Conformally coupled scalar solitons and black holes with negative cosmological constant

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We present arguments for the existence of both globally regular and black hole solutions of the Einstein equations with a conformally coupled scalar field, in the presence of a negative cosmological constant, for space-time dimensions greater than or equal to four. These configurations approach asymptotically anti-de Sitter space-time and are indexed by the central value of the scalar field. We also study the stability of these solutions, and show that, at least for all the solutions studied numerically, they are linearly stable.

I. INTRODUCTION

Within the various theories violating the no-hair conjecture, the case of a conformally coupled scalar field is of particular interest. In asymptotically flat space with no scalar self-interaction potential, this theory admits an exact, closed-form black hole solution, which has a scalar field diverging on the event horizon [1]. However, it is known that this solution is unstable [2], and cannot be considered as a valid example of a black hole with scalar hair. There are also a number of theoretical results as well as numerical evidence against the existence of black holes with scalar field hair (with various couplings to the Ricci scalar curvature) in asymptotically flat space-time (see, for example, [3] for a recent discussion).

In an unexpected development, hairy black hole solutions have been found in both theories with minimally as well as nonminimally coupled scalar fields by considering asymptotically anti-de Sitter (AAdS) boundary conditions [4,5]. Moreover, some of these solutions are found to be stable. Exact four-dimensional black hole solutions of gravity with a minimally coupled self-interacting scalar field have been presented by Martinez, Troncoso, and Zanelli (MTZ) [6] and Zloshchastiev [7].

However, in many other theories admitting hairy black hole solutions, these configurations survive in the limit of zero event horizon radius, yielding particlelike, globally regular configurations. Motivated by this observation, we consider in this paper the case of a conformally coupled scalar field in an n-dimensional AAdS space-time (with $n \geq 4$) and look for both globally regular and black hole solutions. In the black hole case, we extend the results of Ref. [5] by considering higher dimensional configurations. Since a negative cosmological constant allows for the existence of black holes whose horizon has nontrivial topology, we consider, apart from spherically symmetric

solutions, topological black holes also. We find that the spherically symmetric solutions admit a nontrivial regular limit, representing gravitating scalar solitons. These configurations are indexed by the central value of the scalar field and are found to be stable against linear fluctuations.

The outline of this paper is as follows: in Sec. II we introduce our model, the numerical results being presented in Sec. III. The stability of our solutions is addressed in Sec. IV. Our conclusions are presented in Sec. V.

II. THE MODEL

A. The ansatz and field equations

We consider the following action, which describes a self-interacting scalar field ϕ with nonminimal coupling to gravity in n dimensions (throughout this paper we will use units in which $c = 8\pi G = 1$):

$$S = \int d^n x \sqrt{-g} \left[\frac{1}{2}(R - 2\Lambda) - \frac{1}{2}(\nabla \phi)^2 - \frac{1}{2}\xi R \phi^2 - V(\phi) \right], \tag{1}$$

where R is the Ricci scalar curvature, $\Lambda = -(n-2) \times (n-1)/2\ell^2$ is the cosmological constant, and ξ is the coupling constant. For a minimally coupled scalar field $\xi = 0$ and for conformal coupling (which is the focus of this paper) $\xi = \xi_c = (n-2)/4(n-1)$.

The field equations are obtained by varying the action (1) with respect to field variables $g_{\mu\nu}$ and ϕ,

$$(1 - \xi \phi^2)G_{\mu\nu} + g_{\mu\nu}\Lambda = (1 - 2\xi)\nabla_\mu \phi \nabla_\nu \phi$$
$$+ \left(2\xi - \frac{1}{2}\right)g_{\mu\nu}(\nabla \phi)^2$$
$$- 2\xi \phi \nabla_\mu \nabla_\nu \phi$$
$$+ 2\xi g_{\mu\nu}\phi \nabla^2 \phi - g_{\mu\nu}V(\phi), \tag{2}$$

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$$\nabla^2 \phi - \xi R \phi - \frac{dV}{d\phi} = 0. \tag{3}$$

The Ricci scalar expression which follows from these equations is

$$R = -\frac{2(n-1)(\xi - \xi_c)(\nabla\phi)^2 + 2\xi(n-1)\phi \frac{dV}{d\phi} - n(V(\phi) + \Lambda)}{n/2 - 1 + 2(n-1)\xi(\xi - \xi_c)\phi^2}. \tag{4}$$

Since for a negative cosmological constant topological black holes may appear (with a nonspherical topology of the event horizon), we consider a general metric ansatz

$$ds^2 = \frac{dr^2}{H(r)} + r^2 d\sigma^2_{d-2,k} - H(r)e^{2\delta(r)}dt^2, \tag{5}$$

where $d\sigma^2_{n-2,k} = d\psi^2 + f_k^2(\psi)d\Omega_2^2$ denotes the line element of an $(n-2)$-dimensional space Σ_k with constant curvature. The discrete parameter k takes the values 1, 0, and -1 and implies the following form of the function $f_k(\psi)$:

$$f_k(\psi) = \begin{cases} \sin\psi, & \text{for } k = 1, \\ \psi, & \text{for } k = 0, \\ \sinh\psi, & \text{for } k = -1. \end{cases} \tag{6}$$

When $k = 1$, the hypersurface Σ_1 represents a sphere; for $k = -1$, it is a negative constant curvature space and it could be a closed hypersurface with arbitrarily high genus under appropriate identifications. For $k = 0$, the hypersurface Σ_0 is a $(n-2)$-dimensional Euclidean space (see e.g. the discussion in [8]).

A convenient parametrization of the metric function H is

$$H(r) = k - \frac{2m(r)}{r^{n-3}} + \frac{r^2}{\ell^2}. \tag{7}$$

Although a rigorous computation of the solutions' mass and action is a nontrivial task for $\xi \neq 0$, we assume that the asymptotic value of $m(r)$ corresponds to the mass of our configurations, up to some n-dependent factor.

Within this ansatz, we find the resulting equations (where a prime denotes the derivative with respect to r)

$$\frac{n-2}{2r}(1 - \xi\phi^2)\left[H' - \frac{n-3}{r}(k-H)\right] - \left(2\xi - \frac{1}{2}\right)H\phi'^2 + \xi\phi\phi'(H' + 2H\delta') - 2\xi\phi\nabla^2\phi + V(\phi) + \Lambda = 0,$$

$$\frac{n-2}{r}(1 - \xi\phi^2)\delta' - (1 - 2\xi)\phi'^2 - 2\xi\phi(\delta'\phi' - \phi'') = 0, \tag{8}$$

$$H\phi'' + \phi'\left(H\delta' + H' + H\frac{n-2}{r}\right) - \xi R\phi - \frac{dV(\phi)}{d\phi} = 0.$$

B. Boundary conditions

The asymptotic solutions to these equations can be systematically constructed in both regions, near the origin (or event horizon) and for $r \gg 1$.

The corresponding expansion as $r \to 0$ is (globally regular solutions may exist for $k = 1$ only)

$$H(r) = 1 + \frac{2(-2\xi^2\phi_0^2 R_0 + \Lambda + V_0 - 2\xi\phi_0 V_0')}{(n-1)(n-2)(\xi\phi_0^2 - 1)} r^2$$

$$+ O(r^4),$$

$$\delta(r) = \delta_0 + \frac{\xi\phi_0(\xi\phi_0 R_0 + V_0')}{(n-1)(n-2)(\xi\phi_0^2 - 1)} r^2 + O(r^4), \tag{9}$$

$$\phi(r) = \phi_0 + \frac{\xi\phi_0 R_0 + V_0'}{2(n-1)} r^2 + O(r^4),$$

where

$$R_0 = \frac{n(V_0 + \Lambda) - 2\xi(n-1)\phi_0 V_0'}{n/2 - 1 + 2(n-1)\xi(\xi - \xi_c)\phi_0^2} \tag{10}$$

is the Ricci scalar evaluated at the origin and $V_0 = V(\phi_0)$, $V_0' = dV/d\phi|_{\phi=\phi_0}$.

For black hole configurations with a regular, nonextremal event horizon at $r = r_h$, the expression near the event horizon is

$$H(r) = H'(r_h)(r - r_h) + O(r - r_h)^2,$$

$$\delta(r) = \delta_h + \delta'(r_h)(r - r_h) + O(r - r_h)^2,$$

$$\phi(r) = \phi_h + \phi'(r_h)(r - r_h) + \phi_2(r - r_h)^2 + O(r - r_h)^3,$$

$$\tag{11}$$

where

$$H'(r_h) = (n-3)\frac{k}{r_h} + \frac{2r_h(\Lambda + V_h - \xi\phi_h(\xi\phi_h R_h + V_h'))}{(n-2)(\xi\phi_h^2 - 1)}, \qquad \phi'(r_h) = \frac{\xi R_h \phi_h + V_h'}{H'(r_h)},$$

$$\phi_2 = \frac{\phi'(r_h)}{2r_h}\left\{-1 + \frac{1}{2H_h' r_h}[r_h^2(\xi R_h + V_h'') + 2k(n-3)]\right\}$$

$$+ \frac{1}{(\xi\phi_h^2 - 1)}\left\{\frac{(2\xi-1)\phi'(r_h)^2 r_h V_h'}{2(n-2)H'(r_h)} + \xi\phi_h\phi'(r_h)\left[1 - \frac{k(n-3)}{2r_h H'(r_h)} + \frac{r_h}{2(n-2)H'(r_h)}(3\xi R_h + V_h'')\right]\right\},$$

$$\delta'(r_h) = \frac{r_h[(2\xi-1)\phi'(r_h)^2 + 4\xi\phi_h\phi_2]}{(n-2)(\xi\phi_h^2 - 1) + 2\xi\phi_h\phi'(r_h)r_h}, \tag{12}$$

and

$$R_h = \frac{n(V_h + \Lambda) - 2\xi(n-1)\phi_h V_h'}{n/2 - 1 + 2(n-1)\xi(\xi - \xi_c)\phi_h^2} \tag{13}$$

is the Ricci scalar evaluated at the event horizon; noting that $V_h^{(k)} = V^{(k)}(\phi)|_{\phi=\phi_h}$.

To analyze the $r \gg 1$ region, we assume that the geometry approaches asymptotically the AdS space-time and that the function $m(r)$ does not diverge in the same limit. These assumptions imply that $\lim_{r\to\infty}\phi(r) = 0$ and

$$\phi = \frac{c_1}{r^{n/2-1}} + \frac{c_2}{r^{n/2}} - \frac{c_3}{r^{n/2+1}} + \dots,$$

$$H(r) = k - \frac{2M}{r^{n-3}} - \frac{2\Lambda r^2}{(n-2)(n-1)} + \dots, \tag{14}$$

$$\delta(r) = \frac{\delta_2}{r^n} + \dots,$$

and also impose some constraints on the scalar potential (for example, in $n = 4$, the potential should satisfy as $r \to \infty$ the condition $V = V' = V'' = V''' = 0$). In the above relations c_1, c_2, and M are real constants which fix the values of the other coefficients in the asymptotic expansion. In the simplest case of a vanishing self-interaction potential, we find

$$c_3 = -\frac{c_1 k(n-1)(n-2)^2(n-4)}{16\Lambda},$$

$$\delta_2 = -\frac{c_1^2(n-1)(n-2)^3(n-4)k + 8c_2^2 n\Lambda}{16n(n-1)(n-2)\Lambda}. \tag{15}$$

C. Conformal transformation

The conformal transformation [1,9]

$$\bar{g}_{\mu\nu} = \Omega^{2/(n-2)}g_{\mu\nu}, \quad \text{with } \Omega = 1 - \xi\phi^2 \tag{16}$$

maps the original system (1) onto a much simpler one involving just a minimally coupled scalar field, but with a more complicated potential (this transformation is valid only for those solutions with a nonvanishing Ω). For a conformally coupled scalar field, the new action principle takes the form

$$S = \int d^n x \sqrt{-\bar{g}}\left[\frac{\bar{R} - 2\Lambda}{2} - \frac{1}{2}(\bar{\nabla}\Phi)^2 - U(\Phi)\right], \tag{17}$$

where a bar denotes quantities calculated using the transformed metric $\bar{g}$ and we define a new scalar field Φ as (for $\xi = \xi_c$)

$$\Phi = \frac{1}{\sqrt{\xi_c}}\text{arctanh}\sqrt{\xi_c}\phi, \tag{18}$$

with a nonvanishing potential

$$U(\Phi) = [\Lambda + V(\phi)][\cosh(\sqrt{\xi_c}\Phi)]^{2n/(n-2)} - \Lambda. \tag{19}$$

The main advantage of the rescaled frame is that the field equations are much simpler. However, the potential (19) is unphysical (for example, with $V \equiv 0$, it is negative everywhere).

In the transformed frame, the metric $\bar{g}_{\mu\nu}$ takes the form (5), but with the quantities H and δ replaced by $\bar{H}$ and $\bar{\delta}$, respectively. In addition, there is a new radial coordinate $\bar{r} = \Omega^{1/(n-2)}r$, which is a good coordinate as long as

$$\mathcal{A} = (n-2)\Omega - 2r\xi_c\phi\phi' > 0. \tag{20}$$

This is an additional constraint on the scalar field ϕ required for the conformal transformation to be valid. For all our numerical solutions, the conditions Ω, $\mathcal{A} > 0$ are satisfied.

III. NUMERICAL RESULTS

As analytic solutions to the coupled nonlinear equations (8) appear to be intractable for every dimension, except for the $n = 4$, $k = -1$ black hole solution found in [6], the resulting system has to be solved numerically.

In this section we discuss mainly the case of a conformally coupled scalar field without a self-interaction potential. Since for $V(\phi) = 0$ the field equations are invariant under the transformation $\phi \to -\phi$, only positive values of ϕ_i are considered. Here ϕ_i denotes the initial value of the scalar field at the $r = r_0$ [with $r_0 = (0, r_h)$ for regular and black hole solutions, respectively]. Also, by rescaling the radial coordinate [together with $m(r)$], we can set $\Lambda = -(n-1)(n-2)/2$ (i.e. $\ell = 1$) without any loss of generality.
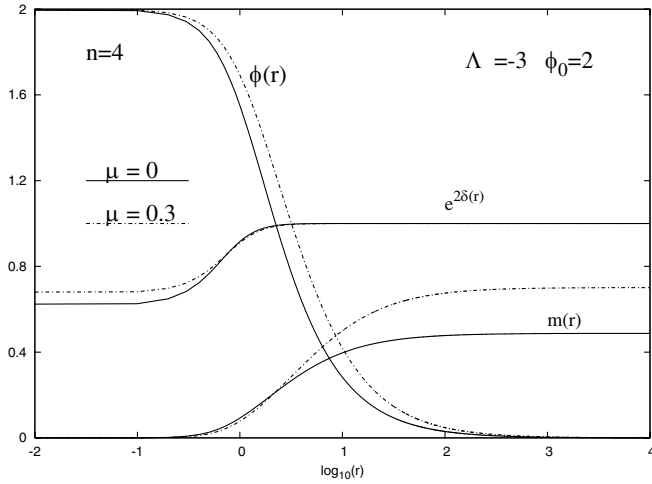
FIG. 1. The functions $m(r)$, $e^{2\delta(r)}$, and $\phi(r)$ are plotted as functions of radius for two typical $n = 4$ regular solutions. The scalar potential in one case is $V(\phi) = \mu/2\phi^4$, with $\mu = 0.3$, while $V(\phi) = 0$ for the second solution. The central value of the scalar field is $\phi_0 = 2$.

We follow the usual approach and, by using a standard ordinary differential equation solver, we evaluate the initial conditions at $r = r_0 + 10^{-6}$ for global tolerance 10^{-12}, and integrate towards $r \to \infty$. In this way we find that nontrivial solutions may exist in any dimension $n \geq 4$ (both black hole and regular solutions exist also in three space-time dimensions; however, their properties are somewhat special and we do not discuss them here). Black hole solutions seem to exist for any values of the parameters (k, ϕ_h, r_h) satisfying $H'(r_h) > 0$.

Typical profiles are presented in Fig. 1 for regular configurations and in Figs. 2 and 3 for black hole solutions. The dependence of the mass parameter M and $e^{2\delta(r_0)}$ on ϕ_i is plotted in Figs. 4–6 (note the occurrence of negative
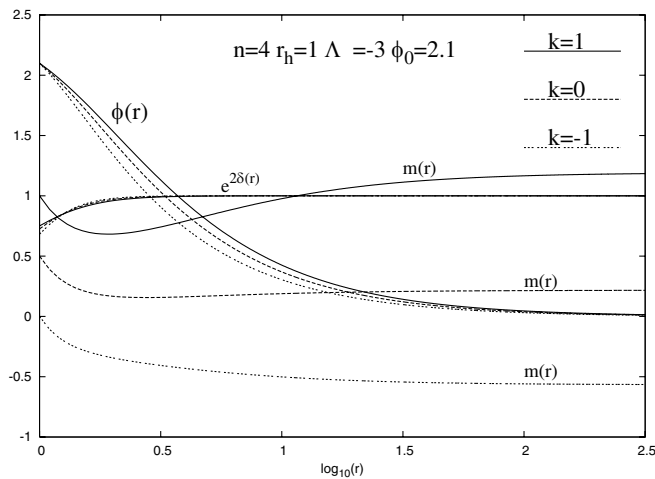

FIG. 2. The functions $m(r)$, $e^{2\delta(r)}$, and $\phi(r)$ are plotted as functions of radius for $n = 4$ black hole solutions with $\phi_0 = 2.1$ and $V(\phi) = 0$.
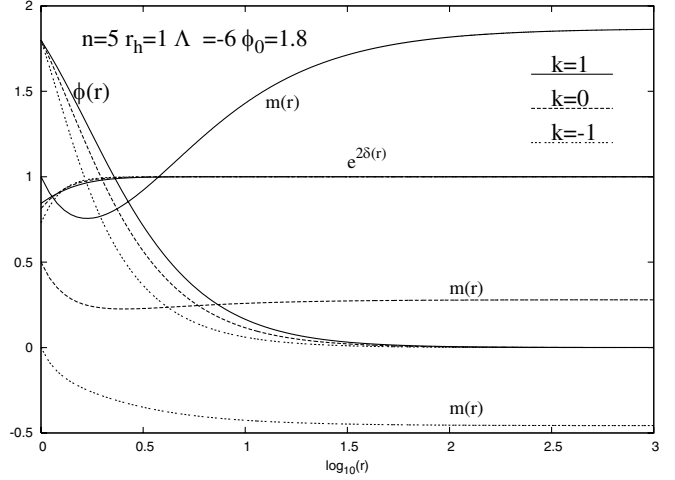

FIG. 3. The functions $m(r)$, $e^{2\delta(r)}$, and $\phi(r)$ are plotted as functions of radius for $n = 5$ black hole solutions with $\phi_0 = 1.8$ and $V(\phi) = 0$.

values of M for $k = -1$ black holes, a common situation in topological black hole physics).

The properties of the configurations can be summarized as follows:
(1) For $n \geq 4$, AAdS solutions exist for any values of ϕ_i in the interval $0 < \phi_i < 1/\sqrt{\xi_c}$;
(2) The scalar field interpolates monotonically between ϕ_i and zero and has no nodes;
(3) The value of the metric function $e^{2\delta}$ at the origin (event horizon, respectively) decreases for an increasing ϕ_i and approaches zero in the limit $\phi_i \to 1/\sqrt{\xi_c}$.

As seen in Fig. 4, for regular solutions M approaches a finite value in the same limit. In the black hole case, the
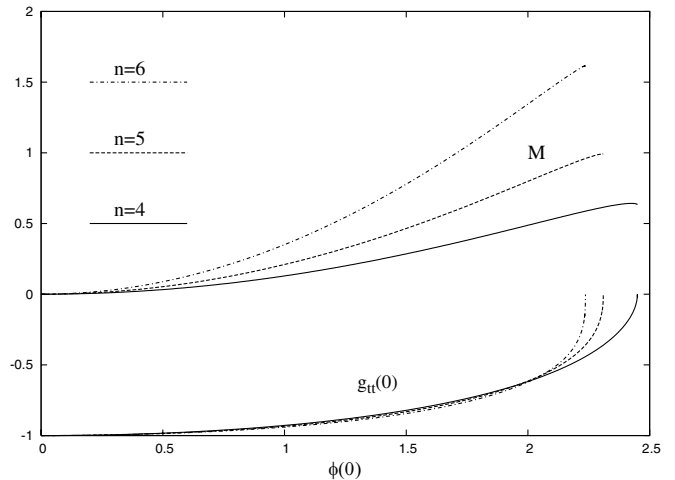

FIG. 4. The parameters M and $g_{tt}(0) = -e^{2\delta(0)}$ are shown as a function of $\phi(0)$ for spherically symmetric regular solutions in several space-time dimensions. Here and in Figs. 5 and 6 the value of the cosmological constant is $\Lambda = -(n-1)(n-2)/2$ and $V(\phi) = 0$.
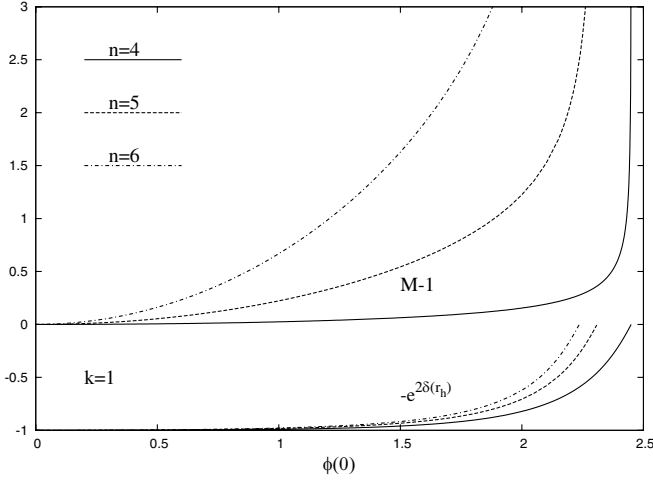
FIG. 5. The parameters M and $e^{2\delta(r_h)}$ are shown as a function of $\phi(r_h)$ for spherically symmetric black hole solutions in several space-time dimensions.

critical value of M increases very rapidly with r_h, which makes its accurate determination a difficult task for large r_h. In the $k = -1$ case, the condition $H'(r_h) > 0$ implies the existence of a minimal value of the event horizon radius,

$$r_h > \frac{(n-1)(n-3)(1 - \xi\phi_h^2)}{[1 + n(n-2)(1 - \phi_h^2/8)]^{1/2}}. \qquad (21)$$

We have also found that nontrivial configurations may exist in the presence of a nonzero scalar potential. In this case the scalar field equation (3) in a fixed AdS background has two exact solutions,

$$\phi = \left[1 + \frac{r^2}{\ell^2}\right]^p, \qquad V(\phi) = c\phi^s, \qquad (22)$$
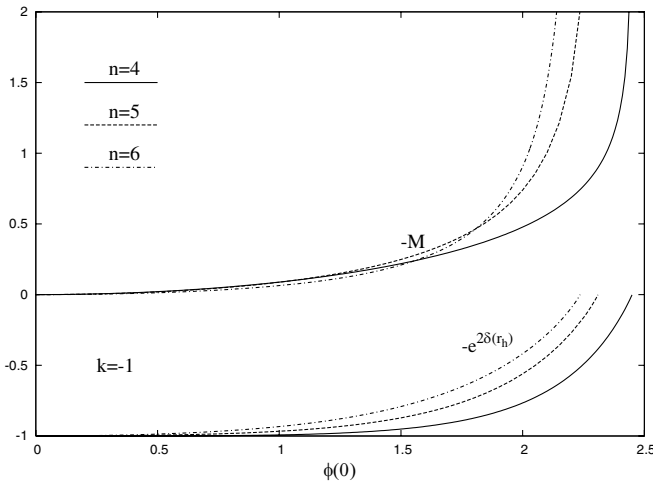

FIG. 6. The parameters M and $e^{2\delta(r_h)}$ are shown as a function of $\phi(r_h)$ for $k = -1$ topological black hole solutions in several space-time dimensions.

with $p = (2 - n)/4$, $c = -(n - 2)^3/8n\ell^2$, $s = 2n/(n - 2)$ in one case, and $p = -n/4$, $c = -n^3/8\ell^2(n + 2)$, $s = 2(n + 2)/n$ in the other.

However, we have restricted our analysis to the particular form $V(\phi) = 1/2\mu^2\phi^n$, which for $n = 4$ corresponds to the case considered in [6]. These solutions share many properties with the zero-potential case, being also indexed by the initial value of the scalar field ϕ_i. The shape of the solutions is similar to the $\mu = 0$ case and we again found no nodes in the scalar function. In Fig. 1 we plotted a typical $n = 4$ regular solution with $\mu = 0.3$. Similar solutions exist also in the black hole case. In this context, we have found that the black hole solution found in [6] corresponds to a $n = 4$, $k = -1$ configuration with a particular choice of (μ, ϕ_h).

As expected, the mass M of the solutions increases with μ while the maximal value of ϕ_i decreases. We will not address here the question of the limiting solution for $\mu \neq 0$, which seems to be an involved problem and a different metric parametrization appears to be necessary. Our preliminary results indicate that for $\mu \neq 0$ the metric function $\delta(r_i)$ remains finite in this limit, while the value of M diverges.

IV. ON THE STABILITY OF SOLUTIONS

A. Stability of the numerical solutions

Following the standard method, we consider spherically symmetric, linear perturbations of our equilibrium solutions, keeping the metric ansatz as in (5), but now the functions H, δ, and ϕ depend on t as well as r. The algebra is simplest if we work in the transformed frame (see Sec. II C), where we have a minimally coupled scalar field, and, once the perturbation equations have been derived, transform back to the frame with a conformally coupled scalar field. The metric perturbations can be eliminated to yield a single perturbation equation for

$$\Psi = r^{(n-2)/2}(1 - \xi_c\phi^2)^{-1/2}\delta\phi, \qquad (23)$$

where $\delta\phi$ is the perturbation in the conformally coupled scalar field. For periodic perturbations ($\delta\phi(t, r) = e^{i\sigma t}\delta\phi(r)$, etc.), the perturbation equation takes the standard Schrödinger form

$$\sigma^2\Psi = -\frac{d^2}{dr_*^2}\Psi + \mathcal{V}\Psi, \qquad (24)$$

where we have defined the "tortoise" coordinate r_* by

$$\frac{dr_*}{dr} = \frac{1}{He^\delta}. \qquad (25)$$

For regular solutions, r_* has values in a finite interval $[0, r_{*1}]$ for some $r_{*1} < \infty$, while for black holes, $r_* \in (-\infty, 0]$. The perturbation potential $\mathcal{V}$ is given as follows (we write the formula explicitly only for the vanishing self-interaction potential case, for simplicity):
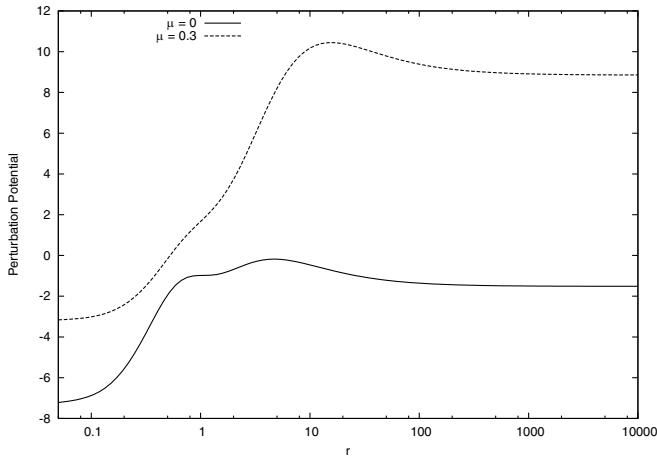
FIG. 7.   The perturbation potential (26) is plotted for the typical $n = 4$ regular solutions shown in Fig. 1.
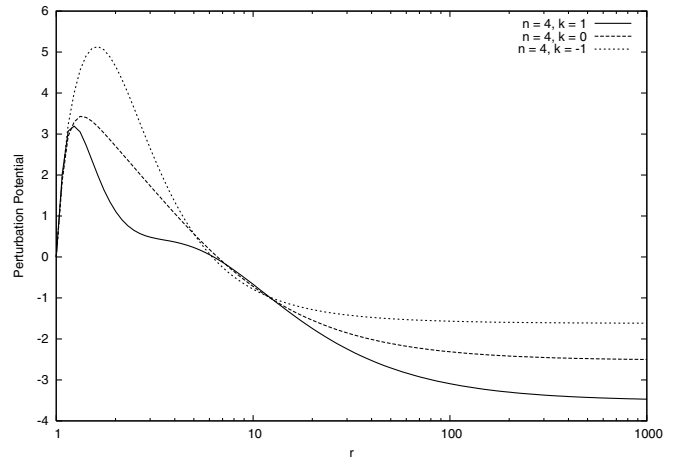

FIG. 9.   The perturbation potential (26) is plotted for the typical $n = 4$ black hole solutions shown in Fig. 2.

$$\mathcal{V} = \frac{He^{2\delta}}{r^2} \left\{ \frac{k}{2}(n-2)(n-3) - \frac{\mathcal{A}^2 H}{(n-2)^2 \Omega^2} - \frac{\Lambda r^2}{\Omega} \right.$$

$$+ \frac{2\Lambda \xi_c n r^2}{(n-2)^2} \left[\frac{2n}{\Omega} - (n+2) \right] + \frac{4\Lambda \xi_c n \phi \phi' r^3}{\mathcal{A}\Omega}$$

$$\left. - \frac{k}{2}(n-2)^3 (n-3) r^2 \phi'^2 + \frac{\Lambda(n-2)^2 r^4 \phi'^2}{\mathcal{A}^2 \Omega} \right\}, \quad (26)$$

where Ω is given in (16) and $\mathcal{A}$ is given in (20).

As is often the case for AAdS solutions, care is needed in the use of boundary conditions to ensure that there is a self-adjoint operator in the perturbation equation (24) (see, for example, the discussion in [10] for the Einstein-Yang-Mills case). We consider black hole and soliton solutions separately in this regard.

First, for black hole solutions, it is convenient to change the independent variable to $y = -r_*$ so that $y \in [0, \infty)$. In order to have a self-adjoint operator, we need to impose the boundary condition $\Psi = 0$ at $y = 0$, which corresponds to $r \to \infty$. Second, for the regular soliton solutions, boundary conditions need to be imposed at both $r_* = 0$ and $r_* = r_{*1}$ as we are working on a finite interval. Suitable boundary conditions are $\Psi = 0$ at both the end points, namely, at the origin and at infinity. In both these cases, it is straightforward to check that these boundary conditions are sufficient to enable a self-adjoint operator to be constructed from the differential operator in (24) using the standard techniques found, for example, in section XIII:2 of [11].

Some typical perturbation potentials (26) are shown in Figs. 7–10. We find a complicated behavior of the perturbation potential depending on the values of the parameters Λ, ϕ_0, and μ and the number of space-time dimensions. As $r \to 0$, we have
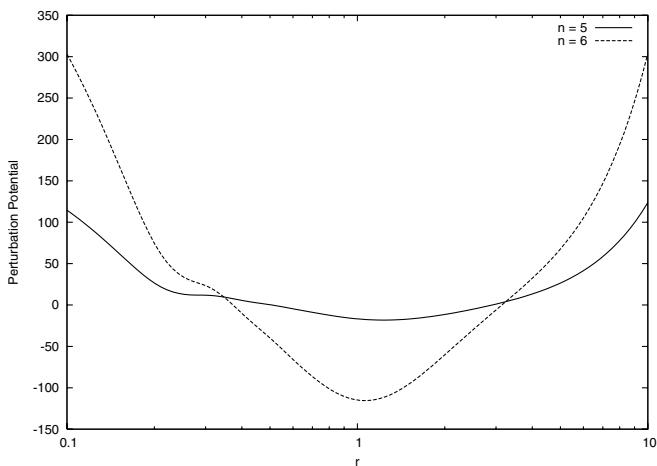

FIG. 8.   The perturbation potential (26) is plotted for two typical $n = 5, 6$ regular solutions. For both the solutions, the scalar potential $V(\phi) = 0$, and the central value of the scalar field is $\phi_0 = 2$.
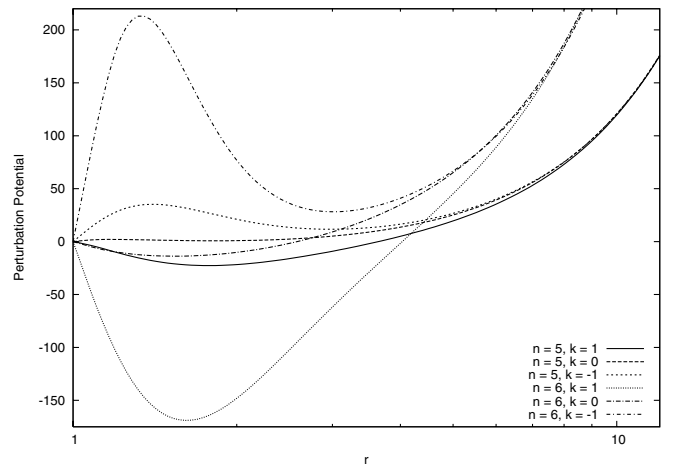

FIG. 10.   The perturbation potential (26) is plotted for typical $n = 5, 6$ black hole solutions with the value of the scalar field on the event horizon $\phi_0 = 1.8$.

$$\mathcal{V} \sim \frac{(n-4)(n-1)}{2r^2} + O(1), \qquad (27)$$

so for $n > 4$ the potential diverges to infinity, like a standard central well potential with angular momentum (see Fig. 8). The potential $\mathcal{V}$ can also be seen to vanish at the black hole event horizon (provided $\mathcal{A} > 0$ and $\Omega > 0$ there), and, at infinity, the leading order behavior is

$$\mathcal{V} \sim \frac{r^2 n(n-4)}{4\ell^4} + O(1), \qquad (28)$$

so for $n > 4$ the potential again diverges to infinity (see Figs. 8 and 10). Turning on the self-interaction potential $V(\phi)$ tends to increase the perturbation potential (Fig. 7), as also observed in [5].

In a limited number of cases we find that the perturbation potential (26) is positive everywhere (see, for example, some of the plots in Fig. 10). In these cases, we can immediately conclude that the corresponding solutions are (linearly) stable, since we have a self-adjoint operator in (24). However, for the majority of the solutions examined, the potential is not positive everywhere, and in these cases we examine the *zero mode* solution of the perturbation equation (24), namely, the time-independent solution when $\sigma^2 = 0$ (see [3] for further details of the zero mode method applied to scalar field perturbations). For all the solutions we examined, the zero modes have no nodes (zeros). As the operator in (24) is self-adjoint, we can apply standard theorems (see, for example, section XIII:7 of [11], or [12]), which state that the number of nodes of the zero mode equals the number of eigenvalues σ^2 of (24) which are less than zero. Therefore there are no eigenmodes which grow exponentially in time and the solutions are linearly stable. Our conclusion that our numerical solutions are stable is in accordance with previous work [3–5], in which solutions with a scalar field having no zeros were linearly stable.

B. Stability of the closed-form solutions

We can also study the stability of the exact, closed-form solutions on pure AdS space, given by (22). In this case we keep the background AdS metric fixed and perturb simply the scalar field. The equation for the tortoise coordinate r_* (25) can be explicitly integrated to give $x = r_*/\ell = \tan^{-1}(r/\ell)$, where $x \in [0, \pi/2]$, and the perturbation equation takes the form (24), for periodic perturbations, with r_* replaced by x, and the perturbation potential is now

$$\mathcal{V} = -\frac{1}{4}(n-2)^2\csc^2(x) - \frac{1}{2}(n-2)(n-3)\sec^2(x)$$
$$- 4p(p-1); \qquad (29)$$

where $4p(p-1) = (n-2)(n+2)/4$ for the first type of solutions, and $4p(p-1) = n(n+4)/4$ for the second type of solutions. Since the perturbation potential is so simple in this case, we use an analytic, variational method

to study the stability. We define a functional

$$\mathcal{F}[\Psi] = \int_{x=0}^{\pi/2} \left[-\Psi \frac{d^2\Psi}{dx^2} + \mathcal{V}\Psi^2 \right] dx$$
$$= \left[-\Psi \frac{d\Psi}{dx} \right]_{x=0}^{\pi/2} + \int_{x=0}^{\pi/2} \left[\left(\frac{d\Psi}{dx} \right)^2 + \mathcal{V}\Psi^2 \right] dx, \qquad (30)$$

where in the second line we have integrated by parts. If we can find a test function Ψ_0 such that

$$\int_{x=0}^{\pi/2} \Psi_0^2 dx \qquad (31)$$

is finite, the boundary term in (30) vanishes, and $\mathcal{F}[\Psi_0] < 0$, then there must be at least one bound state solution of the perturbation equation (24) with $\sigma^2 < 0$, rendering the solutions unstable. Using $\Psi_0(x) = \sin(2x)$ as our test function, the integral (31) is finite, the required boundary term does indeed vanish, and

$$\mathcal{F}[\Psi_0] = \begin{cases} \frac{1}{16}[-13n^2 + 56n - 44]; \\ \frac{1}{16}[-13n^2 + 52n - 48]; \end{cases} \qquad (32)$$

for the first and second type of exact solutions, respectively. Both the quadratics above are negative for all $n \geq 4$, so our exact, closed-form solutions are unstable. This instability is perhaps a little surprising given the stability of the numerical solutions; however this may be understood as a result of the particular form of the (negative-) scalar field potential. If we perturb the scalar field slightly, there is no nearby solution of the form (22) for it to become. The nearest numerical solution will require a finite change in the scalar field at some value of r, and this shows up as a linear instability. However, it might be reasonable to expect that the scalar field would settle on a stable, nontrivial solution rather than radiating away to infinity.

V. CONCLUSIONS

In this paper we have studied the Einstein-scalar system in various space-time dimensions, with a conformally coupled scalar field and a negative cosmological constant. We find both regular and black hole solutions, generalizing the black hole solutions of [5].

Both types of solutions are shown to be linearly stable, apart from some exact, discrete, closed-form solutions on pure AdS, which are linearly unstable. As with previous solutions [3,5], this stability can be readily understood in terms of the Breitenlohner-Freedman bound [13], which, in n dimensions, states that scalar fields in pure AdS are stable if their mass squared satisfies the inequality [13]

$$m_{\text{BF}}^2 > \frac{\Lambda(n-1)}{2(n-2)}, \qquad (33)$$

noting that $\Lambda < 0$ so the bound is for negative mass squared. In our case, with zero self-interaction potential,

the "effective" mass is given by

$$\xi_c R(r \to \infty) = \frac{\Lambda n}{2(n-1)}. \tag{34}$$

For all $n \geq 3$, it is the case that $\xi_c R(r \to \infty) > m_{\mathrm{BF}}^2$, implying the stability of our solutions.

There are various interesting applications of these solutions. The soliton solutions may be of interest in the gravitational collapse of scalar fields in AAdS. Critical collapse of a conformally coupled scalar field has been studied in flat space [14], but not, to date at least, in $3 + 1$ dimensions in AAdS. Soliton solutions of the type found in this paper do not occur in flat space, so their presence in AAdS may change the phenomenology of gravitational collapse, since there are no longer just the end-point possibilities of empty space or a black hole. However, since the solitons we have found here are stable, they cannot be the critical solutions, unlike the situation for Einstein-Yang-Mills solitons in asymptotically flat space [15].

It would also be of interest to calculate the mass and action of these solutions, and to look for possible applications within the context of the AdS/conformal field theories correspondence. The thermodynamics of black holes with a conformally coupled scalar field with a positive or zero cosmological constant has already yielded some surprises (see, for example, [16] for a review), prompting a detailed study of the thermodynamics when the cosmological constant is negative.

Finally, it would be interesting to extend the results derived in [17] within the isolated horizon formalism, to the more general case considered in this work.

We hope to return to these questions in a future publication.

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