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Instantons in the Higgs phase

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When instantons are put into the Higgs phase, vortices are attached to instantons. We construct such composite solitons as 1/4 BPS states in five-dimensional supersymmetric $U(N_C)$ gauge theory with $N_F(\geq N_C)$ fundamental hypermultiplets. We solve the hypermultiplet BPS equation and show that all 1/4 BPS solutions are generated by an $N_C \times N_F$ matrix which is holomorphic in two complex variables, assuming the vector multiplet BPS equation does not give additional moduli. We determine the total moduli space formed by topological sectors patched together and work out the multi-instanton solution inside a single vortex with complete moduli. Small instanton singularities are interpreted as small sigma-model lump singularities inside the vortex. The relation between monopoles and instantons in the Higgs phase is also clarified as limits of calorons in the Higgs phase. Another type of instantons stuck at an intersection of two vortices and dyonic instantons in the Higgs phase are also discussed.

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I. INTRODUCTION

Instantons have attracted much attention and have been applied to a wide variety of subjects in physics and mathematics since their discovery [1]. The method to construct multiple instanton solutions was established by Atiyah, Hitchin, Drinfeld, and Manin (ADHM) [2] and it was shown that the moduli space of instantons is a hyper-Kähler manifold. In supersymmetric (SUSY) gauge theories, instantons play crucial roles as a tool to study nonperturbative effects. Instanton calculus determines the exact superpotential in the low-energy effective action of $\mathcal{N} = 1$ SUSY QCD [3]. Seiberg and Witten presented the exact effective action of $\mathcal{N} = 2$ SUSY QCD whose prepotential contains nonperturbative terms as instanton corrections to all orders [4].

The instanton solutions that are used in these studies of nonperturbative effects are called constrained instantons, which become solutions of the field equation only with scale-fixing source terms [5]. This method is motivated by the necessity to fix the scale for instantons in the presence of vacuum expectation values of the Higgs field. This complication arises as a result of a generalized version of the well-known theorem by Derrick [6], which states that gauge theories coupled to nontrivial scalar fields do not allow any finite energy solution with four codimensions [7]. Therefore instantons have to possess an infinite amount of energy as solutions of the source-free field equation, if the gauge fields are in the Higgs phase. This situation is quite similar to monopoles in the Higgs phase which have to accompany vortices because of the Meissner effect [8–12]. In the presence of the Fayet-Iliopoulos term for the $U(1)$ factor gauge group, vortices are allowed to exist. Then it is expected to be energetically favorable for instantons in the Higgs phase to accompany vortices. Recently it has been suggested by Hanany and Tong [13] that there should be a solution as a composite state of an instanton and vortices, quite similar to the monopole in the Higgs phase. Such a composite state is expected to be realized as a lump [14,15] (or a sigma-model instanton [16,17]) in the effective field theory on the vortex world volume [13].

Instantons (without vortices accompanied) in SUSY gauge theories become 1/2 Bogomol'nyi-Prasad-Sommerfield (BPS) states, preserving a half of SUSY, if it is embedded in the Euclidean four space of the $d = 4 + 1$ space-time. In string theory these BPS instantons can be realized as Dp-branes on D$(p + 4)$-branes in type IIA/IIB string [18], and this brane configuration gives a clear physical interpretation of the ADHM constraints as the F- and D-flatness conditions in the SUSY gauge theory on Dp-brane world volume. Compactification of the small instanton singularity in the ADHM moduli space [19] was understood by Nekrasov and Schwartz [20] as noncommutative instantons. In the brane picture this phenomenon corresponds to the presence of a self-dual NS-NS B-field background on the D$(p + 4)$-brane world volume. Moreover, a direct calculation of $\mathcal{N} = 2$ Seiberg-Witten prepotential was given by Nekrasov using the instanton counting [21].

The purpose of this paper is to discuss instantons attached to vortices, when instantons are placed in the Higgs phase of the five-dimensional SUSY $U(N_C)$ gauge theory with $N_F(\geq N_C)$ flavors of hypermultiplets in the fundamental representation. We show that composite states of instantons and vortices[1] are 1/4 BPS states. We solve the

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[1]Since vortices (instantons) are defined as solitons with codimension two (four), they are membranes (particles) with $2 + 1$ $(0 + 1)$ dimensional world volume in $d = 4 + 1$ space-time.

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hypermultiplet BPS equations for these states. Assuming the vector multiplet BPS equation has a unique solution without additional moduli, we find that solutions are completely generated by a holomorphic $N_C \times N_F$ matrix function of two complex variables made of four codimensions of solitons. We call this matrix the *moduli matrix*. We find the total moduli space containing all topological sectors with all possible boundary conditions. We also find that other 1/4 BPS configurations containing walls, vortices and monopoles [22] are completely included in the 1/4 BPS states of instantons and vortices. The moduli matrix for multiple instantons inside a single vortex is specified by using the effective theory on the vortex. We also obtain calorons (periodic instantons) in the Higgs phase and clarify their relation to instantons and monopoles in the Higgs phase. We also find another type of instantons which are stuck at an intersection of two vortices. Dyonic instantons in the Higgs phase and other related issues are also discussed.

A key point of our discussion is to consider a 1/2 BPS vortex as a host soliton for certain class of composite 1/4 BPS states. The effective theory on the world volume of a single vortex is the SUSY $\mathbf{C}P^N$ model [23–25]. An instanton in the Higgs phase is realized as a 1/2 BPS lump in this effective theory on the vortex [13] as stated above. This is similar to a recent discovery of a monopole in the Higgs phase (a confined monopole) [8–13] as a 1/4 BPS state, which turns out to be a composite state of a monopole attached to vortices and is realized as a 1/2 BPS kink [26] in the vortex effective action [8]. In [22] we have shown all the solutions are generated by the moduli matrix which is holomorphic with respect to a single holomorphic coordinate on the wall world volume. The moduli matrix contains all moduli parameters in all the different topological sectors of the solitons.

The moduli matrix for instantons also contains all moduli parameters in all the different topological sectors. Since the moduli matrix as a function of two holomorphic variables contains infinitely many moduli parameters, it is now difficult to specify all of them corresponding to all the solutions. Instead, we specify a moduli matrix for multiple instantons on a single vortex by interpreting them as 1/2 BPS multiple lumps in the effective theory on the world volume of the single 1/2 BPS vortex, similarly to the case of monopoles in the Higgs phase. Monopoles in the Higgs phase can be obtained in $\mathcal{N} = 2$ (eight SUSY) massive SUSY QCD (SQCD) in $d = 3 + 1$ dimensions. They can be promoted to monopole strings in $d = 4 + 1$. The $\mathcal{N} = 2$ (eight SUSY) massive SQCD in $d = 3 + 1$ dimensions can be obtained from our eight SUSY massless SQCD in $d = 4 + 1$ dimensions by a Scherk-Schwarz dimensional reduction [27] preserving SUSY. It has been found that in

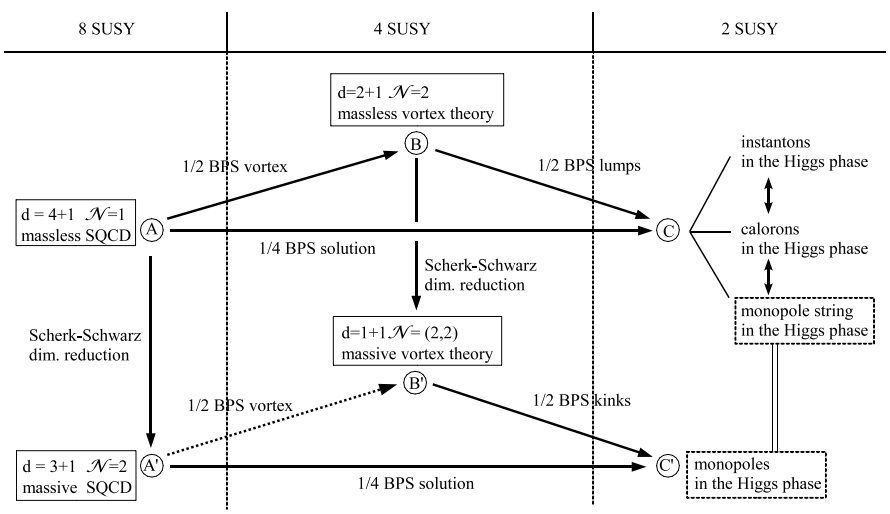


FIG. 1. BPS states and their relations. Relations discussed in this paper are denoted by solid lines with arrows and the one not discussed by a dashed line with an arrow. The upper triangle ($\triangle$ ABC) describes the $d = 4 + 1$ gauge theory with massless hypermultiplets, and the lower one ($\triangle$ A′B′C′) the $d = 3 + 1$ gauge theory with massive hypermultiplets. Theories A and A′ contain eight supercharges, 1/2 BPS vortices and their effective theories B and B′ four supercharges and the 1/4 BPS composite states C and C′ two supercharges. The lower theory A′ is obtained from the upper theory A by the Scherk-Schwarz dimensional reduction (A → A′) preserving SUSY. The effective theory on a single vortex in $d = 4 + 1$ (B) or $d = 3 + 1$ (B′) is the SUSY $\mathbf{C}P^{N-1}$ model without or with a potential, respectively. Here the latter B′ coincides with the one obtained from the former B by the Scherk-Schwarz dimensional reduction (B → B′) preserving SUSY. Instantons in C and monopoles C′ attached by vortices as 1/4 BPS states can be interpreted as 1/2 BPS lumps (B → C) and kinks (B′ → C′) in the effective theories on the vortex in $d = 4 + 1$ and $d = 3 + 1$, respectively. In $d = 4 + 1$ calorons interpolates between instantons within a vortex and a monopole string in the Higgs phase (see C). The monopole string in C in $d = 4 + 1$ can be dimensionally reduced to a monopole C′ in the Higgs phase in $d = 3 + 1$.

the Coulomb phase there is a soliton called caloron that interpolates between an instanton and a monopole [28,29]. We clarify relations between a monopole string and instantons in the Higgs phase and show that they can be obtained as particular limits in a wider class of solutions, namely, calorons in the Higgs phase. Various BPS states and their relations considered in this paper are illustrated in Fig. 1. We hope that the present work opens a new direction in the research of instantons and monopoles.

This paper is organized as follows. In Sec. II we derive the 1/4 BPS equations both by requiring the preservation of SUSY in the SUSY transformation laws on the fermions and by performing a Bogomol'nyi completion of the energy density. We solve them and show that all solutions are generated by the moduli matrix. The moduli space with all possible boundary conditions is also clarified. In Sec. III we first identify the moduli matrix for a single vortex where we determine the coefficient (the Kähler class) of the Kähler potential in the effective theory on a single vortex (A $\to$ B). Promoting moduli parameters in that matrix to functions of the world-volume coordinates we obtain the moduli matrix for a certain class of 1/4 BPS states, given as multiple instantons inside a single vortex (A $\to$ C). We also determine the topology of the moduli space for that topological sector. In Sec. IV we discuss the relation between theories and solitons in $d = 4 + 1$ (the triangle ABC) and $d = 3 + 1$ (the triangle A′B′C′). It is shown that the 1/4 BPS equations and the projection operators for instantons and vortices in $d = 4 + 1$ reduce to those for monopoles, vortices and walls in $d = 3 + 1$. The instanton charge reduces to the monopole charge. We find that a particular form of the moduli matrix (in $d = 4 + 1$) reproduces composite states made of walls, vortices and monopoles, uniformly distributed to one direction. Therefore the total moduli space of these composite states found in [22] has one-to-one correspondence with a subset of the total moduli space for composite states of vortices and instantons. We also clarify the relations between these total moduli spaces for 1/4 BPS states and those for 1/2 BPS walls found in [30,31] and for 1/2 BPS vortices. Then we specify the moduli matrix for a monopole in the Higgs phase in $d = 3 + 1$. As a by-product we give a new way to obtain the potential in the effective theory on a single vortex in the massive theory. (That potential was originally discussed by Tong [32].) We then discuss the calorons in the Higgs phase using the vortex effective theory. Section V is devoted to conclusion and discussion. We discuss classification of all solutions of our 1/4 BPS equations. There we construct another interesting solution, intersecting vortices whose intersecting point carries instanton charges. We thus conclude that there exist two kinds of instantons in the Higgs phase; the one is an instanton inside a vortex and the other is an instanton stuck at the intersection of vortices. 1/4 BPS dyonic instantons are also discussed.

II. 1/4 BPS EQUATIONS AND SOLUTIONS

A. 1/4 BPS equations for vortices and instantons

We work with a $U(N_C)$ gauge theory with N_F massless hypermultiplets in the fundamental representation in $d = 4 + 1$ dimensions[2] as the minimum dimension with four spacial dimensions allowing instantons (A in Fig. 1). We consider the minimum number of supersymmetry (SUSY) which is eight in our case. Since we consider massless hypermultiplet, we have an $SU(N_F)$ flavor symmetry. We consider the case of $N_F \geq N_C$. The physical fields contained in the vector multiplet are a $U(N_C)$ gauge field W_M ($M = 0, 1, \ldots, 4$), symplectic Majorana spinors λ^i with $SU(2)_R$ indices $i = 1, 2$ and a real adjoint scalar field Σ. The physical fields contained in the hypermultiplets are complex scalars H^{irA} ($r = 1, 2, \ldots, N_C, A = 1, 2, \ldots, N_F$) and Dirac spinors ψ^{rA}. We express $N_C \times N_F$ matrix of hypermultiplets by H^i. The bosonic Lagrangian takes the form of

$$\mathcal{L} = \mathrm{Tr}\Big[-\frac{1}{2g^2}F_{MN}F^{MN} + \frac{1}{g^2}\mathcal{D}_M\Sigma\mathcal{D}^M\Sigma + \mathcal{D}^M H^i(\mathcal{D}_M H^i)^\dagger - \frac{1}{g^2}(Y^a)^2 - H^i(H^i)^\dagger\Sigma^2\Big], \tag{2.1}$$

where the trace is taken over the color indices, g is the gauge coupling constant taken common for $U(1)$ and $SU(N_C)$ parts of the $U(N_C)$ gauge group, in order to allow simple solutions later. The covariant derivatives and the field strength are defined by $\mathcal{D}_M\Sigma = \partial_M\Sigma + i[W_M, \Sigma]$, $\mathcal{D}_M H^i = \partial_M H^i + iW_M H^i$ and $F_{MN} = -i[\mathcal{D}_M, \mathcal{D}_N] = \partial_M W_N - \partial_N W_M + i[W_M, W_N]$, respectively. Here Y^a are auxiliary fields of the vector multiplet which are determined by their equations of motion as

$$Y^a = \frac{g^2}{2}(c^a \mathbf{1}_{N_C} - (\sigma^a)^j{}_i H^i H^{j\dagger}), \qquad (a = 1, 2, 3) \tag{2.2}$$

with the Pauli matrices σ^a for $SU(2)_R$ and real parameters c^a called the Fayet-Iliopoulos (FI) parameters.

The SUSY transformation of the fermionic fields are given by

$$\delta_\varepsilon\lambda^i = \Big(\frac{1}{2}\gamma^{MN}F_{MN} + \gamma^M\mathcal{D}_M\Sigma\Big)\varepsilon^i + iY^a(\sigma^a)^i{}_j\varepsilon^j, \tag{2.3}$$

$$\delta_\varepsilon\psi = \sqrt{2}(-i\gamma^M\mathcal{D}_M H^i + \Sigma H^i)\epsilon_{ij}\varepsilon^j. \tag{2.4}$$

Here the antisymmetric tensor is defined by $\epsilon_{12} = \epsilon^{12} = 1$. The $SU(2)_R$ rotation allows us to choose the FI parameters as $c^a = (0, 0, c > 0)$ without loss of generality. Then con-

[2]Since we consider massless hypermultiplet, our consideration applies equally well for theories in $d = 5 + 1$ dimensions. Our convention for the metric is $\eta_{MN} = \mathrm{diag.}(+1, -1, \ldots, -1)$.

ditions for supersymmetric vacua are obtained as

$$\begin{aligned} H^1(H^1)^\dagger - H^2(H^2)^\dagger &= c\mathbf{1}_{N_C}, \\ H^1(H^2)^\dagger &= 0, \\ \Sigma H^i &= 0. \end{aligned} \tag{2.5}$$

Since the nonvanishing FI parameter in the first equation does not allow $H^i = 0$ for both i, the third equation requires Σ to vanish. Hence the vacua are in the Higgs branch with completely broken $U(N_C)$ gauge symmetry. For $N_F > N_C$ the moduli space of vacua is the cotangent bundle over the complex Grassmann manifold, $T^*G_{N_F,N_C} = T^*[SU(N_F)/(SU(N_F - N_C) \times SU(N_C) \times U(1))]$ [33].

Recently it has been suggested that this model admits BPS states containing both non-Abelian vortices and instantons [13]. The Bogomol'nyi completion for energy density in static configurations can be performed as [13]

$$\begin{aligned} \mathcal{E} &= \mathrm{Tr}\left[\frac{1}{2g^2}F_{mn}F_{mn} + \mathcal{D}_m H(\mathcal{D}_m H)^\dagger + \frac{1}{g^2}(Y^3)^2\right] \\ &= \mathrm{Tr}\left[\frac{1}{g^2}\{(F_{13} + F_{24} + Y^3)^2 + (F_{12} - F_{34})^2 \right. \\ &\quad \left. + (F_{14} - F_{23})^2\} + 4\bar{\mathcal{D}}_z H(\bar{\mathcal{D}}_z H)^\dagger + 4\bar{\mathcal{D}}_w H(\bar{\mathcal{D}}_w H)^\dagger \right. \\ &\quad \left. - c(F_{13} + F_{24}) + \frac{1}{2g^2}F_{mn}\tilde{F}_{mn} + \partial_m J_m\right] \\ &\geq \mathrm{Tr}\left[-c(F_{13} + F_{24}) + \frac{1}{2g^2}F_{mn}\tilde{F}_{mn} + \partial_m J_m\right], \end{aligned} \tag{2.6}$$

where we define $\tilde{F}_{mn} \equiv (1/2)\varepsilon_{mnkl}F^{kl}$ with $m, n(= 1, 2, 3, 4)$ which denote spatial indices for the four codimensional coordinates of solitons. We also define two complex coordinates and the covariant derivatives as

$$\begin{aligned} z &\equiv x^1 + ix^3, \qquad w \equiv x^2 + ix^4, \\ \bar{\mathcal{D}}_z &\equiv \frac{\mathcal{D}_1 + i\mathcal{D}_3}{2}, \qquad \bar{\mathcal{D}}_w \equiv \frac{\mathcal{D}_2 + i\mathcal{D}_4}{2}, \end{aligned} \tag{2.7}$$

respectively. In Eq. (2.6) we have assumed $H^2 = 0$ for simplicity,[3] and we have simply denoted $H \equiv H^1$. Here we have also ignored Σ because it vanishes for our 1/4 BPS states except in Sec. V where we restore Σ in order to discuss more general solution including dyonic instantons. The last line of Eq. (2.6) gives the BPS bound for the energy density. Its first term counts topological charges for vortices in the 1-3 plane and the 2-4 plane extending to the 2-4 plane and the 1-3 plane, respectively, and the second term for the instantons. The current J_m is defined by $J_1 = \mathrm{Re}(-i\mathcal{D}_3 HH^\dagger)$, $J_3 = \mathrm{Re}(i\mathcal{D}_1 HH^\dagger)$, and similarly for 2, 4 directions. It gives a surface term which does not contribute to the energy of solitons integrated over the entire space. By using the BPS equations given below, it can be rewritten as $J_m \equiv (1/2)\mathcal{D}_m(HH^\dagger)$.

The BPS equations minimizing the energy density can be obtained from (2.6) as [13]:

$$\begin{aligned} F_{12} &= F_{34}, \qquad F_{23} = F_{14}, \\ \bar{\mathcal{D}}_z H &= 0, \qquad \bar{\mathcal{D}}_w H = 0, \end{aligned} \tag{2.8}$$

$$F_{13} + F_{24} = -\frac{g^2}{2}[c\mathbf{1}_{N_C} - HH^\dagger]. \tag{2.9}$$

The first two equations in Eq. (2.8) give an integrability condition for differential operators $\bar{\mathcal{D}}_z$ and $\bar{\mathcal{D}}_w$

$$[\bar{\mathcal{D}}_z, \bar{\mathcal{D}}_w] = \frac{i}{4}[(F_{12} - F_{34}) + i(F_{14} - F_{23})] = 0. \tag{2.10}$$

If we turn off the FI parameter c and set $H = 0$, these equations reduce to the self-dual equation for instantons. On the other hand, if we ignore the x^2, x^4 (x^1, x^3) dependence and W_2, W_4 (W_1, W_3), these equations reduce to the BPS equations for vortices in the 1-3 (2-4) plane.

We now show that all configurations satisfying the BPS equations (2.8) and (2.9) preserve 1/4 (but not 1/8) SUSY.[4] To this end we introduce projections on the fermionic supertransformation parameters ε: it is specified by the subspace with positive eigenvalues of gamma matrices Γ ($\Gamma^2 = 1$) in the form of $\Gamma\varepsilon = \varepsilon$. The gamma matrices Γ_{v} for the projection allowing vortices in the 1-3 plane, $\Gamma_{\mathrm{v}'}$ for vortices in the 2-4 plane and Γ_{i} for instantons are given by

$$\begin{aligned} \Gamma_{\mathrm{v}} &= -\gamma^{13} \otimes i\sigma^3, \qquad \Gamma_{\mathrm{v}'} = -\gamma^{24} \otimes i\sigma^3, \\ \Gamma_{\mathrm{i}} &= \gamma^0 \otimes \mathbf{1}_2, \end{aligned} \tag{2.11}$$

respectively. Each projection operator projects out different sets of four supercharges among eight supercharges, and therefore it is a projection for 1/2 BPS states. By requiring 1/2 SUSY specified by Γ_{v} ($\Gamma_{\mathrm{v}'}$) in the supertransformations (2.3) and (2.4) to be conserved, we obtain the BPS equations allowing vortices in the 1-3 (2-4) plane. Similarly Γ_{i} leads to another 1/2 BPS (self-dual) equations admitting instantons. Since a projection is defined by the subspace with positive eigenvalues of a gamma matrix Γ, two projections are compatible if and only if two gamma matrices commute with each other. In our case of vortices in the 1-3 and the 2-4 planes and instantons, any two of all three gamma matrices Γ_{v}, $\Gamma_{\mathrm{v}'}$ and Γ_{i} commute with each other. Therefore we can impose all three projections simultaneously to preserve 1/4 SUSY. By requiring the supertransformation (2.3) and (2.4) to be conserved for the 1/4 SUSY, we obtain the BPS equations (2.8) and (2.9) again. Note that any of the three projections can be derived from the product of the other two, for example $\Gamma_{\mathrm{v}'} = \Gamma_{\mathrm{v}}\Gamma_{\mathrm{i}}$.

[3] If there are vortices and $H^2 \neq 0$, we can show that fields increase indefinitely away from the vortex and energy density diverges, at least for simple cases of $U(1)$ gauge theory.

[4] The authors in Ref. [13] suspected that solutions of Eqs. (2.8) and (2.9) preserve 1/8 SUSY, but it is not the case.

Therefore we conclude that all solutions of the BPS equations (2.8) and (2.9) preserve 1/4 SUSY.

B. Solutions and their moduli space

Let us solve the BPS equations (2.8) and (2.9) by generalizing the method introduced in [30] (A → C in Fig. 1). The four equations in Eq. (2.8) can be formally solved as

$$\bar{W}_z = -iS^{-1}\bar{\partial}_z S, \qquad \bar{W}_w = -iS^{-1}\bar{\partial}_w S, \qquad H = S^{-1}H_0(z, w), \tag{2.12}$$

with W_z and W_w defined by

$$\bar{W}_z \equiv \frac{W_1 + iW_3}{2}, \qquad \bar{W}_w \equiv \frac{W_2 + iW_4}{2} \tag{2.13}$$

and an $N_C \times N_C$ nonsingular matrix function $S(x^m)$ is defined as a solution of the first two equations in (2.12). Then the last two equations in Eq. (2.8) is solved by Eq. (2.12) with an $N_C \times N_F$ matrix $H_0(z, w)$ whose components are arbitrary holomorphic functions with respect to z and w. The matrix $H_0(z, w)$ should have rank N_C in generic points (z, w). We call H_0 the *moduli matrix* because all moduli parameters of solutions are expected to be contained in this matrix.[5] There is an important symmetry, which we call the *world-volume symmetry* [30], defined by

$$H_0 \to H_0' = VH_0, \qquad S \to S' = VS \tag{2.14}$$

with $V(z, w)$ an element of $GL(N_C, \mathbf{C})$ whose components are holomorphic with respect to z and w. The world-volume symmetry (2.14) relates sets of (H_0, S) and (H_0', S') which give the same physical quantities and defines an equivalence relation [22,30,31]. Then the total moduli space $\mathcal{M}_{\mathrm{vv'i}}$ including *all* topological sectors with different boundary conditions can be identified as a quotient of the holomorphic maps defined by

$$\mathcal{M}_{\mathrm{vv'i}} = \mathcal{H}\backslash\mathcal{G}, \tag{2.15}$$

$$\mathcal{G} \equiv \{H_0|\mathbf{C}^2 \longrightarrow M(N_C \times N_F, \mathbf{C}), \bar{\partial}_z H_0 = \bar{\partial}_w H_0 = 0\},$$
$$\mathcal{H} \equiv \{V|\mathbf{C}^2 \longrightarrow GL(N_C, \mathbf{C}), \bar{\partial}_z V = \bar{\partial}_w V = 0\},$$

where $M(N_C \times N_F, \mathbf{C})$ is an $N_C \times N_F$ complex matrix. The dimension of this moduli space is of course infinite because it contains topological sectors with arbitrary numbers of topological charges. By enforcing a boundary condition properly we can obtain a topological sector with finite dimension, as shown in the next section.[6] All topological sectors are patched together to form the total moduli space.[7] We would like to emphasize that the same thing occurs in the case of the composite states made of monopoles, vortices and walls [22] [see Eqs. (4.13) and (4.14) below]. If we put the same requirement with footnote 6, all vortices end on walls with a perpendicular angle. However once ignoring such a requirement, we were able to obtain tilted walls where vortices end with an angle. Changing boundary conditions produces new solutions. *It is very important to consider the total moduli space, in order to exhaust all the solutions of BPS equations for composite states.*

Once H_0 is given, the matrix function $S(x^m)$ can be determined by the last two equations in Eq. (2.8) up to the $U(N_C)$ gauge transformation. To solve it, it is useful to introduce a gauge invariant matrix

$$\Omega \equiv SS^\dagger \tag{2.16}$$

which transforms as $\Omega \to V\Omega V^\dagger$ under the world-volume transformation (2.14). Then the remaining BPS equation (2.9) can be reexpressed in terms of Ω as

$$4\bar{\partial}_z(\partial_z\Omega\Omega^{-1}) + 4\bar{\partial}_w(\partial_w\Omega\Omega^{-1}) = cg^2(\mathbf{1}_{N_C} - \Omega_0\Omega^{-1}), \qquad c\Omega_0 \equiv H_0H_0^\dagger. \tag{2.17}$$

We call this the master equation for our 1/4 BPS system.

The energy density for the 1/4 BPS states consists of contributions with the vorticity densities ρ_{v} in the 1-3 plane and $\rho_{\mathrm{v}'}$ in the 2-4 plane, the instanton number density ρ_{i}, and the current divergence for the correction term $\partial_m J_m$ as

$$\mathcal{E} = 2\pi c(\rho_{\mathrm{v}} + \rho_{\mathrm{v}'}) + \frac{8\pi^2}{g^2}\rho_{\mathrm{i}} + \partial_m \mathrm{Tr} J_m. \tag{2.18}$$

The vorticity densities ρ_{v} in the 1-3 plane and $\rho_{\mathrm{v}'}$ in the 2-4 plane are given in terms of Ω as

[5] In the next section, we explicitly show that the moduli matrix contains all the moduli parameters in the case of the single non-Abelian vortex. In the presence of at least one vortex, the equation $H^2 = 0$ holds as explained in footnote 3 and therefore no moduli parameters appear in H^2. There exists the possibility such that Ω defined in Eq. (2.16) below contains additional moduli parameters. We have to prove the index theorem for our 1/4 BPS states to clarify this point.

[6] One should recall that the total moduli space (in our language) of the sigma-model instanton is the whole space of the holomorphic map from $\mathbf{C}$ to the target space M [16,17]. Requiring that infinity should be mapped into a single point in M, the total moduli space is decomposed into topological sectors, according to the homotopy class of the map $S^2 \to M$. Then each topological sector contains a finite number of moduli parameters.

[7] We have not yet clarified how the total moduli space is decomposed into different topological sectors in the case of instantons. It was, however, completely clarified [30,34] in the case of the moduli space of the domain walls given in Eq. (4.16), below, which is obtained by the dimensional reduction of this system.

$$\rho_{\rm v} \equiv -\frac{1}{2\pi}{\rm Tr}F_{13} = \frac{1}{\pi}\bar{\partial}_z\partial_z \log\det\Omega,$$
$$\rho_{\rm v'} \equiv -\frac{1}{2\pi}{\rm Tr}F_{24} = \frac{1}{\pi}\bar{\partial}_w\partial_w \log\det\Omega. \tag{2.19}$$

For finite energy configurations, Ω must approach Ω_0 at asymptotic spacial infinity in the codimensions (along the direction perpendicular to the vortex). Therefore topological charge such as the vorticity in the 1-3 plane $\nu_{\rm v}$ is determined by boundary conditions encoded in Ω_0 as

$$\nu_{\rm v} \equiv \int dx^1 dx^3 \rho_{\rm v} = \frac{1}{2\pi i}\oint_\infty dz\partial_z \log\det\Omega_0, \tag{2.20}$$

and similarly for vorticity $\nu_{\rm v'}$ in the 2-4 plane. Then the maximal power of $|z|^2$ and $|w|^2$ of the determinant of Ω_0 gives the vorticity $\nu_{\rm v}$ and $\nu_{\rm v'}$, respectively. On the other hand, the instanton density $\rho_{\rm i}$ is given in terms of Ω as

$$\begin{aligned}\rho_{\rm i} &\equiv \frac{1}{16\pi^2}{\rm Tr}(F_{mn}\tilde{F}_{mn})\\ &= \frac{1}{\pi^2}{\rm Tr}[\bar{\partial}_z(\partial_w\Omega\Omega^{-1})\bar{\partial}_w(\partial_z\Omega\Omega^{-1})\\ &\quad - \bar{\partial}_z(\partial_z\Omega\Omega^{-1})\bar{\partial}_w(\partial_w\Omega\Omega^{-1})].\end{aligned} \tag{2.21}$$

To obtain the instanton number $\nu_{\rm i}$, we should just integrate over the density $\rho_{\rm i}$ over the Euclidean four space as $\nu_{\rm i} = \int d^4x\rho_{\rm i}$. In the case of instantons in the Higgs phase, Ω approaches to Ω_0 at the infinity ($|z| \to \infty$) perpendicular to the host vortices in the 1-3 plane. On the other hand, it does not approach to Ω_0 at infinities ($|w| \to \infty$) along the host vortices, but to the solution of 1/2 BPS vortices $\Omega_{\rm v}(z, \bar{z})$ which does not depend on $w, \bar{w}$. Instanton charges can be calculated by evaluating asymptotic values of Ω, $\partial_w\Omega$ and $\bar{\partial}_w\Omega$ at infinities ($|w| \to \infty$). This can be correctly performed by using the Manton's effective action [35] of the host vortices as we will show in the next section. Finally the current divergence for the correction term $\partial_m J_m$ is given in terms of Ω as

$$\partial_m {\rm Tr}J_m = -\frac{2}{g^2}\partial_m^2{\rm Tr}[\bar{\partial}_z(\Omega^{-1}\partial_z\Omega) + \bar{\partial}_w(\Omega^{-1}\partial_w\Omega)]. \tag{2.22}$$

Theories with $N_{\rm F} > N_{\rm C}$ are called semilocal theories, and vortices in these theories are called semilocal vortices [36]. In this case of $N_{\rm F} > N_{\rm C}$ we can consider the strong gauge coupling limit $g^2 \to \infty$ in which the model reduces to the nonlinear sigma model whose target space is the cotangent bundle over the complex Grassmann manifold, $T^*(G_{N_{\rm F},N_{\rm C}})$ [33], and semilocal vortices become Grassmannian sigma-model lumps. In this limit the master equation (2.17) becomes the algebraic equation as [22,30,31]

$$\Omega^{g^2\to\infty} = \Omega_0 = c^{-1}H_0H_0^\dagger. \tag{2.23}$$

Equation (2.23) requires the moduli matrix H_0 to have rank $N_{\rm C}$ for the entire complex (z, w) plane, in order for Ω to be invertible. Therefore the moduli space in this limit becomes simply the space of all the holomorphic maps from the complex two plane to the complex Grassmann manifold

$$\mathcal{M}_{\rm vv'i}^{g^2\to\infty} = \{\varphi|\mathbf{C}^2 \to G_{N_{\rm F},N_{\rm C}}, \bar{\partial}_z\varphi = \bar{\partial}_w\varphi = 0\}. \tag{2.24}$$

Let us recall that the moduli space $\mathcal{M}_{\rm vv'i}$ in Eq. (2.15) at finite g^2 admits isolated points where the rank of H_0 is less than $N_{\rm C}$. Such isolated points correspond to Abrikosov-Nielsen-Olesen (ANO) vortices [37] sizes of whose cores are of order $L_{\rm v} \sim 1/(g\sqrt{c})$. In the infinite gauge coupling limit, the ANO vortices tend to zero-size singular configurations of the delta function. In the sigma models ($g^2 \to \infty$) such singular configurations give small lump singularities and are no longer points in the moduli space $\mathcal{M}_{\rm vv'i}^{g^2\to\infty}$. In other words, the small lump singularities in $\mathcal{M}_{\rm vv'i}^{g^2\to\infty}$ are blown up in the moduli space $\mathcal{M}_{\rm vv'i}$ for finite gauge coupling by inserting the degrees of freedom of the ANO vortices. However not the all singularities in the moduli space $\mathcal{M}_{\rm vv'i}^{g^2\to\infty}$ in the strong gauge coupling limit are smoothed out in the moduli space $\mathcal{M}_{\rm vv'i}$ for finite gauge coupling. The moduli space $\mathcal{M}_{\rm vv'i}$ still has singularities interpreted as small instanton singularities as shown in the next section.

For the case of $N_{\rm F} = N_{\rm C}$ where we cannot take the infinite gauge coupling limit the moduli space purely contains degrees of freedom of the ANO vortices and instantons as studied in the next section.

III. INSTANTONS IN THE HIGGS PHASE

We have derived the master equation (2.17) of our 1/4 BPS system and have shown that the total moduli space is given by $\mathcal{M}_{\rm vv'i} \simeq \mathcal{H}\backslash\mathcal{G}$ in Eq. (2.15) (A → C in Fig. 1). It is, however, difficult to clarify what configuration each point of the moduli space gives, because we cannot solve the master equation (2.17) in its full generality. In order to overcome this problem partially, we here restrict ourselves to consider 1/4 BPS solutions which can be interpreted as 1/2 BPS lumps (B → C in Fig. 1) on the world volume of 1/2 BPS vortices in the 1-3 plane (A → B in Fig. 1).[8] Such restricted solutions constitute a moduli subspace in the total moduli space $\mathcal{M}_{\rm vv'i}$ defined as the space of all the holomorphic maps from complex plane to the vortex moduli space $\mathcal{M}_{\rm v} \simeq \mathbf{C} \times \hat{\mathcal{M}}_{\rm v}$ [given in Eq. (3.2) below]:

$$\begin{aligned}&\{\varphi|\mathbf{C} \to \mathcal{M}_{\rm v}, \bar{\partial}_w\varphi = 0\}\\ &\quad \simeq \mathbf{C} \times \{\varphi|\mathbf{C} \to \hat{\mathcal{M}}_{\rm v}, \bar{\partial}_w\varphi = 0\} \subset \mathcal{M}_{\rm vv'i},\end{aligned} \tag{3.1}$$

[8]We cannot exhaust all 1/4 BPS states by this method. For example, vortices in the 2-4 plane cannot be expressed in the effective theory on the world volume of the vortex. We will return to discuss this problem in Sec. V.

where the factor $\mathbf{C}$ representing the center of positions of the vortices is factored out from the target space because lumps cannot wrap it. In the case of a single vortex the reduced moduli space $\hat{\mathcal{M}}_{\rm v}$ coincides with the target space of lumps and the moduli space (3.1) reduces to that of the lump as will be shown below. The further study is required for the case of multiple vortices.

This section consists of two subsections. In the first subsection we give the effective action on 1/2 BPS vortices, which we call the vortex theory (B in Fig. 1), and explain a relation between the 1/4 BPS states and the 1/2 BPS states in the vortex theory (A $\to$ B $\to$ C in Fig. 1). In particular we work out the vortices in the theory with $N_{\rm C} = N_{\rm F} \equiv N$ focusing on $N = 2$, but not the semilocal vortices with $N_{\rm F} > N_{\rm C}$. In the second subsection we find the 1/4 BPS solutions for instantons in the Higgs phase by embedding the lump solution holomorphically into the moduli matrix for the vortex of 1/4 BPS solutions (A $\to$ C in Fig. 1).

A. Instantons as lumps on vortices

Let us first work out the 1/2 BPS non-Abelian vortex (A $\to$ B in Fig. 1). The 1/2 BPS equations for the vortices in the 1-3 plane can be derived under 1/2 SUSY condition with the projection defined by $\Gamma_{\rm v}$. They are obtained by ignoring dependence on w in Eqs. (2.8) and (2.9). Then the master equation for vortices is also obtained by throwing away the w dependence in Eq. (2.17). The moduli matrix $H_{\rm v0}$ for vortices does not depend on w and is holomorphic with respect to z. The total moduli space of the non-Abelian vortices is also obtained from $\mathcal{M}_{\rm vv'i}$ in Eq. (2.15) by ignoring w dependence:

$$\mathcal{M}_{\rm v} = \mathcal{H}_{\rm v} \backslash \mathcal{G}_{\rm v}, \tag{3.2}$$

$$\mathcal{G}_{\rm v} \equiv \{H_{\rm v0} | \mathbf{C} \longrightarrow M(N_{\rm C} \times N_{\rm F}, \mathbf{C}), \bar{\partial}_z H_{\rm v0} = 0\},$$
$$\mathcal{H}_{\rm v} \equiv \{V | \mathbf{C} \longrightarrow GL(N_{\rm C}, \mathbf{C}), \bar{\partial}_z V = 0\}.$$

The effective Lagrangian using the method of Manton [35] is obtained by promoting the moduli parameters ϕ^i in the background solutions with $\Omega_{\rm v}(H_0(\phi), H_0^*(\phi^*))$ to fields $\phi^i(x^u)$ depending on the world-volume coordinates x^u ($u = 0, 2, 4$) on vortices. After a lengthy calculation taking the Gauss's law into account, we find the following effective Lagrangian on the world volume of the vortices [38] in terms of $\Omega_{\rm v}$ (B in Fig. 1):

$$\begin{aligned}\mathcal{L}_{\rm v} = \int d^2x \Big[& \delta^u \delta_u^\dagger c \log\det\Omega_{\rm v} \\ & + \frac{4}{g^2} {\rm Tr}\{\bar{\partial}_z(\delta^u \Omega_{\rm v} \Omega_{\rm v}^{-1}) \delta_u^\dagger (\partial_z \Omega_{\rm v} \Omega_{\rm v}^{-1}) \\ & - \bar{\partial}_z(\partial_z \Omega_{\rm v} \Omega_{\rm v}^{-1}) \delta_u^\dagger (\delta^u \Omega_{\rm v} \Omega_{\rm v}^{-1})\}\Big],\end{aligned} \tag{3.3}$$

where the variation δ_u and its conjugate $\delta_u^\dagger$ act on complex moduli fields as $\delta_u \Omega_{\rm v} = \sum_i \partial_u \phi^i (\delta\Omega_{\rm v}/\delta\phi^i)$ and $\delta_u^\dagger \Omega_{\rm v} = \sum_i \partial_u \phi^{i*} (\delta\Omega_{\rm v}/\delta\phi^{i*})$, respectively.

The original theory with $N_{\rm F} > N_{\rm C}$ reduces to the nonlinear sigma model on $T^* G_{N_{\rm F}, N_{\rm C}}$ in the strong gauge coupling limit $g^2 \to \infty$ as stated in the last section. Semilocal vortices become 1/2 BPS Grassmannian sigma-model lumps. The second term in the effective Lagrangian (3.3) for the vortices vanishes in this limit and we get the Kähler potential of the effective action for the Grassmannian lumps as $K_{\rm lumps} = c \int d^2x \log\det\Omega^{g^2\to\infty}$ with $\Omega^{g^2\to\infty}$ in Eq. (2.23). This form of the Kähler potential is well known in the case of the $\mathbf{C}P^{N_{\rm F}-1}$ lumps corresponding to the case of $N_{\rm C} = 1$ [14,15].

Let us now clarify the correspondence between 1/4 BPS states of the parent theory (A $\to$ C in Fig. 1) and 1/2 BPS states on the vortex theory (A $\to$ B $\to$ C in Fig. 1). To this end, it is very important to observe a relation between the effective Lagrangian (3.3) of the vortex theory and the energy density (2.18) of 1/4 BPS states with Eqs. (2.19) and (2.21). Similarly to the last equation in Eq. (2.8), the 1/2 BPS equation for lumps on the vortex theory is obtained as

$$\bar{\partial}_w \phi^i = 0. \tag{3.4}$$

Assuming a static solution, we obtain

$$\begin{aligned}\delta^u \times \delta_u^\dagger &= \partial^u \phi^i \frac{\delta}{\delta\phi^i} \times \partial_u \phi^{j*} \frac{\delta}{\delta\phi^{j*}} \\ &= -2\Big(\partial_w \phi^i \frac{\delta}{\delta\phi^i} \times \bar{\partial}_w \phi^{j*} \frac{\delta}{\delta\phi^{j*}} \\ &\quad + \bar{\partial}_w \phi^i \frac{\delta}{\delta\phi^i} \times \partial_w \phi^{j*} \frac{\delta}{\delta\phi^{j*}}\Big).\end{aligned} \tag{3.5}$$

The 1/2 BPS equation (3.4) on the vortex theory implies that the variation $\delta^u \times \delta_u^\dagger$ can be identified with $-2\partial_w \times \bar{\partial}_w$ on the 1/2 BPS states

$$\delta^u \times \delta_u^\dagger |\text{BPS on vortex}\rangle = -2\partial_w \times \bar{\partial}_w |\text{BPS on vortex}\rangle. \tag{3.6}$$

Thus the effective Lagrangian (3.3) evaluated on the 1/2 BPS states correctly gives the minus of the energy density (2.18) omitting the contribution of vortices in the 1-3 plane. More explicitly, the first term in Eq. (3.3) corresponds to the vortices in the 2-4 plane and the second term to the instantons. This assures that instantons as 1/4 BPS states can be identified as 1/2 BPS states on the vortex theory.

To avoid inessential complications, let us consider the theory in the case of $N = 2$. The moduli space for a single vortex in this theory was found to be $\mathbf{C} \times \mathbf{C}P^1$ in Refs. [23,24] where $\mathbf{C}$ parametrizes zero modes for broken translational symmetries in the two codimensions and $\mathbf{C}P^1$ for broken global $SU(2)_{\rm F}$ symmetry in the internal space, respectively. Let us first find out the moduli matrices for the single vortex and recover the previous results in Refs. [23,24] in terms of the moduli matrices (A $\to$ B in

Fig. 1). As was mentioned in the previous section, $\det\Omega_0 \propto \det H_0 H_0^\dagger$ for the single vortex has to be proportional to $|z - z_0|^2$ with z_0 the position of the vortex. We find that general moduli matrices for the single vortex can be transformed by the world-volume symmetry (2.14) to either of the following two matrices:

$$H_{\rm v0}^{\rm single}(z; z_0, b) \equiv \sqrt{c}\begin{pmatrix} z - z_0 & 0 \\ b & 1 \end{pmatrix}, \qquad H_{\rm v0}^{\prime\,\rm single}(z; z_0, b') \equiv \sqrt{c}\begin{pmatrix} 1 & b' \\ 0 & z - z_0 \end{pmatrix}, \tag{3.7}$$

with $b, b' \in \mathbf{C}$. These two matrices can be transformed to each other with the relation $b' = 1/b$ by a world-volume transformation (2.14) [9] except for specific points $b = 0$ and $b' = 0$. Clearly, the moduli space of the single vortex can be identified as $\mathbf{C} \times S^2 \simeq \mathbf{C} \times \mathbf{C}P^1$. More explicitly, $\mathbf{C}$ and $S^2 \simeq \mathbf{C}P^1$ are covered by z_0 and by two patches b and b' in Eq. (3.7), respectively. The moduli parameter b can be identified as the orientational moduli parameter which is associated with the spontaneously broken $SU(2)_{\rm F}$ flavor symmetry [23,24]. In fact, the above moduli matrix can be rederived from that with $b = 0$ by acting $U \in SU(2)_{\rm F}$ combined with a world-volume transformation V_U:

$$H_{\rm v0}^{\rm single}(z; z_0, b) = V_U H_{\rm v0}^{\rm single}(z; z_0, 0) U, \tag{3.8}$$

with

$$U \equiv \begin{pmatrix} \phi_1 & \phi_2 \\ -\phi_2^* & \phi_1^* \end{pmatrix}, \qquad V_U \equiv \begin{pmatrix} \phi_1^* & -\phi_2(z - z_0) \\ 0 & 1/\phi_1^* \end{pmatrix}. \tag{3.9}$$

Here $|\phi_1|^2 + |\phi_2|^2 = 1$ and we have identified $b = -\phi_2^*/\phi_1^*$. Note that $H_{\rm v0}^{\rm single}(z; z_0, 0)$ breaks $SU(2)_{\rm F}$ into $U(1)_{\rm F}$, so the orientational moduli space is found to be $\mathbf{C}P^1 \simeq SU(2)_{\rm F}/U(1)_{\rm F}$ parametrized by homogeneous coordinates ϕ_1 and ϕ_2 or an inhomogeneous coordinate b or b'. $H_{\rm v0}^{\rm single}(z; z_0, b)$ is the most general for a single vortex in the sense that it contains all solutions with a single vortex. This is consistent with the results in Ref. [23], where the real dimension of the moduli space of the non-Abelian k vortices was obtained as $2kN$ by making use of the index theorem.

Let us next solve the master equation for the single vortex with the moduli matrix (3.7). To do this, we first recall the case of $N = 1$, where we obtain the well-known ANO vortex. In our formulation, the ANO vortex is given by $\Omega_\star$ satisfying

$$4\bar\partial_z \partial_z \log\Omega_\star = cg^2(1 - |z - z_0|^2 \Omega_\star^{-1}). \tag{3.10}$$

Returning to the $N = 2$ case, let us first take the diagonal moduli matrix $H_{\rm v0}^{\rm single}(z; z_0, 0)$. The solution of the master equation (2.17) for the 1/2 BPS vortex with this moduli matrix $H_{\rm v0}^{\rm single}(z; z_0, 0)$ is obtained by embedding the ANO vortex solution $\Omega_\star$ as $\Omega_{\rm v}|_{b=0} = {\rm diag.}(\Omega_\star, 1)$ [23]. The solutions corresponding to the general moduli matrix $H_{\rm v0}$ in Eq. (3.7) can be obtained by using the world-volume transformation V_U in Eq. (3.9) as

$$\Omega_{\rm v} = V_U(\Omega_{\rm v}|_{b=0})V_U^\dagger = \begin{pmatrix} \frac{\Omega_\star + |b|^2|z - z_0|^2}{1 + |b|^2} & \bar b(z - z_0) \\ b(\bar z - \bar z_0) & 1 + |b|^2 \end{pmatrix}. \tag{3.11}$$

We now reach the place where the effective theory of the single vortex can be exactly obtained (B in Fig. 1). This can be achieved by promoting the moduli parameters z_0 and b in the solution (3.11) into fields on the vortex world volume $z_0(t, w, \bar w)$ and $b(t, w, \bar w)$ and by plugging them into the effective Lagrangian (3.3) [35]. We thus find the Kähler potential with the coefficient of the Kähler metric (Kähler class) $4\pi/g^2$ for the full moduli fields $z_0(t, w, \bar w)$ and $b(t, w, \bar w)$:

$$K_{\rm v} = c\pi|z_0|^2 + \frac{4\pi}{g^2}\log(1 + |b|^2). \tag{3.12}$$

The first term comes from the first term of the effective Lagrangian (3.3) and the second term from the second term of (3.3) corresponding to the instantons. This Lagrangian with the Kähler class $4\pi/g^2$ was also determined in [23] by the brane configuration and in [10] by using the 1/4 BPS states with monopoles in the Higgs phase.

Following the prescription given in the introduction, next we consider the 1/2 BPS lumps in the effective theory on the 2 + 1 dimensional world volume of the vortex (B → C in Fig. 1). The BPS equation (3.4) can be solved for k lumps using rational functions of degree k [14,16] as

$$b(w) = \frac{P_k(w)}{\alpha P_k(w) + aQ_{k-1}(w)}, \tag{3.13}$$

with

$$P_k(w) \equiv \prod_{i=1}^{k}(w - p_i), \quad Q_{k-1}(w) \equiv \prod_{j=1}^{k-1}(w - q_j). \tag{3.14}$$

The moduli parameters $\{p_1, p_2, \ldots, p_k\}$ have one-to-one correspondence with the positions of the k lumps in the host vortex, a with the total size of the configurations and $\{q_1, q_2, \ldots, q_{k-1}\}$ with the relative sizes of the k lumps. The remaining modulus α parametrizes $\mathbf{C}P^1$ at the boundary (the infinity) of w since $b(w) \to 1/\alpha$ as $|w| \to \infty$. Especially, $\{p_i, a, q_j\}$ can precisely be identified with positions and sizes of k lumps when $\alpha = 0$.[10] Notice that zeros of the denominator in Eq. (3.13) are not true singularities but mere coordinate singularities. This is an artifact caused by the fact that b is an inhomogeneous coordinate of the $\mathbf{C}P^1$ manifold. Namely the corresponding configurations are smooth and continuous at these coordinate singularities. On the other hand, the point $a = 0$ and the points

[9] $H_{\rm v0}^{\prime\,\rm single}(z; z_0, b' = 1/b) = \begin{pmatrix} 0 & 1/b \\ -b & z - z_0 \end{pmatrix} H_{\rm v0}^{\rm single}(z; z_0, b)$.

[10] For $\alpha \neq 0$ we should redefine these parameters to describe the physical positions and sizes. For the case of $k = 1$ see Eq. (3.25).

$p_i = q_j$ are true singularities of the moduli space of the lumps since the degree of the solution (3.13) decreases and the corresponding configurations become singular. These singularities are called small lump singularities.

As was expected, *the mass of k lumps precisely agrees with that of the k instantons*, namely $8\pi^2 k/g^2$. This mass comes from the second term in Eq. (3.12) which originally corresponds to the instanton charges as was shown in the previous section. We thus can identify the 1/4 BPS instantons in the original theory in $d = 4 + 1$ dimensions (A → C in Fig. 1) as the 1/2 BPS lumps in the effective theory on the $d = 2 + 1$ dimensional world volume of the vortex (A → B → C in Fig. 1).

Returning to the vortex, orientational moduli space for spontaneous symmetry breaking by the single non-Abelian vortex for the case of $N > 2$ was shown to be $SU(N)/[SU(N-1) \times U(1)] \simeq \mathbf{C}P^{N-1}$ [23–25]. One of the patch for moduli space of the single vortex is given by

$$H_{\rm v0} = \sqrt{c}\begin{pmatrix} z - z_0 & 0 & 0 & \cdots & 0 \\ b_1 & 1 & 0 & \cdots & 0 \\ b_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_{N-1} & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{3.15}$$

with b_i complex parameters. There exist other $N - 1$ patches for $\mathbf{C}P^{N-1}$ given through the world-volume transformation (2.14). There exist N complex moduli parameters z_0 and b_i. Here z_0 is the position of the vortex and b_i are orientational moduli parametrizing $\mathbf{C}P^{N-1}$. The Kähler potential for the orientational moduli parameters b_i can be determined up to the constant factor as $K \propto \log(1 + \sum |b_i|^2)$ by discussing only symmetry. The factor can be precisely determined without exact solutions or any calculations by recognizing an equivalence between the mass of the 1/4 BPS objects in the original theory and the 1/2 BPS objects in the vortex theory. Then we get

$$K_{\rm v} = \pi c|z_0|^2 + \frac{4\pi}{g^2}\log\left(1 + \sum_{i=1}^{N-1}|b_i|^2\right). \tag{3.16}$$

The multilump solution for the $\mathbf{C}P^{N-1}$ model is also known [15].

B. 1/4 BPS solutions of the instantons in the Higgs phase

The aim of this subsection is to specify the moduli matrix $H_0(z, w)$ for the instantons in the Higgs phase as the 1/4 BPS states (A → C in Fig. 1), which have been found to be the 1/2 BPS lumps on the vortex theory in the previous subsection (A → B → C in Fig. 1). We will also specify the moduli space of the instantons in the Higgs phase. Our basic strategy is to replace the moduli parameter b in the moduli matrix $H_{\rm v0}^{\rm single}(z; z_0, b)$ in Eq. (3.7) for a single vortex by the lump solution $b(w)$ in Eq. (3.13):[11]

$$H_0(z, w) \sim H_{\rm v0}^{\rm single}(z; z_0, b(w)) = \sqrt{c}\begin{pmatrix} z - z_0 & 0 \\ \frac{P_k}{\alpha P_k + aQ_{k-1}} & 1 \end{pmatrix}. \tag{3.17}$$

Although this procedure is very simple, there exists a technical complication; a solution $b(w)$ is not holomorphic at some points in w where $b(w)$ diverges, whereas all components in the moduli matrix $H_0(z, w)$ have to be holomorphic with respect to both z and w at any point $(z, w) \in \mathbf{C}^2$ to cover the whole solutions consistently. This can be overcome by noting that the lump solution $b(w)$ is given in an inhomogeneous coordinate b on $\mathbf{C}P^1$. Therefore we should transform the moduli matrix $H_{\rm v0}^{\rm single}(z; z_0, b(w))$ written in the inhomogeneous coordinate b into the one in homogeneous coordinates. This can be achieved by

$$H_0(z, w) = \sqrt{c}\begin{pmatrix} (z - z_0)A_{k-1}(w) & (z - z_0)(\alpha A_{k-1}(w) + aB_{k-2}(w)) \\ P_k(w) & \alpha P_k(w) + aQ_{k-1}(w) \end{pmatrix}, \tag{3.18}$$

with A_{k-1} and B_{k-2} being the polynomial functions of order $k - 1$ and $k - 2$ in w, given by

$$\begin{aligned} A_{k-1}(w) &= \sum_{i=1}^{k}\frac{1}{Q_{k-1}(p_i)}\prod_{i'(\neq i)=1}^{k}\left(\frac{w - p_{i'}}{p_i - p_{i'}}\right), \\ B_{k-2}(w) &= \sum_{j=1}^{k-1}\frac{-1}{P_k(q_j)}\prod_{j'(\neq j)=1}^{k-1}\left(\frac{w - q_{j'}}{q_j - q_{j'}}\right). \end{aligned} \tag{3.19}$$

These $A_{k-1}(w)$ and $B_{k-2}(w)$ have been uniquely determined by the condition[12]

$$A_{k-1}Q_{k-1} - B_{k-2}P_k = 1 \tag{3.20}$$

requiring that the vorticity of the solution should coincide with the one in Eq. (3.17), namely, that the solutions should have a single vortex in the 1-3 plane and no vortices in the 2-4 plane. Now the relation between the right hand side of Eq. (3.17) and Eq. (3.18) is shown to be

$$H_0(z, w) = V(P_k(w), Q_{k-1}(w))H_{\rm v0}^{\rm single}(z; z_0, b(w)), \tag{3.21}$$

with the matrix $V(P_k, Q_{k-1})$ defined by

[12]We can consider the polynomial functions $A_{k'}$ of order k' and $B_{k'-1}$ of order $k' - 1$ for $k' > k - 1$. However, we can always set $k' = k - 1$ by use of the world-volume transformation without loss of generality.

[11]The exact relation between these matrices is given in Eq. (3.21), below.

$$V(P_k, Q_{k-1}) \equiv \begin{pmatrix} \frac{a}{\alpha P_k + aQ_{k-1}} & (z - z_0)(\alpha A_{k-1} + aB_{k-2}) \\ 0 & \alpha P_k + aQ_{k-1} \end{pmatrix}. \tag{3.22}$$

This matrix is a valid world-volume transformation (2.14) only in a particular region of w with $\alpha P_k + aQ_{k-1}$ nonzero, since it has a singularity in w. Although $V(P_k, Q_{k-1})$ is not a valid world-volume transformation (2.14) because of singularities in w, it is needed to obtain the regular moduli matrix (3.18) by compensating singularities in $H_{\mathrm{v}0}^{\mathrm{single}}(z; z_0, b(w))$.

Next let us examine the moduli parameters of the k instantons in the Higgs phase in detail. No new parameters appear in A_{k-1} and B_{k-2}, and therefore the configuration of k instantons in the Higgs phase has the $2k + 2$ complex moduli parameters $(z_0, \{p_i\}, \{q_j\}, a, \alpha)$. Here z_0 is the position of the single vortex on the 1-3 plane. As was mentioned in the first of this section, this decouples with other moduli parameters. So the moduli space of this configuration can be simply written as

$$\begin{aligned} \mathcal{M}^{k\text{-instantons}} &\simeq \mathbf{C} \times \mathcal{M}^{k\text{-lumps}} \\ &\simeq \mathbf{C} \times \{\varphi | \mathbf{C} \to \hat{\mathcal{M}}^{1\text{-vortex}}, \bar{\partial}_w \varphi = 0\}. \end{aligned} \tag{3.23}$$

Note that this decoupling property of z_0 from $(\{p_i\}, \{q_j\}, a, \alpha)$ can also be read from the Kähler potential (3.12). From the discussion given in the previous subsection, we realize that $\{p_i\}$ corresponds to the positions of k instantons inside the vortex, a to the total size and the orientation of the configurations and $\{q_j\}$ to the relative sizes and orientations of the instantons. It is very interesting to observe that the *small lump singularities* with $a = 0$ or $p_i = q_j$ in Eq. (3.13) are now interpreted as the *small instanton singularities* in the Higgs phase. In fact, in the limit with a tending to zero the rank of the moduli matrix (3.18) reduces by one and its determinant vanishes. Then the point $a \to 0$ is singular in the moduli space. On the other hand, the small lump singularities coming from $p_i = q_j$ in Eq. (3.13) appear as the divergences of $1/P_k$ and $1/Q_{k-1}$ in A_{k-1} and B_{k-2} in Eq. (3.19).[13] The remaining parameter α parametrizes $\mathbf{C}P^1$ similarly to the lump solutions. In summary we find $z_0 \in \mathbf{C}$, $p_i \in \mathbf{C}$, $a \in \mathbf{C}^* \equiv \mathbf{C} - \{0\} \simeq \mathbf{R} \times S^1$, $q_j \in \mathbf{C} - \{p_1, p_2, \ldots, p_k\}$ and $\alpha \in \mathbf{C}P^1$.

Now we discuss the simplest case of a single instanton ($k = 1$) with $A_0 = 1$ and $B_{-1} = 0$ in more detail. Then we have

$$b(w) = \frac{w - p_1}{\alpha(w - p_1) + a}$$
$$\Rightarrow H_0^{1\text{-instanton}} = \sqrt{c}\begin{pmatrix} z - z_0 & \alpha(z - z_0) \\ w - p_1 & \alpha(w - p_1) + a \end{pmatrix}. \tag{3.24}$$

To clarify the physical significance of these four complex moduli parameters z_0, p_1, a, α, let us transform the moduli matrix in Eq. (3.24) into that with $\alpha = 0$ by the $SU(2)_{\mathrm{F}}$ rotation. This can be performed by choosing $\phi_2 = -\alpha\phi_1^*$ of U in Eq. (3.9) and factor out the world-volume symmetry in Eq. (2.14). Then we get the physical position p_0 and the size $|a_0|$ of the instanton in the vortex

$$p_0 = p_1 - \frac{\alpha^*}{1 + |\alpha|^2} a, \qquad |a_0| = \frac{|a|}{1 + |\alpha|^2}. \tag{3.25}$$

These are invariant under the $SU(2)_{\mathrm{F}}$ transformation. We illustrate this configuration in four Euclidean space schematically in Fig. 2.

Let us discuss the global structure (topology) of the moduli space $\mathcal{M}^{1\text{-instanton}}$ of one instanton. The moduli matrix in one patch H_0' corresponding to the second one in Eq. (3.7) is related to that in the other patch H_0 in Eq. (3.24) by a world-volume transformation V as

$$\begin{aligned} H_0'^{1\text{-instanton}} &\equiv \sqrt{c}\begin{pmatrix} \alpha'(w - p_1') + a' & w - p_1' \\ \alpha'(z - z_0) & z - z_0 \end{pmatrix} \\ &= V H_0^{1\text{-instanton}}, \\ V &= \begin{pmatrix} 0 & 1/\alpha \\ 1/\alpha & 0 \end{pmatrix}, \end{aligned} \tag{3.26}$$

with the following relation between coordinates in two patches

$$\alpha' = \frac{1}{\alpha}, \qquad a' = -\frac{a}{\alpha^2}, \qquad p_1' = p_1 - \frac{a}{\alpha}. \tag{3.27}$$

Here both α and α' are the patches of the standard inhomogeneous coordinates of the $\mathbf{C}P^1$ and they are enough to cover the whole manifold. We see that p_1 and a transform in the union of the two patches α and α'. We find that a requires a nontrivial transition function $-1/\alpha^2$ between two patches, showing that it is a tangent vector as a fiber on

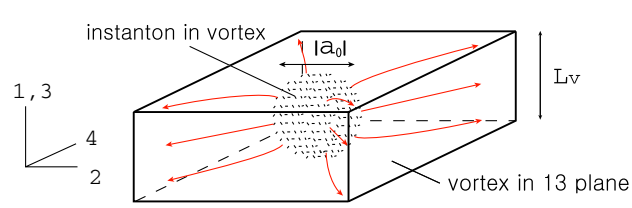


FIG. 2 (color online). Single instanton in the Higgs phase. The size of the vortex is given by $L_{\mathrm{v}} \sim 1/g\sqrt{c}$.

[13] In Eq. (3.19) these also appear to diverge when $p_i = p_{i'}$ for $i \neq i'$ and $q_j = q_{j'}$ for $j \neq j'$, respectively, but it is not the case; We can show that the factors $p_i - p_{i'}$ and $q_j - q_{j'}$ in denominators are always canceled with numerators after the summation. Hence the points $p_i = p_{i'}$ and $q_i = q_{j'}$ are not singular in the moduli space.

the $\mathbf{C}P^1$. However, instead of p_1 we can use the coordinate p_0 in Eq. (3.25) which is an invariant global coordinate for two patches, indicating that the space $\mathbf{C}$ parametrized by p_0 is a direct product to the $\mathbf{C}P^1$. Therefore we obtain the topology of the moduli space of one $U(2)$ instanton in the Higgs phase as

$$(z_0, p_0, a, \alpha) \in \mathbf{C} \times \mathbf{C} \times (\mathbf{C}^* \ltimes \mathbf{C}P^1) \simeq \mathcal{M}^{1\text{-instanton}}, \tag{3.28}$$

with $F \ltimes B$ denoting a fiber bundle over a base space B with a fiber F. More precisely $\mathbf{C}^* \ltimes \mathbf{C}P^1$ is the tangent bundle without zero section.[14]

Here we make a comment on (non)normalizability of zero modes. Some zero modes corresponding to these moduli parameters in (3.28) are not normalizable under four-dimensional integration over the all codimensions. For instance the modulus z_0 for the position of the vortex is normalizable in two dimensions perpendicular to the vortex but is apparently nonnormalizable in four dimensions. The modulus α parametrizes the boundary condition of the sigma-model instanton in the effective theory on the vortex. It is also nonnormalizable in the effective theory and therefore in the original theory. Nevertheless we emphasize that all of the moduli parameters in (3.28) are needed to determine the configuration of this composite state, and that dynamics of the composite state is described by these parameters.[15]

For the case of $N > 2$ we can specify the moduli matrix $H_0(z, w)$ for a particular class of 1/4 BPS solutions identified as 1/2 BPS states in the vortex theory, similarly to the case of $N = 2$. We could also obtain the moduli matrices and the moduli space for 1/4 BPS states corresponding to the $U(N)$ instantons in the Higgs phase by repeating the same discussion.

[14]It is interesting to observe that the moduli space (3.28) with zero section added is homeomorphic to that of single $U(2)$ noncommutative instanton, $\mathbf{C}^2 \times T^*\mathbf{C}P^1$ [20].

[15]This phenomenon of nonnormalizable modes is a commonplace issue in composite solitons, and has been observed in the case of the wall junction [39]. Let us explain the inevitability of nonnormalizable modes in composite solitons by taking the junction as the simplest example. The 1/4 BPS junction can be formed if three or more half-infinite 1/2 BPS walls meet at a junction. Nambu-Goldstone (NG) fermions necessarily arise corresponding to the broken 3/4 of supercharges. One might hope that these NG fermions are localized around the junction and are normalizable in the codimension two plane of the junction of walls. One can show that they cannot be normalizable as follows. Take any one of the constituent walls. Those NG fermions corresponding to the supercharges broken by that wall have support on the wall, which extends to infinity along the wall. Hence they are nonnormalizable as modes on the codimension two composite soliton (junction of walls). We can choose the remaining NG modes which do not have support on that particular wall. However, these NG fermions correspond to supercharges broken by at least one of the other walls. Then they have to have support along the walls which break the supercharge. Consequently all of the NG fermions should have support infinitely extending along at least one of the walls, and are nonnormalizable. One can easily recognize that this feature of nonnormalizable modes is a usual phenomenon of composite solitons, and care should be exercised when we discuss effective theories on the composite solitons.

IV. MONOPOLES AND CALORONS IN THE HIGGS PHASE

Recently, 1/4 BPS states of the monopoles in the Higgs phase were studied [8–12] in 3 + 1 dimensional massive SQCD with nondegenerate masses for hypermultiplets. Unlike the case of the massless model (or the massive model in which all the mass parameters are degenerate), the $SU(N_F)$ flavor symmetry is explicitly broken to $U(1)^{N_F-1}$ in the massive case. Therefore vortices in the model with nondegenerate masses are essentially Abelian (ANO) vortices, and moduli fields corresponding to the orientational moduli parameters [9,13,25] are not exactly massless moduli in the effective theory of vortices. It has been found that the effective theory of the non-Abelian vortices can be constructed in the case where the mass difference is much smaller than the FI parameter $|\Delta m| \ll \hat{g}\sqrt{\hat{c}}$ [8,32]. Here $\hat{g}$ and $\hat{c}$ are the gauge coupling and the FI parameter in 3 + 1 dimensions, respectively. The effective theory has a potential $\hat{V} \sim k^2$ where k is a Killing vector on the moduli space of the non-Abelian vortices in the massless model [8,32]. In the following of this section we call the vortex theory with the potential *massive* vortex theory and the vortex theory without any potential *massless* vortex theory.

It has been shown in Ref. [8] that the 1/4 BPS state of the monopoles in the Higgs phase can be realized as the 1/2 BPS kinks in the massive vortex theory ($\mathrm{A}' \to \mathrm{B}' \to \mathrm{C}'$ in Fig. 1). In this section we will find the 1/4 BPS solution corresponding to the monopoles in the Higgs phase directly. Namely, we specify the moduli matrix for the 1/4 BPS state of the monopoles in the Higgs phase ($\mathrm{A}' \to \mathrm{C}'$ in Fig. 1). To achieve this, we find that it is very useful to promote the 3 + 1 dimensional massive theory to the 4 + 1 dimensional massless theory ($\mathrm{A}' \to \mathrm{A}$ in Fig. 1). By this procedure a monopole in 3 + 1 dimensions becomes a monopole string in 4 + 1 dimensions. This 4 + 1 dimensional point of view (the triangle ABC in Fig. 1) not only gives a nice realization of the monopoles but also leads to calorons in the Higgs phase which interpolate between the instantons and the monopoles in the Higgs phase.

A. Walls, vortices and monopoles revisited

The four-dimensional massive model with nondegenerate masses

$$M \equiv \mathrm{diag}\,(m_1, m_2, \ldots, m_{N_F}), \tag{4.1}$$

for hypermultiplets can be derived from our five-

dimensional massless SQCD[16] by performing the Scherk-Schwarz (SS) dimensional reduction [27] ($\mathrm{A} \to \mathrm{A}'$ in Fig. 1), in which the fifth direction x^4 is compactified on S^1 with radius R using a twisted boundary condition

$$H(x^\mu, x^4 + 2\pi R) = H(x^\mu, x^4)e^{i2\pi RM}, \tag{4.2}$$

with $0 \le m_A < 1/R$. If we ignore the infinite towers of the Kaluza-Klein modes, we have the lightest mass field $\hat{H}(x^\mu)$ as a function of the four-dimensional space-time coordinates

$$H(x^\mu, x^4) = \frac{1}{\sqrt{2\pi R}}\hat{H}(x^\mu)e^{iMx^4}, \tag{4.3}$$

with $\mu = 0, 1, 2, 3$. Other fields neutral under the $SU(N_\mathrm{F})$ flavor symmetry are of the form

$$\begin{aligned} W_\mu(x^\mu, x^4) &= W_\mu(x^\mu), \\ \Sigma(x^\mu, x^4) - iW_4(x^\mu, x^4) &= \Sigma(x^\mu) + i\hat{\Sigma}(x^\mu). \end{aligned} \tag{4.4}$$

The 1/4 BPS equations (we have ignored H^2 and Σ) in (2.8) and (2.9) reduce to those in four dimensions [22]

$$\begin{aligned} F_{12} = -\mathcal{D}_3\hat{\Sigma}, \qquad F_{23} = -\mathcal{D}_1\hat{\Sigma}, \qquad \bar{\mathcal{D}}_z\hat{H} = 0, \\ \mathcal{D}_2\hat{H} = \hat{H}M - \hat{\Sigma}\,\hat{H}, \end{aligned} \tag{4.5}$$

$$F_{13} - \mathcal{D}_2\hat{\Sigma} = -\frac{\hat{g}^2}{2}(\hat{c}\mathbf{1}_{N_\mathrm{C}} - \hat{H}\hat{H}^\dagger). \tag{4.6}$$

Here the gauge coupling $\hat{g}$ and the FI parameter $\hat{c}$ in four dimensions are given by

$$\begin{aligned} \frac{1}{\hat{g}^2} &\equiv \int_0^{2\pi R} dx^4 \frac{1}{g^2} = \frac{2\pi R}{g^2}, \\ \hat{c} &\equiv \int_0^{2\pi R} dx^4 c = 2\pi R c, \end{aligned} \tag{4.7}$$

respectively. It was known that these BPS equations admit walls, vortices and monopoles as 1/4 BPS states [8,22] ($\mathrm{A}' \to \mathrm{C}'$ in Fig. 1). The supercharges preserved by the above BPS equations are summarized in Table I. Table I shows that vortices in the 1-3 plane, vortices in the 2-4 plane and instantons in five dimensions are the BPS states with the conserved supercharge specified by the same projection as vortices in the 1-3 plane, walls transverse to the x^2 direction and monopoles in four dimensions, respectively. Therefore after the SS dimensional reduction, these BPS solitons in five dimensions reduce to the respective BPS solitons in four dimensions. In fact, the instanton charge coincides with the monopole charge under the SS dimensional reduction as

$$\frac{1}{2g^2}\int d^3x \int_0^{2\pi R} dx^4 \mathrm{Tr}(F_{mn}\tilde{F}^{mn}) = \frac{2}{\hat{g}^2}\int d^3x \vec{\partial}\cdot \mathrm{Tr}({}^*\vec{F}\hat{\Sigma}). \tag{4.8}$$

The BPS equations in Eqs. (4.5) and (4.6) in 3 + 1 dimensional massive theory have been solved in terms of the moduli matrix of the system [22]. Especially all the exact solutions were obtained in the strong gauge coupling limit $\hat{g}^2 \to \infty$ in the semilocal case with $N_\mathrm{F} > N_\mathrm{C}$. Here we reconsider Eqs. (4.5) and (4.6) in a general case of $N_\mathrm{F} \ge N_\mathrm{C}$ from the five-dimensional point of view. For that purpose, let us consider a restricted sector of the moduli space which is specified by the moduli matrix $H_0(z, w)$ in the form of

$$H_0(z, w) = \frac{1}{\sqrt{2\pi R}}\hat{H}_0(z)e^{Mw}, \tag{4.9}$$

where an $N_\mathrm{C} \times N_\mathrm{F}$ matrix $\hat{H}_0(z)$ does not depend on w, $\bar{w}$, and is holomorphic with respect to z. The matrix $\hat{H}_0(z)$ should have rank N_C in generic points of z (namely, apart from isolated points). Note that this restriction (4.9) is up to the world-volume transformation (2.14). For the restricted moduli matrix given above the "source" Ω_0 of the master equation (2.17) is independent of the x^4 coordinate $\Omega_0(x^M) = \hat{c}^{-1}\hat{H}_0 e^{2Mx^2}\hat{H}_0^\dagger \equiv \hat{\Omega}_0(x^\mu)$. Then the solution Ω of Eq. (2.17) is also independent of the x^4 coordinate: $S(x^M) = \hat{S}(x^\mu)$ and $\Omega(x^M) = SS^\dagger = \hat{S}\hat{S}^\dagger \equiv \hat{\Omega}(x^\mu)$. At this stage, the master equation (2.17) reduces to

$$4\bar{\partial}_z(\partial_z\hat{\Omega}\hat{\Omega}^{-1}) + \partial_2(\partial_2\hat{\Omega}\hat{\Omega}^{-1}) = \hat{c}\hat{g}^2(\mathbf{1}_{N_\mathrm{C}} - \hat{\Omega}_0\hat{\Omega}^{-1}). \tag{4.10}$$

Their solution (2.12) can also be rewritten as follows:

TABLE I. Gamma matrices of the projection operators for 1/4 BPS states in 4 and 5 dimensions. Here we define $\gamma^5 \equiv -i\gamma^4 = i\gamma^0\gamma^1\gamma^2\gamma^3$. In our previous paper [22], we used another set of gamma matrices related by the redefinition $\gamma^m \to -\gamma^m i\gamma^5$, and chose the wall profile along x^3 instead of the present choice of x^2.

$d = 4 + 1$	World volume	Γ
Vortex	0,2,4	$-\gamma^{13} \otimes i\sigma^3$
Vortex	0,1,3	$-\gamma^{24} \otimes i\sigma^3$
Instanton	0	$\gamma^0 \otimes \mathbf{1}_2$
$d = 3 + 1$	World volume	Γ
Vortex	0,2	$-\gamma^{13} \otimes i\sigma^3$
Wall	0,1,3	$-i\gamma^2\gamma^5 \otimes i\sigma^3$
Monopole	0	$\gamma^0 \otimes \mathbf{1}_2$

[16]In Ref. [22], we studied the *massive* SQCD in $d = 4 + 1$ dimensions. This can be derived from the six-dimensional *massless* SQCD by the Scherk-Schwarz dimensional reduction, in exactly the same manner.

$$H = \frac{1}{\sqrt{2\pi R}}\hat{S}^{-1}\hat{H}_0 e^{Mw}, \qquad \bar{W}_z = -i\hat{S}^{-1}\bar{\partial}_z\hat{S},$$
$$W_2 - i\hat{\Sigma} = -i\hat{S}^{-1}\partial_2\hat{S}. \tag{4.11}$$

Notice that the above solution automatically satisfies the condition of the SS dimensional reduction (4.3) if we identify

$$\hat{H}(x^\mu) = \hat{S}^{-1}(x^\mu)\hat{H}_0(z)e^{Mx^2}. \tag{4.12}$$

The master equation (4.10) and its solutions (4.11) and (4.12) completely agree with those for the 1/4 BPS states containing walls, vortices and monopoles [22]. Therefore the restriction (4.9) to the form of the moduli matrix gives a map from 1/4 BPS solutions in 3 + 1 dimensions to those in 4 + 1 dimensions (C → C′ in Fig. 1).

We now realize that all the 1/4 BPS states of the walls, vortices and monopoles in the massive SQCD [22] have one-to-one correspondence with those in the restricted sector (4.9) of our 1/4 BPS states (vortices and instantons) in the five-dimensional massless SQCD. The moduli space of the former can also be understood from the five-dimensional point of view as

$$\mathcal{M}_{\rm wvm} = \mathcal{H}_{\rm wvm}\backslash\mathcal{G}_{\rm wvm}, \tag{4.13}$$

$$\mathcal{G}_{\rm wvm} \equiv \{\hat{H}_0|\mathbf{C} \longrightarrow M(N_{\rm C}\times N_{\rm F}, \mathbf{C}),\ \bar{\partial}_z\hat{H}_0 = 0\},$$

$$\mathcal{H}_{\rm wvm} \equiv \{V|\mathbf{C} \longrightarrow GL(N_{\rm C}, \mathbf{C}),\ \bar{\partial}_z V = 0\},$$

where $\hat{H}_0(z)$ must have the maximal rank $N_{\rm C}$ in generic z except for several points. It is interesting to observe that this total moduli space agrees with that for the non-Abelian vortices in Eq. (3.2), $\mathcal{M}_{\rm wvm} \simeq \mathcal{M}_{\rm v}$, although the former is for 1/4 BPS states and the latter is for 1/2 BPS states. In the strong gauge coupling limit the moduli space $\mathcal{M}_{\rm wvm}$ becomes

$$\mathcal{M}_{\rm wvm}^{g^2\to\infty} = \{\varphi|\mathbf{C} \to G_{N_{\rm F},N_{\rm C}},\ \bar{\partial}_z\varphi = 0\}, \tag{4.14}$$

in the case of $N_{\rm F} > N_{\rm C}$. This coincides with the moduli space of the Grassmannian sigma-model lumps [17] which can be classified by $\pi_2(G_{N_{\rm F},N_{\rm C}}) = \mathbf{Z}$ if we compactify $\mathbf{C}$ to $\mathbf{C}P^1$.

Moreover, if we push forward this descent relation from $\mathcal{M}_{\rm vv'i}$ to $\mathcal{M}_{\rm wvm}$, we arrive at solutions of the non-Abelian walls and their moduli space $\mathcal{M}_{\rm w}$ which have been extensively studied in Refs. [30,40]. To achieve this, we ignore z dependence in $\mathcal{M}_{\rm wvm}$:

$$\mathcal{M}_{\rm w} = \mathcal{H}_{\rm w}\backslash\mathcal{G}_{\rm w} \simeq G_{N_{\rm F},N_{\rm C}}, \tag{4.15}$$

$$\mathcal{G}_{\rm w} \equiv \{\tilde{H}_0|\mathbf{C}^0 \longrightarrow M(N_{\rm C}\times N_{\rm F}, \mathbf{C}), \text{rank } N_{\rm C}\}$$
$$= \{M(N_{\rm C}\times N_{\rm F}, \mathbf{C}), \text{rank } N_{\rm C}\},$$

$$\mathcal{H}_{\rm w} \equiv \{V|\mathbf{C}^0 \longrightarrow GL(N_{\rm C}, \mathbf{C})\} = GL(N_{\rm C}, \mathbf{C}),$$

where $\mathbf{C}^0$ is just a point. The condition on the constant matrix $\tilde{H}_0$ to have the maximal rank $N_{\rm C}$ has been deduced from the condition on $H_0(z, w)$ or $\hat{H}_0(z)$ in generic points of z or (z, w) in the case of instantons or monopoles, respectively. It comes from the fact that the moduli matrix must have rank $N_{\rm C}$ in the vacuum. It is interesting that in the strong gauge coupling ($N_{\rm F} > N_{\rm C}$) the total moduli space is unchanged

$$\mathcal{M}_{\rm w}^{g^2\to\infty} \simeq \mathcal{M}_{\rm w}, \tag{4.16}$$

unlike the case of other solitons because the moduli matrix $\tilde{H}_0$ is a constant matrix here.

We thus have found that solutions of our BPS equations (2.8) and (2.9) can give all four kinds of solitons: walls, vortices, monopoles and instantons. The relations between their total moduli spaces are given by

$$\mathcal{M}_{\rm w} \subset \mathcal{M}_{\rm wvm}(\simeq \mathcal{M}_{\rm v}) \subset \mathcal{M}_{\rm vv'i}. \tag{4.17}$$

B. Monopoles in the Higgs phase

Let us next find the solution of Eqs. (4.5) and (4.6) (A′ → C′ in Fig. 1) corresponding to one monopole in the Higgs phase, namely, a single monopole attached to a vortex. To be precise, we restrict ourselves into the simplest case with $N_{\rm C} = N_{\rm F} \equiv N = 2$ in the following of this section. As was explained in Sec. III, the most general moduli matrix containing a single vortex can be written in the form of

$$\hat{H}_0(z) = \sqrt{\hat{c}}\begin{pmatrix} z - z_0 & 0 \\ b_c & 1 \end{pmatrix}, \tag{4.18}$$

where b_c is a constant complex parameter. Notice that this is the same form with the moduli matrix (3.7) generating only a single vortex in the massless theory. However in the massive theory the moduli matrix (4.18) gives not only a vortex but also a monopole, where b_c gives the position and the phase of a monopole inside the vortex, as will be shown below. The difference between the massless theory and the massive theory appears as the factor e^{Mx^2} in Eq. (4.12) which is absent in Eq. (2.12). In the massless limit $M \to 0$, the moduli matrix (4.18) gives a single vortex only as expected.

Similarly to instantons in the Higgs phase, we can calculate charge of the monopole in the Higgs phase in terms of the massive vortex theory. To this end, let us recall the following two facts. One is that the instantons can be realized as lumps in the massless vortex theory with the Kähler potential (3.12) (A → B → C in Fig. 1). The other is that 1/4 BPS states in the four-dimensional massive theory can be obtained by the restriction (4.9) on 1/4 BPS states in the five-dimensional massless theory (C → C′ in Fig. 1). Combining these facts together, we naturally arrive at a notion of the SS dimensional reduction for the vortex theory (B → B′ in Fig. 1). In order to achieve this, we should first find what the action of the SS dimensional

reduction to the vortex theory is. In terms of the moduli matrix $H_0(z, w)$ the above twisted boundary condition (4.2) can be translated in terms of the moduli matrix as $H_0(z, w + 2\pi iR) = VH_0(z, w)e^{2\pi iRM}$ where $V(z, w)$ is an element of world-volume transformation (2.14). This naturally induces the following twisted boundary condition on the moduli fields $z_0(t, x^2, x^4)$ and $b(t, x^2, x^4)$ in the moduli matrix $H_{\rm v0}^{\rm single}(z; z_0, b)$ in the effective theory of the host vortex

$$H_{\rm v0}^{\rm single}(z; z_0(t, x^2, x^4 + 2\pi R), b(t, x^2, x^4 + 2\pi R)) = VH_{\rm v0}^{\rm single}(z; z_0(t, x^2, x^4), b(t, x^2, x^4))e^{i2\pi RM}, \quad (4.19)$$

with $V = e^{-2\pi iRM}$. Hence we can identify the twisted boundary condition to the moduli fields as $z_0(t, x^2, x^4 + 2\pi R) = z_0(t, x^2, x^4)$ and $b(t, x^2, x^4 + 2\pi R) = e^{i2\pi R\Delta m}b(t, x^2, x^4)$ with $\Delta m \equiv m_1 - m_2$. We thus get the action of the SS dimensional reduction for z_0 and b as

$$z_0(t, x^2, x^4) = \hat{z}_0(t, x^2), \qquad b(t, x^2, x^4) = e^{i\Delta m x^4}\hat{b}(t, x^2). \quad (4.20)$$

Plugging (4.20) into the effective Lagrangian with the Kähler potential (3.12) of the 2 + 1 dimensional massless vortex theory, we can obtain the effective Lagrangian for the massive vortex theory after integrating over x^4 (B → B′ in Fig. 1). The resulting 1 + 1 dimensional massive vortex theory consists of the scalar potential $\hat{V}_{\rm v}$ arising from the kinetic term in extra dimension x^4 and the Kähler potential $\hat{K}_{\rm v}$, given by

$$\hat{K}_{\rm v} = \hat{c}\pi|\hat{z}_0|^2 + \frac{4\pi}{\hat{g}^2}\log(1 + |\hat{b}|^2), \qquad \hat{V}_{\rm v} = \frac{4\pi}{\hat{g}^2}\frac{\Delta m^2|\hat{b}|^2}{(1 + |\hat{b}|^2)^2}. \quad (4.21)$$

These exactly agree with the results in Ref. [8] including the Kähler class of the Kähler potential and the coefficient of the scalar potential.

The scalar potential in Eq. (4.21) admits two discrete SUSY vacua at $\hat{b} = 0, \infty$. Thus 1/2 BPS states on the vortex theory become kinks (B′ → C′ in Fig. 1). In fact, the 1/2 BPS equation (3.4) reduces to

$$\partial_2\hat{b} - \Delta m\hat{b} = 0, \quad (4.22)$$

and the solution interpolating between $\hat{b} = 0$ and $\hat{b} = \infty$ is found to be

$$\hat{b}(t, x^2) = e^{-i\theta}e^{\Delta m(x^2 - x_0^2)}, \quad (4.23)$$

where $e^{-i\theta}$ is a phase and a real parameter x_0^2 is a position of the kink [26]. Although this configuration exponentially grows, the energy density is localized around $x^2 = x_0^2$

$$\mathcal{E} = \frac{4\pi}{g^2}\frac{\Delta m^2}{\cosh^2\Delta m(x^2 - x_0^2)}, \quad (4.24)$$

implying usual wall profile in suitable coordinates. The mass of the kink can be obtained by integrating this over the x^2 coordinate and we get $4\pi\Delta m/\hat{g}^2$. This coincides with the mass of a monopole in the Coulomb phase.[17] Thus monopoles in the Higgs phase can be seen as the 1/2 BPS kinks in the vortex theory [8].

Before closing this subsection, let us clarify the relation between b_c in the moduli matrix (4.18) and θ and x_0^2 in the 1/2 BPS kink solution (4.23). For that purpose the five-dimensional point of view (triangle ABC in Fig. 1) gives us a very nice picture. Taking Eq. (4.20) into account, the monopole(kink) solution (4.23) is understood in the massless vortex theory (3.12) as [(B′ → C′) → (B → C) in Fig. 1]

$$b(t, x^2, x^4) = e^{i\Delta m x^4}\hat{b}(t, x^2) = e^{-(\Delta m x_0^2 + i\theta)}e^{\Delta m w}. \quad (4.25)$$

Note that the Kähler potential (3.12) with this solution substituted is independent of x^4 because it is a function of $|b| = e^{\Delta m(x^2 - x_0^2)}$. Therefore the energy density of this configuration extends along the x^4 axis to infinity. Then this configuration is understood as a 1/4 BPS state of the monopole string in the Higgs phase. Let us next find the moduli matrix $H_0(z, w)$ in five dimensions corresponding to the moduli matrix given in Eq. (4.18) for the monopole in the Higgs phase in four dimensions. This can be obtained from the first equation in Eq. (4.11) with $S(x^M) = \hat{S}(x^\mu)$ as [(A′ → C′) → (A → C) in Fig. 1]

$$H_0(z, w) = \sqrt{c}\begin{pmatrix} z - z_0 & 0 \\ b(w) & 1 \end{pmatrix} = \sqrt{c}V\begin{pmatrix} z - z_0 & 0 \\ b_c & 1 \end{pmatrix}e^{Mw}, \qquad V = e^{-Mw}. \quad (4.26)$$

We thus find $b(w) = b_c e^{\Delta m w}$. Comparing this with the kink solution in (4.25), the complex parameter b_c can be identified as [(A → C) = (A → B → C) in Fig. 1]

$$b_c = e^{-(\Delta m x_0^2 + i\theta)}. \quad (4.27)$$

Hence, we conclude that the 1/4 BPS moduli matrix (4.18) describes a monopole in the Higgs phase and the complex parameter b_c therein is the position and the phase of the monopole. In the massless limit $\Delta m \to 0$ with $\Delta m x_0^2$ fixed, b_c becomes the orientational moduli of the non-Abelian vortex since the $SU(2)_{\rm F}$ flavor symmetry, which is explicitly broken to $U(1)_{\rm F}$ by nonzero Δm, is restored when $\Delta m = 0$.

[17]The symmetry breaking $SU(2)_{\rm F} \to U(1)_{\rm F}$ is given by Δm in the Higgs phase, and by the vacuum expectation value (VEV) of the adjoint scalar in the case of the Coulomb phase ('t Hooft-Polyakov monopole). In fact, by replacing Δm by the VEV of the adjoint scalar, we correctly reproduce the mass of the monopole in the Coulomb phase.

C. Calorons in the Higgs phase

In the Coulomb phase (unbroken gauge symmetry) there is well known way to get ordinary monopole solution independent of x^4 from instanton solutions [28,29]. Range the instantons with equal size along the x^4 axis periodically. After taking the limit where the size parameter of instantons goes to infinity the configuration becomes one BPS monopole-string solution extending to the x^4 axis [28,29]. In the Higgs phase, instantons, a monopole string and calorons which interpolate between instantons and the monopole string can be understood in terms of the deformations of the lump solutions in the vortex theory. In this subsection we will concentrate on 1/2 BPS states in the 2 + 1 dimensional massless vortex theory with the Kähler potential (3.12) (B → C in Fig. 1). However we can easily find the moduli matrix for the corresponding 1/4 BPS states (A → C in Fig. 1).

Let us first examine the monopole-string solution (4.25) as the sigma-model lump in the massless vortex theory in more detail. We first note that the solution $b = b_c e^{\Delta m w}$ in Eq. (4.25) is a 1/2 BPS state in the vortex theory since this is holomorphic in w and is a solution of the BPS equation (3.4) in the vortex theory. Although this solution has one codimension in the vortex theory, this is not a domain wall which is a topological soliton supported by the homotopy group π_0 because there is no scalar potential here. Rather, we should realize this solution as a topological object which consists of an infinite number of 1/2 BPS lumps supported by the homotopy group π_2. To see this, we decompose the solution as $b = b_c e^{\Delta m x^2} e^{i\Delta m x^4}$. Then it is clear that a strip $\sigma_1^{(k)} = \{(x^2, x^4) | x^2 \in (-\infty, \infty), x^4 \in [2\pi k/\Delta m, 2\pi(k+1)/\Delta m]\}$ is mapped to the $\mathbf{C}P^1$ manifold once by this configuration. Then the solution $b = b_c e^{\Delta m w}$ has infinite winding number $\pi_2(\mathbf{C}P^1) = \infty$. Hence this can be realized as a topological object which has an infinite number of lump charges. The energy density of the solution is the same form as that in Eq. (4.24). If we integrate this in a strip $\sigma_{\rm m}^{(k)} = \{(x^2, x^4) | x^2 \in (-\infty, \infty), x^4 \in [2\pi k R, 2\pi(k+1)R]\}$ (R is a compactification radius associated with the SS dimensional reduction), we find that the tension of the solution in the strip $\sigma_{\rm m}^{(k)}$ coincides with to the mass of the monopole $4\pi\Delta m/\hat{g}^2$. So this solution is suitable to be called the monopole string.

Let us next consider 1/4 BPS calorons in the Higgs phase as periodic lumps in the vortex effective theory (3.12)

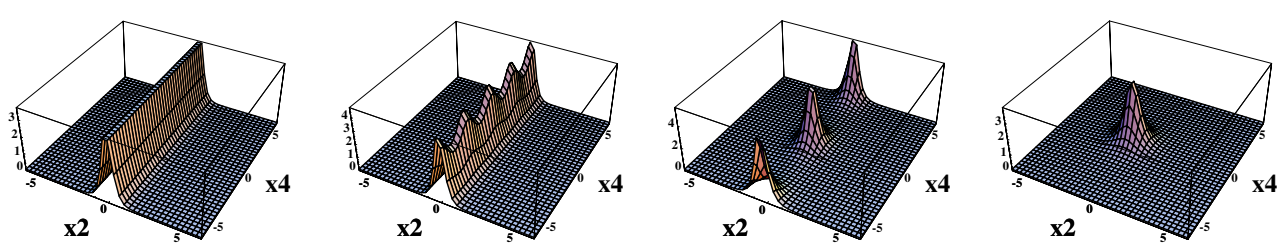
$$b(w) = (1+\theta)e^{\mu(w-w_0)} - \theta, \qquad \theta \equiv \frac{1}{a\mu}. \tag{4.28}$$

Here, μ and a are arbitrary real parameters with mass dimension one and minus one, respectively. Similarly to the ordinary calorons in the Coulomb phase [28,29], the calorons in the Higgs phase can be continuously deformed to the monopole string or instantons. In fact, in the limit $\theta \to 0$ with μ fixed, this solution reduces to the 1/4 BPS monopole string $b \to e^{\mu(w-w_0)}$ as shown in Fig. 3(a). There is an another limit $\theta \to \infty$ with a fixed. In this limit this solution reduces to 1/4 BPS states of an instanton in the Higgs phase as shown in Fig. 3(d). For general θ, we find periodic lump solutions inside a vortex which can be understood as the 1/4 BPS caloron as shown in Figs. 3(b) and 3(c). The parameter a is the size of the instanton and $1/\mu$ is the period of the caloron. By simply replacing b in (3.7) by (4.28), we can actually obtain the correct moduli matrix for the 1/4 BPS calorons without encountering singularities in w.

In view of this solution, we can guess that the monopole strings with large instanton charges in the compactified theory are unstable as follows. If we compactify the x^4 direction with radius R, the mass of the monopole-string solution $b = e^{\mu(w-w_0)}$ is $8\pi^2 R\mu/g^2$. To be precise, let us represent $\mu = (k+\tau)/R$ with $k \in \mathbf{Z}$ and $\tau \in [0, 1)$. Then the above mass of the monopole string can be rewritten as

$$\frac{8\pi^2}{g^2}k + \frac{4\pi}{\hat{g}^2}\Delta m, \qquad \left(\frac{\tau}{R} \equiv \Delta m\right). \tag{4.29}$$

This mass corresponds to the mass of k instantons and a monopole with a "fractional" instanton charge τ. Then the monopole string with mass $8\pi^2 R\mu/g^2$ can be decomposed into these solitons by continuous deformation like (4.28).



(a) moneopole string $\mu = 2,\ \theta = 5\times 10^{-4}$

(b) caloron $\mu = 2,\ \theta = 0.1$

(c) caloron $\mu = 1,\ \theta = 1$

(d) instanton $\mu = 0.2,\ \theta = 5$

FIG. 3 (color online). Energy density of the calorons in terms of the vortex theory.

Since the aggregate of the decomposed solitons has larger entropy, the monopole string with instanton charge greater than unity is unstable and may decay into instantons and a monopole string with the fractional instanton charge, if this system is put at finite temperatures.

V. CONCLUSION AND DISCUSSION

Previously monopoles in the Higgs phase were obtained merely as 1/2 BPS kinks in the effective theory [23] on a 1/2 BPS vortex [8]. Similarly instantons in the Higgs phase were suggested as 1/2 BPS lumps in the vortex effective theory [13]. In our previous paper [22] we have constructed monopoles in the Higgs phase (together with walls) as 1/4 BPS composite states in the fundamental theory in five dimensions. In this paper we have generalized this method to construct instantons in the Higgs phase as another type of 1/4 BPS composite states, not just a 1/2 BPS state in the vortex effective theory.

We have solved 1/4 BPS equations for composite states made of instantons and vortices. We have shown that all solutions are generated by the moduli matrix which is a holomorphic function of $z = x^1 + ix^3$ and $w = x^2 + ix^4$. The moduli matrix contains all solutions with different boundary conditions and/or different topological charges. As a first step toward the complete classification of all solutions, we have specified the moduli matrix for 1/4 BPS states which can be interpreted as lumps on a single vortex. We have found that the mass of k lumps precisely agrees with that of the k instantons. Small instanton singularities have been shown to correspond to small lump singularities. We have determined the moduli space for a single instanton in the Higgs phase to be the direct product of $\mathbf{C}^2$ and the tangent bundle over $\mathbf{C}P^1$ without zero section. We have clarified the relations between the moduli spaces of 1/4 BPS states for vortices and instantons, of 1/4 BPS states for walls, vortices and monopoles, of 1/2 BPS vortices, and of 1/2 BPS walls. We also have constructed calorons in the Higgs phase which interpolate between instantons and a monopole string in the Higgs phase.

We did not exhaust all solutions in this paper: our moduli matrix contains more varieties of solutions. The complete classification of all solutions is a very important open problem. Let us discuss this issue. First of all we could consider multiple vortices as host solitons. However the moduli matrix for multiple vortices is not available yet. It is now in progress to specify the moduli parameters in the moduli matrix, and therefore we have to wait for the completion of that work [38] to discuss the multiple lumps on multiple vortices.

Second, if we do not restrict ourselves to solitons which can be understood as lumps on vortices, we can obtain more varieties of solitons. The intersection of two or more vortices cannot be understood as solitons in the effective theory on a host vortex, because the energy of such solitons diverges in the effective theory in general. Instead, we can directly construct solutions of intersecting vortices as follows.[18] In the same model with $N = 2$, the following two moduli matrices give configurations with $\nu_{\rm v} = k_z (\geq 0)$ vortices in the 1-3 plane and $\nu_{\rm v'} = k_w (\geq 0)$ vortices in the 2-4 plane

$$H_0 = \begin{pmatrix} z^{k_z} & 0 \\ 0 & w^{k_w} \end{pmatrix}, \qquad H_0 = \begin{pmatrix} z^{k_z} w^{k_w} & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.1}$$

The vortices intersect at a point for both cases. It is, however, a trivial intersection for the former case, and they carry no instanton charge. On the other hand, they intersect nontrivially for the latter case, and the intersecting point carries the instanton charge $\nu_{\rm i} = -k_z k_w$. In this case the instantons give a negative energy contribution. However, there is no inconsistency, since the total energy including vortices is always positive.[19] Since the instanton is stuck at the intersecting point of vortices, it may be called an "intersecton." It cannot move once the vortices are fixed. We thus conclude that there exist two kinds of instantons; one is what lives inside a vortex and the other is an instanton stuck at the intersection point of vortices. As we have seen, there also exist trivially intersecting vortices. We expect that the most general solution is given by the mixture of these configurations.

Here we show that the intersections found above essentially exist in $U(1)$ gauge theories. To this end we consider multiple semilocal vortices in the theory with $N_{\rm F} > N_{\rm C}$. We take the strong gauge coupling limit $g^2 \to \infty$ to obtain an exact solution. In this limit the model reduces to a nonlinear sigma model whose target space is the cotangent bundle over the complex Grassmann manifold, $T^*(G_{N_{\rm F},N_{\rm C}})$ [33]. Then the master equation (2.17) can be solved algebraically as Eq. (2.23). For definiteness we consider a model with $N_{\rm C} = 1$ and $N_{\rm F} = 4$. The following moduli matrix gives nontrivially intersecting vortices with $\nu_{\rm v} = k_z$, $\nu_{\rm v'} = k_w$

$$H_0 = (z^{k_z} w^{k_w}, z^{k_z}, w^{k_w}, 1). \tag{5.2}$$

The Ω can be calculated as

$$\Omega^{g\to\infty} = \Omega_0 = (|z|^2 + 1)^{k_z} (|w|^2 + 1)^{k_w} \tag{5.3}$$

and the exact solution can be obtained as $H = (1/\sqrt{\Omega^{g\to\infty}}) H_0$. The instanton number can be calculated to be the product of vorticities, namely $\nu_{\rm i} = -k_z k_w$. This

[18] Similar 1/4 BPS states of intersecting vortices were discussed in [41].

[19] A similar situation occurs in a domain wall junction [39,42]. The energy of the intersecting wall receives contributions from constituent walls and from the junction. The known analytic solution in Ref. [39] shows that the contribution from the junction is negative. This phenomenon may naturally be understood as a kind of binding energy of constituent walls.

solution explicitly shows that the $U(1)$ instantons are stuck at the intersection of vortices. We can show that the instanton charge $\nu_{\rm i}$ changes its sign under the duality transformation $N_{\rm C}$, $\tilde{N}_{\rm C} \equiv N_{\rm F} - N_{\rm C}$ [30]. Therefore, there also exist intersections with positive instanton charges.

We discuss some more issues in the following. Although we ignore Σ and the zeroth component of gauge potential W_0 in this paper, we can also construct electrically charged solitons whose charge is $Q_e \propto g^2 \int d^4x \partial_m \mathrm{Tr}(\Sigma F_{0m})$ by restoring these fields. The Bogomol'nyi completion gives the most general BPS equations

$$\mathcal{D}_0\Sigma = 0, \qquad F_{m0} + \mathcal{D}_m\Sigma = 0, \qquad \mathcal{D}_0 H^i + i\Sigma H^i = 0, \tag{5.4}$$

added to Eqs. (2.8) and (2.9). These equations have to be solved with the Gauss's law

$$\frac{1}{g^2}\mathcal{D}_m F_{m0} = \frac{i}{g^2}[\Sigma, \mathcal{D}_0\Sigma] + \frac{i}{2}[H\mathcal{D}_0 H^\dagger - \mathcal{D}_0 H H^\dagger]. \tag{5.5}$$

We can show that solutions of these equations are $1/4$ BPS states. If we turn off the FI parameter c and set $H^i = 0$, these BPS solutions reduce to that for the $1/2$ BPS dyonic instanton [43]. The time-independent solutions of the dyonic instantons in the Higgs phase can be obtained as follows. First we solve the BPS equations (2.8) and (2.9) as shown in this paper. Next we set $\partial_0 = 0$ and $W_0 = -\Sigma$ to solve additional Eqs. (5.4). Finally Σ can be obtained by solving the Gauss's law under a given solution of instantons in the Higgs phase as the background

$$\mathcal{D}_m\mathcal{D}_m\Sigma = -\frac{g^2}{2}(HH^\dagger\Sigma + \Sigma HH^\dagger). \tag{5.6}$$

To obtain a nontrivial solution, we may need mass differences for hypermultiplets. In superstring theory dyonic instantons in the pure SUSY Yang-Mills theory were found to be supertubes [44] (and see also references in [45]) between parallel D4-branes [46]. Brane constructions for the dyonic instantons in the Higgs phase is an open problem.

Our instantons in the Higgs phase share some properties with noncommutative instantons. First $U(1)$ instantons can exist in the Higgs phase as shown in Eq. (5.2) like noncommutative $U(1)$ instantons [20]. Second their topologies are similar (but not identical) as stated in footnote 14. The moduli space of (noncommutative) instantons has the hyper-Kähler structure. Although the moduli space of the former has the Kähler structure at least and has real dimensions four multiplied by $\mathbf{Z}$, we do not know if it has the hyper-Kähler structure. It may be not the case because there remain only two SUSY in the former, but their moduli space may be obtained as a deformation of the moduli space of the latter preserving only the Kähler structure. It is very interesting to explore more similarities between instantons in the Higgs phase and noncommutative instantons [20]. It is also desired to obtain the ADHM construction for our instantons in the Higgs phase.

Dynamics of instantons within a vortex is equivalent to the dynamics of lumps [14,15]. Further study in dynamics of sigma-model lumps would clarify dynamics of instantons in more general configurations. Not only classical dynamics but also quantum effects in these solitons are important subjects. It was found by Dorey in [47] that the BPS spectra in $d = 4$ $\mathcal{N} = 2$ SUSY gauge theory with $N_{\rm F} > N_{\rm C}$ and $d = 2$ $\mathcal{N} = (2, 2)$ SUSY $\mathbf{C}P^N$ model with twisted masses completely coincide. One explanation for this coincidence has been given in [10,13] by considering a monopole inside a vortex. In conformity with these observations, there exist similarities between $d = 4$ Yang-Mills instantons and $d = 2$ sigma-model instantons (lumps). Our results in the present paper give further evidence for the relation because we have realized instantons inside a vortex as sigma-model lumps on the vortex theory. For instance, small instanton singularities are understood as small lump singularities. It is also quite interesting to generalize the instanton counting [21] to the case of the instantons in the Higgs phase.

Non-Abelian walls found in [30] have been recently realized as D-brane configurations in string theory [40]. By doing this the diverse phenomena of non-Abelian walls can be easily understood by the dynamics of D-branes. Monopoles in the Higgs phase are also realized by the same brane configuration [11,13]. Hence we would like to realize the instantons in the Higgs phase by some D-brane configuration, and we expect that the relation between monopoles and instantons in the Higgs phase can be interpreted as a T-duality in such a D-brane configuration, as in the case of the ordinary monopoles and instantons.

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