i\partial_0 - V_S^{(0)}]S + O^{A\dagger} [iD_0 - V_{O^A}^{(0)}]O^A + O^{S\dagger} [iD_0 - V_{O^S}^{(0)}]O^S + \Delta^\dagger [iD_0 - V_\Delta^{(0)}]\Delta, \quad (42)$$

$$\begin{aligned} \mathcal{L}_{\text{pNRQCD}}^{(0,1)} = & \int d^3\rho d^3\lambda V_{S\rho E O^S}^{(0,1)} \sum_{a=1}^8 \frac{1}{2\sqrt{2}} [S^\dagger \boldsymbol{\rho} \cdot g\mathbf{E}^a O^{Sa} + O^{Sa\dagger} \boldsymbol{\rho} \cdot g\mathbf{E}^a S] - V_{O^A \boldsymbol{\rho} \cdot E O^S}^{(0,1)} \sum_{abc=1}^8 \left(\frac{if^{abc} + 3d^{abc}}{4\sqrt{3}} \right) \\ & \times [O^{Aa\dagger} \boldsymbol{\rho} \cdot g\mathbf{E}^b O^{Sc} + O^{Sa\dagger} \boldsymbol{\rho} \cdot g\mathbf{E}^b O^{Ac}] + V_{O^A \boldsymbol{\rho} \cdot E \Delta}^{(0,1)} \sum_{ab=1}^8 \sum_{\delta=1}^{10} \left[\left(\sum_{i'j'k=1}^3 \epsilon_{ijk} T_{i'i'}^a T_{j'j}^b \underline{\mathbf{\Delta}}_{i'j'k}^\delta \right) O^{Aa\dagger} \boldsymbol{\rho} \cdot g\mathbf{E}^b \Delta^\delta \right. \\ & \left. - \left(\sum_{i'j'k=1}^3 \underline{\mathbf{\Delta}}_{ijk}^\delta T_{i'i'}^b T_{j'j}^a \epsilon_{i'j'k} \right) \Delta^{\delta\dagger} \boldsymbol{\rho} \cdot g\mathbf{E}^b O^{Aa} \right] - V_{S\boldsymbol{\lambda} \cdot E O^A}^{(0,1)} \sum_{a=1}^8 \frac{1}{\sqrt{6}} [S^\dagger \boldsymbol{\lambda} \cdot g\mathbf{E}^a O^{Aa} + O^{Aa\dagger} \boldsymbol{\lambda} \cdot g\mathbf{E}^a S] \\ & - V_{O^A \boldsymbol{\lambda} \cdot E O^A}^{(0,1)} \sum_{abc=1}^8 \left(if^{abc} \frac{2m - m_3}{2m_R} + \frac{d^{abc}}{2} \right) O^{Aa\dagger} \boldsymbol{\lambda} \cdot g\mathbf{E}^b O^{Ac} \\ & + V_{O^S \boldsymbol{\lambda} \cdot E O^S}^{(0,1)} \sum_{abc=1}^8 \left(if^{abc} \frac{5m_3 - 2m}{6m_R} + \frac{d^{abc}}{2} \right) O^{Sa\dagger} \boldsymbol{\lambda} \cdot g\mathbf{E}^b O^{Sc} \\ & + V_{O^S \boldsymbol{\lambda} \cdot E \Delta}^{(0,1)} \sum_{ab=1}^8 \sum_{\delta=1}^{10} \left[\left(\frac{2}{\sqrt{3}} \sum_{i'j'k=1}^3 \epsilon_{ijk} T_{i'i'}^a T_{j'j}^b \underline{\mathbf{\Delta}}_{i'j'k}^\delta \right) O^{Sa\dagger} \boldsymbol{\lambda} \cdot g\mathbf{E}^b \Delta^\delta \right. \\ & \left. - \left(\frac{2}{\sqrt{3}} \sum_{i'j'k=1}^3 \underline{\mathbf{\Delta}}_{ijk}^\delta T_{i'i'}^b T_{j'j}^a \epsilon_{i'j'k} \right) \Delta^{\delta\dagger} \boldsymbol{\lambda} \cdot g\mathbf{E}^b O^{Sa} \right] \\ & - V_{\Delta \boldsymbol{\lambda} \cdot E \Delta}^{(0,1)} \frac{2m - 2m_3}{m_R} \sum_{a=1}^8 \sum_{\delta\delta'=1}^{10} \left(\sum_{i'j'k=1}^3 \underline{\mathbf{\Delta}}_{ijk}^\delta T_{i'i'}^a \underline{\mathbf{\Delta}}_{i'j'k}^{\delta'} \right) \Delta^{\delta\dagger} \boldsymbol{\lambda} \cdot g\mathbf{E}^a \Delta^{\delta'}, \quad (43) \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{\text{pNRQCD}}^{(1,0)} = & \int d^3\rho d^3\lambda S^\dagger \left[\frac{\nabla_R^2}{2m_R} + \frac{\nabla_\rho^2}{2m_\rho} + \frac{\nabla_\lambda^2}{2m_\lambda} \right] S + O^{A\dagger} \left[\frac{D_R^2}{2m_R} + \frac{\nabla_\rho^2}{2m_\rho} + \frac{\nabla_\lambda^2}{2m_\lambda} \right] O^A + O^{S\dagger} \left[\frac{D_R^2}{2m_R} + \frac{\nabla_\rho^2}{2m_\rho} + \frac{\nabla_\lambda^2}{2m_\lambda} \right] O^S \\ & + \Delta^\dagger \left[\frac{D_R^2}{2m_R} + \frac{\nabla_\rho^2}{2m_\rho} + \frac{\nabla_\lambda^2}{2m_\lambda} \right] \Delta + \dots, \end{aligned} \quad (44)$$

where the gauge fields in the covariant derivatives acting on the octets, $O^A = (O^{A1}, O^{A2}, \dots, O^{A8})$ and $O^S = (O^{S1}, O^{S2}, \dots, O^{S8})$, and the decuplet, $\Delta = (\Delta^1, \Delta^2, \dots, \Delta^{10})$, are understood in the octet and decuplet representations, respectively. The dots in the last line of Eq. (44) stand for terms that appear at orders higher than tree level and other $1/m$ terms, similar to those discussed for the doubly heavy-baryon case. These terms are suppressed in the power counting with respect to the kinetic energy and the terms shown in Eqs. (42) and (43).

The functions V are the Wilson coefficients of pNRQCD. They encode the contributions coming from gluons of energy or momentum of order $m\nu$. They are nonanalytic functions of $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$. As we will discuss in the next section, at tree level we have

$$\begin{aligned} V_{S\rho \cdot E O^S}^{(0,1)} = V_{O^A \rho \cdot E O^S}^{(0,1)} = V_{O^A \rho \cdot E \Delta}^{(0,1)} = 1, \\ V_{S\lambda \cdot E O^A}^{(0,1)} = V_{O^A \lambda \cdot E O^A}^{(0,1)} = V_{O^S \lambda \cdot E O^S}^{(0,1)} = V_{O^S \lambda \cdot E \Delta}^{(0,1)} = V_{\Delta \lambda \cdot E \Delta}^{(0,1)} = 1, \end{aligned} \quad (45)$$

while $V_S^{(0)}$, $V_{O^A}^{(0)}$, $V_{O^S}^{(0)}$ and $V_\Delta^{(0)}$ get the first nonvanishing contribution at order α_s . The coefficients in front of the operators D_R^2 , ∇_ρ^2 and ∇_λ^2 are equal to 1, due to Poincaré invariance or dynamical considerations similar to those developed in [31].

The power counting of the Lagrangian (41) in the center-of-mass frame goes as follows: ∇_λ , $\nabla_\rho \sim m\nu$, $\boldsymbol{\rho}$, $\boldsymbol{\lambda} \sim 1/(m\nu)$, $D_R \sim \Lambda_{\text{QCD}}$, $m\nu^2$, $V_{S,O^A,S,\Delta}^{(0)} \sim m\nu^2$ and \mathbf{E} , $\mathbf{B} \sim \Lambda_{\text{QCD}}^2$, $(m\nu^2)^2$. The pNRQCD Lagrangian at leading order reads

$$\begin{aligned} \mathcal{L}_{\text{pNRQCD}}^{\text{LO}} = & \int d^3\rho d^3\lambda \left\{ S^\dagger \left[i\partial_0 + \frac{\nabla_\rho^2}{2m_\rho} + \frac{\nabla_\lambda^2}{2m_\lambda} - V_S^{(0)} \right] S + O^{A\dagger} \left[iD_0 + \frac{\nabla_\rho^2}{2m_\rho} + \frac{\nabla_\lambda^2}{2m_\lambda} - V_{O^A}^{(0)} \right] O^A \right. \\ & \left. + O^{S\dagger} \left[iD_0 + \frac{\nabla_\rho^2}{2m_\rho} + \frac{\nabla_\lambda^2}{2m_\lambda} - V_{O^S}^{(0)} \right] O^S + \Delta^\dagger \left[iD_0 + \frac{\nabla_\rho^2}{2m_\rho} + \frac{\nabla_\lambda^2}{2m_\lambda} - V_\Delta^{(0)} \right] \Delta \right\} - \frac{1}{4} \sum_{a=1}^8 F_{\mu\nu}^a F^{a\mu\nu} + \sum_{f=1}^3 \bar{q}_f i \not{D} q_f. \end{aligned} \quad (46)$$

2. Matching

The matching from NRQCD to pNRQCD is performed by calculating Green functions in the two theories and imposing that they are equal order by order in the inverse of the mass and in the multipole expansion. Since we are working here in the situation where the typical momentum transfer between the heavy quarks is larger than Λ_{QCD} , we can in addition perform the matching order by order in α_s . The procedure is analogous to the one discussed previously for the doubly heavy-baryon case, which we follow closely.

The matching at tree level may be performed by projecting the NRQCD Hamiltonian on the three-quark Fock space spanned by

$$\begin{aligned} & \int d^3x_1 d^3x_2 d^3x_3 \sum_{ijk=1}^3 \Phi_{Q_1 Q_2 Q_3}^{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) Q_1^\dagger(\mathbf{x}_1) Q_2^\dagger(\mathbf{x}_2) \\ & \times Q_3^\dagger(\mathbf{x}_3) |0\rangle, \end{aligned} \quad (47)$$

where $\Phi_{Q_1 Q_2 Q_3}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is a $3 \otimes 3 \otimes 3$ tensor in color space and a $2 \otimes 2 \otimes 2$ tensor in spin space. After projection, all gluon fields are multipole expanded in $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$. In order to make gauge invariance explicit at the Lagrangian level, we decompose the three-quark fields into a field $S(\boldsymbol{\rho}, \boldsymbol{\lambda}, \mathbf{R}, t)$, which transforms like a color singlet, two fields $O^A(\boldsymbol{\rho}, \boldsymbol{\lambda}, \mathbf{R}, t)$ and $O^S(\boldsymbol{\rho}, \boldsymbol{\lambda}, \mathbf{R}, t)$, which transform like octets, and a field $\Delta(\boldsymbol{\rho}, \boldsymbol{\lambda}, \mathbf{R}, t)$, which transforms like a decuplet:

$$\begin{aligned} \Phi_{Q_1 Q_2 Q_3}^{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = & \sum_{i'j'k'=1}^3 \phi_{i'j'}(\mathbf{x}_1, \mathbf{R}, t) \phi_{j'j'}(\mathbf{x}_2, \mathbf{R}, t) \phi_{kk'}(\mathbf{x}_3, \mathbf{R}, t) \left(S(\boldsymbol{\rho}, \boldsymbol{\lambda}, \mathbf{R}, t) \underline{S}_{i'j'k'} + \sum_{a=1}^8 O^{Aa}(\boldsymbol{\rho}, \boldsymbol{\lambda}, \mathbf{R}, t) \underline{O}_{i'j'k'}^{Aa} \right. \\ & \left. + \sum_{a=1}^8 O^{Sa}(\boldsymbol{\rho}, \boldsymbol{\lambda}, \mathbf{R}, t) \underline{O}_{i'j'k'}^{Sa} + \sum_{\delta=1}^{10} \Delta^\delta(\boldsymbol{\rho}, \boldsymbol{\lambda}, \mathbf{R}, t) \underline{\Delta}_{i'j'k'}^\delta \right), \end{aligned} \quad (48)$$

where \underline{S}_{ijk} , \underline{O}_{ijk}^{Aa} , \underline{O}_{ijk}^{Sa} and $\underline{\Delta}_{ijk}^\delta$ have been defined in Appendix B 2 and the Wilson string ϕ_{ij} in Eq. (15). After projecting on (48) the Lagrangian (42)–(44) with the matching conditions (45) follows.

The perturbative matching of the static potentials $V_S^{(0)}$, $V_{O^A}^{(0)}$, $V_{O^S}^{(0)}$ and $V_\Delta^{(0)}$ goes as follows. In NRQCD we compute static Green functions, whose initial and final states overlap with the singlet, octet and decuplet fields in pNRQCD. A possible choice, working in a nongauge invariant framework, is

$$I_{\mathcal{M}}^{uv} \equiv \sum_{ijk'i'j'k'=1}^3 \langle 0 | \underline{\mathcal{M}}_{ijk}^u Q_i(\mathbf{R}, \mathbf{x}_1, T/2) Q_j(\mathbf{R}, \mathbf{x}_2, T/2) Q_k(\mathbf{R}, \mathbf{x}_3, T/2) \underline{\mathcal{M}}_{i'j'k'}^{v*} Q_{i'}^\dagger(\mathbf{R}, \mathbf{y}_1, -T/2) Q_{j'}^\dagger(\mathbf{R}, \mathbf{y}_2, -T/2) \times Q_{k'}^\dagger(\mathbf{R}, \mathbf{y}_3, -T/2) | 0 \rangle, \quad (49)$$

$$\begin{aligned} (1) \quad & \text{if } \mathcal{M} = S, & I_{\mathcal{M}}^{uv} &= I_S, & \underline{\mathcal{M}}_{ijk}^u &= \underline{S}_{ijk}, \\ (2) \quad & \text{if } \mathcal{M} = O^A, & I_{\mathcal{M}}^{uv} &= I_{O^A}^{uv}, & \underline{\mathcal{M}}_{ijk}^u &= \underline{O}_{ijk}^{Au}, & u, v &= 1, 2, \dots, 8, \\ (3) \quad & \text{if } \mathcal{M} = O^S, & I_{\mathcal{M}}^{uv} &= I_{O^S}^{uv}, & \underline{\mathcal{M}}_{ijk}^u &= \underline{O}_{ijk}^{Su}, & u, v &= 1, 2, \dots, 8, \\ (4) \quad & \text{if } \mathcal{M} = \Delta, & I_{\mathcal{M}}^{uv} &= I_\Delta^{uv}, & \underline{\mathcal{M}}_{ijk}^u &= \underline{\Delta}_{ijk}^u, & u, v &= 1, 2, \dots, 10, \end{aligned}$$

where $Q(\mathbf{R}, \mathbf{x}, t)$ has been defined in Eq. (23). Integrating out the heavy-quark fields from $I_{\mathcal{M}}^{uv}$ we obtain

$$I_{\mathcal{M}}^{uv} = \delta^3(\mathbf{x}_1 - \mathbf{y}_1) \delta^3(\mathbf{x}_2 - \mathbf{y}_2) \delta^3(\mathbf{x}_3 - \mathbf{y}_3) \langle 0 | (W_{QQQ}^{\mathcal{M}})^{uv} | 0 \rangle, \quad (50)$$

with $W_{QQQ}^{\mathcal{M}}$ diagrammatically represented in Fig. 2 and explicitly given by

$$(W_{QQQ}^{\mathcal{M}})^{uv} \equiv P \sum_{ijk'i'j'k'rst'r's't'=1}^3 \underline{\mathcal{M}}_{ijk}^u \phi_{i'i'}(\mathbf{R}, \mathbf{x}_1, T/2) \phi_{i'r'}(T/2, -T/2, \mathbf{x}_1) \phi_{r'r}(\mathbf{x}_1, \mathbf{R}, -T/2) \phi_{j'j'}(\mathbf{R}, \mathbf{x}_2, T/2) \times \phi_{j's'}(T/2, -T/2, \mathbf{x}_2) \phi_{s's}(\mathbf{x}_2, \mathbf{R}, -T/2) \phi_{k'k'}(\mathbf{R}, \mathbf{x}_3, T/2) \phi_{k't'}(T/2, -T/2, \mathbf{x}_3) \phi_{t't}(\mathbf{x}_3, \mathbf{R}, -T/2) \underline{\mathcal{M}}_{rst'}^{v*}. \quad (51)$$

In the large T limit, the Green functions I_S , $I_{O^A}^{uv}$, $I_{O^S}^{uv}$ and I_Δ^{uv} are reduced to the singlet, octet and decuplet propagators of pNRQCD, respectively. If we neglect subleading loop corrections to the pNRQCD side of the matching we obtain

$$\begin{aligned} \lim_{T \rightarrow \infty} \langle 0 | (W_{QQQ}^{\mathcal{M}})^{uv} | 0 \rangle &= \lim_{T \rightarrow \infty} Z_{\mathcal{M}}(\boldsymbol{\rho}, \boldsymbol{\lambda}) \exp(-iV_{\mathcal{M}}^{(0)}(\boldsymbol{\rho}, \boldsymbol{\lambda})T) \\ &\times \langle 0 | \sum_{ijk'i'j'k'=1}^3 \underline{\mathcal{M}}_{ijk}^u \phi_{i'i'}(T/2, -T/2, \mathbf{R}) \phi_{j'j'}(T/2, -T/2, \mathbf{R}) \phi_{k'k'}(T/2, -T/2, \mathbf{R}) \underline{\mathcal{M}}_{i'j'k'}^{v*} | 0 \rangle, \quad (52) \end{aligned}$$

where $Z_{\mathcal{M}}$ is a normalization factor. At order α_s , the result is

$$V_S^{(0)}(\boldsymbol{\rho}, \boldsymbol{\lambda}) = -\frac{2}{3} \alpha_s \left(\frac{1}{|\boldsymbol{\rho}|} + \frac{1}{|\boldsymbol{\lambda} + \boldsymbol{\rho}/2|} + \frac{1}{|\boldsymbol{\lambda} - \boldsymbol{\rho}/2|} \right), \quad (53)$$

$$V_{O^S}^{(0)}(\boldsymbol{\rho}, \boldsymbol{\lambda}) = \frac{\alpha_s}{3} \left(\frac{1}{|\boldsymbol{\rho}|} - \frac{5}{4} \frac{1}{|\boldsymbol{\lambda} + \boldsymbol{\rho}/2|} - \frac{5}{4} \frac{1}{|\boldsymbol{\lambda} - \boldsymbol{\rho}/2|} \right), \quad (55)$$

$$V_\Delta^{(0)}(\boldsymbol{\rho}, \boldsymbol{\lambda}) = \frac{\alpha_s}{3} \left(\frac{1}{|\boldsymbol{\rho}|} + \frac{1}{|\boldsymbol{\lambda} + \boldsymbol{\rho}/2|} + \frac{1}{|\boldsymbol{\lambda} - \boldsymbol{\rho}/2|} \right). \quad (56)$$

B. pNRQCD for strongly coupled QQQ baryons

1. Lagrangian and degrees of freedom

$$V_{O^A}^{(0)}(\boldsymbol{\rho}, \boldsymbol{\lambda}) = -\frac{2}{3} \alpha_s \left(\frac{1}{|\boldsymbol{\rho}|} - \frac{1}{8} \frac{1}{|\boldsymbol{\lambda} + \boldsymbol{\rho}/2|} - \frac{1}{8} \frac{1}{|\boldsymbol{\lambda} - \boldsymbol{\rho}/2|} \right), \quad (54)$$

In the situation in which the typical distances ρ and λ in the baryon are of the order $1/\Lambda_{\text{QCD}}$, the matching from NRQCD to pNRQCD cannot rely on perturbation theory anymore. Also, it is more difficult to identify the effective degrees of freedom of pNRQCD. Despite these difficulties,

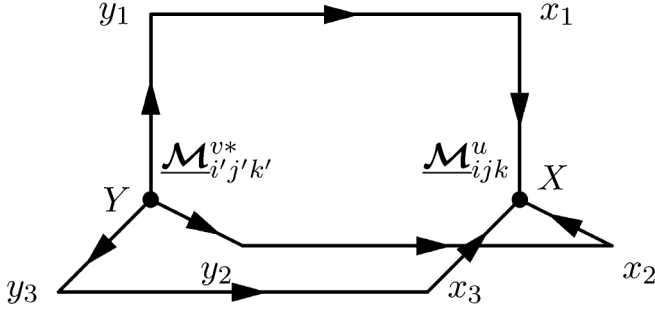


FIG. 2. Static Wilson loop with edges $x_1 = (x_1, T/2)$, $x_2 = (x_2, T/2)$, $x_3 = (x_3, T/2)$, $y_1 = (x_1, -T/2)$, $y_2 = (x_2, -T/2)$, $y_3 = (x_3, -T/2)$ and insertions of the tensors $\underline{M}_{ij'k}^u$ and $\underline{M}_{i'j'k'}^{v*}$ in $X = (\mathbf{R}, T/2)$ and $Y = (\mathbf{R}, -T/2)$ respectively.

the situation appears pretty much similar to the one described for strongly coupled quarkonium in [16,23]. From the available lattice simulations (e.g. [19], see Fig. 3), it appears that the gluonic excitations between three static quarks develop an energy gap of about $1 \text{ GeV} \approx \Lambda_{\text{QCD}}$ with respect to the lowest static energy. This means that all gluonic excitations between the heavy quarks are integrated out once we go to pNRQCD. pNRQCD in its simplest formulation, i.e. without light-quark degrees of freedom, is, therefore, as simple as a potential model.⁴ It has the three-quark singlet field $S(\boldsymbol{\rho}, \boldsymbol{\lambda}, \mathbf{R}, t)$ as the only degree of freedom and is described by a Lagrangian $\mathcal{L}_{\text{pNRQCD}} = \mathcal{L}_{\text{pNRQCD}}(\mathbf{R}, t)$, which reads

$$\mathcal{L}_{\text{pNRQCD}} = \int d^3\rho d^3\lambda S^\dagger \left[i\partial_0 + \frac{\nabla_{\mathbf{R}}^2}{2m_{\mathbf{R}}} + \frac{\nabla_{\boldsymbol{\rho}}^2}{2m_{\boldsymbol{\rho}}} + \frac{\nabla_{\boldsymbol{\lambda}}^2}{2m_{\boldsymbol{\lambda}}} - V_S \right] S. \quad (57)$$

The potential V_S may be organized in an expansion (not necessarily analytic [37]) in the inverse of the heavy-quark masses. In the following, we will consider the matching of the $1/m$ potential, which, in the nonperturbative regime, may, in principle, be of the same order as the static potential, and the $1/m^2$ spin-dependent potentials.

2. Matching

The nonperturbative matching goes as in the quarkonium case discussed in [16,23] to which we refer for further details. Here we only list some results.

The singlet static potential is given by

$$V_S^{(0)}(\boldsymbol{\rho}, \boldsymbol{\lambda}) = \lim_{T \rightarrow \infty} \frac{i}{T} \ln \langle 0 | W_{\text{QQQ}}^S | 0 \rangle, \quad (58)$$

where W_{QQQ}^S is the singlet Wilson loop defined in Eq. (51)

⁴The relevance of the energy gap in relation to the success of the quark model has also been stressed in [19].

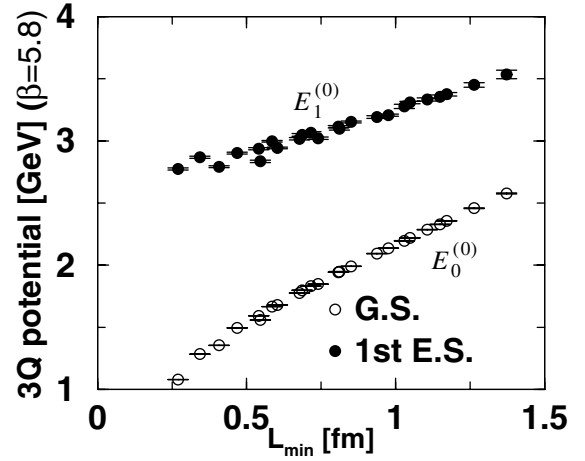


FIG. 3. Lattice measurements of the three-quark static energies of the lowest state, $E_0^{(0)}$, and of the first gluonic excitation, $E_1^{(0)}$, as a function of L_{min} , the minimal total length of the flux tubes linking the three quarks, from [19].

and shown in Fig. 2. Lattice evaluations of $V_S^{(0)}$ may be found in [19,38]. A plot is shown in Fig. 3.

The order $1/m$ potential is given by

$$V_S^{(1)} = \frac{V_S^{(1,1)}}{m} + \frac{V_S^{(1,3)}}{m_3}, \quad (59)$$

with

$$V_S^{(1,1)}(\boldsymbol{\rho}, \boldsymbol{\lambda}) = -\frac{1}{2} \sum_{i=1}^2 \int_0^\infty dt \text{tr} \langle \langle g\mathbf{E}(\mathbf{x}_i, t) \cdot g\mathbf{E}(\mathbf{x}_i, 0) \rangle \rangle_{c, \text{QQQ}}^S, \quad (60)$$

$$V_S^{(1,3)}(\boldsymbol{\rho}, \boldsymbol{\lambda}) = -\frac{1}{2} \int_0^\infty dt \langle \langle g\mathbf{E}(\mathbf{x}_3, t) \cdot g\mathbf{E}(\mathbf{x}_3, 0) \rangle \rangle_{c, \text{QQQ}}^S, \quad (61)$$

where the double brackets stand for the gauge field average in the presence of a static Wilson loop of infinite time length:

$$\langle \langle \dots \rangle \rangle_{\text{QQQ}}^S \equiv \lim_{T \rightarrow \infty} \frac{\langle 0 | \dots W_{\text{QQQ}}^S | 0 \rangle}{\langle 0 | W_{\text{QQQ}}^S | 0 \rangle}, \quad (62)$$

$$\begin{aligned} \langle \langle O_1(t_1) O_2(t_2) \rangle \rangle_{c, \text{QQQ}}^S &\equiv \langle \langle O_1(t_1) O_2(t_2) \rangle \rangle_{\text{QQQ}}^S \\ &\quad - \langle \langle O_1(t_1) \rangle \rangle_{\text{QQQ}}^S \langle \langle O_2(t_2) \rangle \rangle_{\text{QQQ}}^S \end{aligned} \quad (63)$$

with

$$\frac{T}{2} \geq t_1 \geq t_2 \geq -\frac{T}{2}.$$

As in the quarkonium case [23], in the nonperturbative regime the $1/m$ potential may, in principle, be of order

mv^2 , and therefore as important as the static potential. There are no available lattice data for this quantity.

For the potentials responsible for the spin splittings of the heavy baryons, we obtain at order $1/m^2$:

$$\begin{aligned}
V_S^{(2,\text{spin dep.})} = & \sum_{i=1}^3 \frac{c_S^{(i)}}{4m_i^2} \boldsymbol{\sigma}^{(i)} \cdot [(\nabla_{\mathbf{x}_i} V_S^{(0)}) \times (-i\nabla_{\mathbf{x}_i})] + \sum_{i,i'=1}^3 i \frac{c_F^{(i)}}{m_i m_{i'}} \int_0^\infty dt \sum_{kl=1}^3 \langle\langle g \mathbf{B}^k(\mathbf{x}_i, t) g \mathbf{E}^l(\mathbf{x}_{i'}, 0) \rangle\rangle_{c,QQQ}^S \boldsymbol{\sigma}_k^{(i)} (-i\nabla_{\mathbf{x}_{i'}}^l) \\
& - \sum_{i>i'=1}^3 i \frac{c_F^{(i)} c_F^{(i')}}{2m_i m_{i'}} \int_0^\infty dt \sum_{kl=1}^3 \langle\langle g \mathbf{B}^k(\mathbf{x}_i, t) g \mathbf{B}^l(\mathbf{x}_{i'}, 0) \rangle\rangle_{c,QQQ}^S \boldsymbol{\sigma}_k^{(i)} \boldsymbol{\sigma}_l^{(i')} \\
& - \sum_{i>i'=1}^3 (d_{Q_i Q_{i'}}^{sv} + d_{Q_i Q_{i'}}^{vv} \langle\langle T^{a(i)} T^{a(i')} \rangle\rangle_{c,QQQ}^S) \boldsymbol{\sigma}^{(i)} \cdot \boldsymbol{\sigma}^{(i')} \delta^3(\mathbf{x}_i - \mathbf{x}_{i'}), \tag{64}
\end{aligned}$$

where $T^{a(i)} T^{a(i')}$ stands for two color matrices T^a inserted at the same time in the Wilson lines of spatial coordinates \mathbf{x}_i and $\mathbf{x}_{i'}$ respectively, and the matching coefficients $d_{Q_i Q_{i'}}^{sv}$ and $d_{Q_i Q_{i'}}^{vv}$ have been calculated in Appendix A. The above expressions give at order α_s the well-known one-gluon exchange results [39]. The spin-dependent potentials have not been calculated on the lattice yet, differently from the quarkonium case, where such calculations have instead a long history [18]. Model dependent predictions may be found in [40–42]. It is expected that these potentials satisfy some exact relations due to Poincaré invariance of the type studied in [31,43,44] for the quarkonium case.

V. CONCLUSION AND OUTLOOK

This work is a first step in the direction of a complete study of baryons made of two or three heavy quarks in the framework of nonrelativistic EFTs of QCD. For both types of baryons, we identify the degrees of freedom and write the pNRQCD Lagrangian appropriate to describe the system in the heavy-quark sector. In the doubly heavy-baryon case this represents an update of Ref. [13], which, however, used a HQET framework. In the case of baryons made of three heavy quarks, we also provide nonperturbative expressions for some of the potentials. Relevantly for both types of systems, we calculate the one-loop matching of the 4-quark operators of lowest dimensionality.

Several further developments are possible. For doubly heavy baryons, where data are already available, an important step forward would consist in providing pNRQCD with a light-quark sector that fully implements chiral symmetry and chiral symmetry breaking effects. One could then study, for instance, isospin splittings and transitions and also address a variety of decay and production processes. The pursuit of such a program of phenomenological studies will, however, very much depend on the future of the experimental searches for these states.

For what concerns heavy baryons made of three heavy quarks, in the absence of a discovery, lattice studies will remain the main source of information. First of all, it will be important to have at least the one-loop expressions for the heavy-baryon static potentials $V_S^{(0)}$, $V_{O^A}^{(0)}$, $V_{O^S}^{(0)}$ and $V_\Delta^{(0)}$

(also for $V_T^{(0)}$ and $V_\Sigma^{(0)}$). This may lead to a precise comparison of short-range lattice data with perturbative QCD in the heavy-baryon sector. At three loop, the heavy-baryon static potentials exhibit an ultrasoft running like in the heavy-quarkonium case [45]. The ultrasoft running of the singlet static potential $V_S^{(0)}$ comes from the coupling with the octets O^A and O^S . The leading logarithmic contribution at order α_s^4 is

$$\begin{aligned}
\delta V_S^{(0)} = & \frac{4}{9} \frac{\alpha_s}{\pi} \lambda^2 (V_{O^A}^{(0)} - V_S^{(0)})^3 \ln \frac{(V_{O^A}^{(0)} - V_S^{(0)})^2}{4\pi\mu^2} \\
& + \frac{1}{3} \frac{\alpha_s}{\pi} \rho^2 (V_{O^S}^{(0)} - V_S^{(0)})^3 \ln \frac{(V_{O^S}^{(0)} - V_S^{(0)})^2}{4\pi\mu^2}, \tag{65}
\end{aligned}$$

where μ is the ultrasoft factorization scale.

Let us comment on the renormalon singularities affecting the perturbative series of the baryonic static potentials. These must cancel in physical observables. In the quarkonium case, the renormalon of the static potential cancels against twice the renormalon affecting the heavy-quark pole masses (see e.g. [46]). From Eq. (53) one can read that the order Λ_{QCD} renormalon affecting $V_S^{(0)}$ is $3 \times 1/2$ that one of the static potential in the quarkonium case. Indeed, in the expression of the baryon mass it cancels against 3 times the renormalon affecting the heavy-quark pole masses. Similarly, in the doubly heavy-baryon case, from Eq. (27) we have that the renormalon of order Λ_{QCD} affecting $V_T^{(0)}$ is $1/2$ that one of the static potential in the quarkonium case. In the expression of the baryon mass, it cancels against the renormalon affecting the $\bar{\Lambda}$ parameter of the HQET and the two heavy-quark masses.

Concerning the energies of gluonic excitations from three static sources, in the short range they are expected to behave like the singlet potential (53), if they are singlet plus glueball states, or like the octet or decuplet potentials (54)–(56) if they are hybrid states. Only if $E_1^{(0)}$ corresponds to the first case, the Coulomb contribution is expected to cancel in $E_1^{(0)} - E_0^{(0)}$, which is the difference between the energy of the first excited state and the ground state. This could be in contradiction with a statement in Ref. [19],

where the Coulomb contribution is said to cancel in the difference without any further specification. Like in the quarkonium case [29], it is expected that the ordering of the levels of the gluonic excitations in the short range is dictated by the correlation lengths of some gluonic operators. If we assume that correlation lengths of operators of higher dimensions are suppressed and if we consider that there is a singlet channel only in $8 \otimes 8$ but not in $10 \otimes 8$, then, in the short range, the leading gluonic excitation is expected to come from the coupling of an octet heavy-quark state with a gluon field. It would be interesting to investigate if the first gluonic excitation shown in Fig. 3 is such an octet hybrid, and in this case what kind of octet. If it is not an octet hybrid, then likely it exists a lower gluonic excitation that still needs to be identified.

Finally, in the perspective of a future spectroscopy of baryons made of three heavy quarks, it may become important to have a lattice determination of the spin-dependent potentials. Moreover, as the history of quarkonium suggests, spin-dependent observables may provide an excellent insight into the quark-confinement mechanism in the baryonic sector.

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APPENDIX A: 1-LOOP MATCHING OF 4-QUARK OPERATORS OF DIMENSION 6

The only graphs contributing to the 1-loop matching of the 4-quark operators of dimension 6 are displayed in Fig. 4. The situation is similar to the quark-antiquark case with different masses studied in [47].

In the case of two different quarks of masses m_h and $m_{h'}$ ($h \neq h'$) we obtain in the $\overline{\text{MS}}$ scheme:

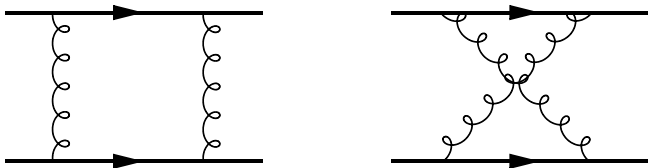


FIG. 4. Feynman diagrams contributing to the 1-loop matching of the 4-quark operators of dimension 6.

$$d_{Q_h Q_{h'}}^{ss} = C_F \left(\frac{C_A}{2} - C_F \right) \frac{\alpha_s^2}{m_h^2 - m_{h'}^2} \left\{ m_h^2 \left(\ln \frac{m_{h'}^2}{\mu^2} + \frac{1}{3} \right) - m_{h'}^2 \left(\ln \frac{m_h^2}{\mu^2} + \frac{1}{3} \right) \right\}, \quad (\text{A1})$$

$$d_{Q_h Q_{h'}}^{sv} = C_F \left(\frac{C_A}{2} - C_F \right) \frac{\alpha_s^2}{m_h^2 - m_{h'}^2} m_h m_{h'} \ln \frac{m_h^2}{m_{h'}^2}, \quad (\text{A2})$$

$$d_{Q_h Q_{h'}}^{vs} = \frac{2C_F \alpha_s^2}{m_h^2 - m_{h'}^2} \left\{ m_h^2 \left(\ln \frac{m_{h'}^2}{\mu^2} + \frac{1}{3} \right) - m_{h'}^2 \left(\ln \frac{m_h^2}{\mu^2} + \frac{1}{3} \right) \right\} - \frac{C_A \alpha_s^2}{4(m_h^2 - m_{h'}^2)} \left[3 \left\{ m_h^2 \left(\ln \frac{m_{h'}^2}{\mu^2} + \frac{1}{3} \right) - m_{h'}^2 \left(\ln \frac{m_h^2}{\mu^2} + \frac{1}{3} \right) \right\} - \frac{1}{m_h m_{h'}} \left\{ m_h^4 \left(\ln \frac{m_{h'}^2}{\mu^2} + \frac{10}{3} \right) - m_{h'}^4 \left(\ln \frac{m_h^2}{\mu^2} + \frac{10}{3} \right) \right\} \right], \quad (\text{A3})$$

$$d_{Q_h Q_{h'}}^{vv} = \frac{2C_F \alpha_s^2}{m_h^2 - m_{h'}^2} m_h m_{h'} \ln \frac{m_h^2}{m_{h'}^2} - \frac{C_A \alpha_s^2}{4(m_h^2 - m_{h'}^2)} \times \left[\left\{ m_h^2 \left(\ln \frac{m_{h'}^2}{\mu^2} + 5 \right) - m_{h'}^2 \left(\ln \frac{m_h^2}{\mu^2} + 5 \right) \right\} + 3 m_h m_{h'} \ln \frac{m_h^2}{m_{h'}^2} \right], \quad (\text{A4})$$

where $C_A = N_c = 3$ and $C_F = (N_c^2 - 1)/(2N_c) = 4/3$. For $m_h = m_{h'} = m$ the above formulas become

$$d_{QQ}^{ss} = C_F \left(\frac{C_A}{2} - C_F \right) \alpha_s^2 \left(\ln \frac{m^2}{\mu^2} - \frac{2}{3} \right), \quad (\text{A5})$$

$$d_{QQ}^{sv} = C_F \left(\frac{C_A}{2} - C_F \right) \alpha_s^2, \quad (\text{A6})$$

$$d_{QQ}^{vs} = 2C_F \alpha_s^2 \left(\ln \frac{m^2}{\mu^2} - \frac{2}{3} \right) - \frac{1}{4} C_A \alpha_s^2 \left(\ln \frac{m^2}{\mu^2} - \frac{23}{3} \right), \quad (\text{A7})$$

$$d_{QQ}^{vv} = 2C_F \alpha_s^2 - \frac{C_A \alpha_s^2}{4} \left(\ln \frac{m^2}{\mu^2} + 7 \right). \quad (\text{A8})$$

Working in D dimensions, we have used the prescription $\epsilon_{ijk} \epsilon_{ijk} = (D-1)(D-2)(D-3)$. If the prescription $\epsilon_{ijk} \epsilon_{ijk} = (D-1)(D-2)$ of [47] is instead used, this amounts to changing $d_{Q_h Q_{h'}}^{vv} \rightarrow d_{Q_h Q_{h'}}^{vv} + C_A \alpha_s^2/2$.

APPENDIX B: GROUP FACTORS

1. Multiplet tensors: $3 \otimes 3$

The product of two triplet representations of $SU(3)$ may be decomposed into the sum of an antitriplet and a sextet representation: $3 \otimes 3 = \bar{3} \oplus 6$. A possible matrix represen-

tation for the antitriplet ($\underline{\mathbf{T}}_{ij}^\ell$, $\ell, i, j = 1, 2, 3$) and the sextet ($\underline{\mathbf{\Sigma}}_{ij}^\sigma$, $\sigma = 1, 2, \dots, 6$ and $i, j = 1, 2, 3$) is

$$\underline{\mathbf{T}}_{ij}^\ell = \frac{1}{\sqrt{2}} \epsilon_{lij}, \quad (\text{B1})$$

$$\begin{aligned} \underline{\mathbf{\Sigma}}_{11}^1 &= \underline{\mathbf{\Sigma}}_{22}^4 = \underline{\mathbf{\Sigma}}_{33}^6 = 1, \\ \underline{\mathbf{\Sigma}}_{12}^2 &= \underline{\mathbf{\Sigma}}_{21}^2 = \underline{\mathbf{\Sigma}}_{13}^3 = \underline{\mathbf{\Sigma}}_{31}^3 = \underline{\mathbf{\Sigma}}_{23}^5 = \underline{\mathbf{\Sigma}}_{32}^5 = \frac{1}{\sqrt{2}}, \end{aligned}$$

all other entries are zero. (B2)

Both $\underline{\mathbf{T}}_{ij}^\ell$ and $\underline{\mathbf{\Sigma}}_{ij}^\sigma$ are real; $\underline{\mathbf{T}}_{ij}^\ell$ is totally antisymmetric and $\underline{\mathbf{\Sigma}}_{ij}^\sigma$ totally symmetric. They satisfy the orthogonality and normalization relations:

$$\begin{aligned} \sum_{ij=1}^3 \underline{\mathbf{T}}_{ij}^\ell \underline{\mathbf{T}}_{ij}^{\ell'} &= \delta^{\ell\ell'}, & \sum_{ij=1}^3 \underline{\mathbf{\Sigma}}_{ij}^\sigma \underline{\mathbf{\Sigma}}_{ij}^{\sigma'} &= \delta^{\sigma\sigma'}, \\ \sum_{ij=1}^3 \underline{\mathbf{T}}_{ij}^\ell \underline{\mathbf{\Sigma}}_{ij}^\sigma &= 0. \end{aligned} \quad (\text{B3})$$

2. Multiplet tensors: $3 \otimes 3 \otimes 3$

The product of three triplet representations of $SU(3)$ may be decomposed into the sum of a singlet, two octet and a decuplet representation: $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$. A possible matrix representation for the singlet ($\underline{\mathbf{S}}_{ijk}$, $i, j, k = 1, 2, 3$), the octets ($\underline{\mathbf{O}}_{ijk}^{Aa}$ and $\underline{\mathbf{O}}_{ijk}^{Sa}$, $a = 1, 2, \dots, 8$, $i, j, k = 1, 2, 3$) and the decuplet ($\underline{\mathbf{\Delta}}_{ijk}^\delta$, $\delta = 1, 2, \dots, 10$, $i, j, k = 1, 2, 3$) is

$$\underline{\mathbf{S}}_{ijk} = \frac{1}{\sqrt{6}} \epsilon_{ijk}, \quad (\text{B4})$$

$$\underline{\mathbf{O}}_{ijk}^{Aa} = \frac{1}{2} \sum_{n=1}^3 \epsilon_{ijn} \lambda_{kn}^a, \quad (\text{B5})$$

$$\underline{\mathbf{O}}_{ijk}^{Sa} = \frac{1}{2\sqrt{3}} \sum_{n=1}^3 (\epsilon_{jkn} \lambda_{in}^a + \epsilon_{ikn} \lambda_{jn}^a), \quad (\text{B6})$$

$$\begin{aligned} \underline{\mathbf{\Delta}}_{111}^1 &= \underline{\mathbf{\Delta}}_{222}^4 = \underline{\mathbf{\Delta}}_{333}^{10} = 1, \\ \underline{\mathbf{\Delta}}_{112}^2 &= \underline{\mathbf{\Delta}}_{121}^2 = \underline{\mathbf{\Delta}}_{211}^2 = \underline{\mathbf{\Delta}}_{122}^3 = \underline{\mathbf{\Delta}}_{212}^3 = \underline{\mathbf{\Delta}}_{221}^3 = \frac{1}{\sqrt{3}}, \\ \underline{\mathbf{\Delta}}_{113}^5 &= \underline{\mathbf{\Delta}}_{131}^5 = \underline{\mathbf{\Delta}}_{311}^5 = \underline{\mathbf{\Delta}}_{223}^7 = \underline{\mathbf{\Delta}}_{232}^7 = \underline{\mathbf{\Delta}}_{322}^7 = \frac{1}{\sqrt{3}}, \\ \underline{\mathbf{\Delta}}_{133}^8 &= \underline{\mathbf{\Delta}}_{313}^8 = \underline{\mathbf{\Delta}}_{331}^8 = \underline{\mathbf{\Delta}}_{233}^9 = \underline{\mathbf{\Delta}}_{323}^9 = \underline{\mathbf{\Delta}}_{332}^9 = \frac{1}{\sqrt{3}}, \\ \underline{\mathbf{\Delta}}_{123}^6 &= \underline{\mathbf{\Delta}}_{132}^6 = \underline{\mathbf{\Delta}}_{213}^6 = \underline{\mathbf{\Delta}}_{231}^6 = \underline{\mathbf{\Delta}}_{312}^6 = \underline{\mathbf{\Delta}}_{321}^6 = \frac{1}{\sqrt{6}}, \end{aligned} \quad (\text{B7})$$

all other entries are zero,

where λ^a are the Gell-Mann matrices. $\underline{\mathbf{S}}_{ijk}$ and $\underline{\mathbf{\Delta}}_{ijk}^\delta$ are

real; $\underline{\mathbf{S}}_{ijk}$ is totally antisymmetric and $\underline{\mathbf{\Delta}}_{ijk}^\delta$ totally symmetric. The octets $\underline{\mathbf{O}}_{ijk}^{Aa}$ and $\underline{\mathbf{O}}_{ijk}^{Sa}$ have been chosen to be, respectively, antisymmetric and symmetric in the first two indices. The matrices satisfy the orthogonality and normalization relations:

$$\begin{aligned} \sum_{ijk=1}^3 \underline{\mathbf{S}}_{ijk} \underline{\mathbf{S}}_{ijk} &= 1, & \sum_{ijk=1}^3 \underline{\mathbf{O}}_{ijk}^{Aa*} \underline{\mathbf{O}}_{ijk}^{Aa'} &= \delta^{aa'}, \\ \sum_{ijk=1}^3 \underline{\mathbf{O}}_{ijk}^{Sa*} \underline{\mathbf{O}}_{ijk}^{Sa'} &= \delta^{aa'}, & \sum_{ijk=1}^3 \underline{\mathbf{\Delta}}_{ijk}^\delta \underline{\mathbf{\Delta}}_{ijk}^{\delta'} &= \delta^{\delta\delta'}, \\ \sum_{ijk=1}^3 \underline{\mathbf{S}}_{ijk} \underline{\mathbf{O}}_{ijk}^{Aa} &= \sum_{ijk=1}^3 \underline{\mathbf{S}}_{ijk} \underline{\mathbf{O}}_{ijk}^{Sa} = \sum_{ijk=1}^3 \underline{\mathbf{S}}_{ijk} \underline{\mathbf{\Delta}}_{ijk}^\delta = 0, \\ \sum_{ijk=1}^3 \underline{\mathbf{O}}_{ijk}^{Aa*} \underline{\mathbf{O}}_{ijk}^{Sa'} &= \sum_{ijk=1}^3 \underline{\mathbf{O}}_{ijk}^{Aa*} \underline{\mathbf{\Delta}}_{ijk}^\delta = \sum_{ijk=1}^3 \underline{\mathbf{O}}_{ijk}^{Sa*} \underline{\mathbf{\Delta}}_{ijk}^\delta = 0. \end{aligned} \quad (\text{B8})$$

3. $SU(3)$ representations

Here we list our choice of matrix representations for the $SU(3)$ generators in the 3, $\bar{3}$, 6, 8, 10 representations:

$$T^a \equiv T_3^a \equiv \frac{\lambda^a}{2}, \quad (\text{B9})$$

$$T_3^a \equiv -\frac{\lambda^{aT}}{2}, \quad (\text{B10})$$

$$(T_6^a)_{\sigma\sigma'} \equiv \sum_{ijk=1}^3 \underline{\mathbf{\Sigma}}_{ij}^\sigma \lambda_{jk}^a \underline{\mathbf{\Sigma}}_{ki}^{\sigma'}, \quad \sigma, \sigma' = 1, 2, \dots, 6, \quad (\text{B11})$$

$$(T_8^a)_{bc} \equiv i f^{bac}, \quad b, c = 1, 2, \dots, 8, \quad (\text{B12})$$

$$(T_{10}^a)_{\delta\delta'} \equiv \frac{3}{2} \sum_{i'jk=1}^3 \underline{\mathbf{\Delta}}_{ijk}^\delta \lambda_{i'i}^a \underline{\mathbf{\Delta}}_{j'k}^{\delta'}, \quad \delta, \delta' = 1, 2, \dots, 10, \quad (\text{B13})$$

where $a = 1, 2, \dots, 8$.

APPENDIX C: SPIN STATES

Let us consider a meson made by a heavy antiquark Q and a light quark q . We denote by $|P^*; \mathbf{S}_{\bar{Q}q}^z\rangle$ the lowest $S_{\bar{Q}q} = 1$ states ($\mathbf{S}_{\bar{Q}q}^z = 1, 0, -1$), and by $|P; 0\rangle$ the lowest $S_{\bar{Q}q} = 0$ state. The heavy antiquark content of the states may be made explicit by writing

$$|P^*; 1\rangle = \int d^3R Q_+^\dagger(\mathbf{R}) |S_i^z = 1/2\rangle, \quad (\text{C1})$$

$$|P^*; 0\rangle = \int d^3R \frac{1}{\sqrt{2}} (Q_+^\dagger(\mathbf{R})|\mathbf{S}_i^z = -1/2\rangle + Q_-^\dagger(\mathbf{R})|\mathbf{S}_i^z = 1/2\rangle), \quad (\text{C2})$$

$$|P^*; -1\rangle = \int d^3R Q_-^\dagger(\mathbf{R})|\mathbf{S}_i^z = -1/2\rangle, \quad (\text{C3})$$

$$|P; 0\rangle = \int d^3R \frac{1}{\sqrt{2}} (Q_+^\dagger(\mathbf{R})|\mathbf{S}_i^z = -1/2\rangle - Q_-^\dagger(\mathbf{R})|\mathbf{S}_i^z = 1/2\rangle). \quad (\text{C4})$$

In the case of the lowest doubly heavy-baryon states, we denote by $|\Xi^*; \mathbf{S}_{QQq}^z\rangle$ the $S_{QQq} = 3/2$ states ($\mathbf{S}_{QQq}^z = \pm 3/2, \pm 1/2$), and by $|\Xi; \pm 1/2\rangle$ the $S_{QQq} = 1/2$ states. The heavy antitriplet content of the states may be made explicit by writing

$$|\Xi^*; 3/2\rangle = \int d^3R d^3r \varphi_{QQ}(\mathbf{r}) T_+^\dagger(\mathbf{r}, \mathbf{R}) |\mathbf{S}_i^z = 1/2\rangle, \quad (\text{C5})$$

$$|\Xi^*; 1/2\rangle = \int d^3R d^3r \varphi_{QQ}(\mathbf{r}) \left(\sqrt{\frac{1}{3}} T_+^\dagger(\mathbf{r}, \mathbf{R}) |\mathbf{S}_i^z = -1/2\rangle + \sqrt{\frac{2}{3}} T_0^\dagger(\mathbf{r}, \mathbf{R}) |\mathbf{S}_i^z = 1/2\rangle \right), \quad (\text{C6})$$

$$|\Xi^*; -1/2\rangle = \int d^3R d^3r \varphi_{QQ}(\mathbf{r}) \left(\sqrt{\frac{2}{3}} T_0^\dagger(\mathbf{r}, \mathbf{R}) |\mathbf{S}_i^z = -1/2\rangle + \sqrt{\frac{1}{3}} T_-^\dagger(\mathbf{r}, \mathbf{R}) |\mathbf{S}_i^z = 1/2\rangle \right), \quad (\text{C7})$$

$$|\Xi^*; -3/2\rangle = \int d^3R d^3r \varphi_{QQ}(\mathbf{r}) T_-^\dagger(\mathbf{r}, \mathbf{R}) |\mathbf{S}_i^z = -1/2\rangle, \quad (\text{C8})$$

$$|\Xi; 1/2\rangle = \int d^3R d^3r \varphi_{QQ}(\mathbf{r}) \left(\sqrt{\frac{2}{3}} T_+^\dagger(\mathbf{r}, \mathbf{R}) |\mathbf{S}_i^z = -1/2\rangle - \sqrt{\frac{1}{3}} T_0^\dagger(\mathbf{r}, \mathbf{R}) |\mathbf{S}_i^z = 1/2\rangle \right), \quad (\text{C9})$$

$$|\Xi; -1/2\rangle = \int d^3R d^3r \varphi_{QQ}(\mathbf{r}) \left(\sqrt{\frac{1}{3}} T_0^\dagger(\mathbf{r}, \mathbf{R}) |\mathbf{S}_i^z = -1/2\rangle - \sqrt{\frac{2}{3}} T_-^\dagger(\mathbf{r}, \mathbf{R}) |\mathbf{S}_i^z = 1/2\rangle \right), \quad (\text{C10})$$

where

$$\int d^3r \varphi_{QQ}^*(\mathbf{r}) \varphi_{QQ}(\mathbf{r}) = 1. \quad (\text{C11})$$

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