Generalized parton distributions in the impact parameter space with nonzero skewedness

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We investigate the generalized parton distributions (GPDs) with nonzero ξ and $\Delta^\perp$ for a relativistic spin-1/2 composite system, namely, for an electron dressed with a photon, in light-front framework by expressing them in terms of overlaps of light-cone wave functions. The wave function provides a template for the quark spin-one diquark structure of the valence light-cone wave function of the proton. We verify the inequalities among the GPDs with different helicities and show the qualitative behavior of the fermion and gauge boson GPDs in the impact parameter space.

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I. INTRODUCTION

Generalized parton distributions (GPDs) have attracted a considerable amount of theoretical and experimental attention recently. An interesting physical interpretation of GPDs has been obtained in [1,2] by taking their Fourier transform with respect to the transverse momentum transfer. When the longitudinal momentum transfer $\xi = 0$, this gives the distribution of partons in the nucleon in the transverse plane. They are called impact parameter dependent parton distributions $q(x, b^\perp)$. In fact they obey certain positivity constraints which justify their physical interpretation as probability densities. This interpretation holds in the infinite momentum frame (even the forward parton distribution functions have a probabilistic interpretation only in this frame) and there is no relativistic correction to this identification because in light-front formalism, as well as in the infinite momentum frame, the transverse boosts act like nonrelativistic Galilean boosts. It is to be remembered that the GPDs, being off-forward matrix elements of light-front bilocal currents do not have a probabilistic interpretation, rather they have interpretation as probability amplitudes. $q(x, b^\perp)$ is defined in a proton state with a sharp plus momentum p^+ and localized in the transverse plane such that the transverse center of momentum $R^\perp = 0$ (normally, one should work with a wave packet state which is very localized in transverse position space, in order to avoid the state to be normalized to a delta function [2,3]). $q(x, b^\perp)$ gives simultaneous information about the longitudinal momentum fraction x and the transverse distance b of the parton from the center of the proton and thus gives a new insight to the internal structure of the proton. The impact parameter space representation also has been extended to the spin-dependent GPDs [1] and chiral odd ones [4].

GPDs $H_q(x, 0, t)$ have been investigated in the impact parameter space in several approaches, for example, in the transverse lattice formalism for the pion [5], in a two component (spectator) model [6] for the nucleon, in the chiral quark model for the pion [7], and using a power law wave function for the pion [8]. The spin-flip GPD E_q has not been addressed in these. The connection of E_q in the impact parameter space and the Siver's effect has been shown in [9] within the framework of the scalar diquark model of the proton. In a previous work [10], we have calculated both $H(x, 0, t)$ and $E(x, 0, t)$ in the impact parameter space for a spin-1/2 composite relativistic system, namely, for an electron dressed with a photon in QED. The state can be expanded in Fock space in terms of light-cone wave functions. The GPDs are expressed as overlaps of light-cone wave functions [11]. The wave functions in this case can be obtained from perturbation theory, and thus their correlations are known at a certain order in the coupling constant. Their general form provides a template for the effective quark spin-one diquark structure of the valence light-cone wave function of the proton [12]. Such a model is self consistent and has been used to investigate the helicity structure of a composite relativistic system [12]. An interesting advantage is that the two-body Fock component contains a gauge boson as one of its constituents and so it is possible to investigate the gauge boson GPDs H_g and E_g. Studies of the deep inelastic scattering structure functions in this approach and for a dressed quark state also yield interesting results [13,14].

So far we have discussed GPDs in impact parameter space for $\xi = 0$. However, deeply virtual Compton scattering experiments probe GPDs at nonzero ξ. In this case, a Fourier transform with respect to the transverse momentum transfer $\Delta^\perp$ is not enough to diagonalize the GPDs and thus giving a density interpretation. As the longitudinal momentum in the final state is different from that in the initial state, the resulting matrix element would still be off diagonal. Recently, certain reduced Wigner distributions, when integrated over the transverse momenta of the partons, were shown to be the Fourier transforms of GPDs [15], and they can be interpreted as the 3D density in the

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rest frame of the proton for the quarks with light-cone momentum fraction x. In fact, integration over the z coordinate relates them to the impact parameter dependent parton distributions with $\xi = 0$. In [3] it has been shown that for nonzero ξ, the Fourier transform of the GPDs with respect to $\Delta^\perp$ probes partons at transverse position $b^\perp$, with the initial and final protons localized around $0^\perp$ but shifted from each other by an amount of order $\xi b^\perp$. At the same time, the longitudinal momentum of the protons are specified. This difference of the transverse position of the protons depends on ξ but not on x and thus this information should be present in the scattering amplitudes measurable in experiments where the GPDs enter through a convolution in x. This aspect makes it interesting to investigate the GPDs in the impact parameter space for nonzero ξ. Here also, a useful approach is based on the overlap representation of GPDs in terms of light-cone wave functions [11]. The overlap representation also can be formulated directly in the impact parameter space, in terms of overlaps of light-cone wave functions $\psi(x, b^\perp)$, which are the Fourier transforms of the wave functions with definite transverse momenta $\psi(x, k^\perp)$.

Here, we calculate the GPDs $H_{q,g}(x, \xi, t)$ and $E_{q,g}(x, \xi, t)$ for an effective spin-1/2 system of an electron dressed with a photon in QED and we investigate them in the impact parameter space. The plan of the paper is as follows: The definitions of the fermion and gauge boson GPDs are given in Sec. II. The fermion and gauge boson GPDs are calculated, respectively, in Secs. III and IV for a dressed electron state. The GPDs are expressed in the impact parameter space in Sec. V. The issue of certain inequalities among the GPDs in the impact parameter space is addressed in Sec. VI. The summary and discussions are given in Sec. VII.

II. GENERALIZED PARTON DISTRIBUTIONS

The GPDs are defined in terms of off-forward matrix elements of light-front bilocal currents. In the light-front gauge $A^+ = 0$ we have

$$\begin{aligned}F^{+q}_{\lambda'\lambda} &= \int \frac{dy^-}{8\pi} e^{(i/2)x\bar{P}^+y^-} \left\langle P'\lambda' \,|\, \bar{\psi}\left(-\frac{y^-}{2}\right)\gamma^+\psi\left(\frac{y^-}{2}\right) | \, P\lambda \right\rangle \\ &= \frac{1}{2\bar{P}^+}\bar{U}_{\lambda'}(P')\left[H_q(x,\xi,t)\gamma^+ + E_q(x,\xi,t)\frac{i}{2M}\sigma^{+\alpha}\Delta_\alpha\right] \\ &\quad \times U_\lambda(P) + \ldots, \end{aligned} \tag{2.1}$$

$$\begin{aligned}F^{+g}_{\lambda'\lambda} &= \frac{1}{8\pi x\bar{P}^+}\int dy^-\, e^{(i/2)\bar{P}^+y^-x} \\ &\quad \times \left\langle P'\lambda' | F^{+\alpha}\left(-\frac{y^-}{2}\right)F^+_\alpha\left(\frac{y^-}{2}\right) | \, P\lambda\right\rangle \\ &= \frac{1}{2\bar{P}^+}\bar{U}_{\lambda'}(P')\Big[H_g(x,\xi,t)\gamma^+ + E_g(x,\xi,t) \\ &\quad \times \frac{i}{2M}\sigma^{+\alpha}\Delta_\alpha\Big]U_\lambda(P) + \ldots, \end{aligned} \tag{2.2}$$

where the ellipses indicate higher twist terms. The momenta of the initial (final) state is $P(P')$ and helicity $\lambda(\lambda')$. $U_\lambda(P)$ is the light-front spinor for the proton. The momentum transfer is given by $\Delta^\mu = P'^\mu - P^\mu$, skewedness $\xi = -\frac{\Delta^+}{2\bar{P}^+}$. The average momentum of the initial and final state proton is $\bar{P}^\mu = \frac{P^\mu + P'^\mu}{2}$. We take the frame where $\bar{P}^\perp = 0$. Without any loss of generality, we take $\xi > 0$. t is the invariant momentum transfer in the process, $t = \Delta^2$. For simplicity we suppress the flavor indices. Following [3] we define

$$D^\perp = \frac{P'^\perp}{1-\xi} - \frac{P^\perp}{1+\xi} = \frac{\Delta^\perp}{1-\xi^2}. \tag{2.3}$$

$H_{q,g}$ and $E_{q,g}$ are the twist two fermion and gauge boson GPDs. Using the light-cone spinors [16] we get

$$\begin{aligned}F^{+q,g}_{++} &= F^{+q,g}_{--} \\ &= \sqrt{1-\xi^2}H_{q,g}(x,\xi,t) - \frac{\xi^2}{\sqrt{1-\xi^2}}E_{q,g}(x,\xi,t),\end{aligned} \tag{2.4}$$

$$F^{+q,g}_{+-} = F^{+q,g}_{-+} = \frac{-\Delta^1 + i\Delta^2}{2M\sqrt{1-\xi^2}}E_{q,g}(x,\xi,t). \tag{2.5}$$

Note that $E_{q,g}$ appear both in helicity flip and helicity nonflip parts. For $\xi = 0$, $H_{q,g}$ correspond to nucleon helicity nonflip and $E_{q,g}$ correspond to the helicity flip part.

III. FERMION GPDS

We take the state $| \, P, \sigma\rangle$ to be a dressed electron consisting of bare states of an electron and an electron plus a photon:

$$\begin{aligned}| \, P, \sigma\rangle &= \phi_1 b^\dagger(P,\sigma) \\ &\quad | \, 0\rangle + \sum_{\sigma_1,\lambda_2}\int\frac{dk_1^+ d^2k_1^\perp}{\sqrt{2(2\pi)^3k_1^+}}\int\frac{dk_2^+ d^2k_2^\perp}{\sqrt{2(2\pi)^3k_2^+}} \\ &\quad \times\sqrt{2(2\pi)^3P^+}\,\delta^3(P-k_1-k_2)\phi_2(P,\sigma \\ &\quad | \, k_1,\sigma_1; k_2,\lambda_2)b^\dagger(k_1,\sigma_1)a^\dagger(k_2,\lambda_2) \, | \, 0\rangle. \end{aligned} \tag{3.1}$$

Here $a^\dagger$ and $b^\dagger$ are bare photon and electron creation operators, respectively, and ϕ_1 and ϕ_2 are the multiparton wave functions. They are the probability amplitudes to find one bare electron and one electron plus photon inside the dressed electron state, respectively.

We introduce Jacobi momenta x_i, $q_i^\perp$ such that $\sum_i x_i = 1$ and $\sum_i q_i^\perp = 0$. They are defined as

$$x_i = \frac{k_i^+}{P^+}, \qquad q_i^\perp = k_i^\perp - x_iP^\perp. \tag{3.2}$$

Also, we introduce the wave functions,

$$\psi_1 = \phi_1, \qquad \psi_2(x_i, q_i^\perp) = \sqrt{P^+}\phi_2(k_i^+, k_i^\perp); \tag{3.3}$$

which are independent of the total transverse momentum $P^\perp$ of the state and are boost invariant. The state is normalized as

$$\langle P', \lambda' \mid P, \lambda\rangle = 2(2\pi)^3 P^+ \delta_{\lambda,\lambda'}\delta(P^+ - P'^+)\delta^2(P^\perp - P'^\perp). \tag{3.4}$$

The two-particle wave function depends on the helicities of the electron and photon. Using the eigenvalue equation for the light-cone Hamiltonian, this can be written as [13]

$$\psi^\sigma_{2\sigma_1,\lambda}(x, q^\perp) = -\frac{x(1-x)}{(q^\perp)^2 + m^2(1-x)^2}\frac{1}{\sqrt{(1-x)}} \times \frac{e}{\sqrt{2(2\pi)^3}}\chi^\dagger_{\sigma_1}\Big[2\frac{q^\perp}{1-x} + \frac{\tilde{\sigma}^\perp\cdot q^\perp}{x}\tilde{\sigma}^\perp - im\tilde{\sigma}^\perp\frac{(1-x)}{x}\Big]\chi_\sigma\epsilon_\lambda^{\perp *}\psi_1. \tag{3.5}$$

m is the bare mass of the electron, $\tilde{\sigma}^2 = -\sigma^1$ and $\tilde{\sigma}^1 = \sigma^2$. ψ_1 actually gives the normalization of the state [13]:

$$|\psi_1|^2 = 1 - \frac{\alpha}{2\pi}\int_\epsilon^{1-\epsilon} dx\Big[\frac{1+x^2}{1-x}\log\frac{\Lambda^2}{m^2(1-x)^2} - \frac{1+x^2}{1-x} + (1-x)\Big], \tag{3.6}$$

within order α. Here ϵ is a small cutoff on x, the longitudinal momentum fraction carried by the fermion. We have taken the cutoff of the transverse momenta to be Λ^2 [13]. This gives the large scale of the process. The above expression is derived using Eqs. (3.4), (3.1), and (3.5).

The helicity nonflip part of the matrix element F^{+q}_{++} gives information about both H_q and E_q, as can be seen from (2.4). In terms of the wave function this can be written as

$$F^{+q}_{++} = |\psi_1|^2\,\delta(1-x) + \int d^2q^\perp \psi^*_{2+}\Big(\frac{x-\xi}{1-\xi}, q^\perp + (1-x)D^\perp\Big)\psi_{2+}\Big(\frac{x+\xi}{1+\xi}, q^\perp\Big). \tag{3.7}$$

We restrict ourselves to the Dokshitsa-Gribov-Lipatov-Altarelli-Parisi region $1 > x > \xi$. As we are considering no antiparticles $0 < x < 1$ in our case. It is known that in the Efremov-Radyushkin-Brodsky-Lepage region, $-\xi < x < \xi$ the GPDs are expressed as off-diagonal overlaps of light-cone wave functions involving higher Fock components [11].

The normalization of the state $|\psi_1|^2$ given by Eq. (3.6) gives another $O(\alpha)$ contribution. The $q^\perp$ integral in the above expression is divergent and can be performed using the same cutoffs as discussed above. We get, using Eq. (3.5),

$$F^{+q}_{++} = |\psi_1|^2\,\delta(1-x) + \frac{e^2}{2(2\pi)^3}|\psi_1|^2\frac{1}{\sqrt{(1-x_1)(1-x_2)}}\Big\{2(1+x_1x_2)\Big[\pi\log\Big[\frac{\Lambda^2}{(1-x)^2D_\perp^2 + m^2(1-x_1)^2}\Big] - m^2(1-x_2)^2 I\Big] + (1+x_1x_2)\Big[\pi\log\Big[\frac{\Lambda^2}{m^2(1-x_2)^2}\Big] - \pi\log\Big[\frac{\Lambda^2}{(1-x)^2D_\perp^2 + m^2(1-x_1)^2}\Big] - ((1-x)^2D_\perp^2 + m^2[(1-x_1)^2 - (1-x_2)^2])I\Big] + 2m^2(1-x_1)^2(1-x_2)^2 I\Big\}, \tag{3.8}$$

where $I = \int\frac{d^2q^\perp}{L_1L_2}$, $L_1 = (q^\perp + (1-x)D^\perp)^2 + m^2(1-x_1)^2$, and $L_2 = (q^\perp)^2 + m^2(1-x_2)^2$; $x_1 = \frac{x-\xi}{1-\xi}$, $x_2 = \frac{x+\xi}{1+\xi}$.

It is especially interesting to investigate (3.7) in the forward limit. For simplicity, we consider the massless case. We get

$$F^{+q}_{++}(x,0,0) = |\psi_1|^2\,\delta(1-x) + \int d^2q^\perp\psi^*_{2+}(x,q^\perp)\psi_{2+}(x,q^\perp) = |\psi_1|^2\,\delta(1-x) + |\psi_1|^2\frac{\alpha}{2\pi}\frac{1+x^2}{1-x}\log\frac{\Lambda^2}{\mu^2}; \tag{3.9}$$

here μ is a scale, $\mu << \Lambda$.

The normalization in this case gives

$$|\psi_1|^2 = 1 - \frac{\alpha}{2\pi}\int_\epsilon^{1-\epsilon} dx\frac{1+x^2}{1-x}\log\frac{\Lambda^2}{\mu^2}. \tag{3.10}$$

Thus we have

$$F^{+q}_{++}(x,0,0) = \delta(1-x) + \frac{\alpha}{2\pi}\log\frac{\Lambda^2}{\mu^2}\Big[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\Big]. \tag{3.11}$$

Here the plus prescription is defined in the usual way. Now we know that in the forward limit $H_q(x,0,0) = q(x)$ which is the (unpolarized) quark distribution of a given flavor in the proton. From (2.4) we then get $F^{+q}_{++}(x,0,0) = q(x)$, so from (3.11) we get the splitting function for the leading order evolution of the fermion distribution

$$P_{qq}(x) = \frac{1 + x^2}{1 - x}. \tag{3.12}$$

Note that the ϵ dependence is no longer there in Eq. (3.11). This result is obtained also in [14] by calculating the structure function of a quark dressed with a gluon (here one also would get the color factor C_f). In the nonforward case, one obtains the splitting function for the LO evolution of the GPDs [17]. Finally, it can be shown from (3.11) that

$$\int_0^1 dx H_q(x, 0, 0) = F_1(0) = 1, \tag{3.13}$$

where $F_1(0)$ is the Dirac form factor at zero momentum transfer.

Next we calculate the helicity flip matrix element

$$F^{+q}_{+-} = \int \frac{dy^-}{8\pi} e^{(i/2)x\bar{P}^+y^-} \times \left\langle P' + \left| \bar{\psi}\left(-\frac{y^-}{2}\right)\gamma^+\psi\left(\frac{y^-}{2}\right) \right| P- \right\rangle. \tag{3.14}$$

Contribution to (3.14) comes from the two-particle sector of the state. The mass cannot be neglected here. It can be written as

$$F^{+q}_{+-} = \int d^2q^\perp \psi^*_{2+}\left(\frac{x - \xi}{1 - \xi}, q^\perp + (1 - x)D^\perp\right) \times \psi_{2-}\left(\frac{x + \xi}{1 + \xi}, q^\perp\right). \tag{3.15}$$

Using Eq. (3.5) we get

$$F^{+q}_{+-} = \frac{e^2}{(2\pi)^3}(im) \int \frac{d^2q^\perp}{L_1L_2} \frac{1}{\sqrt{(1 - x_1)(1 - x_2)}}[(iq^1 + q^2) \times [x_1(1 - x_2)^2 - x_2(1 - x_1)^2] + (iD^1 + D^2) \times (1 - x)x_1(1 - x_2)^2]. \tag{3.16}$$

The $q^\perp$ integration can be performed either using the Feynman parameter method or by using

$$\frac{1}{A^k} = \frac{1}{\Gamma(k)} \int_0^\infty d\beta \beta^{k-1} e^{-\beta A}. \tag{3.17}$$

Here we use the latter method and obtain

$$\int d^2q^\perp \frac{(iq^1 + q^2)}{L_1L_2} = -i\pi \int_0^\infty d\sigma \int_0^\infty d\beta \frac{\beta}{(\sigma + \beta)^2}(1 - x)D^\perp_V e^{-(\sigma\beta/\sigma+\beta)(1-x)^2D^2_\perp} e^{-m^2[\sigma(1-x_2)^2+\beta(1-x_1)^2]}, \tag{3.18}$$

where $D^\perp_V = D^1 - iD^2$. We introduce the variables λ and y defined as

$$\lambda = \sigma + \beta, \qquad \sigma = y\lambda, \qquad \beta = (1 - y)\lambda. \tag{3.19}$$

The above integral can be written as

$$\begin{aligned}\int d^2q^\perp \frac{(iq^1 + q^2)}{L_1L_2} &= -i\pi \int_0^1 dy(1 - y)(1 - x)D^\perp_V \int_0^\infty d\lambda e^{-\lambda y(1-y)(1-x)^2D^2_\perp} e^{-\lambda m^2[y(1-x_2)^2+(1-y)(1-x_1)^2]} \\ &= -i\pi \int_0^1 dy \frac{(1 - y)(1 - x)D^\perp_V}{y(1 - y)(1 - x)^2D^2_\perp + m^2[y(1 - x_2)^2 + (1 - y)(1 - x_1)^2]}.\end{aligned} \tag{3.20}$$

The other integral can be done in a similar way:

$$\int d^2q^\perp \frac{iD^\perp_V}{L_1L_2} = i\pi D^\perp_V \int_0^1 dy \frac{1}{y(1 - y)(1 - x)^2D^2_\perp + m^2[y(1 - x_2)^2 + (1 - y)(1 - x_1)^2]}. \tag{3.21}$$

Substituting in Eq. (3.18) we obtain

$$F^{+q}_{+-} = \frac{e^2}{(2\pi)^3}\pi(im)\frac{(1 - x)}{\sqrt{(1 - x_1)(1 - x_2)}}(-iD^1 - D^2) \int_0^1 dy \frac{[x_1(1 - x_2)^2 - x_2(1 - x_1)^2](1 - y) - x_1(1 - x_2)^2}{y(1 - y)(1 - x)^2D^2_\perp + m^2[y(1 - x_2)^2 + (1 - y)(1 - x_1)^2]}. \tag{3.22}$$

In the forward limit, $\xi = 0$, $x_1 = x_2 = x$, $D^\perp = 0$, and we obtain, using Eq. (2.5),

$$E_q(x, 0) = \frac{\alpha}{\pi} \int_0^1 dy \frac{x(1 - x)^2}{y(1 - x)^2 + (1 - y)(1 - x)^2} = \frac{\alpha}{\pi}x, \tag{3.23}$$

which gives the Schwinger value for the anomalous magnetic moment of an electron in QED [10]:

$$\int_0^1 E_q dx = F_2(0) = \frac{\alpha}{2\pi}. \tag{3.24}$$

IV. GAUGE BOSON GPDS

We calculate the helicity nonflip part of the gauge boson matrix element F^{+g}_{++} for the same state as before. This gives information on the gauge boson GPDs H_g and E_g. Contribution comes from the two-body wave function, which has one fermion and one gauge boson as constituents. This can be written as an overlap

$$F^{+g}_{++} = \int d^2q^{\perp}\psi^*_{2+}\left(\frac{1-x}{1-\xi}, q^{\perp}\right)\psi_{2+}\left(\frac{1-x}{1+\xi}, q^{\perp} + (1-x)D^{\perp}\right). \tag{4.1}$$

The $q^{\perp}$ integral is divergent. The above can be calculated using (3.5):

$$\begin{aligned} F^{+g}_{++} = {}& \frac{e^2}{2(2\pi)^3}\frac{1}{\sqrt{(1-x_1)(1-x_2)}}\frac{\sqrt{x^2-\xi^2}}{x}\left\{2(1+x_1x_2)\left[\pi\log\left[\frac{\Lambda^2}{(1-x)^2D^2_{\perp} + m^2(1-x_2)^2}\right] - m^2(1-x_1)^2 I\right]\right. \\ & + (1+x_1x_2)\left[\pi\log\left[\frac{\Lambda^2}{m^2(1-x_1)^2}\right] - \pi\log\left[\frac{\Lambda^2}{(1-x)^2D^2_{\perp} + m^2(1-x_2)^2}\right] - ((1-x)^2D^2_{\perp} + m^2[(1-x_2)^2\right. \\ & \left.\left. - (1-x_1)^2])I\right] + 2m^2(1-x_2)^2(1-x_1)^2 I\right\}, \end{aligned} \tag{4.2}$$

where $I = \int\frac{d^2q^{\perp}}{L_1L_2}$, $L_2 = (q^{\perp} + (1-x)D^{\perp})^2 + m^2(1-x_2)^2$, and $L_1 = (q^{\perp})^2 + m^2(1-x_1)^2$; $x_1 = \frac{1-x}{1-\xi}$, $x_2 = \frac{1-x}{1+\xi}$.

Like the quark case, it is again interesting to look at the forward limit of the above expression. We get

$$F^{+g}_{++}(x, 0, 0) = \frac{\alpha}{2\pi}\log\frac{\Lambda^2}{\mu^2}\frac{1+(1-x)^2}{x}. \tag{4.3}$$

Here we have neglected the electron mass for simplicity. $F^{+g}_{++}(x, 0, 0)$ gives the (unpolarized) gluon distribution in the nucleon and the above expression gives the splitting function [14]

$$P_{gq}(x) = \frac{1+(1-x)^2}{x}. \tag{4.4}$$

In the off-forward case, the splitting functions can be found using the same approach [17].

We now calculate the helicity flip gauge boson GPD E_g given in (2.2) for the same state. The matrix element is given by

$$F^{+g}_{+-} = \frac{1}{8\pi x\bar{P}^+}\int dy^- e^{(i/2)\bar{P}^+y^-x}\left\langle P' + \left| F^{+\alpha}\left(-\frac{y^-}{2}\right)F^+_{\alpha}\left(\frac{y^-}{2}\right)\right| P-\right\rangle. \tag{4.5}$$

Contribution comes from the two-particle sector. As in the fermion case, the mass cannot be neglected here. This can be written as

$$F^{+g}_{+-} = \int d^2q^{\perp}\psi^*_{2+}\left(\frac{1-x}{1-\xi}, q^{\perp}\right)\psi_{2-}\left(\frac{1-x}{1+\xi}, q^{\perp} + (1-x)D^{\perp}\right). \tag{4.6}$$

Using Eq. (3.5) we get

$$F^{+g}_{+-} = \frac{e^2}{(2\pi)^3}(im)\frac{\sqrt{x^2-\xi^2}}{x}\left[\int d^2q^{\perp}\frac{iq^{\perp}_V}{L_1L_2}[x_1(1-x_2)^2 - x_2(1-x_1)^2] - (1-x)x_2(1-x_1)^2 iD^{\perp}_V\int\frac{d^2q^{\perp}}{L_1L_2}\right]. \tag{4.7}$$

The $q^{\perp}$ integration can be performed in a similar way as for the fermions and we get

$$F^{+g}_{+-} = \frac{\alpha}{2\pi}m\frac{\sqrt{1-\xi^2}}{x}D^{\perp}_V(1-x)\int_0^1 dy\frac{[x_1(1-x_2)^2 - x_2(1-x_1)^2](1-y) + x_2(1-x_1)^2}{y(1-y)(1-x)^2D^2_{\perp} + m^2[y(1-x_1)^2 + (1-y)(1-x_2)^2]}. \tag{4.8}$$

In the forward limit, $x_1 = x_2 = 1-x$ and we get, in agreement with [10],

$$E_g(1-x, 0) = -\frac{\alpha}{\pi}\frac{(1-x)^2}{x}; \tag{4.9}$$

here E_g is given by Eq. (2.5). Note that x in the forward case is the momentum fraction of the gauge boson. The second moment of $E_{q,g}(x, 0)$, $\int dx x E_{q,g}(x, 0)$ gives in units of $\frac{1}{2m}$ by how much the transverse center of momentum of the parton q, g is shifted away from the origin in the transversely polarized state. When summed over all partons, the transverse center of momentum would still be at the origin. Indeed it is easy to check for a dressed electron [10]

$$\int_0^1 dx x E_q(x, 0) + \int_0^1 dx(1-x)E_g(x, 0) = 0, \qquad (4.10)$$

which is due to the fact that the anomalous gravitomagnetic moment of the electron has to vanish [1]. Note that in the second term, $(1 - x)$ is the momentum fraction of the gauge boson.

V. GPDS IN THE IMPACT PARAMETER SPACE

Fourier transform of the GPDs with respect to the transverse momentum transfer $\Delta^\perp$ brings them to the impact parameter space. When the longitudinal momentum transfer $\xi = 0$, this gives the density of partons with longitudinal momentum fraction x and transverse distance b from the center of the proton. For nonzero ξ, the Fourier transforms are defined as [3],

$$I^{q,g}_{++}(x, \xi, b^\perp) = \int \frac{d^2D^\perp}{(2\pi)^2} e^{-iD^\perp \cdot b^\perp} F^{+q,g}_{++}(x, \xi, D^\perp) = \frac{1}{4\pi}\int_0^\infty d(D^\perp)^2 J_0(|D||b|) \times \left(H_{q,g} - \frac{\xi^2}{1-\xi^2}E_{q,g}\right), \qquad (5.1)$$

$$I^{q,g}_{+-}(x, \xi, b^\perp) = \int \frac{d^2D^\perp}{(2\pi)^2} e^{-iD^\perp \cdot b^\perp} F^{+q,g}_{+-}(x, \xi, D^\perp) = \frac{1}{4\pi}\frac{b^2 - ib^1}{|b^\perp|}\int_0^\infty d(D^\perp)^2 J_1(|D||b|) \times \frac{|D|}{2m}E_{q,g}; \qquad (5.2)$$

where J_0 and J_1 are Bessel functions and $b^\perp$ is called the impact parameter. In order to avoid infinities in the intermediate steps, we take a wave packet state

$$\int \frac{d^2p^\perp}{16\pi^3}\phi(p^\perp) \mid p^+, p^\perp, \lambda\rangle, \qquad (5.3)$$

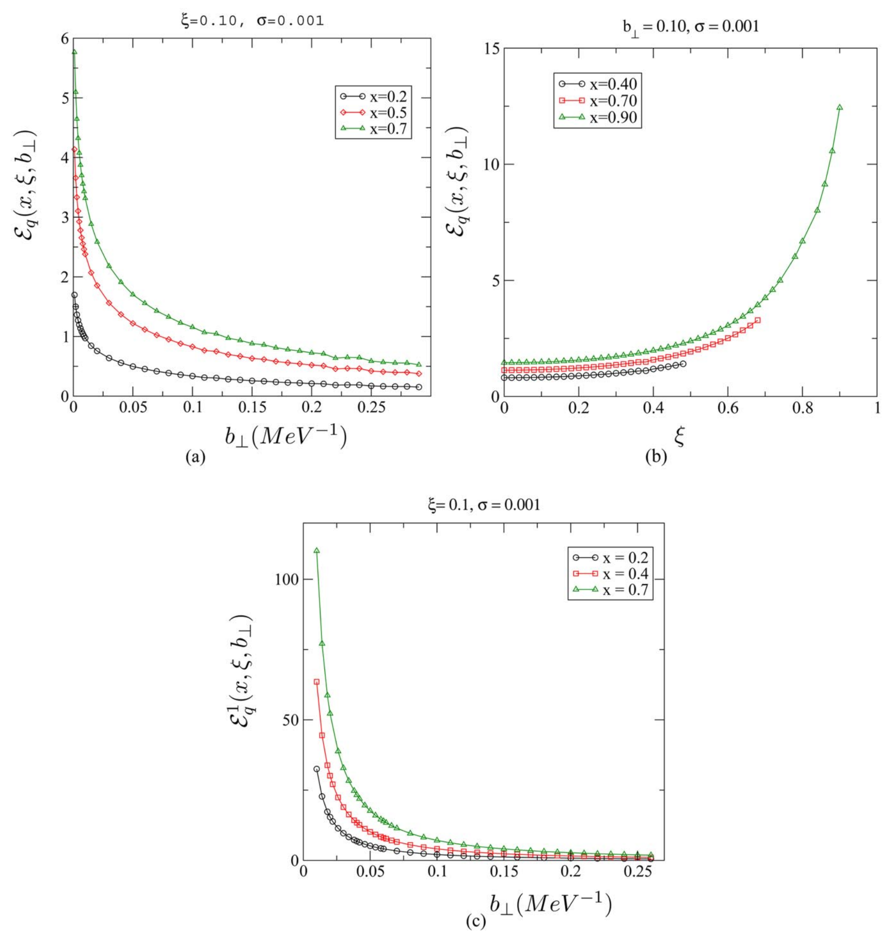

FIG. 1 (color online). (a) $\mathcal{E}_q$ vs $b^\perp$ for $\xi = 0.1$, (b) $\mathcal{E}_q$ vs ξ for $b^\perp = 0.1$ MeV^{-1}, (c) $\mathcal{E}^1_q$ vs $b^\perp$ for $\xi = 0.1$ and three different values of x. We have taken $\sigma = 0.001$.

which have a definite plus momentum. Following [3] we take a Gaussian wave packet,

$$\phi(p^\perp) = G(p^\perp, \sigma^2), \tag{5.4}$$

where

$$G(p^\perp, \sigma^2) = e^{[-(p^\perp)^2/2\sigma^2]}. \tag{5.5}$$

σ gives the width of the wave packet. It is the accuracy to which one can localize information in the impact parameter space [3]. The states centered around b_0 with an accuracy σ are normalized as

$$\langle p'^+, b'^\perp, \lambda' \mid p^+, b^\perp, \lambda\rangle = \frac{1}{16\pi^2\sigma^2}\frac{1}{p^+}G(b'^\perp - b^\perp, 2\sigma^2) \times \delta(p^+ - p'^+). \tag{5.6}$$

Fourier transform of the matrix elements in Eqs. (5.1) and (5.2) with the Gaussian wave packet probe partons in the nucleon at transverse position b but when the initial and final state protons are centered around 0 but shifted from each other by an amount of the order of $\xi \mid b^\perp \mid$. In our case, they probe a bare electron or a photon in an electron dressed with a photon.

It is interesting to look at the qualitative behavior of the helicity-flip GPDs E_q and E_g in the impact parameter space. The contribution in this case comes purely from the two-body sector, that is it involves the wave function ψ_2 of the relativistic spin-1/2 system. The overlap is given in terms of the light-cone wave functions whose orbital angular momentum differ by $\Delta L_z = \pm 1$ [12]. The scale dependence, as mentioned before, is suppressed here, unlike H_q and H_g. We use the notations

$$\mathcal{E}_{q,g}(x, \xi, b^\perp) = \frac{1}{4\pi}\int_0^\infty d(D^\perp)^2 J_0(\mid D \mid\mid b \mid)E_{q,g}; \tag{5.7}$$

$$\mathcal{E}^1_{q,g}(x, \xi, b^\perp) = \frac{1}{4\pi}\int_0^\infty d(D^\perp)^2 J_1(\mid D \mid\mid b \mid)\frac{\mid D \mid}{2m}E_{q,g}, \tag{5.8}$$

which contribute to Eq. (5.1) and (5.2), respectively. Figure 1(a) shows $\mathcal{E}_q$ vs $b^\perp$ for fixed $\xi = 0.1$ and three different values of $x > \xi$. We have plotted for positive $b^\perp$.

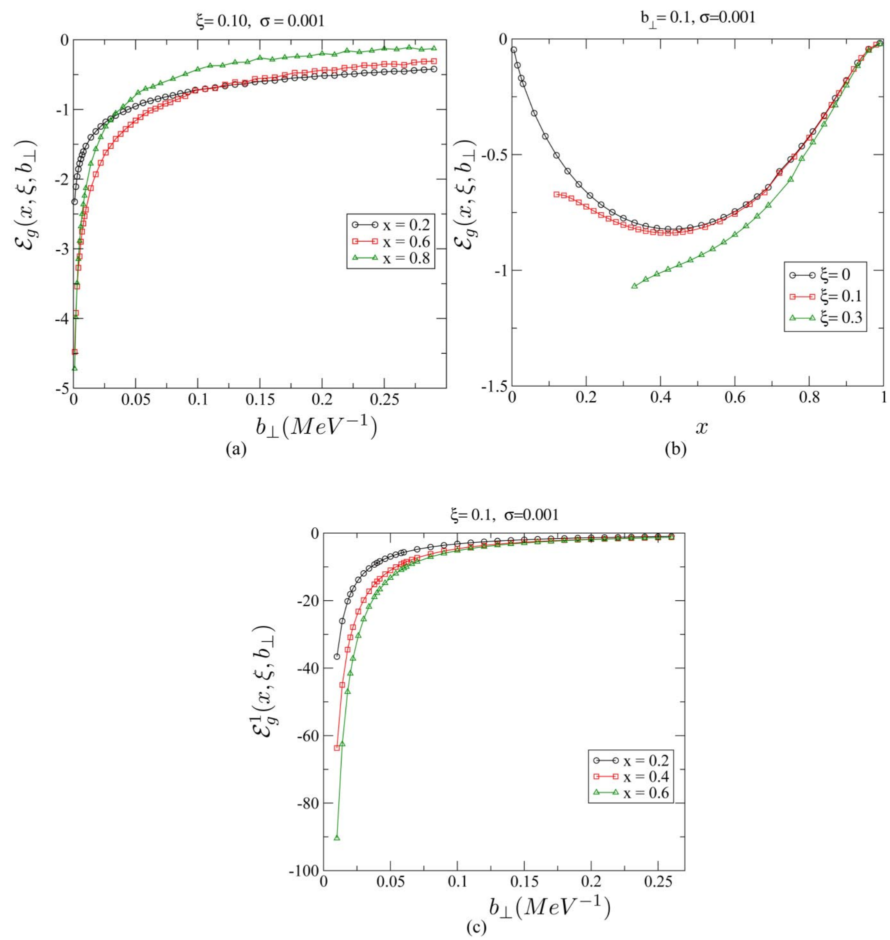

FIG. 2 (color online). (a) $\mathcal{E}_g$ vs $b^\perp$ for $\xi = 0.1$ and three different values of x. (b) $\mathcal{E}_g$ vs x for $b^\perp = 0.1$ MeV^{-1} and three different values of ξ. (c) $\mathcal{E}^1_g$ vs $b^\perp$ with $\xi = 0.1$ and three different values of x. We have taken $\sigma = 0.001$.

The functions are symmetrical in $| b^\perp |$. For all numerical studies, we have taken a Gaussian wave packet. $\mathcal{E}_q$ is positive and has a maximum for $| b^\perp |= 0$ and decreases smoothly with increasing $| b^\perp |$. For fixed $| b^\perp |$, $\mathcal{E}_q$ is higher in magnitude for higher x. The qualitative behavior for nonzero ξ is the same as $\xi = 0$. We have taken the normalization to be $\frac{\alpha}{2\pi} = 1$ and $m = 0.5$ MeV. Figure 1(b) shows $\mathcal{E}_q$ vs ξ for fixed $b^\perp = 0.1\ \mathrm{MeV}^{-1}$ and three different values of $x > \xi$. $\mathcal{E}_q$ is again a smooth function of ξ and increases as ξ increases for given $b^\perp$ and x. For fixed ξ, $\mathcal{E}_q$ increases as x increases. We have plotted $\mathcal{E}_q^1$ in Fig. 1(c) as a function of $b^\perp$ for $\xi = 0.1$ and three different values of x. The rise near $b^\perp = 0$ is much more sharp here than in Fig. 1(a).

In Fig. 2(a) we have shown $\mathcal{E}_g(x, \xi, b^\perp)$ vs $b^\perp$ for fixed $\xi = 0.1$ and for three different values of x. $\mathcal{E}_g$ is negative for positive $b^\perp$. It has a negative maximum at $b^\perp = 0$ and smoothly decreases in magnitude as $b^\perp$ increases. Again, the qualitative behavior is the same as $\xi = 0$. For a given $b^\perp$, $\mathcal{E}_g$ decreases in magnitude for increasing x. Figure 2(b) shows $\mathcal{E}_g$ as a function of x for fixed $b^\perp = 0.1\ \mathrm{MeV}^{-1}$ and three different values of $\xi < x$. $\mathcal{E}_g$ vanishes at $x = 1$. For $\xi = 0$, it also vanishes at $x = 0$. For $\xi > 0$, $\mathcal{E}_g$ increases in magnitude as ξ increases. The curves cannot be continued for $x < \xi$ as there will be contribution from the higher Fock components in this region. Figure 2(c) shows $\mathcal{E}_g^1$ vs $b^\perp$ for $\xi = 0.1$ and three different values of x. The qualitative behavior is the same as in 2(a)—again the rise near $b^\perp = 0$ is much sharp like the fermion case. For large $b^\perp$, $\mathcal{E}_g^1$ behaves in the same way independent of x. For completeness, in Fig. 3(a) and 3(b) we show $H_q(x, 0, t)$ and $H_g(x, 0, t)$ in the impact parameter space, $\mathcal{H}_q(x, b^\perp)$ and $\mathcal{H}_g(x, b^\perp)$, respectively, for a definite scale $\Lambda = 5$ GeV. The width of the Gaussian is $\sigma = 0.1$. Both of them are smooth functions of $b^\perp$, increasing as $| b^\perp |$ decreases. We have omitted the very small $b^\perp$ region in order to show the resolution of the different curves at higher $b^\perp$. For a given $b^\perp$, $\mathcal{H}_q(x, b^\perp)$ increases as x becomes closer to 1 and at $x \to 1$ it becomes a delta function [10] which can be seen analytically. $\mathcal{H}_g(x, b^\perp)$, on the other hand, decreases in magnitude for a given $b^\perp$ as x goes closer to 1.

VI. INEQUALITIES

GPDs in the impact parameter space obey certain inequalities, which impose severe constraints on phenomenological models of GPDs. The most general forms of these inequalities were derived in [18] from positivity constraints. The spin-flip GPDs $\mathcal{E}_{q,g}(x, b^\perp)$ defined as the Fourier transform of $E_{q,g}(x, b^\perp)$ for $\xi = 0$ obey two inequalities given in [19]; both of them can be shown to hold for a dressed electron state. For nonzero ξ, a general inequality can be derived [3],

$$(1 - \xi^2)^3 \,| I^{q,g}_{\lambda'\lambda}(x, \xi, b^\perp) |^2 \le I^{q,g}_{++}\left(\frac{x - \xi}{1 - \xi}, 0, \frac{b^\perp}{1 - \xi}\right) I^{q,g}_{++}\left(\frac{x + \xi}{1 - \xi}, 0, \frac{b^\perp}{1 + \xi}\right), \tag{6.1}$$

for $\xi \le x \le 1$ and for any combinations of helicities λ', λ. For $\lambda' \neq \lambda$, the inequality is easy to prove, as there are large logarithmic contributions to $I^{q,g}_{++}$ from the scale dependent part. For $\lambda' = \lambda$, the inequality is nontrivial and can be verified numerically. Figure 4 shows the left-hand side and right-hand side of Eq. (6.1) vs $b^\perp$ for the fermions

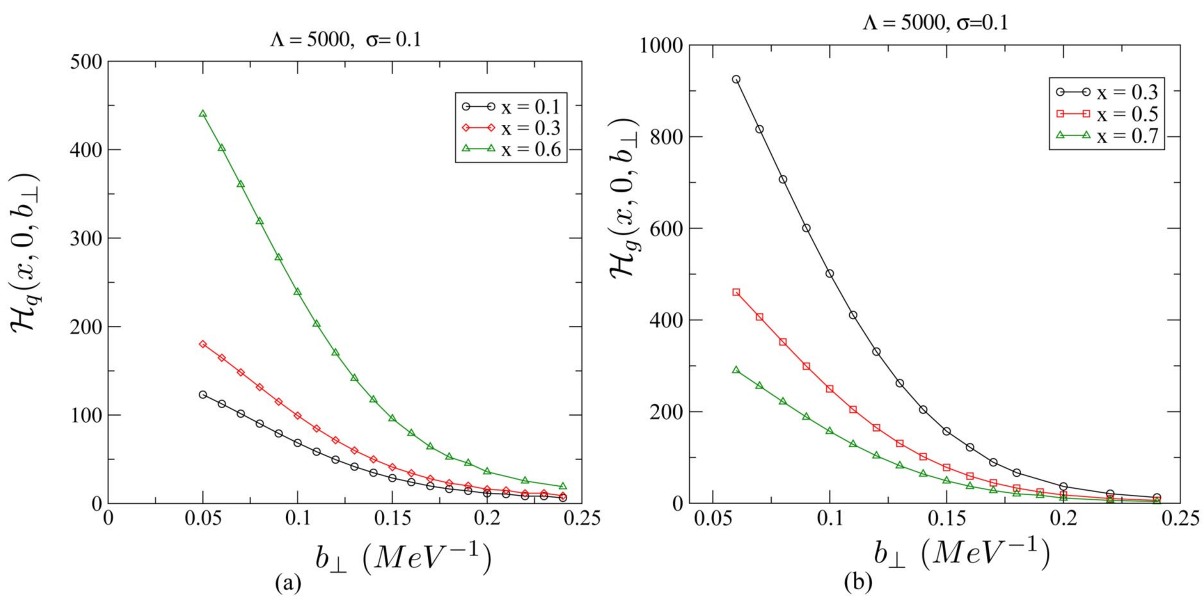

FIG. 3 (color online). (a) $\mathcal{H}_q(x, b^\perp)$ vs $b^\perp$, (b) $\mathcal{H}_g(x, b^\perp)$ vs $b^\perp$ for three different values of x. Λ is given in MeV.

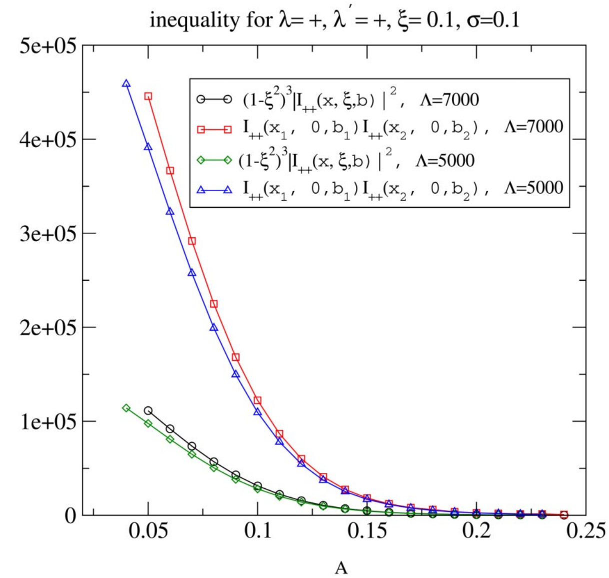

FIG. 4 (color online). Inequality for I_{++}. Λ is given in MeV. Here $x_1 = \frac{x-\xi}{1-\xi}$, $b_1 = \frac{b^\perp}{1-\xi}$, $x_2 = \frac{x+\xi}{1+\xi}$, $b_2 = \frac{b^\perp}{1+\xi}$.

for two different choices of the scale $\Lambda = 7$ GeV and $\Lambda = 5$ GeV. The scale dependence comes entirely from H_q. In the plot $x_1 = \frac{x-\xi}{1-\xi}$, $x_2 = \frac{x+\xi}{1-\xi}$, $b_1 = \frac{b^\perp}{1-\xi}$, and $b_2 = \frac{b^\perp}{1+\xi}$. We have taken $\xi = 0.1$ and $x = 0.5$ and $\sigma = 0.1$. For large $b^\perp$, the magnitudes of both the left-hand side and the right-hand side decrease but the inequality does not saturate, even for higher $b^\perp$ not plotted in the figure.

VII. SUMMARY AND DISCUSSIONS

In this work, we have investigated the GPDs for an effective spin-$1/2$ composite relativistic system, namely, for an electron dressed with a photon in QED. It is known [12] that the light-cone wave function of this two-body state gives a template for the effective quark spin-one diquark structure of the proton light-cone wave function, which provides the phenomenological relevance of our study. The GPDs are expressed as overlaps of the light-cone wave functions, which in this case are known order by order in perturbation theory. We keep both the skewedness ξ and the transverse momentum transfer $\Delta^\perp$ nonzero, which is relevant for deeply virtual Compton scattering experiments to probe the GPDs. Fourier transform with respect to the transverse momentum transfer brings the GPDs to the impact parameter space. We showed that the GPDs for the effective state obey the necessary inequalities, and we investigated the qualitative behavior of the fermion and the gauge boson GPDs in the impact parameter space.

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