Yukawa scalar self-mass on a conformally flat background

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We compute the one loop self-mass-squared of a massless, minimally coupled scalar which is Yukawa-coupled to a massless Dirac fermion in a general conformally flat background. Dimensional regularization is employed and a fully renormalized result is obtained. For the special case of a locally de Sitter background our result is manifestly de Sitter invariant. By solving the effective field equations we show that the scalar mode functions acquire no significant one loop corrections. In particular, the phenomenon of superadiabatic amplification is not affected. One consequence is that the scalar-catalyzed production of fermions during inflation should not be reduced by changes in the scalar sector before it has time to go to completion.

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I. INTRODUCTION

Essentially all quantum field theoretic effects can be understood through classical interactions of the virtual particles whose existence is required by the uncertainty principle. In general one expects quantum field theoretic effects to become stronger the longer virtual particles live and the more probable it is for them to emerge from the vacuum. For example, vacuum polarization arises due to the polarization of charged virtual particles in an external electric field. The largest effect derives from electron-positron pairs because they are the lightest charged particles and therefore live the longest. One can even understand the running of the electromagnetic force from the incomplete polarization of the longest wavelength virtual pairs in the field provided by two very close sources.

The expansion of spacetime affects quantum field theory by lengthening the time virtual particles can exist, and sometimes by altering the probability with which they emerge from the vacuum. The first effect can be understood from the energy-time uncertainty principle. In comoving coordinates the geometry of a homogeneous, isotropic and spatially flat universe is

$$ds^2 = -dt^2 + a^2(t)d\vec{x}\cdot d\vec{x}. \tag{1}$$

Although t measures physical time, physical distance is Δx multiplied by the scale factor $a(t)$. Because spatial translation invariance is still a good symmetry, one can label particles by their wave vectors $\vec{k}$ as in flat space. However, the physical momentum of such a particle is $\vec{k}/a(t)$, and one can think of the corresponding energy as

$$E(\vec{k}, t) = \sqrt{m^2 + \|\vec{k}\|^2/a^2(t)}. \tag{2}$$

The spontaneous emergence of a pair with wave numbers $\pm\vec{k}$ at time t will not lead to a detectable violation of energy conservation provided the pair persists no longer than a time Δt defined by the equation

$$\int_t^{t+\Delta t} dt' 2E(\vec{k}, t') = 1. \tag{3}$$

Hence we conclude that the expansion of spacetime always increases the time which a virtual pair can persist.

Just as in flat space the persistence time Δt is longest for massless particles. In an expanding geometry this means $m \ll H(t)$, where the Hubble and deceleration parameters are

$$H(t) \equiv \frac{\dot{a}}{a}, \qquad q(t) \equiv -\frac{a\ddot{a}}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2}. \tag{4}$$

In this case the equation for persistence time takes the form

$$m = 0 \longrightarrow 2\|\vec{k}\| \int_t^{t+\Delta t} \frac{dt'}{a(t')} = 1. \tag{5}$$

The integral in (5) is just the conformal time interval $\Delta\eta$, defined by the change of variables $d\eta = dt/a(t)$. The dependence of $\Delta\eta$ upon Δt is controlled by sign of $q(t)$. For $q(t) > 0$ (decelerating expansion) $\Delta\eta$ grows without bound; while for $q(t) < 0$ (inflation) it approaches a finite constant. Hence we conclude that any sufficiently long wavelength virtual particles which are produced during inflation can persist forever.

Whether or not effectively massless particles actually engender stronger quantum effects during inflation depends upon the probability with which they emerge from the vacuum. Almost all massless particles possess a symmetry known as conformal invariance which means that physical processes are the same as in flat space when expressed in conformal coordinates,

$$d\eta = \frac{dt}{a(t)} \longrightarrow ds^2 = a^2(-d\eta^2 + d\vec{x}\cdot d\vec{x}). \tag{6}$$

Hence the number of virtual particles which emerge from the vacuum per conformal time is the same as the constant flat space rate we might call Γ. It follows that the rate per physical time falls off,

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$$\frac{dN}{dt} = \frac{d\eta}{dt}\frac{dN}{d\eta} = \frac{\Gamma}{a(t)}. \tag{7}$$

Hence we conclude that while any sufficiently long wavelength, massless virtual particles which happen to emerge from the vacuum can persist forever during inflation, very few conformally invariant particles will emerge.

The two massless particles which are not conformally invariant are minimally coupled scalars and gravitons. For these particles it turns out that the rate of emergence per unit physical time is unsuppressed, so we expect quantum field theoretic effects from them to be stronger than in flat space. In fact it is quantum fluctuations of these fields which are responsible for the scalar [1] and tensor [2] perturbations predicted by inflation [3,4]. These are tree order effects. At two loop order it can be shown that a massless, minimally coupled scalar with a quartic self-interaction experiences violations of the weak energy condition on cosmological scales [5,6] and that quantum gravitational back-reaction slows inflation [7].

Conformal invariance in the free theory need not rule out significant quantum corrections if the conformally invariant particle couples to one which is not conformally invariant. Two examples of this have been studied recently. In the first, electrodynamics—which is conformally invariant in $D = 4$ spacetime dimensions—is coupled to a charged, massless, minimally coupled scalar. The one loop vacuum polarization [8,9] induced by the latter causes superhorizon photons to behave, in some ways, as though they had nonzero mass [10]. This engenders no physical photon creation during inflation but leads instead to a vast enhancement of the 0-point energy of superhorizon photons which may seed cosmic magnetic fields after the end of inflation [11,12].

The other example consists of a massless fermion which is Yukawa-coupled to a massless, minimally coupled scalar [13]. Massless fermions are conformally invariant (in any dimension) so they are not much produced by themselves. However, the fermion self-energy (Fig. 1), and the effect it has on the quantum-corrected fermion field equations, show that the massless, minimally coupled scalar catalyzes the emission of scalar-fermion-anti-fermion from the vacuum. In this paper we study what effect the fermion has on the scalar through the one loop self-mass-squared (Fig. 2). This can hardly alter the nonconformal coupling but it *might* induce a nonzero scalar mass. If this mass became large enough sufficiently quickly it could cut off the scalar-induced fermion creation. We will show that this does not happen. The one loop scalar self-mass-squared can be renormalized so that there is no significant change to the scalar mode functions, and higher loop corrections cannot become significant soon enough.

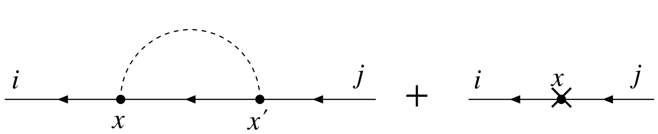

FIG. 1. : One loop contributions to $[{}_i\Sigma_j](x;x')$.

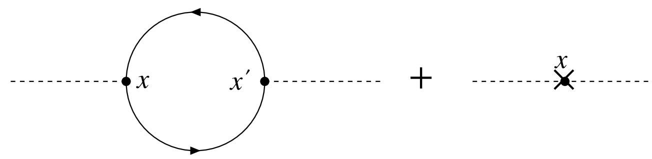

FIG. 2. : One loop contributions to $M^2(x;x')$.

In the next section we use the Yukawa Lagrangian to derive the relevant Feynman rules for an arbitrary scale factor $a(t)$. In Sec. III we compute the renormalized scalar self-mass-squared at one loop order. Although our result is valid for any $a(t)$ we show that it reduces to a manifestly de Sitter invariant form for the locally de Sitter case of $a(t) = e^{Ht}$. In Sec. IV we use the self-mass-squared to study one loop corrections to the scalar mode functions. The consequences of this result are discussed in Sec. V.

II. FEYNMAN RULES

We begin by reviewing the conventions appropriate to Dirac fields in a nontrivial geometry. In order to facilitate dimensional regularization we make no assumption about the spacetime dimension D. The gamma matrices γ^b_{ij} ($b = 0, 1, \dots, D-1$) anticommute in the usual way, $\{\gamma^b, \gamma^c\} = -2\eta^{bc}I$. One interpolates between local Lorentz indices $(b, c, d, \dots)$ and vector indices (lower case Greek letters) with the vierbein field, $e_{\mu b}(x)$. The metric is obtained by contracting two vierbeins with the Minkowski metric, $g_{\mu\nu}(x) = e_{\mu b}(x)e_{\nu c}(x)\eta^{bc}$. The vierbein's vector index is raised and lowered by the metric ($e^\mu_b = g^{\mu\nu}e_{\nu b}$) while the local Lorentz index is raised and lowered with the Minkowski metric ($e^b_\mu = \eta^{bc}e_{\mu c}$). The spin connection and the Lorentz representation matrices are

$$A_{\mu bc} \equiv e^\nu_b(e_{\nu c,\mu} - \Gamma^\rho_{\mu\nu}e_{\rho c}), \qquad J^{bc} \equiv \frac{i}{4}[\gamma^b, \gamma^c]. \tag{8}$$

Let $\phi(x)$ represent a real scalar field and let $\psi_i(x)$ stand for a Dirac field. In a general background metric the Lagrangian we wish to study would be

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\sqrt{-g} + \overline{\psi}e^\mu_b\gamma^b\Big(i\partial_\mu - \frac{1}{2}A_{\mu cd}J^{cd}\Big) \times \psi\sqrt{-g} - \frac{1}{2}\xi_0\phi^2R\sqrt{-g} - f_0\phi\overline{\psi}\psi\sqrt{-g}, \tag{9}$$

where $\overline{\psi} \equiv \psi^\dagger\gamma^0$ is the usual Dirac adjoint, ξ_0 is the bare conformal coupling and f_0 is the bare Yukawa coupling constant. Note that we do not require mass counterterms because mass is multiplicatively renormalized in dimensional regularization.

The geometry of interest is the very special form associated with the homogeneous and isotropic element. By defining a new time coordinate $d\eta \equiv dt/a(t)$, the metric of this geometry can be made conformal to the Minkowski metric,

$$ds^2 = a^2(-d\eta^2 + d\vec{x}\cdot d\vec{x}). \tag{10}$$

A convenient choice for the associated vierbein is, $e_{\mu b} = a\eta_{\mu b}$. With these simplifications the spin connection assumes the form

$$e_{\mu b} = a\eta_{\mu b} \longrightarrow A_{\mu cd} = (\eta_{\mu c}\partial_d - \eta_{\mu d}\partial_c)\ln(a). \tag{11}$$

And our Lagrangian reduces to

$$\begin{aligned}\mathcal{L} \rightarrow & -\frac{1}{2}a^{D-2}\partial_\mu\phi\partial_\nu\phi\eta^{\mu\nu} \\ & + (a^{(D-1)/2}\overline{\psi})i\gamma^\mu\partial_\mu(a^{(D-1)/2}\psi) \\ & + \frac{1}{2}\xi_0(D-1)(2a_{,\mu\nu}a^{D-3} + (D-4)a_{,\mu}a_{,\nu}a^{D-4}) \\ & \times \eta^{\mu\nu}\phi^2 - f_0 a^D\phi\overline{\psi}\psi. \end{aligned} \tag{12}$$

Renormalization is facilitated by introducing the renormalized fields

$$\phi \equiv \sqrt{Z}\phi_R \quad \text{and} \quad \psi \equiv \sqrt{Z_2}\psi_R. \tag{13}$$

This brings the Lagrangian to the form

$$\begin{aligned}\mathcal{L} \rightarrow & -\frac{1}{2}Za^{D-2}\partial_\mu\phi_R\partial^\mu\phi_R \\ & + Z_2(a^{(D-1)/2}\overline{\psi}_R)i\not{\partial}(a^{(D-1)/2}\psi_R) \\ & + \frac{1}{2}Z\xi_0(D-1)a^{D-2}\Big(2\frac{\partial^2 a}{a} + (D-4)\frac{\partial_\mu a}{a}\frac{\partial^\mu a}{a}\Big) \\ & \times \phi_R^2 - \sqrt{Z}Z_2 f_0 a^D\phi_R\overline{\psi}_R\psi_R. \end{aligned} \tag{14}$$

Note the Dirac slash notation ($\not{\partial} \equiv \gamma^\mu\partial_\mu$) and the convention—used henceforth—that indices are raised and lowered with the Lorentz metric (e.g., $\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$). Note also that $\partial^2 \equiv \partial_\mu\partial^\mu$. We now define the counterterms,

$$Z \equiv 1 + \delta Z, \qquad Z_2 \equiv 1 + \delta Z_2, \tag{15}$$

$$\sqrt{Z}Z_2 f_0 \equiv f + \delta f, \qquad Z\xi_0 \equiv 0 + \delta\xi. \tag{16}$$

Note that the conformal coupling enters only as a counterterm because we want the scalar to be minimally coupled. We can now express the Lagrangian in terms of primitive interactions and counterterms,

$$\begin{aligned}\mathcal{L} \rightarrow & -\frac{1}{2}a^{D-2}\partial_\mu\phi_R\partial^\mu\phi_R + (a^{(D-1)/2}\overline{\psi}_R)i\not{\partial}(a^{(D-1)/2}\psi_R) - fa^D\phi_R\overline{\psi}_R\psi_R - \frac{1}{2}\delta Za^{D-2}\partial_\mu\phi_R\partial^\mu\phi_R \\ & + \frac{1}{2}\delta\xi(D-1)a^{D-2}\Big(2\frac{\partial^2 a}{a} + (D-4)\frac{\partial_\mu a}{a}\frac{\partial^\mu a}{a}\Big)\phi_R^2 + \delta Z_2(a^{(D-1)/2}\overline{\psi}_R)i\not{\partial}(a^{(D-1)/2}\psi_R) - \delta f a^D\phi_R\overline{\psi}_R\psi_R. \end{aligned} \tag{17}$$

We shall need the counterterms proportional to δZ and $\delta\xi$; we will not require those proportional to δZ_2 and δf.

The diagrams of Fig. 2 do not require the scalar propagator. The fermion propagator can be determined by noting from the second term of (17) that the combination $a^{(D-1)/2}\psi_R$ behaves like a massless Dirac field in flat space. It follows that the Feynman propagator of ψ is just a conformal rescaling of the flat space result,

$$\begin{aligned} i[{}_iS_j](x;x') = & (aa')^{(1-D)/2}\gamma^\mu_{ij}i\partial_\mu\Big\{\frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}} \\ & \times[\Delta x^2(x;x')]^{1-(D/2)}\Big\}, \end{aligned} \tag{18}$$

$$= \frac{\Gamma(\frac{D}{2})}{2\pi^{D/2}}(aa')^{(1-D)/2}\frac{-i\gamma^\mu_{ij}\Delta x_\mu}{[\Delta x^2(x;x')]^{D/2}}. \tag{19}$$

Here $\Delta x_\mu \equiv \eta_{\mu\nu}(x^\nu - x'^\nu)$ and the conformal coordinate interval is,

$$\Delta x^2(x;x') \equiv \|\vec{x} - \vec{x}'\|^2 - (|\eta - \eta'| - i\delta)^2. \tag{20}$$

Note that we label the spacetime position with the D-vector $x^\mu = (\eta, \vec{x})$. The split index notation in $i[{}_iS_j](x;x')$ indicates that the first index (i) transforms according to the local Lorentz group at the first coordinate argument (x^μ) whereas the second index (j) transforms at the second argument (x'^μ).

The interaction vertex derives from the $-fa^D\phi_R\overline{\psi}_R\psi_R$ term of (17),

$$-ifa^D\delta_{ij}. \tag{21}$$

We also require the scalar field strength renormalization and conformal counterterms,

$$i\delta Z\partial_\mu(a^{D-2}\partial^\mu\delta^D(x-x')), \tag{22}$$

and

$$i\delta\xi(D-1)a^{D-2}\Big(2\frac{\partial^2 a}{a} + (D-4)\frac{\partial_\mu a}{a}\frac{\partial^\mu a}{a}\Big)\delta^D(x-x'). \tag{23}$$

Let us note, for future reference, that the choice

$$\delta\xi = \frac{1}{4}\Big(\frac{D-2}{D-1}\Big)\delta Z, \tag{24}$$

makes the two counterterms sum to a simple form,

$$i\delta Z\partial_\mu(a^{D-2}\partial^\mu\delta^D(x-x')) + i\delta Z a^{D-2}\Big(\frac{D}{2}-1\Big)\times\Big\{\frac{\partial^2 a}{a}+\Big(\frac{D}{2}-2\Big)\frac{\partial_\mu a}{a}\frac{\partial^\mu a}{a}\Big\}\delta^D(x-x') = i\delta Z(aa')^{(D/2)-1}\partial^2\delta^D(x-x'). \tag{25}$$

Because the "in" ($t \to -\infty$) vacuum is not equal to the "out" ($t \to +\infty$) vacuum in this background, it is desirable to compute true expectation values rather than in-out matrix elements. This can be done covariantly using a simple extension of the Feynman rules known as the Schwinger-Keldysh formalism [14–17]. Briefly, the end of each line has a polarity which can be "+" or "−." Vertices are either all + or all −. A + vertex is the familiar one from the Feynman rules whereas the − vertex is its negative. Propagators can be $++$, $+-$, $-+$ or $--$. Each propagator can be obtained from the Feynman propagator by replacing the conformal coordinate interval, $\Delta x^2(x;x')$, with the interval of appropriate polarization,

$$\Delta x^2_{++}(x;x') \equiv \|\vec{x}-\vec{x}'\|^2 - (|\eta-\eta'| - i\delta)^2, \tag{26}$$

$$\Delta x^2_{+-}(x;x') \equiv \|\vec{x}-\vec{x}'\|^2 - (\eta-\eta' + i\delta)^2, \tag{27}$$

$$\Delta x^2_{-+}(x;x') \equiv \|\vec{x}-\vec{x}'\|^2 - (\eta-\eta' - i\delta)^2, \tag{28}$$

$$\Delta x^2_{--}(x;x') \equiv \|\vec{x}-\vec{x}'\|^2 - (|\eta-\eta'| + i\delta)^2. \tag{29}$$

External lines can be either + or − in the Schwinger-Keldysh formalism. Hence every N-point 1-particle-irreducible (1PI) function of the in-out formalism gives rise to 2^N 1PI functions in the Schwinger-Keldysh formalism. For every field $\phi(x)$ of an in-out effective action, a Schwinger-Keldysh effective action must depend upon two fields—call them $\phi_+(x)$ and $\phi_-(x)$—in order to access the appropriate 1PI function [18–20]. If external fermions are suppressed, the effective action for our model takes the form

$$\Gamma[\phi_+,\phi_-] = S[\phi_+] - S[\phi_-] - \frac{1}{2}\int d^4x\int d^4x'\left\{\begin{array}{c}\phi_+(x)M^2_{++}(x;x')\phi_+(x') + \phi_+(x)M^2_{+-}(x;x')\phi_-(x')\\ +\phi_-(x)M^2_{-+}(x;x')\phi_+(x') + \phi_-(x)M^2_{--}(x;x')\phi_-(x')\end{array}\right\} + O(\phi^3_\pm), \tag{30}$$

where S is the free scalar action. The effective field equations are obtained by varying with respect to ϕ_+ and then setting both fields equal [18–20]

$$\frac{\delta\Gamma[\phi_+,\phi_-]}{\delta\phi_+(x)}\bigg|_{\phi_\pm=\phi} = \partial_\mu(a^2\partial^\mu\phi(x)) - \int d^4x'[M^2_{++}(x;x') + M^2_{+-}(x;x')]\times\phi(x') + O(\phi^2). \tag{31}$$

It follows that the two 1PI 2-point functions we need are $M^2_{++}(x;x')$ and $M^2_{+-}(x;x')$. Their sum in (31) gives effective field equations which are causal in the sense that the two 1PI functions cancel unless x'^μ lies on or within the past light-cone of x^μ. Their sum is also real, which neither 1PI function is separately.

III. RENORMALIZED ONE LOOP SELF-MASS

In this section we compute and fully renormalize the scalar self-mass-squared at one loop order. Our result applies for any scale factor $a(t)$. For the special case of de Sitter ($a(t) = e^{Ht}$, with constant H) we give a manifestly de Sitter invariant form for $M^2_{++}(x;x')$ and $M^2_{+-}(x;x')$.

Using the Feynman rules of the previous section we see that the $++$ and $+-$ contributions from the first diagram in Fig. 2 are,

$$-(-ifa^D)i[{}_iS_j]_{+\pm}(x;x')(\mp ifa'^D)i[{}_jS_i]_{\pm+}(x';x) = \mp\frac{f^2\Gamma^2(\frac{D}{2})}{\pi^D}\frac{aa'}{\Delta x^{2(D-1)}_{+\pm}}, \tag{32}$$

where a is the scale factor at conformal time η and a' at η'. One cannot yet take the spacetime dimension to $D = 4$ because this term is too singular to give a well-defined integral in (31). To make it less singular we extract derivatives with respect to the unintegrated coordinate x^μ, which can be pulled outside the integral. The key identity can be stated without regard to $\pm$ variations,

$$\frac{1}{\Delta x^{2\alpha}} = \frac{1}{4(\alpha-1)(\alpha-\frac{D}{2})}\partial^2\Big(\frac{1}{\Delta x^{2(\alpha-1)}}\Big). \tag{33}$$

Two applications of this identity give us

$$\mp\frac{f^2\Gamma^2(\frac{D}{2})}{\pi^D}\frac{aa'}{\Delta x^{2(D-1)}_{+\pm}} = \mp\frac{f^2aa'\Gamma^2(\frac{D}{2}-1)}{16\pi^D(D-3)(D-4)}\times\partial^4\Big(\frac{1}{\Delta x^{2(D-3)}_{+\pm}}\Big). \tag{34}$$

At this point we could take the limit $D = 4$ were it not for the explicit factor of $(D-4)$ in the denominator.

It is now necessary to distinguish the $++$ case, which has a one loop counterterm, and $+-$ case, which is free of primitive divergences. The trick for obtaining the renormalized result in each case involves adding zero using the identities

$$\partial^2\left(\frac{1}{\Delta x_{++}^{D-2}}\right) = \frac{i4\pi^{D/2}}{\Gamma(\frac{D}{2}-1)}\delta^D(x-x') \quad \text{and} \quad \partial^2\left(\frac{1}{\Delta x_{+-}^{D-2}}\right) = 0. \tag{35}$$

We can therefore write the two self-mass-squared's as

$$-iM^2_{++}(x;x') = -\frac{f^2aa'}{16\pi^D}\frac{\Gamma^2(\frac{D}{2}-1)}{(D-3)(D-4)}\partial^4\left(\frac{1}{\Delta x_{++}^{2(D-3)}} - \frac{\mu^{D-4}}{\Delta x_{++}^{2((D/2)-1)}}\right) - \frac{if^2}{8\pi^{D/2}}\frac{\Gamma(\frac{D}{2}-2)}{(D-3)}\mu^{D-4}aa'\partial^2\delta^D(x-x') + i\delta Z\partial_\mu(a^{D-2}\partial^\mu\delta^D(x-x')) + i\delta\xi(D-1)a^{D-2}\left(2\frac{\partial^2 a}{a} + (D-4)\frac{\partial_\mu a}{a}\frac{\partial^\mu a}{a}\right)\delta^D(x-x') + O(f^4), \tag{36}$$

$$-iM^2_{+-}(x;x') = \frac{f^2aa'}{16\pi^D}\frac{\Gamma^2(\frac{D}{2}-1)\partial^4}{(D-3)(D-4)}\left(\frac{1}{\Delta x_{+-}^{2(D-3)}} - \frac{\mu^{D-4}}{\Delta x_{+-}^{2((D/2)-1)}}\right) + O(f^4). \tag{37}$$

Note the appearance of the dimensional regularization mass scale μ.

By comparing the primitive divergence in (36) with the simple counterterm (25) that results from the relation (24), we settle on the following choice of counterterms,

$$\delta\xi = \frac{1}{4}\left(\frac{D-2}{D-1}\right)\delta Z + \delta\xi_{\rm fin} + O(f^4) \quad \text{and} \quad \delta Z = \frac{f^2\mu^{D-4}}{8\pi^{D/2}}\frac{\Gamma(\frac{D}{2}-2)}{(D-3)} + O(f^4). \tag{38}$$

Here $\delta\xi_{\rm fin}$ is a finite, order f^2 contribution we shall fix later. With this choice the $++$ result becomes

$$-iM^2_{++}(x;x') = -\frac{f^2aa'}{16\pi^D}\frac{\Gamma^2(\frac{D}{2}-1)}{(D-3)(D-4)}\partial^4\left(\frac{1}{\Delta x_{++}^{2(D-3)}} - \frac{\mu^{D-4}}{\Delta x_{++}^{2((D/2)-1)}}\right) - \frac{if^2}{8\pi^{D/2}}\frac{\Gamma(\frac{D}{2}-2)}{(D-3)}\mu^{D-4}(aa' - (aa')^{(D/2)-1})\partial^2\delta^D(x-x') + i\delta\xi_{\rm fin}6a\partial^2a\delta^4(x-x') + O(f^4). \tag{39}$$

Even though the bare scalar is minimally coupled, the divergent parts of the one loop counterterms are interpretable as the field strength renormalization of the conformal kinetic operator. We might have anticipated this from the fact that only fermion propagators enter the primitive diagram at one loop order, and massless fermions are conformally invariant in any dimension. Higher loop diagrams such as those of Fig. 3 involve the scalar propagator, which breaks conformal invariance, so we do not expect the conformal relation (24) between $\delta\xi$ and δZ to persist at higher loops.

At this stage we take the limit $D=4$ facilitated by the identities

$$\frac{1}{\Delta x^{2(D-3)}} - \frac{\mu^{D-4}}{\Delta x^{2((D/2)-1)}} = -\left(\frac{D}{2}-2\right)\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2} + O((D-4)^2), \tag{40}$$

$$aa' - (aa')^{(D/2)-1} = -\left(\frac{D}{2}-2\right)aa'\ln(aa') + O((D-4)^2). \tag{41}$$

The factor of $\ln(aa')$ in the second relation is reminiscent of the nonlocal conformal anomaly [21] and derives from the same dimensional mismatch between primitive divergence and counterterm. Putting everything together and taking the limit $D=4$ gives

$$M^2_{++}(x;x') = \frac{if^2aa'}{32\pi^4}\partial^4\left(\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}}\right) - \frac{f^2aa'}{8\pi^2}\ln(aa')\partial^2\delta^4(x-x') - \delta\xi_{\rm fin}6a\partial^2a\delta^4(x-x') + O(f^4), \tag{42}$$

$$M^2_{+-}(x;x') = -\frac{if^2aa'}{32\pi^4}\partial^4\left(\frac{\ln(\mu^2\Delta x^2_{+-})}{\Delta x^2_{+-}}\right) + O(f^4). \tag{43}$$

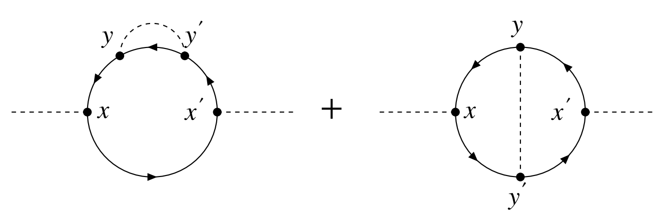

FIG. 3. Two loop contributions to $M^2(x;x')$.

We now specialize to de Sitter background, i.e. $a(\eta) = -1/H\eta$ with H constant, to show that the self-mass-squared can be expressed in a manifestly de Sitter invariant form. The de Sitter invariant, conformal d'Alembertian is

$$\mathcal{D}_c \equiv \partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - \frac{1}{6}\sqrt{-g}R \longrightarrow a\partial^2 a. \quad (44)$$

A simple function of the invariant length $\ell(x;x')$ is,

$$y(x;x') \equiv 4\sin^2\left(\frac{1}{2}H\ell(x;x')\right) = aa'H^2\Delta x^2(x;x'). \quad (45)$$

The polarized forms of this length function are obtained by simply replacing the coordinate interval, $\Delta x^2(x;x')$, with the interval of appropriate polarization from (26) through (29). Using these invariants, (42) and (43) can be rewritten as

$$M^2_{++}(x;x') = \frac{if^2H^2}{32\pi^4}\mathcal{D}_c\mathcal{D}'_c\left(\frac{\ln[y_{++}(x;x')\mu^2/H^2]}{y_{++}(x;x')}\right) - \delta\xi_{\rm fin}R\sqrt{-g}\delta^4(x-x') + O(f^4), \quad (46)$$

$$M^2_{+-}(x;x') = -\frac{if^2H^2}{32\pi^4}\mathcal{D}_c\mathcal{D}'_c\left(\frac{\ln[y_{+-}(x;x')\mu^2/H^2]}{y_{+-}(x;x')}\right) + O(f^4). \quad (47)$$

IV. EFFECTIVE FIELD EQUATIONS

In this section we substitute our results (42) and (43) for $M^2_{+\pm}(x;x')$ into the effective field equation (31) and work out the result for a spatial plane wave. Most of the analysis is valid for arbitrary scale factor $a(t)$. Only at the end do we specialize to the locally de Sitter case and make a choice for $\delta\xi_{\rm fin}$ which keeps corrections to the wave functions small at one loop order.

The linearized, effective field equation is

$$\partial_\mu(a^2\partial^\mu\phi(x)) + \frac{f^2a}{8\pi^2}[\ln(a)\partial^2(a\phi(x)) + \partial^2(\ln(a)a\phi(x))] + \delta\xi_{\rm fin}6a(\partial^2a)\phi(x) - \frac{if^2a}{32\pi^4}\partial^4\int d^4x'\theta(\eta'-\eta_i)a'\phi(x')\left[\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2\Delta x^2_{+-})}{\Delta x^2_{+-}}\right] + O(f^4) = 0. \quad (48)$$

Here η_i is the initial conformal time (corresponding to $t = 0$) at which the state is released in free Bunch-Davies vacuum. The first step in simplifying this equation is to extract another d'Alembertian from the nonlocal term in square brackets,

$$\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2\Delta x^2_{+-})}{\Delta x^2_{+-}} = \frac{\partial^2}{8}\begin{bmatrix} \ln^2(\mu^2\Delta x^2_{++}) - 2\ln(\mu^2\Delta x^2_{++}) \\ -\ln^2(\mu^2\Delta x^2_{+-}) + 2\ln(\mu^2\Delta x^2_{+-}) \end{bmatrix}. \quad (49)$$

We now define the coordinate intervals $\Delta\eta \equiv \eta - \eta'$ and $\Delta x \equiv \|\vec{x} - \vec{x}'\|$ and recall the $++$ and $+-$ intervals,

$$\Delta x^2_{++} = \Delta x^2 - (|\Delta\eta| - i\delta)^2 \quad \text{and} \quad \Delta x^2_{+-} = \Delta x^2 - (\Delta\eta + i\delta)^2. \quad (50)$$

When $\eta' > \eta$ we have $\Delta x^2_{++} = \Delta x^2_{+-}$, so the $++$ and $+-$ terms in (49) cancel. When $\eta' < \eta$ and $\Delta x > \Delta\eta$ (spacelike separation) the arguments of the logarithms become positive, real numbers for $\delta \longrightarrow 0$, so there is also cancellation. Only for $\eta' < \eta$ and $\Delta x < \Delta\eta$ (timelike separation) do we acquire a nonzero result through the relation

$$\theta(\Delta\eta - \Delta x)\ln(\mu^2\Delta x^2_{+\pm}) = \theta(\Delta\eta - \Delta x) \times \{\ln[\mu^2(\Delta\eta^2 - \Delta x^2)] \pm i\pi\}. \quad (51)$$

Hence the square bracketed term in (48) can be written as

$$\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2\Delta x^2_{+-})}{\Delta x^2_{+-}} = \frac{i\pi}{2}\partial^2\{\theta(\Delta\eta - \Delta x)(\ln[\mu^2(\Delta\eta^2 - \Delta x^2)] - 1)\}. \quad (52)$$

Substituting this relation into (48) gives the manifestly real and causal equation

$$\partial_\mu(a^2\partial^\mu\phi(x)) + \frac{f^2a}{8\pi^2}[\ln(a)\partial^2(a\phi(x)) + \partial^2(\ln(a)a\phi(x))] + \delta\xi_{\rm fin}6a(\partial^2a)\phi(x) + \frac{f^2a}{2^6\pi^3}\partial^6\int_{\eta_i}^{\eta}d\eta'a(\eta')\int_{\Delta x\leq\Delta\eta}d^3x'\phi(\eta',\vec{x}')(\ln[\mu^2(\Delta\eta^2 - \Delta x^2)] - 1) + O(f^4) = 0. \quad (53)$$

Because the background geometry is homogeneous, isotropic and spatially flat, we can build up an arbitrary solution from a superposition of spatial plane waves of the form

$$\phi(\eta, \vec{x}) = g(\eta, k) e^{i\vec{k}\cdot\vec{x}}. \tag{54}$$

Evaluating the derivatives of this in the first two terms of (53) is straightforward. The nonlocal term, involving the integral, is more complicated. To begin we make the change of variable $\vec{y} = \vec{x}' - \vec{x}$ and extract the spatial phase factor,

$$\frac{f^2 a}{2^6 \pi^3} \partial^6 \int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k) \int_{\Delta x \leq \Delta\eta} d^3x' e^{\vec{k}\cdot\vec{x}'} (\ln[\mu^2(\Delta\eta^2 - \Delta x^2)] - 1)$$
$$= -\frac{f^2 a}{2^6 \pi^3} e^{i\vec{k}\cdot\vec{x}} (\partial_0^2 + k^2)^3 \int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k) \int_{\|\vec{y}\| \leq \Delta\eta} d^3y e^{i\vec{k}\cdot\vec{y}} (\ln[\mu^2(\Delta\eta^2 - y^2)] - 1). \tag{55}$$

We next perform the angular integrations and make the change of variables $y \equiv \Delta\eta z$,

$$\frac{f^2 a}{2^6 \pi^3} \partial^6 \int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k) \int_{\Delta x \leq \Delta\eta} d^3x' e^{\vec{k}\cdot\vec{x}'} (\ln[\mu^2(\Delta\eta^2 - \Delta x^2)] - 1)$$
$$= -\frac{f^2 a}{2^4 \pi^2} e^{i\vec{k}\cdot\vec{x}} (\partial_0^2 + k^2)^3 \int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k) \int_0^{\Delta\eta} dy \frac{y}{k} \sin(ky) \ln[\mu^2(\Delta\eta^2 - y^2)], \tag{56}$$

$$= -\frac{f^2 a}{2^4 \pi^2} e^{i\vec{k}\cdot\vec{x}} (\partial_0^2 + k^2)^3 \int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k) \int_0^1 dz \Delta\eta^2 \frac{z}{k} \sin(k\Delta\eta z) [\ln(\mu^2 \Delta\eta^2) + \ln(1 - z^2) - 1]. \tag{57}$$

The integral over z is facilitated by the special function

$$\xi(\alpha) \equiv \int_0^1 dz z \sin(\alpha z) \ln(1 - z^2) \tag{58}$$

$$= \frac{2}{\alpha^2} \sin\alpha - \frac{1}{\alpha^2} [\cos\alpha + \alpha \sin\alpha] \left[\text{si}(2\alpha) + \frac{\pi}{2} \right] + \frac{1}{\alpha^2} [\sin\alpha - \alpha \cos\alpha] \left[\text{ci}(2\alpha) - \gamma - \ln\left(\frac{\alpha}{2}\right) \right]. \tag{59}$$

Here γ is the Euler–Mascheroni constant and si(x) and ci(x) stand for the sine integral and cosine integral functions

$$\text{si}(x) = -\int_x^{\infty} dt \frac{\sin t}{t} = -\frac{\pi}{2} + \int_0^x dt \frac{\sin t}{t}, \tag{60}$$

$$\text{ci}(x) = -\int_x^{\infty} dt \frac{\cos t}{t} = \gamma + \ln x + \int_0^x dt \left[\frac{\cos t - 1}{t} \right]. \tag{61}$$

Making use of these relations and performing the elementary integrals gives

$$\frac{f^2 a}{2^6 \pi^3} \partial^6 \int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k) \int_{\Delta x \leq \Delta\eta} d^3x' e^{\vec{k}\cdot\vec{x}'} (\ln[\mu^2(\Delta\eta^2 - \Delta x^2)] - 1)$$
$$= -\frac{f^2 a}{2^4 \pi^2} e^{i\vec{k}\cdot\vec{x}} \left(\frac{\partial_0^2 + k^2}{k} \right)^3 \int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k) \times [(\sin(k\Delta\eta) - k\Delta\eta \cos(k\Delta\eta))(2\ln(\mu\Delta\eta) - 1) + (k\Delta\eta)^2 \xi(k\Delta\eta)]. \tag{62}$$

The next step is acting the derivatives. Because the integrand in (62) vanishes at $\eta' = \eta$ like $\ln(\Delta\eta)\Delta\eta^3$ the first three derivatives commute with the upper limit,

$$\frac{f^2 a}{2^6 \pi^3} \partial^6 \int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k) \int_{\Delta x \leq \Delta\eta} d^3x' e^{\vec{k}\cdot\vec{x}'} (\ln[\mu^2(\Delta\eta^2 - \Delta x^2)] - 1)$$
$$= -\frac{f^2 a}{2^3 \pi^2 k} e^{i\vec{k}\cdot\vec{x}} (\partial_0^2 + k^2)(\partial_0 + ik)(\partial_0 - ik) \int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k) \left\{ \sin(k\Delta\eta) \left[\ln\left(\frac{2\mu^2 \Delta\eta}{k} \right) + \text{ci}(2k\Delta\eta) - \gamma \right] \right.$$
$$\left. - \cos(k\Delta\eta) \left(\text{si}(2k\Delta\eta) + \frac{\pi}{2} \right) \right\}, \tag{63}$$

$$= -\frac{f^2 a}{2^3\pi^2} e^{i\vec{k}\cdot\vec{x}}(\partial_0^2 + k^2)(\partial_0 + ik)\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k) e^{-ik\Delta\eta}\Big\{2\ln(2\mu\Delta\eta) + \int_0^{2k\Delta\eta} dt\Big(\frac{e^{it}-1}{t}\Big)\Big\}. \tag{64}$$

The term containing the logarithm in (64) is divergent for $\eta' = \eta$. We isolate the divergence using

$$\ln(2\mu\Delta\eta) = \ln(-2\mu\eta') + \ln\Big(1 - \frac{\eta}{\eta'}\Big). \tag{65}$$

We can now act the operator $(\partial_0 + ik)$ on the nonsingular terms in (64),

$$\begin{aligned}&\frac{f^2 a}{2^6\pi^3}\partial^6\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k)\int_{\Delta x\le\Delta\eta} d^3x' e^{\vec{k}\cdot\vec{x}'}(\ln[\mu^2(\Delta\eta^2 - \Delta x^2)] - 1)\\ &= -\frac{f^2 a}{4\pi^2} e^{i\vec{k}\cdot\vec{x}}(\partial_0^2 + k^2)\left\{\begin{matrix} a(\eta)g(\eta,k)\ln(-2\mu\eta) + i\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta',k)\frac{1}{\Delta\eta}\sin(k\Delta\eta) \\ +(\partial_0 + ik)\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta',k) e^{-ik\Delta\eta}\ln(1-\frac{\eta}{\eta'})\end{matrix}\right\}.\end{aligned} \tag{66}$$

Had we instead acted $(\partial_0 + ik)$ before $(\partial_0 - ik)$ the result would be

$$\begin{aligned}&\frac{f^2 a}{2^6\pi^3}\partial^6\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k)\int_{\Delta x\le\Delta\eta} d^3x' e^{\vec{k}\cdot\vec{x}'}(\ln[\mu^2(\Delta\eta^2 - \Delta x^2)] - 1)\\ &= -\frac{f^2 a}{4\pi^2} e^{i\vec{k}\cdot\vec{x}}(\partial_0^2 + k^2)\left\{\begin{matrix} a(\eta)g(\eta,k)\ln(-2\mu\eta) - i\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta',k)\frac{1}{\Delta\eta}\sin(k\Delta\eta) \\ +(\partial_0 - ik)\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta',k) e^{ik\Delta\eta}\ln(1-\frac{\eta}{\eta'})\end{matrix}\right\}.\end{aligned} \tag{67}$$

A simpler expression results from adding half of (66) with half of (67),

$$\begin{aligned}&\frac{f^2 a}{2^6\pi^3}\partial^6\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k)\int_{\Delta x\le\Delta\eta} d^3x' e^{\vec{k}\cdot\vec{x}'}(\ln[\mu^2(\Delta\eta^2 - \Delta x^2)] - 1)\\ &= -\frac{f^2 a}{4\pi^2} e^{i\vec{k}\cdot\vec{x}}(\partial_0^2 + k^2)\left\{\begin{matrix} a(\eta)g(\eta,k)\ln(-2\mu\eta) \\ +\partial_0\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta',k)\cos(k\Delta\eta)\ln(1-\frac{\eta}{\eta'}) \\ +k\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta',k)\sin(k\Delta\eta)\ln(1-\frac{\eta}{\eta'})\end{matrix}\right\}.\end{aligned} \tag{68}$$

The top line of (68) is comparable to the local one loop terms in (53). Extracting it is what we have worked so hard to do. The remaining, nonlocal terms make only small contributions at late times. They can be simplified by first expanding the trigonometric functions using the angular addition formulas, then partially integrating to shield the logarithmic singularity, and finally bringing another derivative inside,

$$\begin{aligned}&\partial_0\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k)\cos(k\Delta\eta)\ln\Big(\frac{\Delta\eta}{-\eta'}\Big) + k\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k)\sin(k\Delta\eta)\ln\Big(\frac{\Delta\eta}{-\eta'}\Big)\\ &= \cos(k\eta)\partial_0\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k)\cos(k\eta')\ln\Big(\frac{\Delta\eta}{-\eta'}\Big) + \sin(k\eta)\partial_0\int_{\eta_i}^{\eta} d\eta' a(\eta') g(\eta', k)\sin(k\eta')\ln\Big(\frac{\Delta\eta}{-\eta'}\Big).\end{aligned} \tag{69}$$

$$= \ln\Big(1 - \frac{\eta}{\eta_i}\Big)\frac{\eta_i}{\eta} a(\eta_i)\cos(k(\eta - \eta_i)) g(\eta_i, k) + \frac{1}{\eta}\int_{\eta_i}^{\eta} d\eta'\ln\Big(\frac{\Delta\eta}{-\eta'}\Big)\frac{\partial}{\partial\eta'}\{\eta' a(\eta')\cos(k\Delta\eta) g(\eta', k)\}. \tag{70}$$

It is now time to combine terms and give the general result for the linearized effective field equation (53) specialized to a spatial plane wave (54). We denote conformal time derivatives with a prime,

$$\partial_\mu[a^2\partial^\mu(g(\eta,k)e^{i\vec{k}\cdot\vec{x}})] = -a^2 e^{i\vec{k}\cdot\vec{x}}\Big(g'' + 2\frac{a'}{a}g' + k^2 g\Big), \tag{71}$$

$$\frac{1}{a}(\partial_0^2 + k^2)(ag) = g'' + 2\frac{a}{a}g' + k^2 g, \tag{72}$$

$$(\partial_0^2 + k^2)(\ln(-2\mu\eta)ag) = (\partial_0^2 + k^2)(ag) + \frac{2}{\eta}(ag)' - \frac{1}{\eta^2}ag. \tag{73}$$

Combining everything, and deleting an overall factor of $-a^2 e^{i\vec{k}\cdot\vec{x}}$, we obtain

$$
\begin{aligned}
0 = {} & g'' + 2\frac{a'}{a}g' + k^2 g + \frac{f^2}{4\pi^2}\ln(-2\mu\eta a)\Big[g'' + 2\frac{a'}{a}g' + \Big(\frac{a''}{a} + k^2\Big)g\Big] \\
& + \frac{f^2}{4\pi^2}\Big[\Big(\frac{a'}{a} + \frac{2}{\eta}\Big)g' + \Big(\frac{a'^2}{2a^2} + \frac{a''}{2a} + \frac{2a'}{\eta a} - \frac{1}{\eta^2}\Big)g\Big] + 6\delta\xi_{\rm fin}\frac{a''}{a}g + \frac{f^2}{4\pi^2 a}(\partial_0^2 + k^2) \\
& \times \left\{ \begin{array}{c} \ln(1-\frac{\eta}{\eta_i})\frac{\eta_i}{\eta}a(\eta_i)\cos(k(\eta-\eta_i))g(\eta_i,k) \\ + \int_{\eta_i}^{\eta} d\eta' \ln(\frac{\Delta\eta}{-\eta'})\frac{\partial}{\partial\eta'}[\frac{\eta'}{\eta}a(\eta')\cos(k\Delta\eta)g(\eta',k)] \end{array} \right\} + O(f^4).
\end{aligned} \tag{74}
$$

One can better infer asymptotic behaviors if an extra factor of a^2 is extracted and the equation is converted to physical time t,

$$
\begin{aligned}
& \ddot{g} + 3H\dot{g} + \frac{k^2}{a^2}g + \frac{f^2}{4\pi^2}\ln(-2\mu\eta a)\Big[\ddot{g} + 3H\dot{g} + \Big(2H^2 + \dot{H} + \frac{k^2}{a^2}\Big)g\Big] \\
& + \frac{f^2}{4\pi^2}\Big[\Big(H + \frac{2}{\eta a}\Big)\dot{g} + \Big(\frac{3}{2}H^2 + \frac{3}{4}\dot{H} + \frac{2H}{\eta a} - \frac{1}{\eta^2 a^2}\Big)g\Big] + \delta\xi_{\rm fin}(12H^2 + 6\dot{H})g + \frac{f^2}{4\pi^2}\Big(\frac{\partial^2}{\partial t^2} + 3H\frac{\partial}{\partial t} + 2H^2 + \dot{H} + \frac{k^2}{a^2}\Big) \\
& \quad \times \Big\{\ln\Big(1 - \frac{\eta}{\eta_i}\Big)\frac{\eta_i a_i}{\eta a}\cos(k(\eta-\eta_i))g(\eta_i,k) + \int_0^t dt' \ln\Big(\frac{\Delta\eta}{-\eta'}\Big)\frac{\partial}{\partial t'}\Big[\frac{\eta' a(t')}{\eta a(t)}\cos(k\Delta\eta)g(\eta',k)\Big]\Big\} + O(f^4) = 0.
\end{aligned} \tag{75}
$$

It is also useful to recall the slow roll expansion for the conformal time,

$$
\eta \equiv -\int_t^\infty \frac{dt'}{a(t')} = \frac{-1}{H(t)a(t)}\Big\{1 - \frac{\dot{H}(t)}{H^2(t)} + \ldots\Big\}. \tag{76}
$$

Hence the combination $\eta a(t)$ is only slowly varying during inflation. The tree order mode function approaches a constant at late times so we will keep one loop corrections small by making finite renormalizations to cancel any undifferentiated mode functions which are not suppressed by slow roll parameters. The best choice for this is

$$
\mu = \frac{1}{2}H(t_i) \equiv \frac{1}{2}H_i \quad \text{and} \quad \delta\xi_{\rm fin} = \frac{f^2}{32\pi^2}. \tag{77}
$$

With these choices the effective mode equation becomes

$$
\begin{aligned}
& \ddot{g} + 3H\dot{g} + \frac{k^2}{a^2}g + \frac{f^2}{4\pi^2}\ln\Big(-\frac{\mu}{H_i}\eta a\Big)\Big[\ddot{g} + 3H\dot{g} + \Big(2H^2 + \dot{H} + \frac{k^2}{a^2}\Big)g\Big] \\
& + \frac{f^2}{4\pi^2}\Big[\Big(H + \frac{2}{\eta a}\Big)\dot{g} + \Big(3H^2 + \frac{3}{4}\dot{H} + \frac{2H}{\eta a} - \frac{1}{\eta^2 a^2}\Big)g\Big] + \frac{f^2}{4\pi^2}\Big(\frac{\partial^2}{\partial t^2} + 3H\frac{\partial}{\partial t} + 2H^2 + \dot{H} + \frac{k^2}{a^2}\Big) \\
& \quad \times \Big\{\ln\Big(1 - \frac{\eta}{\eta_i}\Big)\frac{\eta_i a_i}{\eta a}\cos(k(\eta-\eta_i))g(\eta_i,k) + \int_0^t dt' \ln\Big(\frac{\Delta\eta}{-\eta'}\Big)\frac{\partial}{\partial t'}\Big[\frac{\eta' a(t')}{\eta a(t)}\cos(k\Delta\eta)g(\eta',k)\Big]\Big\} + O(f^4) = 0.
\end{aligned} \tag{78}
$$

The next step is to solve the equation perturbatively,

$$
g(\eta,k) = g_0(\eta,k) + \frac{f^2}{4\pi^2}g_1(\eta,k) + \ldots \tag{79}
$$

Because the tree order mode function obeys,

$$
\ddot{g}_0 + 3H\dot{g}_0 + \frac{k^2}{a^2}g_0 = 0, \tag{80}
$$

the equation for the one loop correction is

$$\ddot{g}_1 + 3H\dot{g}_1 + \frac{k^2}{a^2}g_1 = -\ln\Big(-\frac{\mu}{H_i}\eta a\Big)(2H^2+\dot{H})g_0 - \Big(H + \frac{2}{\eta a}\Big)\dot{g}_0 - \Big(3H^2 + \frac{3}{4}\dot{H} + \frac{2H}{\eta a} - \frac{1}{\eta^2 a^2}\Big)g_0 - \Big(\frac{\partial^2}{\partial t^2} + 3H\frac{\partial}{\partial t} + 2H^2 + \dot{H} + \frac{k^2}{a^2}\Big)\left\{\begin{array}{c} \ln(1-\frac{\eta}{\eta_i})\frac{\eta_i a_i}{\eta a}\cos(k(\eta-\eta_i))g_0(\eta_i,k) \\ + \int_0^t dt' \ln(\frac{\Delta\eta}{-\eta'})\frac{\partial}{\partial t'}[\frac{\eta' a(t')}{\eta a(t)}\cos(k\Delta\eta)g_0(\eta',k)] \end{array}\right\}. \tag{81}$$

Although the solution to (80) has been obtained for a general scale factor $a(t)$ [22], the expression is too complicated for the integral in (81) to be evaluated in closed form. For the special case of de Sitter background ($a(t) = e^{Ht} = -1/H\eta$, with H constant) the tree order mode function is

$$g_0(\eta, k) = \frac{H}{\sqrt{2k^3}}\Big(1 - \frac{ik}{Ha}\Big)\exp\Big[\frac{ik}{Ha}\Big] = \frac{H}{\sqrt{2k^3}}(1 + ik\eta)e^{-ik\eta}. \tag{82}$$

Of course many other terms in the effective mode Eq. (75) also simplify in de Sitter background,

$$\ddot{g}_1 + 3H\dot{g}_1 + \frac{k^2}{a^2}g_1 = H\dot{g}_0 - \Big(\frac{\partial^2}{\partial t^2} + 3H\frac{\partial}{\partial t} + 2H^2 + \frac{k^2}{a^2}\Big)\left\{\begin{array}{c} \ln\Big(1-\frac{\eta}{\eta_i}\Big)\cos(k(\eta-\eta_i))g_0(\eta_i,k) \\ + \int_0^t dt' \ln\Big(\frac{\Delta\eta}{-\eta'}\Big)\frac{\partial}{\partial t'}[\cos(k\Delta\eta)g_0(\eta',k)] \end{array}\right\}. \tag{83}$$

To begin evaluating the nonlocal term we note the differential identity

$$\ln\Big(\frac{\Delta\eta}{-\eta'}\Big)\frac{\partial}{\partial\eta'}\{\cos(k\Delta\eta)g_0(\eta',k)\} = \frac{\partial}{\partial\eta'}\Big\{\ln\Big(\frac{\Delta\eta}{-\eta'}\Big)\cos(k\Delta\eta)g_0(\eta',k)\Big\} + \Big[\frac{1}{\eta'} + \frac{1}{\Delta\eta}\Big]\cos(k\Delta\eta)g_0(\eta',k). \tag{84}$$

The $1/\eta'$ term on the right-hand side of (84) can be expressed as

$$\frac{1}{\eta'}\cos(k\Delta\eta)g_0(\eta',k) = \frac{H}{(2k)^{3/2}}\Big\{e^{ik\eta}\Big[\frac{e^{-i2k\eta'}-1}{\eta'}\Big] + \frac{\partial}{\partial\eta'}\Big[2\cos(k\eta)\ln(-H\eta') - \frac{1}{2}e^{ik(\eta-2\eta')} + ik\eta' e^{-ik\eta}\Big]\Big\}. \tag{85}$$

A similar expression can be obtained for the $1/\Delta\eta$ term on the right-hand side of (84),

$$\frac{1}{\Delta\eta}\cos(k\Delta\eta)g_0(\eta',k) = \frac{H}{(2k)^{3/2}}\Big\{(1+ik\eta)e^{-ik\eta}\Big[\frac{e^{i2k\Delta\eta}-1}{\Delta\eta}\Big] + \frac{\partial}{\partial\eta'}\Big[-2(1+ik\eta)e^{-ik\eta}\ln(H\Delta\eta) + \frac{1}{2}e^{ik(\eta-2\eta')} - ik\eta' e^{-ik\eta}\Big]\Big\}. \tag{86}$$

It follows that the bracketed term in (83) is

$$\ln\Big(1-\frac{\eta}{\eta_i}\Big)\cos(k(\eta-\eta_i))g_0(\eta_i,k) + \int_{\eta_i}^{\eta} d\eta' \ln\Big(\frac{\Delta\eta}{-\eta'}\Big)\frac{\partial}{\partial\eta'}[\cos(k\Delta\eta)g_0(\eta',k)] = \ln(1+H\eta)g_0(\eta,k) + \ln(-H\eta)[\cos(k\eta)g_0(0,k) - g_0(\eta,k)] - \frac{1}{2}g_0(0,k)e^{ik\eta}\int_{-2k\eta}^{-2k\eta_i} dz\Big[\frac{e^{iz}-1}{z}\Big] + \frac{1}{2}g_0(\eta,k)\int_0^{2k\Delta\eta_i} dz\Big[\frac{e^{iz}-1}{z}\Big]. \tag{87}$$

The integrals in this expression could be written in terms of the sine and cosine integrals (60) and (61) but it is simpler not to do this.

It remains to substitute the nonlocal source term (87) into the Eq. (83) for the one loop mode function. Acting the derivatives gives a complicated expression which considerable effort brings to the form

$$H^{-2}\ddot{g}_1 + 3H^{-1}\dot{g}_1 + \frac{k^2 g_1}{H^2 a^2} = (g_0^* - g_0)\ln(a) + H^{-1}\dot{g}_0\left\{1 - \left(\frac{e^{2ik\Delta\eta_i} + 1}{a-1}\right)\right\} + g_0\left\{-2ik\eta - 2\ln(1+H\eta) - \frac{(1-ik\eta)e^{2ik\Delta\eta_i} + 1}{a-1} + \frac{(e^{2ik\Delta\eta_i}+1)}{2(a-1)^2}\right\} + \frac{1}{2}g_0^*\int_{-2k\eta}^{-2k\eta_i} dz\left[\frac{e^{iz}-1}{z}\right] - \frac{1}{2}g_0\int_0^{2k\Delta\eta_i} dz\left[\frac{e^{iz}-1}{z}\right]. \tag{88}$$

All the mode functions in this expression are evaluated at η. From the asymptotic late time expansion for the mode function,

$$g_0(\eta, k) = \frac{H}{\sqrt{2k^3}}\left\{1 + \frac{1}{2}\left(\frac{k}{Ha}\right)^2 + \frac{i}{3}\left(\frac{k}{Ha}\right)^3 + O(a^{-4})\right\}, \tag{89}$$

we see that the leading form of the late time source is

$$H^{-2}\ddot{g}_1 + 3H^{-1}\dot{g}_1 + \frac{k^2 g_1}{H^2 a^2} = \frac{H}{\sqrt{2k^3}}\left\{-\frac{2i}{3}\left(\frac{k}{Ha}\right)^3\ln(a) + O(a^{-3})\right\}. \tag{90}$$

The solution at late times is straightforward,

$$g_1(\eta, k) \longrightarrow \frac{H}{\sqrt{2k^3}}\left\{\frac{i}{9}\left(\frac{k}{Ha}\right)^3\ln^2(a) + O(\ln(a)a^{-3})\right\}, \tag{91}$$

except for possible homogeneous terms which can be absorbed into a further finite field strength renormalization.

V. DISCUSSION

We have computed the fully renormalized scalar self-mass-squared at one loop (42) and (43) for a general scale factor $a(t)$. For the special case of de Sitter ($a(t) = e^{Ht}$ with constant H) our results can be written in a manifestly de Sitter invariant form (46) and (47). Although the computation was simple on account of the conformal invariance of Dirac theory, we have not been able to locate a prior result in the literature.

In any case, our real interest lies in the effect the scalar self-mass-squared has on the late time behavior of the scalar mode functions. For that purpose we derived the Schwinger-Keldysh effective field equation (53) at one loop order. When specialized to de Sitter background we were able show that the theory can be renormalized so that there are no significant corrections at late times.

The reason for the null result is that one loop contributions (Fig. 2) involve only the conformally invariant part of the theory: the fermion propagator and the Yukawa coupling. As the introduction explained, significant quantum effects during inflation require the participation of at least one massless particle which is not conformally invariant. The first instance of that for the scalar self-mass-squared would come at two loop order through the diagrams of Fig. 3. Evaluating them would be a formidable undertaking, but perhaps a possible one in view of the fact that the two loop contributions to the scalar self-mass-squared have recently been obtained for massless, minimally coupled ϕ^4 in de Sitter background [23].

Recall from the similar analysis of the one loop fermion self-energy [13] that the leading one loop correction to the effective field equations came in with an extra scale factor compared with the classical terms. It also had a factor of f^2, so we expect the one loop correction to become comparable to the classical mode function at roughly $f^2 a(t) \sim 1$. Two loop corrections would be down by an extra factor of f^2, but should not contain any more scale factors. So this is one case in which it is actually reliable to solve the effective field equations nonperturbatively with only one loop corrections.

Of course the physics of the result is that the massless and not conformally invariant scalar is catalyzing the production of fermions [13]. The process could be throttled if the scalar quickly develops a large enough mass. The work done in this paper shows that no such mass occurs at one loop order. At two loop order we expect the diagrams of Fig. 3 to induce corrections to the effective field equation with an extra factor of a^2 relative to the classical terms. However, they will also contain a factor of f^4, so they would not become important until $f^4 a^2 \sim 1$. That is the same time at which the production of fermions is becoming significant. Because at most one fermion can be produced in each state, we expect the end of inflation to see essentially all superhorizon fermion modes fully populated. This is a profound difference from the usual picture and it seems there must be important consequences.

Although we obtained the effective mode equation for an arbitrary scale factor $a(t)$, we have only solved it for the special case of de Sitter. For more general scale factors there can be stronger asymptotic contributions at late times. However, because these vanish for de Sitter, we conclude that any one loop late time correction to the mode function must be suppressed by at least one factor of a slow roll parameter. The fact that scalar-catalyzed fermion production goes to completion ought therefore to be a general feature of slow roll inflation.

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