Horizon entropy in modified gravity

Peng Wang*

Department of Physics, Stanford University, Stanford, California 94305-4060, USA†

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We present an observation about the proposal that four-dimensional modification of general relativity may explain the observed cosmic acceleration today. Assuming that the thermodynamical nature of gravity theory continues to hold in modified gravity theories, we derive the modified horizon entropy formula from the modified Friedmann equation. We argue that our results imply that there are conceptual problems in some models of four-dimensional modification of general relativity.

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It is a well-known result that in general relativity, we can assign an entropy to black hole horizon [1] and cosmological horizon [2] by the same formula:

$$S = \frac{A}{4G}, \tag{1}$$

where A is the proper area of the horizon.

On the other hand, Jacobson [3] made an interesting observation that Einstein equation can be derived by assuming the universality of Eq. (1) on any local Rindler horizons.

From those two results, we have the observation that *four-dimensional gravitational theory and gravitational entropy formula are very likely in one-to-one correspondence*.

Recently, there are intense discussions on whether the observed cosmic acceleration is due to the fact that general relativity will be modified at large scale so that currently the Friedmann equation is modified [4–8]. There are several different proposals to modify general relativity in the literature, e.g. extra dimensions [6], higher derivative curvature terms [5], etc. In this work, we will consider the possibility that modified Friedmann equation of the form (7) is the result of four-dimensional modification of general relativity in the large scale [4]. In this framework, the Universe is always matter dominated, but due to large-scale modification of general relativity, matter can cause the Universe to accelerate today. Then based on the above observation, it is conceivable that the formula for gravitational entropy will also be modified in large scale. We will show that this is indeed the case. Thus, the main result of this paper is that, when we try to build models of four-dimensional modification of general relativity to explain cosmic acceleration, we should bear in mind that the modified gravity theory should reproduce the modified horizon entropy formula derived in this paper. Interestingly, some of the modified entropy formula is so obviously unphysical (e.g. the $\alpha < 0$ Cardassian model) that it tells us that we had better not spend our time on such endeavor.

As the cosmology driven by modified Friedmann equation is generally not de Sitter, we should first address the question of whether can we assign a gravitational entropy to it? If yes, then since the event horizon, apparent horizon and particle horizon are different, which one should we consider?

We think the proper choice is the *apparent horizon* which is the boundary surface of antitrapped region. In Ref. [9], the particle horizon is taken as the holographic boundary. However, in Ref. [10], it is shown that this choice will violate the holographic bound in inflation. Indeed, let us assume that inflation expands only 10^{30}, it occurs at the GUT scale $H \sim 10^{-6}$ (in Planck units) and the temperature after reheating is $T \sim 10^{-3}$. In this case the size of particle horizon after inflation will be $L_{PH} \sim H^{-1} \times 10^{30} \sim 10^{36}$, the area $A \sim L_{PH}^2 \sim 10^{72}$ and the entropy $S \sim T^3 L_{PH}^3 \sim 10^{99}$, which clearly violates the bound $S/A < 1$. Instead, if we consider the holographic bound as the apparent horizon $L_A \sim H^{-1} \sim 10^6$. Then the area $A \sim L_A^2 \sim 10^{12}$ and the entropy $S \sim T^3 L_A^3 \sim 10^9$ which clearly satisfies the holographic bound. Thus, insisting on the validity of holography during inflation, we should choose the apparent horizon as the holographic boundary.

Moreover, research in black holes also supports the viewpoint that we should focus on apparent horizon. General accelerating cosmological spacetimes are quite similar to dynamical black holes. For dynamical black holes, we also face the question of whether can we assign gravitational entropy to them and how can we define horizon. This problem is analyzed several years ago by Hayward *et al.* [11]. The conclusion is that it is sensible to assign a gravitational entropy to the trapping horizon of dynamical black holes defined as hypersurfaces foliated by marginal surfaces. Moreover, as argued by Jacobson [3], the entire framework of black hole thermodynamics and, in particular, the notion of black hole entropy extends to any causal horizon. In cosmological spacetime, the corresponding object is just the apparent horizon.

Thus, we will focus on the entropy associated with the cosmic apparent horizon in this work. Let us assume that in

*pengwang@stanford.edu
†This address will be valid after Sep. 21, 2005

modified gravity theory, the gravitational entropy is still determined solely by the area of the apparent horizon. So we have

$$S = f(A/4G), \quad \text{when } L_A > \frac{1}{H_0}. \tag{2}$$

where f is an arbitrary function which we will show in the below that it is determined by the modified Friedmann equation, $A = 4\pi L_A^2$ and $L_A = 1/H$ are the area and radius of the apparent horizon. From the discussion below, this assumption is actually equivalent to the requirement that the modified Friedmann equation does not contain *derivatives* of H.

Since temperature of the horizon is derived from the requirement that the wick-rotated metric is smooth on the horizon, it is independent of the gravity theory that leads to the horizon geometry (see e.g. Ref. [12]). Thus the expression for temperature should remain the same in modified gravity theory, which reads $T = 1/(2\pi L_A)$.

The heat flux across the apparent horizon is given by [11]

$$dQ = A(\psi_t dt + \psi_r dr), \tag{3}$$

where ψ is the energy-supply vector defined as

$$\psi_a = T_a^b \partial_b L_A + w \partial_a L_A \tag{4}$$

where w is the work density defined as $w = T_{ab}h^{ab}$. T_{ab} is the projection of the 4d energy-momentum tensor $T_{\mu\nu}$ of a perfect fluid on the normal direction of the two-sphere defined as $r =$ const..

Then we can find that in FRW metric, the heat flux is given by

$$dQ = A(\rho + p)dt. \tag{5}$$

where we have used the fact that $r = 1/(aH)$ on the apparent horizon.

Assuming the thermodynamical nature of gravity continues to hold in modified gravity theory [3], we have the relation $dQ = TdS$. Then from this and Eqs. (5) and (2), we can find that

$$\dot{H} = -\frac{4\pi G}{f'(A/4G)}(\rho + p). \tag{6}$$

Note that for $f(x) = x$, what we did above is exactly the derivation of Einstein equation from gravitational entropy formula by Jacobson in the special case of Firedmann-Robertson-Walker metric. It is also important to notice that we did not use the first law of thermodynamics in deriving Eq. (6) so we circumvented the question of defining gravitational energy in general cosmological spacetime, which is still unresolved.

Now let us consider the general form of modified Friedmann equation studied in Ref. [4],

$$H^2 = H_0^2 g(x), \tag{7}$$

where g is an arbitrary function and $x \equiv \rho/\rho_{c0}$ with ρ_{c0} the current critical density.

An example of the form (7) that has received much discussion in recent literature is the so called "Cardassian" cosmology [8]:

$$H^2 = \frac{8\pi G}{3}\rho + B\rho^\alpha. \tag{8}$$

When $\alpha < 2/3$, this can drive an accelerating Universe without introducing dark energy [8].

Of course, modified gravity may not be the unique way to arrive at Eq. (7). For example, Reference [13] proposed that Eq. (8) may be the result of exotic interaction property of dark matter. In this paper we will focus on the possibility that the modified Friedmann Eq. (7) is the result of large-scale modification of general relativity.

From Eq. (7) and the continuity equation $\dot{\rho} + 3H(\rho + p) = 0$, we can get

$$\dot{H} = -4\pi G g'(x)(\rho + p). \tag{9}$$

Comparing Eq. (9) with Eq. (6), we can find that f and g is related by

$$f'(A/4G) = \frac{1}{g'(x)}. \tag{10}$$

Since x can be expressed in terms of A from Eq. (7), we can find f in terms of g from Eq. (10).

From Eq. (10), we can immediately get an important conclusion: modified Friedmann equations which will give $g'(x) < 0$ are probably physically inconsistent because that means that the horizon entropy $S = f(A/4G)$ will decrease with increasing horizon area. This seems to be in contradiction with the (generalized) second law of thermodynamics. Especially, this will rule out Cardassian expansion model with $\alpha < 0$. Thus, while fitting with cosmological data allows $\alpha < 0$ [14], the $\alpha < 0$ case (if as a result of four-dimensional modification of general relativity), is conceptually problematic. Furthermore, it is also interesting to notice that the $g'(x) < 0$ case just corresponds to "superaccelerating" Universe, i.e. $\dot{H} > 0$. Thus, building models of superaccelerating cosmology from four-dimensional modification of general relativity is problematic. On the other hand, if it is firmly established from observation that our Universe is indeed superaccelerating today. Then based on our analysis, the acceleration of our Universe is probably due to some other mechanisms such as a real dark energy component.

As an example, let us consider Cardassian cosmology with $\alpha > 0$. From Eq. (8) and (10), we can find that when the second term in Eq. (8) begins to dominate, i.e. the Universe begins to accelerate, we have

$$f'(A/4G) = \frac{1}{1 + (C\frac{A}{4G})^{(1/\alpha)-1}} \tag{11}$$

where

$$C = \alpha^{\alpha/(1-\alpha)}\left(\frac{3B\rho_{c0}^{\alpha-1}}{8\pi G}\right)^{1/(1-\alpha)}\frac{H_0^2 G}{\pi}. \quad (12)$$

Integrating Eq. (11), we can get the modified entropy formula. For example, for $\alpha = 1/3$, we can integrate Eq. (11) explicitly to give

$$S = \frac{\arctan(CA/4G)}{C}. \quad (13)$$

while for $\alpha = 1/2$, we have

$$S = \frac{\ln(1 + CA/4G)}{C}. \quad (14)$$

Thus, as commented at the beginning, any four-dimensional theory that intends to explain Cardassian expansion must reproduce the strange entropy formulas above. However, those entropy formulas are so strange (the entropy does not scale like any geometric property of the system) that in our point of view, this actually disfavors the existence of such a theory.

The horizon entropy formula in some other proposed modified Friedmann equation can also be derived in similar ways. For example, let us consider the modified Friedmann equation proposed by Dvali and Turner [7]:

$$H^2 - \frac{H^\alpha}{r_c^{2-\alpha}} = \frac{8\pi G}{3}\rho. \quad (15)$$

Differentiating Eq. (15) with respect to time and using the continuity equation, we can get

$$\dot{H} = -\frac{4\pi G}{1 - \frac{\alpha H^{\alpha-2}}{2r_c^{2-\alpha}}}(\rho + p) \quad (16)$$

Comparing this to Eq. (6) and using the fact that $A = 4\pi/H^2$, we have

$$f'(A/4G) = 1 - \frac{\alpha}{2r_c^{2-\alpha}}\left(\frac{A}{4\pi}\right)^{1-\alpha/2}. \quad (17)$$

which gives

$$S = \frac{A}{4G} - \frac{\alpha}{(4-\alpha)r_c^{2-\alpha}}\left(\frac{\pi}{G}\right)^{\alpha/2-1}\left(\frac{A}{4G}\right)^{2-\alpha/2}. \quad (18)$$

For $\alpha = 0$, Eq. (18) reduce to the standard formula. This is expected as the $\alpha = 0$ case is just the standard Friedmann equation with a cosmological constant. For $\alpha = 1$, S will decrease with cosmic expansion which is unphysical. Thus this case cannot be derived from a four-dimensional theory (but it can be derived in models with extra dimensions and branes [6]). For $\alpha = -1$, S will scale like $A^{5/2}$ with cosmic expansion, which is again very strange based on our current understanding of entropy in thermal field theory.

In conclusion, assuming thermodynamical relation $dQ = TdS$ continues to hold in four-dimensional modification of general relativity, we derived the modified horizon entropy formula in a class of modified Friedmann equation. Because of the strange form of the modified entropy formula, we argue that this actually poses a problem for attempts in this direction.

Finally, we should also comment that our analysis do not apply to all the current four-dimensional modified gravity theories. For example, for the $f(R)$ gravity theory, since the modified Friedmann equation contains derivatives of H, we cannot establish a relationship with entropy formula of the form (2). Thus our analysis cannot be applied to it (entropy and other thermodynamical quantities in $f(R)$ gravity were discussed in Ref. [15]). Furthermore, in braneworld models where we can also get modified Friedmann equation of the form (7), our analysis also cannot apply directly since the heat flux in this context may also contain contributions from the bulk matter. It is interesting to pursue whether can we make an analogous analysis in braneworld models.

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