

$SU(N)$ caloron measure and its relation to instantonsDmitri Diakonov^{1,2} and Nikolay Gromov³¹*NORDITA, Blegdamsvej 17, DK-2100 Copenhagen, Denmark*²*St. Petersburg Nuclear Physics Institute, Gatchina, 188 300, St. Petersburg, Russia*³*St. Petersburg State University, Faculty of Physics, Peterhof, 198 904, St. Petersburg, Russia*

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Calorons of the $SU(N)$ gauge group with nontrivial holonomy, i.e. periodic instantons with arbitrary eigenvalues of the Polyakov line at spatial infinity, can be viewed as composed of N Bogomolnyi-Prasad-Sommerfeld monopoles or dyons. Using the metric of the caloron moduli space found previously we compute the integration measure over caloron collective coordinates in terms of the constituent monopole positions and their $U(1)$ phases. In the limit of small separations between dyons and/or trivial holonomy, calorons reduce locally to the standard instantons whose traditional collective coordinates are the instanton center, size, and orientation in the color space. We show that in this limit the instanton collective coordinates can be explicitly written through dyons' positions and phases, and that the N -dyon measure coincides exactly with the standard instanton one.

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I. INTRODUCTION

Belavin-Polyakov-Schwartz-Tyupkin (BPST) instantons [1,2] are known to play an important role in quantum chromodynamics (QCD); see Refs. [3,4] for reviews. The instanton liquid model [5] is especially helpful in providing a microscopic mechanism of the spontaneous chiral symmetry breaking, which is due to the delocalization of the would-be zero fermion modes in the instanton ensemble [6].

At the same time, instantons do not lead to confinement, at least in the naive dilute limit. In the pure glue version of QCD, there are two well-known criteria of confinement: the area behavior of large Wilson loops, and the zero average of the Polyakov line [7] (which fluctuates about the center-of-group values at temperatures above the deconfinement transition). To be more precise, one *can* formally obtain the area law for large Wilson loops from averaging over the instanton ensemble, provided instanton size distribution drops as $d\rho/\rho^3$ for large-size instantons [8]. However, such distribution implies that large-size instantons overlap, which makes meaningless the description of the vacuum fluctuations in terms of the instanton collective coordinates: one has to use other degrees of freedom.

To study the temperature dependence of the average Polyakov line over the instanton ensemble, one needs first to generalize the zero-temperature BPST instantons to the periodic Harrington-Shepard instantons [9]. The quantum weight of periodic instantons has been found by Gross, Pisarski, and Yaffe [10], and the instanton ensemble at any temperatures has been built in Ref. [11], using the variational principle of Petrov and one of the authors [5]. Averaging the Polyakov line over this ensemble, it has been found that it is near zero at small temperatures and rapidly approaches the center-of-group value at $T \geq \Lambda$ [12]. However, it is neither exactly zero at small T , nor is

there a sharp phase transition. This is the kind of behavior expected from the approximate treatment of large-size instantons. It is another manifestation of the lack of the phase transition with Harrington-Shepard calorons, found in Ref. [11], and the lack of confinement. Again, one concludes that in order to observe mathematically the confinement-deconfinement phase transition, one needs to use the degrees of freedom appropriate for overlapping instantons. In a more simple 2d CP^{N-1} model also possessing instantons, the appropriate degrees of freedom known as “instanton quarks” [13] or “zindons” have long been available—see Ref. [14] for references and for a detailed study of the CP^{N-1} instanton ensemble in terms of their constituents.

For the 4d Yang-Mills theory, a somewhat similar construction of instantons through their “constituents” became available more recently, owing to Kraan and van Baal [15] and Lee and Lu [16], first for the $SU(2)$ gauge group and later for the general $SU(N)$ [17]. These authors have found explicitly an exact self-dual solution of the Yang-Mills equation of motion at any temperature with a unity topological charge and with arbitrary eigenvalues of the Polyakov line (or holonomy) at spatial infinity. We shall call this general solution the KvBLL caloron. The periodic Harrington-Shepard instanton is a limiting case of the KvBLL caloron at trivial holonomy corresponding to the Polyakov line assuming center-of-group values. A caloron with the double topological charge has been constructed in Ref. [18].

The fascinating feature of the $SU(N)$ calorons is that they can be viewed as composed of N Bogomolnyi-Prasad-Sommerfeld monopoles [19] or, more precisely, dyons since they carry both magnetic and electric charges; the composite calorons are electrically and magnetically neutral. Using the Coulomb gas representation based on the chiral effective Lagrangian it was conjectured in Ref. [20] that the adequate description should be in terms of N

monopoles constituting an $SU(N)$ instanton. Apart from $N - 1$ eigenvalues of the Polyakov line at spatial infinity, the $SU(N)$ KvBLL caloron is characterized by $4N$ collective coordinates forming its moduli space. A natural choice of the collective coordinates is to use $3N$ positions of the dyons' centers in space, and N dyons' $U(1)$ phases, $3N + N = 4N$. If all N dyons are spatially far apart, the action density of the KvBLL caloron consists of N time-independent 3d lumps whose profile is the well-known energy density of individual BPS dyons. In the opposite limit when all dyons are within the spatial range $\leq 1/T$ from each other, the KvBLL caloron becomes a single 4d lump whose profile is close to the usual periodic instanton. As the temperature goes to zero with dyons separation fixed, the caloron action density tends to that of the standard BPST instanton. Contrary to the standard instanton, however, the holonomy (or the Polyakov line at infinity) remains nontrivial.

The average eigenvalues of the Polyakov line are determined by the dynamics of the ensemble of calorons with nontrivial holonomy. For example, in the $\mathcal{N} = 1$ supersymmetric version of the Yang-Mills theory the dyon-induced superpotential can be computed exactly [21], and its minimum corresponds to the Polyakov line's eigenvalues

$$\begin{aligned} L &\equiv \text{P exp} \left(\int_0^{1/T} dt A_4 \right)_{|\vec{x}| \rightarrow \infty} \\ &= \text{diag}(e^{i\pi(1/N)}, e^{i\pi(3/N)}, \dots, e^{i\pi[(2N-1)/N]}), \end{aligned} \quad (1)$$

such that $\text{tr}L = 0$ as it should be in the confining phase. Moreover, the known exact vacuum expectation value (v.e.v.) of the gluino condensate corresponds to this particular holonomy, whereas the trivial-holonomy instantons lead to a wrong value [21,22]. This result is even more surprising since at $T \rightarrow 0$ the local difference between gauge fields with trivial and nontrivial holonomy vanishes, implying that long-range fields are critical, at least in the supersymmetric gluinodynamics [22].

In the nonsupersymmetric Yang-Mills theory, the question of which average holonomy is dynamically preferred is open. From lattice simulations we know that in the confining phase $\langle \text{Tr}L \rangle = 0$ but we do not know what dynamics leads to it. Revealing it would be tantamount to understanding the mechanism of confinement. A step in that direction has been taken in Ref. [23] where the quantum weight of the KvBLL caloron has been computed exactly, as a function of the holonomy, dyon separation, and temperature for the $SU(2)$ group. Based on this calculation, an argument has been presented that at $T < T_c = 1.125\Lambda$ a trivial holonomy becomes dynamically unfavorable from the free energy minimization viewpoint. Below T_c , dyons repulse each other, and calorons presumably "ionize" into separate dyons. However, to find out their fate and whether the system prefers the "confining" holonomy (1), one has to study the dynamics of *many* dyons.

To that end one has first of all to find the statistical or quantum weight of a dyon configuration as given by the combination of the collective coordinate measure and the small oscillation quantum determinant.

This paper is devoted to the study of the measure of a single $SU(N)$ KvBLL caloron, written in terms of the dyon coordinates and their $U(1)$ phases. The metric tensor of the moduli space has been first conjectured by Lee, Weinberg, and Yi [24,25] and then derived by Kraan [26] using the explicit Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) construction [27,28] for the $SU(N)$ caloron [17]. We have independently reproduced the same result for the moduli space metric; however we do not present the derivation here as it is lengthy but is not qualitatively different from that by Kraan. Instead, we compute the determinant of the metric tensor, which defines the integration measure over the dyons' collective coordinates for the general $SU(N)$ caloron, and compare it with the long-known instanton measure [29] written in terms of the instanton position, size, and group orientation. We demonstrate that the $SU(N)$ instanton measure written in these terms coincides exactly with the one written in terms of the coordinates and phases of the instanton constituents. This result is not altogether trivial, as in the first case the measure arises from the volume of the $SU(N)/SU(N-2)$ co-set whereas in the second case it follows from the 3d geometry. We also find the direct relation between the instanton group orientation and the dyons' positions and $U(1)$ phases. We believe that it may be an important step in combining the success of the small-size instantons in physics related to the spontaneous chiral symmetry breaking, with the description of large-size instantons in terms of their dyon constituents, which is presumably necessary for the confinement physics. Since this paper concentrates mainly on the mathematical questions, we do not discuss here the very interesting recent studies of the KvBLL calorons on the lattice [30].

II. NOTATIONS

To help navigate and read the paper, we first introduce some notations used throughout. Basically we use the same notations as in Ref. [17]. In what follows we shall measure all quantities in the temperature units and put $T = 1$. The temperature factors can be restored in the final results from dimensions.

Let the Polyakov line at spatial infinity have the following eigenvalues:

$$\begin{aligned} L &= \text{P exp} \left(\int_0^{1/T} dt A_4 \right)_{|\vec{x}| \rightarrow \infty} \\ &= V \text{diag}(e^{2\pi i \mu_1}, e^{2\pi i \mu_2}, \dots, e^{2\pi i \mu_N}) V^{-1}, \end{aligned} \quad (2)$$

$$\sum_{m=1}^N \mu_m = 0.$$

We use anti-Hermitian gauge fields $A_\mu = it^a A_\mu^a = \frac{i}{2} \lambda^a A_\mu^a$,

$[t^a t^b] = i f^{abc} t^c$, $\text{tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$. The eigenvalues μ_m are uniquely defined by the condition $\sum_{m=1}^N \mu_m = 0$. If all eigenvalues are equal up to the integer, implying $\mu_m = k/N - 1$, $m \leq k$ and $\mu_m = k/N$, $m > k$ where $k = 0, 1, \dots, (N-1)$, the Polyakov line belongs to the $SU(N)$ group center, and the holonomy is then said to be ‘‘trivial.’’ By making a global gauge rotation one can always order the Polyakov line eigenvalues such that

$$\mu_1 \leq \mu_2 \leq \dots \leq \mu_N \leq \mu_{N+1} \equiv \mu_1 + 1, \quad (3)$$

which we shall assume done. The eigenvalues of A_4 in the adjoint representation, $A_4^{ab} = i f^{abc} A_4^c$, are $\pm(\mu_m - \mu_n)$ and $N-1$ zero eigenvalues. For the trivial holonomy all adjoint eigenvalues are integers. The difference of the neighbor eigenvalues in the fundamental representation $\nu_m \equiv \mu_{m+1} - \mu_m$ determines the spatial core size $1/\nu_m$ of the m th monopole whose 3-coordinates will be denoted as \vec{y}_m , and the spatial separation between neighbor monopoles in color space will be denoted by

$$\begin{aligned} \vec{\mathcal{Q}}_m &\equiv \vec{y}_m - \vec{y}_{m-1} = \mathcal{Q}_m (\sin\theta_m \cos\phi_m, \sin\theta_m \sin\phi_m, \cos\theta_m), \\ \mathcal{Q}_m &\equiv |\vec{\mathcal{Q}}_m|. \end{aligned} \quad (4)$$

With each 3-vector $\vec{\mathcal{Q}}_m$ we shall associate a 2-component spinor $\zeta_m^{\dagger\alpha}$ built according to the Euler parametrization

$$\begin{aligned} \zeta_m^{\dagger\alpha} &= \sqrt{\frac{\mathcal{Q}_m}{\pi}} \left[\exp\left(-i\phi_m \frac{\tau_3}{2}\right) \exp\left(-i\theta_m \frac{\tau_2}{2}\right) \right. \\ &\quad \left. \times \exp\left(-i\psi_m \frac{\tau_3}{2}\right) \right]_2^\alpha \\ &= \sqrt{\frac{\mathcal{Q}_m}{\pi}} \left[\begin{array}{c} -\sin\frac{\theta_m}{2} \exp\left(i\frac{\psi_m - \phi_m}{2}\right) \\ \cos\frac{\theta_m}{2} \exp\left(i\frac{\psi_m + \phi_m}{2}\right) \end{array} \right]^\alpha. \end{aligned} \quad (5)$$

This spinor, together with its Hermitian conjugate ζ_m^α , forms a 2×2 matrix for any $m = 1 \dots N$:

$$\zeta_m^{\dagger\alpha} \zeta_m^\beta = \frac{1}{2\pi} (1_2 \mathcal{Q}_m - \vec{\tau} \cdot \vec{\mathcal{Q}}_m)_{\alpha\beta}. \quad (6)$$

These spinors are used in the construction of the caloron field. The Euler angle ψ_m is fictitious in parametrizing the 3d vector $\vec{\mathcal{Q}}_m$ but enters explicitly the gauge field of the caloron and belongs to its moduli space, together with $\vec{\mathcal{Q}}_m$. In fact ψ_m has the meaning of the $U(1)$ phase of the m th dyon. We shall also use the following notation for the variation:

$$\frac{i\pi}{\mathcal{Q}_m} \text{tr}(\zeta_m^\dagger \delta \zeta_m - \delta \zeta_m^\dagger \zeta_m) = \delta\psi_m + \cos\theta_m \delta\phi_m \equiv \delta\Sigma_m. \quad (7)$$

For trivial holonomy, the KvBLL caloron reduces to the Harrington-Shepard periodic instanton at nonzero temperatures and to the ordinary Belavin-Polyakov-Schwartz-Tyupkin instanton at zero temperature.

Instantons are usually characterized by the scale parameter (the ‘‘size’’ of the instanton) ρ . It is directly related to the dyons’ positions in space, actually to the perimeter of the polygon formed by dyons,

$$\rho = \sqrt{\frac{1}{2\pi T} \sum_{m=1}^N \mathcal{Q}_m}, \quad \sum_{m=1}^N \vec{\mathcal{Q}}_m = 0. \quad (8)$$

In these notations the KvBLL caloron gauge field can be written as the following $N \times N$ matrix [17]:

$$\begin{aligned} A_\mu^{mn} &= \frac{1}{2} \phi_{mk}^{1/2} \zeta_\alpha^k \bar{\eta}_{\mu\nu}^a (\tau^a)_\beta^{\dagger\alpha} \partial_\nu f_{kl} \phi_{ln}^{1/2} \\ &\quad + \frac{1}{2} (\phi_{mk}^{1/2} \partial_\mu \phi_{kn}^{-1/2} - \partial_\mu \phi_{mk}^{-1/2} \phi_{kn}^{1/2}) \end{aligned} \quad (9)$$

where the summation over k, l is understood and where

$$\phi_{mn}^{-1} = \delta_{mn} - \zeta_m^\alpha \zeta_n^{\dagger\alpha} f_{mn}. \quad (10)$$

The $N \times N$ matrix f_{mn} is in fact the ADHMN Green function $f(\mu_n, \mu_m)$ found in [31]. In Appendix B we derive a simple expression for this quantity used to obtain certain limiting cases of the general equation (9).

III. ZERO MODES IN THE YANG-MILLS THEORY

Here we remind the reader what zero modes are and how the moduli space metrics arises from the path integral. In our notations the partition function for the pure Yang-Mills theory reads

$$\begin{aligned} Z &= \int DA \exp(-S[A]), \\ S[A] &= -\frac{1}{2g^2} \int d^4x \text{tr} F_{\mu\nu} F_{\mu\nu}, \end{aligned} \quad (11)$$

where $A_\mu(x_4, \vec{x})$ must obey the periodicity condition $A_\mu(0, \vec{x}) = A_\mu(1/T, \vec{x})$.

The integration measure in Eq. (11) is defined through the scalar product

$$\langle u, u' \rangle = -2 \int d^4x \text{tr}(u_\mu(x) u'_\mu(x)) \quad (12)$$

by

$$DA = \prod_n \frac{d\alpha_n}{\sqrt{2\pi g}} \quad (13)$$

where $A_\mu(x) = \sum_n \alpha_n u_{n\mu}(x)$ for the complete normalized set of functions $u_{n\mu}(x)$.

We want to compute the contribution to the partition function from some set of solutions of the classical Yang-Mills equation of motion $A_\mu(Y, x)$ parametrized by the collective coordinates Y_p . This means that we have to take into account only small fluctuations about the surface formed by this set of solutions in the configuration space. Usually $S[A(Y)] = S^{\text{cl}}$ is the same for the whole set, and is locally minimal. The integral over fluctuations is Gaussian

only in the directions orthogonal to the surface. We have to separate Gaussian and non-Gaussian variables of integration.

The result in the quadratic order after fixing the background gauge $D_{\mu}^{\text{cl}} a_{\mu} = 0$ is

$$Z_{A(Y)} = e^{-S^{\text{cl}}} \int J \prod_p \frac{dY_p}{\sqrt{2\pi g}} \int D a_{\mu} D \chi D \bar{\chi} \exp\left(-\frac{1}{2g^2} \times \int d^4 x a_{\mu}^a W_{\mu\nu}^{ab} a_{\nu}^b - \int d^4 x \bar{\chi}^a (D^2)^{ab} \chi^b\right) \quad (14)$$

where $a_{\mu} = it^b a_{\mu}^b$ ($\text{tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$) are small fluctuations orthogonal to the zero modes of the operator $W_{\mu\nu}$,

$$W_{\mu\nu}^{ab} = -D^2[A(Y)]^{ab} \delta_{\mu\nu} - 2f^{acb} F_{\mu\nu}^c[A(Y)]; \quad (15)$$

χ and $\bar{\chi}$ are the ghost fields from gauge fixing. The factor $\sqrt{2\pi g}$ comes from the definition of the measure. The Jacobian J is in fact the determinant of the moduli space metric tensor, i.e.

$$J = \sqrt{\det g_{pq}}, \quad g_{pq} = \langle \delta_p A_{\mu} \delta_q A_{\mu} \rangle, \quad (16)$$

where $\delta_p A_{\mu}$ is a zero mode of $W_{\mu\nu}$, associated with the collective coordinate Y_p through

$$\delta_p A_{\mu} = \partial_{Y_p} A_{\mu} + D_{\mu} \Omega_p \quad (17)$$

where Ω_p is chosen such that the background gauge condition is satisfied,

$$D_{\mu} \delta_p A_{\mu} = 0. \quad (18)$$

In the next section we present the result for the metric tensor g_{pq} for the KvBLL caloron of the $SU(N)$ gauge group.

IV. CALORON MODULI SPACE METRIC

As mentioned in the Introduction, the metric of the moduli space of N different BPS monopoles of the $SU(N)$ gauge group has been first conjectured in Refs. [24,25] generalizing the previous work [32], and then confirmed by an explicit calculation in Ref. [26]. In these papers, the metric tensor was expressed in terms of the monopole—electric charge interaction potential $w_i(\varrho)$ satisfying the equation

$$\epsilon_{ijk} \partial_j w_k(\vec{\varrho}) = \partial_i \frac{1}{\varrho} = -\frac{\varrho_i}{\varrho^3}. \quad (19)$$

One introduces also an $N \times N$ matrix

$$S = \begin{pmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & & \dots & & \\ & & & & 1 & -1 \\ -1 & & & & & 1 \end{pmatrix}, \quad (20)$$

$$S^T = \begin{pmatrix} 1 & & & & & -1 \\ -1 & 1 & & & & \\ & & \dots & & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \end{pmatrix},$$

such that the separation between consecutive dyons is $\vec{\rho}_m = \vec{y}_m - \vec{y}_{m-1} = (S^T \vec{y})_m$. In terms of the dyon interaction potential $\vec{w}(\vec{\varrho})$ the metric found in Refs. [24–26] is [33] (see e.g. Eq. (78) in [26])

$$ds^2 = 8\pi^2 \left[d\vec{y}^T G \cdot d\vec{y} + \left(\frac{d\tau}{4\pi} + \vec{W} \cdot d\vec{y} \right)^T G^{-1} \times \left(\frac{d\tau}{4\pi} + \vec{W} \cdot d\vec{y} \right) \right] \quad (21)$$

where

$$\vec{W} = S \vec{W} S^T, \quad \vec{W} = \frac{1}{4\pi} \text{diag}(\vec{w}(\vec{\varrho}_1), \dots, \vec{w}(\vec{\varrho}_N)), \quad (22)$$

$$G = \mathcal{N} + \frac{1}{4\pi} S R^{-1} S^T, \quad \mathcal{N} = \text{diag}(\nu_1, \dots, \nu_N), \quad (23)$$

$$R = \text{diag}(\varrho_1, \dots, \varrho_N),$$

$$\frac{d\tau_m}{4\pi} = \nu_m d\xi_4 + \frac{1}{4\pi} (S\psi)_m. \quad (24)$$

The metric (21) is implicit as it employs the notion of the monopole-electric charge interaction potential \vec{w} which by itself is ambiguous as it does not exist without a Dirac string singularity. The combination $(\vec{w}(\varrho) \cdot d\vec{\varrho})$ is however independent of the way one introduces the Dirac string singularity. Choosing it along the z axis and solving Eq. (19) we find the monopole-electric charge interaction potential

$$\vec{w}(\vec{\varrho}) = \frac{1}{\varrho} (-\cot\theta \sin\phi, \cot\theta \cos\phi, 0) \quad (25)$$

if one parametrizes $\vec{\varrho} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$. Hence, $\vec{W} \cdot d\vec{y}$ in Eq. (21) can be rewritten as

$$\vec{W} \cdot d\vec{y} = S \vec{W} \cdot d\vec{\varrho} = \frac{1}{4\pi} S \text{diag}(\cos\theta_1 d\phi_1, \dots, \cos\theta_N d\phi_N). \quad (26)$$

Being combined with Eq. (24) it gives

$$\frac{d\tau}{4\pi} + \vec{W} \cdot d\vec{y} = \mathcal{N} d\xi_4 + \frac{1}{4\pi} S d\Sigma \quad (27)$$

where $d\Sigma_m = \cos\theta_m d\phi_m + d\psi_m$ according to Eq. (7). Therefore, the second term in Eq. (21) can be written as

$$\left(\mathcal{N} d\xi_4 + \frac{1}{4\pi} S d\Sigma \right)^T G^{-1} \left(\mathcal{N} d\xi_4 + \frac{1}{4\pi} S d\Sigma \right) \quad (28)$$

where, according to Eq. (23),

$$\begin{aligned} G^{-1} &= \left(\mathcal{N} + \frac{1}{4\pi} S R^{-1} S^T \right)^{-1} \\ &= \mathcal{N}^{-1} - \mathcal{N}^{-1} \frac{1}{4\pi} S R^{-1} S^T \mathcal{N}^{-1} + \dots \end{aligned} \quad (29)$$

Since $d\xi_4$ is, in this context, an N vector $(d\xi_4, \dots, d\xi_4)$, one has for all components $(d\xi_4 S)_m = 0$; see the definition of S in Eq. (20). Hence, the term quadratic in $d\xi_4$ in Eq. (28) is simply

$$\mathcal{N} d\xi_4 G^{-1} \mathcal{N} d\xi_4 = d\xi_4^2 \quad (30)$$

where $\text{tr} \mathcal{N} = \sum_m \nu_m = 1$ has been used. Because of $(d\xi_4 S)_m = 0$, the terms linear in $d\xi_4$ in Eq. (29) are zero. In the last term, quadratic in $d\Sigma$, we note that

$$\begin{aligned} S^T G^{-1} S &= S^T \mathcal{N}^{-1} S - \frac{1}{4\pi} S^T \mathcal{N}^{-1} S R^{-1} \mathcal{N}^{-1} S + \dots \\ &= 4\pi R (4\pi R + S^T \mathcal{N}^{-1} S)^{-1} S^T \mathcal{N}^{-1} S. \end{aligned} \quad (31)$$

We introduce $N \times N$ matrices K, L, M :

$$K = \frac{1}{\pi} R + \frac{1}{4\pi^2} S^T \mathcal{N}^{-1} S, \quad (32)$$

$$\begin{aligned} L &= \pi R - R K^{-1} R = R K^{-1} (\pi K - R) \\ &= R K^{-1} \frac{1}{4\pi} S^T \mathcal{N}^{-1} S, \end{aligned} \quad (33)$$

$$M = 4\pi^2 G. \quad (34)$$

With the help of these matrices, the chain of Eq. (31) can be

$$M^{(4)} = \pi \begin{bmatrix} 4\pi\nu_1 + \frac{1}{\varrho_1} + \frac{1}{\varrho_2} & -\frac{1}{\varrho_2} & 0 & -\frac{1}{\varrho_1} \\ -\frac{1}{\varrho_2} & 4\pi\nu_2 + \frac{1}{\varrho_2} + \frac{1}{\varrho_3} & -\frac{1}{\varrho_3} & 0 \\ 0 & -\frac{1}{\varrho_3} & 4\pi\nu_3 + \frac{1}{\varrho_3} + \frac{1}{\varrho_4} & -\frac{1}{\varrho_4} \\ -\frac{1}{\varrho_1} & 0 & -\frac{1}{\varrho_4} & 4\pi\nu_4 + \frac{1}{\varrho_4} + \frac{1}{\varrho_1} \end{bmatrix}. \quad (41)$$

The $SU(2)$ gauge group is too ‘‘small’’ for the general formula (40). In this case the matrix M is simply

$$M^{(2)} = \pi \begin{bmatrix} 4\pi\nu_1 + \frac{1}{\varrho_1} + \frac{1}{\varrho_2} & -\frac{1}{\varrho_1} - \frac{1}{\varrho_2} \\ -\frac{1}{\varrho_1} - \frac{1}{\varrho_2} & 4\pi\nu_2 + \frac{1}{\varrho_1} + \frac{1}{\varrho_2} \end{bmatrix} \quad (42)$$

where $\varrho_1 = \varrho_2 = |\vec{y}_1 - \vec{y}_2|$ and $\nu_1 + \nu_2 = 1$.

continued as

$$S^T G^{-1} S = 4\pi R \frac{1}{4\pi^2} K^{-1} 4\pi (\pi K - R) = 4L. \quad (35)$$

Thus, the last term in Eq. (28) is

$$\frac{1}{4\pi^2} d\Sigma L d\Sigma. \quad (36)$$

Combining all terms from Eq. (21) we obtain finally a simple and explicit expression for the moduli space metric:

$$ds^2 = 8\pi^2 d\xi_4^2 + 2M_{mn} d\vec{y}_m d\vec{y}_n + 2L_{mn} d\Sigma_m d\Sigma_n, \quad (37)$$

where $d\Sigma_m$ is given by Eq. (7). As a matter of fact, we have independently derived the moduli space metric in precisely this form, using the ADHMN construction for the $SU(N)$ caloron. However, since the derivation is lengthy but not qualitatively different from that of Kraan [26] we do not present it here. Explicitly, the K, L, M matrices involved in Eq. (37) are

$$\begin{aligned} K_{mn} &= \left(\frac{\varrho_n}{\pi} + \frac{1}{4\pi^2 \nu_n} + \frac{1}{4\pi^2 \nu_{n-1}} \right) \delta_{mn} - \frac{1}{4\pi^2 \nu_m} \delta_{m+1,n} \\ &\quad - \frac{1}{4\pi^2 \nu_n} \delta_{m,n+1}, \end{aligned} \quad (38)$$

$$L_{mn} = \pi R_{mn} - (R K^{-1} R)_{mn}, \quad R_{mn} \equiv \delta_{mn} \varrho_n, \quad (39)$$

$$\begin{aligned} M_{mn} &\equiv \left(4\pi^2 \nu_n + \frac{\pi}{\varrho_n} + \frac{\pi}{\varrho_{n+1}} \right) \delta_{mn} - \frac{\pi}{\varrho_n} \delta_{m+1,n} \\ &\quad - \frac{\pi}{\varrho_m} \delta_{m,n+1}. \end{aligned} \quad (40)$$

Notably K and M are symmetric and differ only by interchanging $4\pi^2 \nu_m$ and ρ_m/π : this will be used in computing the determinants.

As an example, we give the matrix M for the $SU(4)$ gauge group:

V. THE DETERMINANT OF THE METRIC TENSOR

In the previous section we have written the moduli space metric in the explicit form (37). However only the determinant of the metric is needed in such calculations as the saddle point approximation; see Eq. (14). In this section we derive a compact expression for the volume of the general $SU(N)$ moduli space and then give examples for the specific cases of the $SU(2)$ and $SU(3)$ groups, as well as an asymptotic formula for the general $SU(N)$ group, valid at large separations between the dyons.

First of all, we need to check the dimension or the number of parameters of the moduli space. These are the $3N$ coordinates of dyon centers \vec{y}_i , one overall time position ξ_4 , and $N - 1$ relative color orientations ψ_m entering the metric (37) from Eq. (7). Therefore, the dimension of the caloron moduli space is $4N$ as it should be for a general self-dual solution with unity topological charge. We note that the transformation $\delta\psi_1 = \delta\psi_2 = \dots = \delta\psi_N$ is a global $U(1)$ gauge rotation leaving the gauge field unchanged. As a consequence, the matrix L has one zero eigenvalue

$$L|1, \dots, 1\rangle = 0 \quad (43)$$

which makes the size of the maximal nondegenerate minor of the metric tensor (37) equal to $4N$. The determinant of the metric tensor is

$$g \equiv \det g_{pq} = 8\pi^2 2^{3N} \det^3 M 2^{N-1} \det' L \quad (44)$$

where $\det' L$ is the product of all nonzero eigenvalues of L . The corresponding volume form is

$$\omega = 2^{N-1} \sqrt{N} \sqrt{g} d\xi_4 d^3 y_1 \dots d^3 y_N d\alpha_1 \dots d\alpha_{N-1} \quad (45)$$

where

$$\alpha_m \equiv \frac{\psi_m}{2} - \sum_{n=1}^N \frac{\psi_n}{2N}, \quad m = 1, \dots, N-1, \quad (46)$$

$$\alpha_N \equiv \sum_{n=1}^N \frac{\psi_n}{2N}$$

is a set of variables that parametrize the relative $U(1)$ orientations of the dyons. Note that α_N corresponds to the trivial gauge transformation. The transformation matrix has the form

$$Q_{mn} \equiv \frac{d\psi_m}{d\alpha_n} = \frac{1}{2} \begin{pmatrix} \frac{N-1}{N} & -\frac{1}{N} & \dots & -\frac{1}{N} \\ -\frac{1}{N} & \frac{N-1}{N} & \dots & -\frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \dots & \frac{1}{N} \end{pmatrix}^{-1}.$$

The factor $2^{N-1} \sqrt{N}$ in Eq. (45) comes from the equation

$$\begin{aligned} \det'(Q^T L Q) &= \lim_{\epsilon \rightarrow 0} \frac{\det(Q^T L Q + \epsilon)}{\epsilon} \\ &= \det(Q Q^T) \lim_{\epsilon \rightarrow 0} \frac{\det(L + (Q Q^T)^{-1} \epsilon)}{\epsilon} \end{aligned} \quad (47)$$

$$= \det(Q Q^T) (Q Q^T)^{-1} \det(L) = (2^{N-1} \sqrt{N})^2 \det' L \quad (48)$$

where $(Q Q^T)^{-1} \equiv \langle 1, 1, \dots, 1 | (Q Q^T)^{-1} | 1, 1, \dots, 1 \rangle / N$.

It turns out that $\det' L$ can be expressed through $\det K$. To show this let us introduce

$$V \equiv 4\pi(\pi K - R) \quad (49)$$

such that the matrix L from Eq. (39) can be written as

$$L \equiv \pi R - R K^{-1} R = R K^{-1} (\pi K - R) = \frac{1}{4\pi} R K^{-1} V. \quad (50)$$

Using that $\langle 1, 1, \dots, 1 | R K^{-1} = \pi \langle 1, 1, \dots, 1 |$ we have

$$\det' L = 4^{1-N} \frac{\det R \det' V}{\pi^N \det K} \quad (51)$$

and from a simply calculable $\det' V = N / \prod_{m=1}^N \nu_m$ we obtain

$$\det' L = \frac{N}{2^{2N-2} \pi^N \det K} \prod \frac{\varrho_m}{\nu_m}. \quad (52)$$

In its turn, $\det K$ has a simple relation to $\det M$,

$$\frac{\det M}{\det K} = \frac{\prod 4\pi^2 \nu_m}{\prod \frac{\varrho_m}{\pi}} \quad (53)$$

which follows from the symmetry between the two matrices mentioned at the end of Sec. IV. Thus, the final result for the element of the volume of the moduli space is

$$\omega = (4\pi)^{N+1} 2^{N-1} N \det M d\xi_4 d^3 y_1 \dots d^3 y_N d\alpha_1 \dots d\alpha_{N-1} \quad (54)$$

where M is given by Eq. (40). One can also rewrite it in terms of the ‘‘center-of-mass’’ position $\vec{\xi} = \sum_m \nu_m \vec{y}_m$ and the separations between dyons neighboring in color space, ρ_m :

$$\omega = (4\pi)^{N+1} 2^{N-1} N \det M d^4 \xi d^3 \rho_1 \dots d^3 \rho_{N-1} d\alpha_1 \dots d\alpha_{N-1}. \quad (55)$$

VI. INTEGRATION OVER DYONS' $U(1)$ PHASES

Since for a single-charged caloron the volume element does not depend on the $U(1)$ phases of the dyons ψ_m or, equivalently, α_m , these phases can be integrated out. Fortunately the integration limits in α_m variables are simple. These variables parametrize a general diagonal $SU(N)$ matrix:

$$U(\alpha_m) = \text{diag}\{e^{i\alpha_1}, e^{i\alpha_2}, \dots, e^{i\alpha_{N-1}}, e^{-i\sum \alpha_m}\}. \quad (56)$$

It is clear that $U(\alpha_m) = U(\alpha'_m)$ if and only if $\alpha'_m = \alpha_m + 2\pi n_m$ where n_m are integers. However $U(\alpha_m)$ and $U(\alpha'_m)$ can differ by an element of the center of $SU(N)$, i.e. $U(\alpha_m) = U_Z U(\alpha'_m)$ where $U_Z = e^{2\pi i/N} 1_N$. Since U_Z acts trivially in the adjoint representation we have to choose the fundamental domain of integration, such that if α_m and α'_m are elements of this domain, the condition $U(\alpha_m) = U_Z U(\alpha'_m)$ implies that $\alpha_m = \alpha'_m$. For example, one can choose the fundamental domain to be

$$0 \leq \alpha_1 < \frac{2\pi}{N}, \quad 0 \leq \alpha_{m>1} < 2\pi. \quad (57)$$

We now integrate over α_m in the fundamental domain

specified by (57) and obtain

$$\int d^{N-1} \alpha_m = (2\pi)^{N-1} \frac{1}{N}. \quad (58)$$

Thus, the caloron measure integrated over the $U(1)$ phases (denoted by \mathcal{G}) is

$$\int_{\mathcal{G}} \omega = (4\pi)^{2N} \det M d^4 \xi d^3 \rho_1 \dots d^3 \rho_{N-1}. \quad (59)$$

Below we find $\det M$ in the particular cases of the $SU(2)$ and $SU(3)$ gauge groups and in the limit of large dyon separations in a general $SU(N)$ case.

A. $SU(2)$

The general expression for the $N \times N$ matrix M is given in Eq. (40). Computing its determinant in the case of $N = 2$ we get

$$\omega = 2^9 \pi^6 \frac{1 + 2\pi \varrho_1 \nu_1 \bar{\nu}_1}{\varrho_1} d^4 \xi d^3 \varrho_1 d(\psi_1 - \psi_2) \quad (60)$$

where we use the notation $\bar{\nu}_m = 1 - \nu_m$. Integrating over the $U(1)$ phase $(\psi_1 - \psi_2)$ and over space rotations we get

$$\int d^3 \varrho_1 d(\psi_1 - \psi_2) = (4\pi)^2 \int \varrho_1^2 d\varrho_1 = 2^4 \pi^2 \int \varrho_1^2 d\varrho_1. \quad (61)$$

According to Eq. (46) $\psi_1 - \psi_2 = 4\alpha_1$ and thus $\int d(\psi_1 - \psi_2) = 4 \int_0^\pi d\alpha_1 = 4\pi$. Replacing ϱ_1 by the commonly used instanton size variable according to Eq. (8), $\varrho_1 = \pi \rho^2$, we arrive at the result already known in the $SU(2)$ case [15,23],

$$\int_{\mathcal{G}, \text{rotations}} \omega = 2^{14} \pi^{10} (1 + 2\pi^2 \rho^2 \nu_1 \bar{\nu}_1) \rho^3 d\rho d^4 \xi. \quad (62)$$

At trivial holonomy ($\nu_1 = 0$) it becomes the well-known measure of the (periodic) $SU(2)$ instanton.

B. $SU(3)$

Computing the determinant of the 3×3 matrix M (40) and putting it into Eq. (59) we obtain the caloron measure for $SU(3)$:

$$\int_{\mathcal{G}} \omega = 2^{14} \pi^{10} \left\{ 16\pi^2 \nu_1 \nu_2 \nu_3 + 4\pi \left[\frac{\nu_2 \bar{\nu}_2}{\varrho_1} + \frac{\nu_3 \bar{\nu}_3}{\varrho_2} + \frac{\nu_1 \bar{\nu}_1}{\varrho_3} \right] + \left(\frac{1}{\varrho_1 \varrho_2} + \frac{1}{\varrho_2 \varrho_3} + \frac{1}{\varrho_3 \varrho_1} \right) \right\} d^3 \varrho_1 d^3 \varrho_2 d^4 \xi.$$

C. $SU(N)$, large separations

In the general case, $\det M$ cannot be written in an easy form. However, for large separations between dyons in a caloron, one can derive a simple asymptotic for $\det M$, Eq. (40). We expand it in inverse powers of ϱ_m . Let us write

$$M = 4\pi^2 \mathcal{N} + \pi M_1, \quad \mathcal{N}_{nm} \equiv \delta_{nm} \nu_m, \quad (63)$$

where the matrix M_1 is composed of the inverse powers of ϱ_m . We have

$$\begin{aligned} \det M &= \det(4\pi^2 \mathcal{N}) \exp \text{tr} \log \left(1 + \frac{\mathcal{N}^{-1} M_1}{4\pi} \right) \\ &\simeq (2\pi)^{2N} \left(1 + \sum_m \frac{1}{4\pi \varrho_m \nu_m} + \frac{1}{4\pi \varrho_m \nu_{m-1}} \right) \prod_n \nu_n. \end{aligned} \quad (64)$$

Hence from Eq. (59) we obtain the caloron measure

$$\begin{aligned} \int_{\mathcal{G}} \omega &\simeq 2^{6N} \pi^{4N} \left[1 + \sum_m \frac{1}{4\pi \varrho_m} \left(\frac{1}{\nu_{m-1}} + \frac{1}{\nu_m} \right) \right] \\ &\times \prod_n \nu_n d^3 \varrho_1 \dots d^3 \varrho_{N-1} d^4 \xi. \end{aligned} \quad (65)$$

We remind the reader that periodicity in the indices is assumed; for example $\nu_N \equiv \mu_{N+1} - \mu_N = \mu_1 + 1 - \mu_N$.

Equation (65) can be interpreted as Coulomb repulsion of dyons inside a caloron. However, not all dyons interact with each other but only those that are ‘‘neighbors’’ in the color space. In $SU(3)$ all dyons are neighbors in this sense, while the $SU(2)$ group is too small to see the effect.

VII. RELATION TO THE INSTANTON MEASURE IN THE TRIVIAL-HOLONOMY LIMIT

In this limit, the KvBLL caloron becomes the Harrington-Shepard periodic instanton with the standard BPST instanton moduli space. It is basically an $SU(2)$ configuration embedded into the $SU(N)$ group. The $4N$ -parameter moduli space is usually described as four ‘‘center-of-mass’’ coordinates ξ_μ , one ‘‘size’’ parameter ρ , and $4N - 5$ ‘‘gauge orientation’’ collective coordinates determining the embedding. This has been the traditional parametrization of instantons for 25 years, starting from the work by Bernard [29] who computed the instanton measure and its volume for a general $SU(N)$ group.

At first glance, there is little in common between this moduli space and that of the nontrivial caloron, given in the previous sections in terms of the constituent dyons’ 3d positions and $U(1)$ phases. Our goal is to demonstrate that the measures of the two moduli spaces in fact coincide exactly, including the nontrivial normalization.

We shall do it in two steps. In this section we show that the volume of the dyon moduli space coincides with that found by Bernard in terms of the $SU(2)$ embedding. In the next section we give an explicit construction of the instanton $SU(N)$ gauge orientation matrix [determining the $SU(2)$ embedding] in terms of the 3d positions and $U(1)$ phases of the constituent dyons.

The trivial-holonomy limit corresponds to taking all Polyakov eigenvalues equal, $\mu_m = k/N - 1$, $m \leq k$ and $\mu_m = k/N$, $m > k$, where $k = 0, 1, \dots, (N - 1)$, meaning

all their differences $\nu_m = \mu_{m+1} - \mu_m$ are zero except one which is unity; see Sec. II. First of all we note that in this limit one gets

$$\det M = 4\pi^{N+1} \frac{s}{\prod_m \varrho_m}, \quad s = \sum_{m=1}^N \varrho_m, \quad (66)$$

where s is the perimeter of the polygon formed by the dyons. It is directly related to the instanton size ρ by Eq. (8): $\rho = \sqrt{s/2\pi}$. To find the volume of the dyons' moduli space and relate it to the standard instanton one, we have to integrate Eq. (59) over the dyons' 3d positions with the perimeter s fixed. More concretely, we have to evaluate

$$\begin{aligned} \mu_N(s) &\equiv \int \prod_{i=1}^N d^3 \varrho_m \det M \delta\left(\sum_{i=1}^N \varrho_m - s\right) \delta^3\left(\sum_{i=1}^N \vec{\varrho}_m\right) \\ &= \int \prod_{i=1}^N d^3 \varrho_m \frac{4\pi^{N+1} s}{\prod_m \varrho_m} \delta\left(\sum_{i=1}^N \varrho_m - s\right) \delta^3\left(\sum_{i=1}^N \vec{\varrho}_m\right). \end{aligned} \quad (67)$$

Leaving unintegrated the center-of-mass 4-coordinate ξ_μ and the instanton size ρ , the moduli space volume is, from Eq. (59),

$$\int_{\mathcal{G}} \omega = \int (4\pi)^{2N} \mu_N(s) ds d^4 \xi. \quad (68)$$

The integral (67) is computed in Appendix A with the result

$$\mu_N(s) = \frac{2^3 \pi^{2N} s^{2N-3}}{(N-1)!(N-2)!}. \quad (69)$$

Consequently

$$\int_{\mathcal{G}} \omega = \int \frac{2^{6N+2} \pi^{6N-2}}{(N-1)!(N-2)!} \rho^{4N-5} d\rho d^4 \xi, \quad (70)$$

coinciding exactly with Bernard's result [29]. It is interesting to note that it was obtained there in a completely different way—by computing the group volume for the embedding of $SU(2)$ into $SU(N)$. There seems to be nothing near it in the present derivation.

VIII. LIMITING CASES OF THE CALORON GAUGE FIELD

In this section we give the trivial-holonomy limit of the KvBLL gauge field. It is the Harrington-Shepard $SU(2)$ instanton imbedded into $SU(N)$. The way it is embedded depends on the constituent dyons' color orientations and their relative positions. As a by-product, we give the gauge field of the KvBLL caloron with the exponential precision [i.e. dropping terms of the order of $\mathcal{O}(e^{-2\pi r_m \nu_m})$, where r_m is a distance to the m th dyon].

A. Far from the cores

With the exponential precision, the matrix F_{mn} [see Eq. (B5)] is diagonal and so is the matrix f_{mn} :

$$f_{mn} = 2\pi \delta_{mn} (r_m + r_{m-1} + \varrho_m)^{-1} + \mathcal{O}(e^{-2\pi r_m \nu_m}). \quad (71)$$

From Eq. (6) one has

$$f_{mn} \zeta_m^\dagger \zeta^m = \frac{\varrho_m}{\pi} f_{mn}, \quad f_{mn} \zeta^m \bar{\eta}_{\mu\nu} \zeta_m^\dagger = -\frac{\varrho_m^a}{\pi} f_{mn} \bar{\eta}_{\mu\nu}^a, \quad (72)$$

and from Eq. (10)

$$\phi_{mn} \simeq \delta_{mn} \frac{r_m + r_{m-1} + \varrho_m}{r_m + r_{m-1} - \varrho_m}. \quad (73)$$

The last term in Eq. (9) is zero, and with the exponential precision we can write the gauge field

$$\begin{aligned} A_{\mu}^{mn} &\simeq \frac{1}{2} (\phi \partial_\nu \lambda \bar{\eta}_{\mu\nu} f \lambda^\dagger)_{mn} \simeq -\frac{\varrho_m^a}{2\pi} \bar{\eta}_{\mu\nu}^a \phi_{mk} \partial_\nu f_{kn} \\ &\simeq \bar{\eta}_{\mu\nu}^a \frac{\varrho_m^a}{2\varrho_m} \phi_{mk} \partial_\nu \phi_{kn}^{-1}. \end{aligned} \quad (74)$$

This expression is similar to the one found in [23] for the $SU(2)$ case. It is given in a nonperiodical gauge. To pass to the periodical gauge one has to add $2\pi i \mu_m \delta_{mn}$ to A_4 (see the discussion at the end of Sec. VIII C). A_4 has the Coulomb-like form. In the periodical gauge

$$A_{4mn}^{\text{per}} = 2\pi i \mu_m \delta_{mn} \delta_{\nu 4} + \frac{i}{2} \delta_{mn} \left(\frac{1}{r_m} - \frac{1}{r_{m-1}} \right), \quad (75)$$

$$\begin{aligned} A_{imn}^{\text{per}} &= -\frac{i}{2} \delta_{mn} \left(\frac{1}{r_m} + \frac{1}{r_{m-1}} \right) \\ &\times \sqrt{\frac{(\varrho_m - r_m + r_{m-1})(\varrho_m + r_m - r_{m-1})}{(\varrho_m + r_m + r_{m-1})(r_m + r_{m-1} - \varrho_m)}} (e_{\varphi_m})_i \end{aligned} \quad (76)$$

where $\vec{e}_{\varphi_m} \equiv [(\vec{r}_{m-1} \times \vec{r}_m)/|\vec{r}_{m-1} \times \vec{r}_m|]$.

B. Reduction to the trivial-holonomy case

In the trivial-holonomy limit ($\nu_L = 1$, $\nu_{m \neq L} = 0$) Eq. (B1) simplifies. It becomes a Shrödinger equation on the unit circle with only one delta function in the left-hand-side. The solution is independent of N and can be found in Ref. [17]. It is given by

$$\begin{aligned} f(\mu_m, \mu_n) &\equiv f_0 \\ &= \frac{\pi \sinh(2\pi r)}{\pi \rho^2 \sinh(2\pi r) + r \cosh(2\pi r) - r \cos(2\pi x_0)} \end{aligned} \quad (77)$$

where $2\pi \rho^2 = \sum \rho_m$ and $r \equiv r_L$.

We now introduce a $N \times N$ unitary matrix U which plays the role of the ‘‘color orientation’’ of the (periodic) instanton to which the KvBLL reduces in the trivial-

holonomy case. The first two columns of U are defined through the spinors

$$U_n^m = \frac{1}{\rho} \zeta_\alpha^m, \quad n = \alpha = 1, 2. \quad (78)$$

The rest of the columns are constrained only by the unitarity condition $U^\dagger U = 1_N$; they are not involved in the field construction. Correspondingly, the first two rows of U^\dagger are given by the Hermitian conjugate spinors,

$$U_m^{\dagger n} = \frac{1}{\rho} \zeta_m^{\dagger \alpha}, \quad n = \alpha = 1, 2. \quad (79)$$

This definition is noncontradictory if the two complex N vectors U_1^m and U_2^m are orthogonal and normalized to unity. Indeed, using Eqs. (6) and (8), we obtain

$$\begin{aligned} \sum_{m=1}^N U_m^{\dagger \alpha} U_\alpha^m &= \frac{1}{\rho^2} \sum_{m=1}^N \zeta_m^{\dagger \alpha} \zeta_\alpha^m \\ &= \frac{1}{2\pi\rho^2} \sum_{m=1}^N (\varrho_m \delta_\beta^\alpha - \tilde{\varrho}_m \cdot \tilde{\tau}_\beta^\alpha) = \delta_\beta^\alpha. \end{aligned} \quad (80)$$

To write down the gauge field of the trivial-holonomy instanton from the general expressions (9) and (10) we first replace there $\zeta \rightarrow U$ according to Eqs. (78) and (79):

$$\begin{aligned} f_0 \zeta_\alpha^m \zeta_n^{\dagger \alpha} &= f_0 \rho^2 (U \lambda^0 U^\dagger)_n^m, \\ \partial_\nu f_0 \tilde{\eta}_{\mu\nu}^a \zeta_\alpha^m (\tau^a)_\beta^{\dagger \alpha} \zeta_n^{\dagger \beta} &= \partial_\nu f_0 \rho^2 \tilde{\eta}_{\mu\nu}^a (U \lambda^a U^\dagger)_n^m, \\ (\phi^{-1})_n^m &= \delta_n^m - f_0 \rho^2 (U \lambda^0 U^\dagger)_n^m \\ &= U_\alpha^m (1 - f_0 \rho^2 \lambda^0)_\beta^\alpha U_n^{\dagger \beta}, \end{aligned} \quad (81)$$

where (λ^0, λ^a) are $N \times N$ matrices with $(1_2, \tau^a)$ put into the left-upper corner. The last term in Eq. (9) is again zero, and we arrive at the compact result for the trivial-holonomy caloron:

$$(A_\mu)_n^m = \frac{1}{2} \tilde{\eta}_{\mu\nu}^a (U \lambda^a U^\dagger)_n^m \partial_\nu \log \Pi \quad (83)$$

where

$$\Pi \equiv \frac{1}{1 - f_0 \rho^2} = 1 + \frac{\pi \rho^2 \sinh(2\pi r)}{r[\cosh(2\pi r) - \cos(2\pi x_0)]}. \quad (84)$$

This formula reproduces exactly the Harrington-Shepard instanton with arbitrary ‘‘gauge orientation’’ U defined, as we see from Eq. (78), by the dyon relative coordinates $\tilde{\varrho}_m$ and the relative $U(1)$ orientation angles.

C. Small-size KvBLL caloron

Another important limit when the caloron field has a simple form is the case of small $\rho = \sqrt{\sum_m \varrho_m} / (2\pi T) \ll 1/T$ implying that the dyons’ separations are small, $\varrho_m \ll 1/T$ (in this subsection we restore the temperature factors). We are interested in the caloron field at the distances r from

the center of the group of N dyons, larger than the separations between them, $r \sim \rho \gg \varrho_m$. Therefore, we can put in the leading order $r_m = r$ for all $m = 1, \dots, N$. We also consider the range of $x_4 \sim \rho$ where the field is large. In this range, the ADHM Green function is simply

$$f_{mn} = \frac{1}{r^2 + x_4^2 + \rho^2}. \quad (85)$$

Repeating the calculations from the previous subsection we arrive at a standard expression for the BPST instanton:

$$A_\mu = \frac{1}{2} \tilde{\eta}_{\mu\nu}^a U \lambda^a U^\dagger \partial_\nu \log \Pi, \quad \Pi = 1 + \frac{\rho^2}{r^2 + x_4^2}, \quad (86)$$

where the instanton orientation matrix U is given by Eq. (78). Corrections to Eq. (86) die out as T in the range $r \sim x_4 \sim \rho \gg \varrho_m$ where the field is large.

Equation (86) is the approximate gauge field for small-size calorons in the nonperiodic gauge used in Ref. [17]. To obtain the approximate small- ρ field in the periodic gauge, one has to gauge-transform Eq. (86):

$$A_{\nu mn}^{\text{per}} = 2\pi i \mu_m \delta_{mn} \delta_{\nu 4} + (g^\dagger A_\nu g)_{mn} \quad (87)$$

where $g_{mn} = \delta_{mn} e^{2\pi i \mu_m x_4}$.

We note finally that when all the dyons’ separations ϱ_m are small, the metric determinant is given by Eq. (66) (even though the holonomy can be nontrivial), and the caloron measure coincides with that of the standard instanton, as shown in Sec. VII.

IX. CONCLUSIONS

The metric of the $4N$ -dimensional moduli space of the general $SU(N)$ caloron with arbitrary eigenvalues of the Polyakov line at spatial infinity and at any temperature is given in terms of the spatial coordinates of the N dyons that constitute the caloron, and their $U(1)$ phases.

We have computed the determinant of the metric tensor, which defines the weight of the $SU(N)$ caloron contribution to the partition function. The metric determinant is a function of the 3d separations between dyons and of the Polyakov loop eigenvalues. When all those eigenvalues are equal, it is the ‘‘trivial-holonomy’’ case, and the KvBLL caloron reduces to the usual periodic instanton whose moduli space is usually written in terms of the instanton position, size, and orientation. We have shown that the $SU(N)$ instanton measure written in these variables coincides exactly with the one written in terms of the coordinates and phases of the instanton constituents, the dyons. This result is not altogether trivial, as in the first case the measure arises from the volume of the $SU(N)/SU(N-2)$ co-set whereas in the second case it follows from the 3d geometry. We have also identified the instanton $SU(N)$ orientation matrix through the dyons’ positions and $U(1)$ phases.

The following emerging physical picture may be plausible. The adequate degrees of freedom in the Yang-Mills vacuum are, at any temperature, calorons with nontrivial holonomy, which are more general than the standard periodic instantons with trivial holonomy. The measure should be described in terms of the dyons' positions and phases. The free energy of the ensemble of interacting dyons should be studied; hopefully at low temperatures it has a minimum at the "most nontrivial holonomy" corresponding to $\text{tr}L = 0$; however at $T > T_c$ related to Λ there must be N degenerate minima corresponding to trivial holonomy. An indication that this may indeed be the case has been presented for $SU(2)$ in Ref. [23]. If correct, it would serve as the microscopic mechanism of the confinement-deconfinement transition.

At low temperatures, although the correct description is still in terms of dyons with nontrivial holonomy supporting the confinement, statistical fluctuations will lead to a large portion of dyons that are not widely separated. If a group of N different-type dyons happen to be close to each other, the configuration is locally undistinguishable from the standard $SU(N)$ instanton. Small-size instantons can be described both in the "position-size-orientation" terms, and in terms of dyons. However, for large-size overlapping instantons the former language loses sense while the latter remains valid.

This physical picture (calling, of course, for a detailed mathematical study) may justify the adequacy of the small-size instantons in physics related to the spontaneous chiral symmetry breaking, while simultaneously explaining confinement as presumably due to dyons.

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APPENDIX A: VOLUME OF THE MODULI SPACE

To get the moduli space volume, one has to integrate over the positions of N dyons with the perimeter of the N polygon fixed. More precisely we have to evaluate the following integral:

$$\begin{aligned} \mu_N(s) &\equiv \int \prod_{m=1}^N d^3 \varrho_m \det M \delta\left(\sum_{i=1}^N \varrho_m - s\right) \delta^3\left(\sum_{m=1}^N \vec{\varrho}_m\right) \\ &= \int \prod_{m=1}^N d^3 \varrho_m \frac{4\pi^{N+1}s}{\prod \varrho_m} \delta\left(\sum_{m=1}^N \varrho_m - s\right) \delta^3\left(\sum_{m=1}^N \vec{\varrho}_m\right). \end{aligned} \quad (\text{A1})$$

To reduce the number of integrations in Eq. (A1) we use the following trick. We introduce auxiliary integrals over Feynman parameters to reproduce the δ functions:

$$\begin{aligned} \mu_N(s) &= \int \prod_{m=1}^N d^3 \varrho_m \frac{4\pi^{N+1}s}{\prod \varrho_m} \int \frac{d^3 \alpha d\beta}{(2\pi)^4} \\ &\times \exp\left(i\left(\sum \varrho_m - s\right)(\beta + i\epsilon) + i \sum_{m=1}^N \vec{\varrho}_m \vec{\alpha}\right). \end{aligned} \quad (\text{A2})$$

The infinitesimal $i\epsilon$ is added to ensure convergence. Now we can integrate over $\vec{\varrho}_m$ since the integrals are factorized:

$$\begin{aligned} \mu_N(s) &= \int \frac{d^3 \alpha d\beta}{(2\pi)^4} \\ &\times \prod_{i=1}^N \left(\int \frac{d^3 \varrho_m}{\varrho_m} e^{i\varrho_m(\beta+i\epsilon)+i\vec{\varrho}_m \vec{\alpha}} \right) 4\pi^{N+1} s e^{-i\beta s}. \end{aligned} \quad (\text{A3})$$

The $i\epsilon$ shift makes each integral over ϱ_m finite. One can easily calculate it:

$$\int d\varrho_m d\cos\theta 2\pi \varrho_m e^{i\varrho_m(\beta+i\epsilon)} e^{i\varrho_m \alpha \cos\theta} = \frac{4\pi}{\alpha^2 - (\beta + i\epsilon)^2}. \quad (\text{A4})$$

Now the measure can be written as a 4d integral

$$\mu_N(s) = \int \frac{4\pi \alpha^2 d\alpha d\beta}{(2\pi)^4} \frac{4(4\pi)^N \pi^{N+1} s}{[\alpha^2 - (\beta + i\epsilon)^2]^N} e^{-i\beta s}. \quad (\text{A5})$$

From dimensions, $\mu_N(s) = \alpha_N s^{2N-3}$ where α_N is a constant to be computed; we find it by induction. We first consider the $N = 2$ case where $\mu_2(s)$ can be found directly from Eq. (A1):

$$\mu_2(s) = \int d^3 \varrho_1 \frac{8\pi^3}{\varrho_1} \delta(2\varrho_1 - s) = 2^3 \pi^4 s. \quad (\text{A6})$$

This implies $\alpha_2 = 2^3 \pi^4$. For general N we rotate the integration contour $\alpha \rightarrow -i\alpha$ in Eq. (A5) since the poles are at $\pm(\beta + i\epsilon)$. We can then rewrite Eq. (A5) in an $SO(4)$ invariant form

$$\mu_N(s_\mu) = \frac{is}{\pi} \int \frac{d^4 \alpha}{(2\pi)^4} \frac{(4\pi^2)^{N+1} (-1)^N}{(\alpha_\mu \alpha_\mu)^N} e^{-i\alpha_\mu s_\mu} \quad (\text{A7})$$

where $\alpha_\mu = (\vec{\alpha}, \beta)$. The crucial step is the following recurrent relation:

$$\begin{aligned} \frac{\mu_{N-1}(s_\mu)}{s} &= \frac{1}{4\pi^2} \partial_\mu^2 \frac{\mu_N(s_\mu)}{s} = \frac{\alpha_N}{4\pi^2} \partial_\mu^2 s^{2N-3} \\ &= \frac{\alpha_N}{4\pi^2} \frac{1}{s^3} \partial_s (s^3 \partial_s s^{2N-4}) \\ &= \frac{\alpha_N}{4\pi^2} (2N-4)(2N-2) s^{2N-6} \end{aligned} \quad (\text{A8})$$

where we have used that the radial part of the 4d Laplace operator is $\partial_\mu^2 f(s) = \frac{1}{s^3} \partial_s (s^3 \partial_s f(s))$. The solution to this equation is

$$\alpha_N = \frac{\pi^2 \alpha_{N-1}}{(N-1)(N-2)}. \quad (\text{A9})$$

Since α_2 is known, it immediately follows that

$$\mu_N(s) = \frac{\alpha_2 \pi^{2N-4} s^{2N-3}}{(N-1)!(N-2)!} = \frac{2^3 \pi^{2N} s^{2N-3}}{(N-1)!(N-2)!} \quad (\text{A10})$$

which is used in Sec. VI.

APPENDIX B: GREEN FUNCTION OF THE ADHM CONSTRUCTION

The Green function $f(z, z')$ is a very important object in the ADHM construction and is used in many formulas. We derive here a compact expression for this key quantity. An alternative expression for $f(z, z')$ can be found in Ref. [31]. For the $SU(N)$ caloron it is defined by a Shrödinger equation on the unit circle [17]:

$$\left[\left(\frac{1}{2\pi i} \partial_z - x_0 \right)^2 + r(z)^2 + \frac{1}{2\pi} \sum_m \delta(z - \mu_m) \varrho_m \right] f(z, z') = \delta(z - z') \quad (\text{B1})$$

where $r(z) \equiv |\vec{x} - \vec{y}(z)|$.

To find $f(z, z')$ we first derive a closed system of linear algebraic equations for $f_{mn} \equiv f(\mu_m, \mu_n)$. Assuming f_{mn} and $f_{m+1,n}$ are known we can present $f(z, \mu_n)$ in the interval (μ_m, μ_{m+1}) in a standard way from solving Eq. (B1):

$$\begin{aligned} f(z, \mu_n) &= -e^{2\pi i x_0(z - \mu_m)} f_{mn} \frac{\sinh[2\pi r_m(z - \mu_{m+1})]}{\sinh(2\pi r_m \nu_m)} \\ &+ e^{2\pi i x_0(z - \mu_{m+1})} f_{m+1,n} \frac{\sinh[2\pi r_m(z - \mu_m)]}{\sinh(2\pi r_m \nu_m)}. \end{aligned} \quad (\text{B2})$$

$$s(z, z') = e^{2\pi i x_0(z - z')} \frac{\sinh(2\pi r_{[z]}(\min\{z, z'\} - \mu_{[z]})) \sinh(2\pi r_{[z]}(\mu_{[z]+1} - \max\{z, z'\}))}{r_{[z]} \sinh(2\pi r_{[z]} \nu_{[z]})}. \quad (\text{B8})$$

Equation (B6) is convenient in some calculations since the main dependence on z, z' is factorized.

Taking the derivatives near the discontinuity points one has

$$\begin{aligned} f'(\mu_m + \epsilon, \mu_n) &= 2\pi \left(i x_0 f_{mn} - r_m \coth(2\pi r_m \nu_m) f_{mn} \right. \\ &\quad \left. + e^{-2\pi i x_0 \nu_m} \frac{r_m}{\sinh(2\pi r_m \nu_m)} f_{m+1,n} \right), \\ f'(\mu_m - \epsilon, \mu_n) &= 2\pi \left(i x_0 f_{mn} \right. \\ &\quad \left. + r_{m-1} \coth(2\pi r_{m-1} \nu_{m-1}) f_{mn} \right. \\ &\quad \left. - e^{2\pi i x_0 \nu_{m-1}} \frac{r_{m-1}}{\sinh(2\pi r_{m-1} \nu_{m-1})} f_{m-1,n} \right). \end{aligned}$$

It follows from Eq. (B1) that

$$-\frac{1}{4\pi^2} \text{disc} f'(\mu_m, \mu_n) = \delta_{mn} - \frac{\varrho_m}{2\pi} f_{mn} \quad (\text{B3})$$

and we can conclude that

$$f_{mn} = F^{-1}_{mn} \quad (\text{B4})$$

where

$$\begin{aligned} 2\pi F_{mn} &= \delta_{mn} [\coth(2\pi r_m \nu_m) r_m \\ &\quad + \coth(2\pi r_{m-1} \nu_{m-1}) r_{m-1} + \varrho_n] \\ &\quad - \frac{\delta_{m+1,n} r_m e^{-2\pi i x_0 \nu_m}}{\sinh(2\pi r_m \nu_m)} - \frac{\delta_{m,n+1} r_n e^{2\pi i x_0 \nu_n}}{\sinh(2\pi r_n \nu_n)}. \end{aligned} \quad (\text{B5})$$

Now we can reconstruct $f(z, z')$ for arbitrary z and z' . We look for the solution in the form

$$f(z, z') = s_m(z) f_{mn} s_n^\dagger(z') + 2\pi s(z, z') \delta_{[z][z']} \quad (\text{B6})$$

where $s(\mu_m, z') = 0, s(\mu_n, z') = 0$; we denote $[z] \equiv m$ if $\mu_m \leq z < \mu_{m+1}$. The first term satisfies the homogeneous equation with given boundary conditions; the second term gives $\delta(z - z')$ and vanishes at the boundary. The functions appearing in Eq. (B6) are

$$\begin{aligned} s_m(z) &= e^{2\pi i x_0(z - \mu_m)} \frac{\sinh[2\pi r_m(\mu_{m+1} - z)]}{\sinh(2\pi r_m \nu_m)} \delta_{m[z]} \\ &+ e^{2\pi i x_0(z - \mu_m)} \frac{\sinh[2\pi r_{m-1}(z - \mu_{m-1})]}{\sinh(2\pi r_{m-1} \nu_{m-1})} \delta_{m,[z]+1}, \end{aligned} \quad (\text{B7})$$

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