

Effects of gravitational backreaction on cosmological perturbationsPatrick Martineau^{1,*} and Robert H. Brandenberger^{1,2,†}¹*Dept. of Physics, McGill University, 3600 University Street, Montréal QC, H3A 2T8, Canada*²*Dept. of Physics, Brown University, Providence Rhode Island, 02912, USA*

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Because of the nonlinearity of the Einstein equations, the cosmological fluctuations which are generated during inflation on a wide range of wavelengths do not evolve independently. In particular, to second order in perturbation theory, the first order fluctuations back-react both on the background geometry and on the perturbations themselves. In this paper, the gravitational backreaction of long-wavelength (super-Hubble) scalar metric fluctuations on the perturbations themselves is investigated for a large class of inflationary models. Specifically, the equations describing the evolution of long-wavelength cosmological metric and matter perturbations in an inflationary universe are solved to second order in both the amplitude of the perturbations and in the slow-roll expansion parameter. Assuming that the linear fluctuations have random phases, we show that the fractional correction to the power spectrum due to the leading infrared backreaction terms does not change the shape of the spectrum. The amplitude of the effect is suppressed by the product of the inflationary slow-roll parameter and the amplitude of the linear power spectrum. The nongaussianity of the spectrum induced by backreaction is commented upon.

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I. INTRODUCTION

The study of cosmological fluctuations is one of the cornerstones of modern cosmology. In order for a cosmological model to be considered successful, it must be able to reproduce, among other things, the power spectrum of the perturbations. These perturbations leave their mark as anisotropies in the cosmic microwave background (CMB) and go on to act as seeds for structure formation. The theory of cosmological perturbations establishes the bridge between observations (namely observations of fluctuations in the CMB and in the distribution of structure in the universe) and the physics of the very early universe which is responsible for providing the generation mechanism for the fluctuations.

Arguably, the most successful cosmological paradigm for the early universe is the inflationary scenario [1]. Inflationary cosmology provides a mechanism for generating the primordial fluctuations and predicts a nearly scale-invariant spectrum of Gaussian, adiabatic, scalar fluctuations [2] (see also [3–5]). These predictions have been verified experimentally to high accuracy in the last decade (see, e.g., the most recent WMAP CMB anisotropy maps [6]). Since, in addition, inflationary cosmology solves numerous problems of Standard Big Bang cosmology [1] (for a review of these problems along with a discussion of some conceptual problems inherent to inflation see [7]), inflation is widely considered as a key ingredient of the theory of the early universe.

At the present time, however, inflationary cosmology does not quite have the status of a *theory*. It is best thought of as a successful scenario that resolves many of the

problems that plague Big Bang cosmology. There are a large number of different *models* that result in accelerated (i.e. inflationary) expansion. However, many models share a common feature in that they involve a scalar field (the “inflaton”) which undergoes a period in which it rolls slowly down its potential - leading to what is known as “slow-roll inflation”. Our analysis will be in the context of a general slow-roll inflation model.

The theory of cosmological perturbations (see e.g. [8] for a comprehensive review, and [9] for a recent abbreviated overview), a formalism crucial to the understanding of CMB anisotropies, is usually studied within the framework of linearized gravity. One writes down an ansatz for the form of the perturbed metric about a homogeneous and isotropic background space-time, linearizes the Einstein equations in the amplitude of the perturbations, and solves the resulting equations. In this scheme, all Fourier modes of the fluctuations evolve independently, as must be the case in a linear approximation. The Einstein equations which govern the evolution of space-time and matter are, however, nonlinear. Thus, retaining terms quadratic and higher in the perturbation amplitude leads to interactions between different perturbation modes. These interactions determine the “gravitational backreaction”, the difference between the full evolution of the space-time and what would be obtained in linear theory, and can lead to potentially important modifications of the results obtained at linear order. In particular, they may effect the key qualitative predictions of inflation, namely, the scale invariance of the spectrum and its Gaussianity.

The period of inflation in most scalar field-driven inflationary models is very long (measured in units of the Hubble time during inflation). Thus, the red-shifting of scales leads to the population of a large phase space of long-wavelength modes (modes with a wavelength larger

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than the Hubble radius). The backreaction of such long-wavelength modes on the background space-time was first studied in [10,11] (making use of the concept of an effective energy-momentum tensor of fluctuations first used in studies of short wavelength gravitational waves [12]). It was found that this effective energy-momentum tensor acts like a negative cosmological constant with a magnitude which increases in time as the phase space of long-wavelength modes grows. A physical explanation of this effect in the quasihomogeneous approximation to the evolution equations was recently provided in [13]. The effect can become nonperturbatively large [11] if the period of inflation is sufficiently long and leads to a change in the Hubble expansion rate. This change is physically measurable in models with at least two matter fields [14]. In models with only one matter field, however, the leading infrared backreaction terms are not physically measurable by a local observer [15–17] (see also [18], and see [19] for a review of previous work on gravitational backreaction in inflationary cosmology¹).

Given that the backreaction of first order cosmological perturbations on the background cosmology can become large, it is important to determine whether this gravitational backreaction can also lead to large effects on the fluctuations themselves, potentially also changing their key characteristics like almost scale-invariance and Gaussianity.

It is the intent of this article to solve the perturbed Einstein equations to quadratic order, and determine the modifications to the results of the linear analysis, focusing on the effects on long-wavelength fluctuations. There has been a significant body of previous work devoted to second order cosmological perturbations, see e.g. [27–32] and papers quoted therein. The effect of long-wavelength modes has also been studied in the “separate universe” approach [18,33,34], in which the effect of long-wavelength perturbations is encoded as a change in the background geometry. Our approach is similar. In particular, we will neglect spatial gradients in the equations of motion, and thus are focusing on the leading infrared contributions to backreaction. What differentiates our analysis from previous work is the emphasis on the fact that, in an inflationary universe, modes continuously move into the infrared sector (wavelengths greater than the Hubble radius), and that thus the infrared phase space grows. This leads to the concern that backreaction effects grow without limits, a concern which is the main motivation for our work.

The main result of our study, however, is that the leading infrared contributions of backreaction to the power spec-

trum of cosmological fluctuations is very small. Assuming that the linear fluctuations have random phases, as they do in the simplest inflationary models, the relative contribution of our backreaction terms to the power spectrum is suppressed by the product of a dimensionless inflationary slow-roll parameter and the amplitude of the linear density fluctuations.

We free ourselves of constraints imposed by any specific inflationary model by abstaining from picking a specific form of the inflaton potential. Rather, we assume that the inflationary slow-roll conditions are satisfied and we employ a very general form of the potential, one that allows us to easily interpret our results within the context of an explicit slow-roll realization. Thus, with regards to cosmological models, our approach is extremely general, while still retaining its ability to be specific.

Section II of this paper is concerned with the general setup of the calculation. Here, the background equations are solved to second order in the slow-roll parameter (ϵ), neglecting spatial gradients, and the corrections to both the background scalar and the Hubble constant are determined, again, to second order in ϵ . Section III begins with a brief review of classical, relativistic, cosmological perturbation theory. The linear perturbation equations are solved using the background obtained in the previous section. The non-linear equations make their appearance in Sec. IV, where the next-to-leading-order terms (those due to backreaction) are solved for. Section V interprets these results as modifications to the linear terms, and their effect on the spectrum of perturbations is determined. Finally, the effects of backreaction on the Gaussianity of the perturbations is investigated in Sec. VI.

II. BACKGROUND

Before solving for the perturbations, we must determine the appropriate background for our model. We confine our attention to a model consisting of pure gravity (with vanishing cosmological constant) and a single, homogeneous, scalar field ϕ , which we presume satisfies the slow-roll conditions. We write

$$\phi(x, t) = \phi_0 + \epsilon f_1(t) + \epsilon^2 f_2(t), \quad (1)$$

where ϕ_0 is a constant and ϵ is a dimensionless slow-roll parameter whose value depends on the specific model in question. In consequence of (1), our background space-time will be approximately de Sitter space,

$$a(t) = e^{H(t)t}. \quad (2)$$

Here, the Hubble rate $H(t)$ is slowly time-dependent and can be expanded as

$$H(t) = H_0 + \epsilon h_1(t) + \epsilon^2 h_2(t), \quad (3)$$

where H_0 is a constant.

The dynamics of the scalar field are determined by its potential $V(\phi)$. In order not to limit our results to any

¹There has been a lot of recent interest in the possibility that the leading gradient terms of long-wavelength modes might back-react on local observables in a way that mimics dark energy [20] (see also [21]). However, objections to this possibility have been raised [22–26].

specific model, but, rather, to show them to be a generic feature of slow-roll inflation, we use the following expansion (related to a power series expansion of the potential about the field value ϕ_0):

$$\begin{aligned} V(\phi) &= \mu\phi_0 + \epsilon\lambda\phi(t) + \epsilon^2\Lambda\phi^2(t) \\ &\equiv V^{(0)} + V^{(1)} + V^{(2)}, \end{aligned} \quad (4)$$

where superscripts indicate the order in ϵ and

$$\mu\phi_0 = V(\phi_0) - \phi_0 V'(\phi_0) + \frac{1}{2}\phi_0^2 V''(\phi_0), \quad (5)$$

$$\epsilon\lambda = V'(\phi_0) - \phi_0 V''(\phi_0) \quad (6)$$

$$\epsilon^2\Lambda = \frac{1}{2}V'''(\phi_0), \quad (7)$$

the primes denoting a derivative with respect to the field. Thus, we see that μ, λ , and Λ are dimensionful parameters depending entirely on the form of the potential in the neighborhood of ϕ_0 . In essence, this represents the series expansion (in ϵ) of any potential that can be used to generate slow-roll inflation.

In light of this, we can use the Klein-Gordon

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{\partial V}{\partial\phi} = 0, \quad (8)$$

and the Friedmann equations

$$3\dot{a}^2 + 4\pi a^2 \dot{\phi}^2 + 8\pi a^2 V = 0, \quad (9)$$

$$\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} + 4\pi\dot{\phi}^2 - 8\pi V = 0 \quad (10)$$

(solved order by order in ϵ) to solve for the background, accurate to second order in the slow-roll parameter.

We find that

$$H_0^2 = 8\pi\mu\phi_0, \quad (11)$$

$$f_1(t) = \frac{\lambda}{3H_0} \left(\frac{1}{3H_0} - t - \frac{e^{-3H_0 t}}{3H_0} \right), \quad (12)$$

$$f_2(t) = \frac{\lambda^2}{18H_0^2\mu} (-2 + 3H_0 t + e^{-3H_0 t}(3H_0 t + 2)), \quad (13)$$

$$h_1(t) = \lambda \sqrt{\frac{2\phi_0}{3\mu\pi}}, \quad (14)$$

and

$$\begin{aligned} h_2(t) &= -\frac{1}{1296\pi\mu^2 H_0^4 t} (288H_0^2\lambda^2 t^2 \pi^2 \mu^2 + 162H_0^5\lambda^2 t \pi \\ &\quad - 243\Lambda H_0^7 t + 16\lambda^2 \pi^2 \mu^2 e^{-6H_0 t} \\ &\quad - 128\lambda^2 \pi^2 \mu^2 e^{-3H_0 t} - 288\lambda^2 \pi^2 \mu^2 H_0 t \\ &\quad + 112\lambda^2 \pi^2 \mu^2). \end{aligned} \quad (15)$$

Note that the corrections to the scalar field have the property that they and their first temporal derivatives vanish at $t = 0$, which coincides with the onset of slow-rolling. Despite the overall factor of t^{-1} preceding $h_2(t)$, this term is not singular at the origin, as can be seen by expanding in a power series about $t = 0$.

III. LINEARIZED THEORY

The theory of cosmological perturbations is usually restricted to an analysis of the linearized (in the amplitude of the perturbations) Einstein equations. In this section, we review the salient results (see e.g. [8,9] for comprehensive reviews).

The most general perturbed line element can be written as

$$\begin{aligned} ds^2 &= a^2(\eta)[(1 + 2\Phi)d\eta^2 - 2(B_{;i} + S_i)dx^i d\eta \\ &\quad - [(1 - 2\Psi)\gamma_{ij} + 2E_{;ij} + (F_{i;j} + F_{j;i}) + h_{ij}] \\ &\quad \times dx^i dx^j] \end{aligned} \quad (16)$$

where Φ, Ψ, B, E represent scalar, S_i, F_i vector, and h_{ij} tensor metric perturbations, respectively. These are distinguished by their transformation properties under three dimensional rotations. For inflationary cosmology and in linear perturbation theory, we can discard vector modes since they do not grow, and tensor modes because they grow at a slower rate than scalar metric fluctuations. Hence, we focus on the scalars. It is possible to choose a gauge in which $E = B = 0$, the so-called ‘‘longitudinal’’ or ‘‘conformal Newtonian’’ gauge. This gauge is convenient for calculational purposes. For matter without anisotropic stresses to linear order in the matter field perturbations, it follows from the off-diagonal space-space Einstein equations that $\Phi = \Psi$. Hence, for such matter the metric in longitudinal gauge takes the form

$$\begin{aligned} ds^2 &= (1 + 2\kappa\Phi(\vec{x}, t))dt^2 - a^2(t)(1 - 2\kappa\Phi(\vec{x}, t)) \\ &\quad \times (dx^2 + dy^2 + dz^2). \end{aligned} \quad (17)$$

In the above, κ is a dimensionless parameter which indicates the order of the term in gravitational perturbation theory.

At second order, the perturbed metric is in general much more complicated. In particular, there is mixing between scalar, vector and tensor modes, and an anisotropic stress is generated, leading to $\Phi \neq \Psi$. However, it can be seen

explicitly that all of the complicating terms contain spatial gradients and can hence be neglected in our study of the backreaction effects of long-wavelength modes. Thus, if we work to leading order in the infrared terms, we can neglect vector, tensor modes and anisotropic stress. Thus, we can apply longitudinal gauge also at second order. We will use this gauge in the following.

To incorporate effects due to slow-rolling, we expand Φ in powers of ϵ to obtain

$$\Phi_1(\vec{x}, t) = \Psi_1(\vec{x}, t) + \epsilon\alpha_1(\vec{x}, t) + \epsilon^2\beta_1(\vec{x}, t). \quad (18)$$

The subscript 1 denotes the fact that the effects are linear in κ .

In order to source our first order metric perturbations, matter perturbations $\delta_1(\vec{x}, t)$ must be present, i.e.

$$\phi(\vec{x}, t) \longrightarrow \phi(t) + \kappa\delta_1(\vec{x}, t), \quad (19)$$

where $\phi(t)$ is the background matter solution given by (1)

At the linear level, perturbations decouple in Fourier space. It is thus convenient to track the evolution of each Fourier mode individually. At higher orders in perturbation

theory, there will be mixing between the modes. Typically, cosmological perturbations are classified into two distinct sectors: sub-Hubble (UV) and super-Hubble (IR) modes. During inflation, the phase space associated with super-Hubble modes grows exponentially, while that of the sub-Hubble modes remains constant. In addition, the modes which are important from the point of view of structure formation and CMB anisotropies are super-Hubble during the last 50 e-foldings of inflation. It is for these reasons that we focus our attention on these modes and ignore their UV counterparts. This choice justifies our dropping of spatial derivatives in the equations of motion.

Using the standard result for the energy-momentum tensor of a scalar field

$$T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - g_{\mu\nu}\left(\frac{\phi^{;\alpha}\phi_{;\alpha}}{2} - V(\phi)\right), \quad (20)$$

expanding the Einstein equations in a power series in κ , and truncating after first order leads to the equations of motion for scalar cosmological perturbations, which read (see e.g. [8])

$$(ii): \quad -\frac{\dot{a}}{a}\dot{\Phi}_1 + 8\pi\dot{\phi}\delta_1 - 2\ddot{\Phi}_1 - 8\frac{\ddot{a}}{a}\Phi_1 - 4\Phi_1\frac{\dot{a}^2}{a^2} - 16\pi\Phi_1(\dot{\phi})^2 - 8\pi V^{(1)}(\phi) + 16\pi V^{(0)}(\phi)\Phi_1 = 0, \quad (21)$$

$$(00): \quad 6\dot{\Phi}_1\frac{\dot{a}}{a} + 8\pi\dot{\phi}\delta_1 + 8\pi V^{(1)}(\phi) + 16\pi V^{(0)}(\phi)\Phi_1 = 0, \quad (22)$$

where the overdot denotes a derivative with respect to cosmic time, and the (ii) and (00) indicate the tensor indices of the Einstein equations in question.

These equations can be solved to yield

$$\Psi_1(\vec{x}, t) = 0, \quad (23)$$

$$\alpha_1(\vec{x}, t) = \psi_1 \int d^3\vec{k} f(\vec{k}) e^{i\vec{k}\cdot\vec{x}} e^{i\alpha_{\vec{k}}} V^{1/2} \quad (24)$$

$$\beta_1(\vec{x}, t) = -\psi_1 \frac{2\Lambda\mu\phi_0 - \lambda^2}{\lambda\mu} (-1 + e^{-H_0 t}) \times \int d^3\vec{k} f(\vec{k}) e^{i\vec{k}\cdot\vec{x}} e^{i\alpha_{\vec{k}}} V^{1/2} \quad (25)$$

$$\delta_1(\vec{x}, t) = -2\frac{\mu\phi_0\psi_1}{\lambda} \int d^3\vec{k} f(\vec{k}) e^{i\vec{k}\cdot\vec{x}} e^{i\alpha_{\vec{k}}} V^{1/2}, \quad (26)$$

where ψ_1 a constant representing the amplitude of the spectrum of linear fluctuations, $f(\vec{k})$ is the function describing the shape of the spectrum, $\alpha_{\vec{k}}$ are the phases (assumed later on to be random), and V is the cutoff volume used in the definition of the Fourier transform. Note that \vec{k} denotes the comoving momentum. We have introduced the cutoff volume in the definition of the Fourier transform in order that the dimension of the

Fourier transform $\tilde{\Phi}(\vec{k})$ of $\Phi(\vec{x})$ is $k^3/2$, which in turn ensures that there is no volume arising in the relation between the power spectrum $P_\Phi(k)$ and $\tilde{\Phi}(\vec{k})$ (see Section 5).

Let us comment briefly on the interpretation of this solution. Note first that the fact that Ψ_1 vanishes is a consistency check on our analysis. In the pure de Sitter limit (no rolling of the scalar field), there are no scalar metric fluctuations to first order in perturbation theory (in κ). Since the equation of motion is second order, there are two fundamental solutions for each mode. In the long wavelength (super-Hubble) limit, the dominant solution is constant in time, and the subdominant solution is a decaying mode. We see that α_1 is the constant perturbation sourced by δ_1 , while β_1 is generated by the rolling of the scalar field and is a combination of the constant and decaying mode.

IV. BACKREACTION

To second order in gravitational perturbation theory (expansion in κ), there are interactions between the Fourier modes of the fluctuation variables caused by the nonlinearities of the Einstein equations. In order to determine the effects of nonlinearities, we make the substitutions for the metric perturbation variable

$$\Phi(\vec{x}, t) \longrightarrow \Phi_1(\vec{x}, t) + \kappa\Phi_2(\vec{x}, t), \quad (27)$$

where Φ_2 can be in turn expanded in terms of the slow-roll parameter ϵ as

$$\Phi_2(\vec{x}, t) = \Psi_2(\vec{x}, t) + \epsilon\alpha_2(\vec{x}, t) + \epsilon^2\beta_2(\vec{x}, t), \quad (28)$$

and for the matter field

$$\phi(\vec{x}, t) \longrightarrow \phi(\vec{x}, t) + \kappa^2\delta_2(\vec{x}, t). \quad (29)$$

Note that Φ_2 and δ_2 represent the effects of gravitational backreaction.

In the following we will neglect spatial gradient terms in the equations of motion, since we are interested in the infrared modes and in the coupling between different infrared modes². To order κ^2 in the perturbative expansion, the Einstein equations then become

$$\begin{aligned} (ii): & 2\ddot{\Phi}_2 + 8\dot{\Phi}_2\frac{\dot{a}}{a} + 8\Phi_2\left(\frac{\ddot{a}}{a} + 2\pi\dot{\phi}^2 + \frac{\dot{a}^2}{2a^2} - 2\pi V^{(2)}(\phi)\right) - 8\pi\dot{\phi}\dot{\delta}_2 \\ & = 4\Phi_1\left(\ddot{\Phi}_1 + 2\Phi_1\frac{\dot{a}^2}{a^2} + 6\dot{\Phi}_1\frac{\dot{a}}{a} + 4\Phi_1\frac{\ddot{a}}{a}\right) + 8\pi\Phi_1\dot{\phi}\dot{\delta}_1 + 4\pi V^{(1)}(\phi)\Phi_1, \end{aligned} \quad (30)$$

$$(00): \quad 6\dot{\Phi}_2\frac{\dot{a}}{a} + 8\pi\dot{\phi}\dot{\delta}_2 - 16\pi V^{(0)}(\phi)\Phi_2 = 3\dot{\Phi}_1^2 - 4\pi\dot{\delta}_1^2 - 12\Phi_1\dot{\Phi}_1\frac{\dot{a}}{a} - 16\pi V^{(1)}(\phi)\Phi_1 - 8\pi V^{(2)}(\phi). \quad (31)$$

These equations can be solved to yield

$$\Psi_2(\vec{x}, t) = 0, \quad (32)$$

$$\alpha_2(\vec{x}, t) = 0, \quad (33)$$

$$\beta_2(\vec{x}, t) = \psi_1^2 \frac{(\Lambda\mu\phi_0 - \lambda^2)}{\lambda^2} (-1 + e^{-H_0 t}) g^2(\vec{x}), \quad (34)$$

$$\delta_2(\vec{x}, t) = 0, \quad (35)$$

with

$$g(\vec{x}) = \int d^3\vec{k} f(\vec{k}) e^{i\vec{k}\cdot\vec{x}} e^{i\alpha_{\vec{k}} V^{1/2}}. \quad (36)$$

Let us briefly comment on the physical interpretation of these results. First, the vanishing of Ψ_2 is a consistency check since there are no scalar metric fluctuations in pure de Sitter space. Since the linear fluctuations are first order in ϵ , they will only contribute to the second order perturbations to quadratic order, and hence the vanishing of α_2 is another consistency check on the algebra. From the expression for Φ_2 it is manifest that the second order perturbations are generated by the linear inhomogeneities Φ_1 and δ_1 at quadratic order. The vanishing of δ_2 is an interesting and unexpected result. It says that, to this order, there are no backreaction effects on the evolution of the background scalar field.

V. EFFECTS OF BACKREACTION ON THE POWER SPECTRUM

Having now determined the form of the backreaction terms, it is important to estimate their amplitude. From observations of CMB anisotropies, we know that the linear perturbations are of order $\kappa\psi_1\epsilon \sim 10^{-5}$. The backreaction terms should be expected to be of order $(\kappa\psi_1\epsilon)^2$. However, the second order correction to a fixed Fourier mode re-

ceives contributions from all linear Fourier modes to this order. Hence, one could expect the backreaction effect to be amplified by a phase-space factor which measures the phase space of Fourier modes which contribute. Since in inflationary cosmology, the phase space of infrared modes is growing, and the linear fluctuations do not decrease in amplitude on scales larger than the Hubble radius, the effects of backreaction could be expected to grow in time and become nonperturbatively large. In this section we show that, provided that the linear fluctuations have random phases, the leading infrared quadratic backreaction effects of linear fluctuations on the power spectrum of Φ do not show any large phase-space enhancement.

The total power spectrum $\mathcal{P}_{\text{total}}(\vec{k})$ of Φ , including the leading infrared terms of second order in κ , can be written as

$$\begin{aligned} \mathcal{P}_{\text{total}}(\vec{k}) &= k^3 |\tilde{\Phi}(\vec{k})|^2 = \mathcal{P}_1(\vec{k}) + \mathcal{P}_2(\vec{k}), \\ &= |\tilde{\Phi}_1(\vec{k}) + \tilde{\Phi}_2(\vec{k})|^2 k^3, \end{aligned} \quad (37)$$

where $\tilde{\Phi}_i(\vec{k})$ are the Fourier transforms (using the definition of Fourier transform including the cutoff volume as in Eq. (24)) of $\Phi_i(\vec{x})$. Making use of the results of Secs. III and IV we have

$$\tilde{\Phi}_1(\vec{k}) = \epsilon\psi_1 f(k) e^{i\alpha_{\vec{k}}} [1 - \epsilon E(t)] \tilde{\Phi}_2(\vec{k}) = \epsilon^2 \psi_1^2 E(t) h(k), \quad (38)$$

where $h(k)$ is the Fourier transform of $g^2(x)$ and where we have introduced the symbol $E(t)$ for the function

$$E(t) \equiv \frac{2\Lambda\mu\phi_0 - \lambda^2}{\lambda^2} (-1 + e^{-H_0 t}). \quad (39)$$

²It is also in this approximation that we can justify writing the perturbed metric in the form (17)—see e.g. [18] for a discussion of this point.

The leading backreaction correction to the power spectrum, denoted by the function $\mathcal{P}_{BR}(\vec{k})$, comes from the cross term in (37) and is thus linear in $\tilde{\Phi}_2(\vec{k})$:

$$\mathcal{P}_{BR}(\vec{k}) = 2\kappa |\tilde{\Phi}_1(\vec{k})\tilde{\Phi}_2(\vec{k})|k^3. \quad (40)$$

Thus, the fractional correction to the power spectrum due to the leading backreaction contributions is

$$\frac{\mathcal{P}_{BR}(k)}{\mathcal{P}_1(k)} = 2\kappa \frac{\epsilon^2 \psi_1^2 E(t) h(k)}{\epsilon \psi_1 f(k) + \mathcal{O}(\epsilon^2)} \simeq 2\kappa \epsilon \psi_1 \frac{h(k)}{f(k)}, \quad (41)$$

from which it follows that, modulo the ratio of $h(k)$ over $f(k)$, the backreaction terms are suppressed, as expected, by $\kappa \epsilon \psi_1$. The ratio of $h(k)$ over $f(k)$ is the possible large phase-space enhancement factor.

Before continuing, we specify the linear power spectrum. We choose a normalization wavenumber k_n and choose ψ_1^2 to be the amplitude of the power spectrum at $k = k_n$. The function $f(k)$ describes the spectral shape. We choose a power law with a tilt ζ away from scale-invariance, i.e. we write

$$f(k) = \left(\frac{k}{k_n}\right)^{-3/2-\zeta} k_n^{-3/2}. \quad (42)$$

It can easily be checked that $\mathcal{P}_1(k_n) = \psi_1^2$.

We now evaluate the magnitude of $h(k)$, assuming that the phases $\alpha_{\vec{k}}$ are random:

$$\begin{aligned} h(\vec{k}) &= \frac{1}{(2\pi)^3} V^{-1/2} \int d^3x g^2(x) e^{-i\vec{k}\cdot\vec{x}} \\ &= \int d^3k_1 f(\vec{k}_1) f(\vec{k} - \vec{k}_1) e^{i(\alpha_{\vec{k}_1} + \alpha_{\vec{k}-\vec{k}_1})} V^{1/2}. \end{aligned} \quad (43)$$

Given that we are considering the effects of long-wavelength fluctuations, we must restrict the above integral over \vec{k}_1 to run only over super-Hubble modes, i.e.

$$|\vec{k}_1| \leq H. \quad (44)$$

To estimate the magnitude of $h(\vec{k})$, we insert the spectrum (42) into (43). If we consider the effects of backreaction on modes \vec{k} which are sub-Hubble now, we can apply the approximation

$$\left(\frac{k - k_1}{k_n}\right) \simeq \frac{k}{k_n}, \quad (45)$$

in which case the integral simplifies.

Assuming constant phases for the moment, the integral (43) can be easily estimated

$$h(k) \sim k^{-3/2-\zeta} k_n^{2\zeta} H^{3/2-\zeta} V^{1/2}. \quad (46)$$

Note, in particular, from (46) that the k -dependence of $\mathcal{P}_{BR}(k)$ is the same as that of the linear power spectrum. The leading effect of backreaction thus does not change the power index of the spectrum. However, for wavelengths close to the Hubble radius, the approximation (45) is no

longer good, and the correction terms will yield changes to the index of the power spectrum. The second fact to notice about the result (46) is the cutoff volume divergence. This stems from the fact that as V increases, more and more infrared modes are contributing to the backreaction. For constant phases, the effect is additive. The volume divergence thus represents the phase-space enhancement which is the focus of this investigation.

Let us now consider the more realistic situation - realized in typical inflationary models—in which the phases are random. A simple way to estimate the effects of the random phases in (43) is to add up the amplitudes of the backreaction contributions of all infrared modes \vec{k}_1 as a random walk. This means dividing the amplitude obtained previously by $N(V)^{1/2}$, where $N(V)$ is the number of modes. Since for a finite volume V the wavenumbers are quantized in units of $\Delta k \sim V^{-1/3}$, the number $N(V)$ becomes

$$N(V) \sim \left(\frac{H}{\Delta k}\right)^3 \quad (47)$$

in which case the result (43) becomes

$$h(k) \sim k^{-3/2-\zeta} k_n^{2\zeta} H^{-\zeta}. \quad (48)$$

Inserting this into (41), we obtain our final result

$$\frac{\mathcal{P}_{BR}(k)}{\mathcal{P}_1(k)} \sim 2\kappa \epsilon \psi_1 \left(\frac{k_n}{H}\right)^\zeta. \quad (49)$$

The main conclusion we draw from (49) is that there is no phase-space enhancement of the backreaction of long-wavelength modes on the spectrum of cosmological perturbations, in contrast to the positive enhancement found for the backreaction on the background metric. Given the absence of such a phase-space enhancement, we find—as expected—that the backreaction terms in the power spectrum are suppressed compared to the terms coming from the linear perturbations by $\kappa \epsilon \psi_1$. Thus, they are completely negligible in the case of a COBE-normalized spectrum of almost scale-invariant linear fluctuations. In addition, we find that the leading backreaction terms do not change the spectral index.

VI. NON-GAUSSIANITY OF THE SPECTRUM DUE TO BACKREACTION

Having established that the backreaction of infrared modes cannot substantially modify the amplitude and spectral tilt of the power spectrum of perturbations, we make some comments regarding the effects of backreaction on the Gaussianity of the spectrum.

The inclusion of higher-order terms implies correlations between different modes, thus breaking strict Gaussianity. However, the question remains: how badly broken is it? To estimate this, we turn our attention to the bispectrum (three-point function).

In the case of a purely Gaussian distribution, all odd moments are identically zero. Therefore, the nonvanishing of the bispectrum indicates that the distribution cannot be Gaussian.

We take it for granted that the bispectrum does not vanish (for examples of the three-point function see [32,35]). In the context of higher-order perturbation theory (see e.g. [28]), however, its amplitude is exceedingly small. We estimate it to be no larger than of order $\kappa^4\epsilon$, thus making it quite unlikely to be detected experimentally. Thus, we conclude that, although backreaction modifies the distribution, Gaussianity remains an excellent approximation.

VII. CONCLUSIONS

In this paper, we have studied the backreaction of long-wavelength linear fluctuations on the power spectrum, produced by the mode mixing which occurs as a consequence of the nonlinearity of the Einstein equations. We find that, assuming that the phases of the linear fluctuations are random, there is no phase-space enhancement of the backreaction effect. The leading infrared backreaction contributions are suppressed by $\kappa\epsilon\psi_1$ compared to the contribution of the linear fluctuations, where $\kappa\psi_1$ is a measure of the amplitude of the linearized metric fluctuations, and ϵ is an inflationary slow-roll parameter. These leading backreaction terms do not modify the tilt of the power spectrum on scales substantially smaller than the Hubble radius. Note that in the case of correlated phases of the linear fluctuations, a much larger backreaction effect is possible.

We have also seen that the modifications to the Gaussianity of the leading-order perturbations are negligible. The small size of these modifications goes a long way towards justifying the linear approximation to cosmological perturbation theory. However, our results do not exclude the possibility that large amplitude local fluctuations can effect the measured fluctuations, as very recently suggested in [36] (based on the second order formalism developed in [37])³.

Our work differs from previous work on second order fluctuations in that it emphasizes the fact that, in an accelerating universe, the phase space of super-Hubble modes is increasing in time. In contrast to what occurs in the case of the backreaction on the homogeneous mode, the backreaction on the fluctuating modes themselves does not increase without limits as a function of time. Compared to previous analyses, our work also gives an easier way to derive the leading-order effects of long-wavelength cosmological fluctuations. Our results have been derived in the context of an arbitrary slow-roll inflationary model and are thus valid for a wide range of cosmological scenarios.

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³The possibility that local fluctuations can have a measurable effect on background quantities such as the deceleration parameter has recently been suggested in [38,39].

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