Gribov parameter and the dimension two gluon condensate in Euclidean Yang-Mills theories in the Landau gauge

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The local composite operator A^2_μ is added to the Zwanziger action, which implements the restriction to the Gribov region Ω in Euclidean Yang-Mills theories in the Landau gauge. We prove that Zwanziger's action with the inclusion of the operator A^2_μ is renormalizable to all orders of perturbation theory, obeying the renormalization group equations. This allows us to study the dimension two gluon condensate $\langle A^2_\mu \rangle$ by the local composite operator formalism when the restriction to the Gribov region Ω is taken into account. The resulting effective action is evaluated at one-loop order in the $\overline{\text{MS}}$ scheme. We obtain explicit values for the Gribov parameter and for the mass parameter due to $\langle A^2_\mu \rangle$, but the expansion parameter turns out to be rather large. Furthermore, an optimization of the perturbative expansion in order to reduce the dependence on the renormalization scheme is performed. The properties of the vacuum energy, with or without the inclusion of the condensate $\langle A^2_\mu \rangle$, are investigated. In particular, it is shown that in the original Gribov-Zwanziger formulation, i.e. without the inclusion of the operator A^2_μ, the resulting vacuum energy is always positive at one-loop order, independently from the choice of the renormalization scheme and scale. In the presence of $\langle A^2_\mu \rangle$, we are unable to come to a definite conclusion at the order considered. In the $\overline{\text{MS}}$ scheme, we still find a positive vacuum energy, again with a relatively large expansion parameter, but there are renormalization schemes in which the vacuum energy is negative, albeit the dependence on the scheme itself appears to be strong. Concerning the behavior of the gluon and ghost propagators, we recover the well-known consequences of the restriction to the Gribov region, and this in the presence of $\langle A^2_\mu \rangle$, i.e. an infrared suppression of the gluon propagator and an enhancement of the ghost propagator. Such a behavior is in qualitative agreement with the results obtained from the studies of the Schwinger-Dyson equations and from lattice simulations.

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I. INTRODUCTION

The dimension two condensate $\langle A^2_\mu \rangle$ has received a great deal of attention in the last few years, see for example [1–17]. This condensate was already introduced in [18] in order to analyze the gluon propagator within the operator product expansion (OPE), while in [19] the condensate $\langle A^2_i \rangle$ was considered in the Coulomb gauge. A renormalizable effective potential for $\langle A^2_\mu \rangle$ has been constructed and evaluated in analytic form up to two-loop order in the Landau gauge within the local composite operator (LCO) formalism in [3,12]. The output of these investigations is that a nonvanishing condensate is favored as it lowers the vacuum energy. The renormalizability of the local composite operator formalism, see [20] for an introduction to the method, was proven to all orders of perturbation theory, in the case of $\langle A^2_\mu \rangle$, in [11] using the algebraic renormalization technique [21]. Besides the Landau gauge, the method was extended to other gauges as, for instance, the Curci-Ferrari gauge [22,23], the linear covariant gauges [24,25] and, more recently, the maximal Abelian gauge [26].

As a consequence of the existence of a nonvanishing condensate $\langle A^2_\mu \rangle$, a dynamical mass parameter for the gluons can be generated in the gauge fixed Lagrangian, see [3,12,25]. We mention that a gluon mass has been proven to be rather useful in the phenomenological context, see e.g. [27–29]. Moreover, mass parameters are commonly used in the fitting formulas for the data obtained in lattice simulations, where the gluon propagator has been studied to a great extent in the Landau gauge [30–36].

The lattice results obtained so far have provided firm evidence of the suppression of the gluon propagator in the infrared region, in the Landau gauge. Next to the gluon propagator, the ghost propagator has also been investigated numerically on the lattice [34–38], exhibiting an infrared enhancement. It is worth remarking that, in agreement with lattice results, this infrared behavior of the gluon as well as of the ghost propagator has been obtained in the analysis of the Schwinger-Dyson equations, see [39–46], as well as in a study making use of the exact renormalizaton group technique [47].

The aim of the present work is to further investigate the condensation of the operator A^2_μ in the Landau gauge using the local composite operator formalism. This will be done by taking into account the nonperturbative effects related to the existence of the Gribov ambiguities [48], which are known to affect the Landau gauge fixing condition,

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$\partial_\mu A^a_\mu = 0$. As a consequence of the existence of the Gribov copies, the domain of integration in the path integral has to be restricted in a suitable way. Gribov's original proposal was to restrict the domain of integration to the region Ω whose boundary $\partial\Omega$ is the first Gribov horizon, where the first vanishing eigenvalue of the Faddeev-Popov operator, $-\partial_\mu(\partial_\mu\delta^{ab} + gf^{acb}A^c_\mu)$, appears [48]. Within the region Ω the Faddeev-Popov operator is positive definite, i.e. $-\partial_\mu(\partial_\mu\delta^{ab} + gf^{acb}A^c_\mu) > 0$. One of the main results of Gribov's work [48] was that the gluon, respectively, ghost propagator, got suppressed, respectively, enhanced, in the infrared due to the restriction to the region Ω.

In two previous papers [49,50], we have already worked out the consequences of the restriction to the Gribov region Ω when the dynamical generation of a gluon mass parameter due to $\langle A^2_\mu\rangle$ takes place, also finding an infrared suppression of the gluon and an enhancement of the ghost propagator. In [49], we closely followed the setup of Gribov's paper [48]. In this work, we shall rely on the Zwanziger local formulation of the Gribov horizon. In a series of papers [51,52], Zwanziger has been able to implement the restriction to the Gribov region Ω through the introduction of a nonlocal horizon function appearing in the Boltzmann weight defining the Euclidean Yang-Mills measure. More precisely, according to [51,52], the starting Yang-Mills measure in the Landau gauge is given by

$$d\mu_\gamma = DA\delta(\partial_\mu A^a_\mu)\det(\mathcal{M})e^{-(S_{\rm YM}+\gamma^4 H)}, \tag{1.1}$$

where

$$\mathcal{M}^{ab} = -\partial_\mu(\partial_\mu\delta^{ab} + gf^{acb}A^c_\mu), \tag{1.2}$$

$$S_{\rm YM} = \frac{1}{4}\int d^4x F^a_{\mu\nu}F^a_{\mu\nu}, \tag{1.3}$$

and

$$H = \int d^4x h(x) = g^2\int d^4x f^{abc}A^b_\mu(\mathcal{M}^{-1})^{ad}f^{dec}A^e_\mu, \tag{1.4}$$

is the so-called horizon function, which implements the restriction to the Gribov region. Notice that H is nonlocal. The parameter γ, known as the Gribov parameter, has the dimension of a mass and is not free, being determined by the horizon condition

$$\langle h(x)\rangle = 4(N^2-1), \tag{1.5}$$

where the expectation value $\langle h(x)\rangle$ has to be evaluated with the measure $d\mu_\gamma$. To the first order, the horizon condition (1.5) reads, in d dimensions,

$$1 = \frac{N(d-1)}{4}g^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^4 + 2Ng^2\gamma^4}. \tag{1.6}$$

This equation coincides with the original gap equation derived by Gribov for the parameter γ [48].

Albeit nonlocal, the horizon function H can be localized through the introduction of a suitable set of additional fields. As shown in [51–53], the resulting local action turns out to be renormalizable to all orders of perturbation theory. Remarkably, we shall be able to prove that this feature is preserved when the local operator A^2_μ is introduced in the Zwanziger action. Moreover, the resulting theory turns out to obey a homogeneous renormalization group equation. These important properties will allow us to study the condensation of the operator A^2_μ within a local renormalizable framework when the restriction to the Gribov region Ω is implemented.

It is worth remarking that the Gribov region is not free from gauge copies [54–57], i.e. Gribov copies still exist inside Ω. To avoid the presence of these additional copies, a further restriction to a smaller region Λ, known as the fundamental modular region, should be implemented. At present, a clear understanding of the role played by these additional copies appears to be a very difficult task. Nevertheless, we should mention that, recently, it has been argued in [58] that the additional copies existing inside Ω could have no influence on the expectation values, so that averages calculated over Λ or Ω might give the same value.

The paper is organized as follows. In Sec. II, we give a short account of how the nonlocal horizon functional H can be localized by means of the introduction of additional fields. In Sec. III, we prove the renormalizability, to all orders of perturbation theory, of Zwanziger's action in the presence of the operator A^2_μ, introduced through the local composite operator formalism. As the model has a rich symmetry structure, translated into several Ward identities, it turns out that only three independent renormalization factors are necessary. The resulting quantum effective action obeys a homogeneous renormalization group equation, as explicitly verified at one-loop order. From this effective action, two coupled gap equations, associated to the condensate $\langle A^2_\mu\rangle$ and to the Gribov parameter γ, are derived. Section IV is devoted to the study of these gap equations at one-loop order in the $\overline{\rm MS}$ renormalization scheme. It is worth mentioning that, under certain conditions, we find that it is possible that the condensate $\langle A^2_\mu\rangle$ is positive when the horizon condition is imposed. We prove that in the $\overline{\rm MS}$ scheme, and at one-loop order, the solution of the gap equations is necessarily one with $\langle A^2_\mu\rangle > 0$. We recall that without the restriction to the Gribov region Ω, the value found for $\langle A^2_\mu\rangle$ using the local composite operator formalism is negative, see [3,12,25]. Let us also mention here that in [6–9], a positive estimate for $\langle A^2_\mu\rangle$ was obtained when using the OPE in combination with $\langle A^2_\mu\rangle$. These works were based on the observation of a certain discrepancy at relatively large momentum between the expected perturbative behavior and the obtained lattice

behavior of e.g. the effective strong coupling constant and gluon propagator. This discrepancy could be accounted for by power corrections in $1/q^2$, due to a positive $\langle A_\mu^2 \rangle_{\rm OPE}$ gluon condensate. The presence of such power corrections has also been discussed in [59]. We do not know if there is a direct connection between the condensate $\langle A_\mu^2 \rangle$ that we determine, and $\langle A_\mu^2 \rangle_{\rm OPE}$, as the latter is expected to contain only infrared contributions, according to an OPE treatment, while the gap equations fixing the gluon condensate and the Gribov parameter are evaluated using perturbation theory, implying that reliable results are only to be expected at a sufficiently large scale.

Although the expansion parameter proves to be rather large, an attempt to obtain explicit values for the Gribov and gluon mass parameter is still presented. Also, we shall prove that in the original Gribov-Zwanziger model, the vacuum energy is always positive at one-loop order, irrespective of the choice of renormalization scheme and scale. We outline the importance of the sign of the vacuum energy, as it is related to the gauge invariant gluon condensate $\langle F_{\mu\nu}^2 \rangle$, via the trace anomaly. From

$$\theta_{\mu\mu} = \frac{\beta(g^2)}{2g^2} F_{\mu\nu}^2, \tag{1.7}$$

the vacuum energy can be traced back to the value of the gluon condensate $\langle F_{\mu\nu}^2 \rangle$. In particular, for $N = 3$, from this anomaly one deduces

$$\left\langle \frac{g^2}{4\pi^2} F_{\mu\nu}^2 \right\rangle = -\frac{32}{11} E_{\rm vac}, \tag{1.8}$$

where the one-loop β-function has been used. Hence, a positive vacuum energy implies a negative value for the condensate $\langle g^2/(4\pi^2) F_{\mu\nu}^2 \rangle$. This is in contradiction with what is found. In QCD, with quarks present, one can extract phenomenological values for $\langle g^2/(4\pi^2) F_{\mu\nu}^2 \rangle$ via the sum rules [60], obtaining positive values for this condensate. It was discussed in [61] how to obtain an estimate for it by means of lattice calculations. In the case of $N = 3$ Yang-Mills theory without quarks, it was found that

$$\left\langle \frac{g^2}{4\pi^2} F_{\mu\nu}^2 \right\rangle = 0.14 \pm 0.02 \ {\rm GeV}^4. \tag{1.9}$$

Let us mention here that the Yang-Mills β-function is negative up to the (known) four-loop order [62–64]. Hence, $E_{\rm vac}$ and $\langle g^2/(4\pi^2) F_{\mu\nu}^2 \rangle$ will continue to have opposite sign at higher order. From this viewpoint, it seems to us that it would be an asset that the vacuum energy obtained from any kind of calculation is at least negative.

In Sec. V we present an optimized expansion in order to reduce the dependence on the choice of renormalization scheme to a single parameter b_0, related to the coupling constant renormalization. This is achieved by exchanging the mass parameters by their renormalization scale and scheme invariant counterparts and by re-expanding the series in the one-loop coupling constant. For $b_0 = 0$, which corresponds to the $\overline{\rm MS}$ scheme, we find a positive $\langle A_\mu^2 \rangle$, positive $E_{\rm vac}$ and hence negative $\langle F_{\mu\nu}^2 \rangle$. However, we find that a region of b_0 is existing in which the vacuum energy is negative, but unfortunately the dependence on b_0 in this region happens to be very large. A higher order analysis seems to be required to reach more definite conclusions about the sign of $\langle A_\mu^2 \rangle$, $E_{\rm vac}$ or $\langle F_{\mu\nu}^2 \rangle$.

For the benefit of the reader, we provide in Sec. VI an overview of some important consequences stemming from the presence of the Gribov and gluon mass parameters on the gluon and ghost propagators. We point out a particular renormalization property of the Zwanziger action in order to ensure the enhancement of the ghost propagator. Conclusions are written down in Sec. VII, while the technical details of our work have been collected in the Appendices A and B.

II. LOCAL ACTION FROM THE RESTRICTION TO THE GRIBOV REGION

As explained in [51,52], the nonlocal functional H can be localized by means of the introduction of a suitable set of additional ghost fields. More precisely, for the localized version of the measure $d\mu_\gamma$ we get,

$$d\mu_\gamma = DADbDcD\bar{c}D\varphi D\bar{\varphi}D\omega D\bar{\omega} e^{-S}, \tag{2.1}$$

where S is given by[1]

$$S = S_0 - \gamma^2 g \int d^4x (f^{abc} A_\mu^a \varphi_\mu^{bc} + f^{abc} A_\mu^a \bar{\varphi}_\mu^{bc}), \tag{2.2}$$

while

$$\begin{aligned} S_0 = S_{\rm YM} &+ \int d^4x (b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu (D_\mu c)^a) \\ &+ \int d^4x (\bar{\varphi}_\mu^{ac} \partial_\nu (\partial_\nu \varphi_\mu^{ac} + g f^{abm} A_\nu^b \varphi_\mu^{mc}) \\ &- \bar{\omega}_\mu^{ac} \partial_\nu (\partial_\nu \omega_\mu^{ac} + g f^{abm} A_\nu^b \omega_\mu^{mc}) \\ &- g (\partial_\nu \bar{\omega}_\mu^{ac}) f^{abm} (D_\nu c)^b \varphi_\mu^{mc}). \end{aligned} \tag{2.3}$$

The fields $(\bar{\varphi}_\mu^{ac}, \varphi_\mu^{ac})$ are a pair of complex conjugate bosonic fields. Each field has $4(N^2-1)^2$ components. Similarly, the fields $(\bar{\omega}_\mu^{ac}, \omega_\mu^{ac})$ are anticommuting. The local action (2.2) is renormalizable by power counting. More precisely, it has been shown in [51–53] that the Green functions obtained with the action S_0 with the insertion of the local composite operators $f^{abc} A_\mu^a \varphi_\mu^{bc}$ and $f^{abc} A_\mu^a \bar{\varphi}_\mu^{bc}$ are renormalizable, the action S_0 being indeed renormalizable by a multiplicative renormalization of the coupling constant g and of the fields [51–53]. We remark that the action S_0 displays a global $U(f)$ symmetry, $f = 4(N^2-1)$, with respect to the composite index

[1]Our conventions are different from those originally used by Zwanziger. These can be obtained from ours by setting $\varphi \to -\varphi$ and $\omega \to -\omega$.

$i = (\mu, c) = 1, \ldots, f$, of the additional fields $(\overline{\varphi}^{ac}_{\mu}, \varphi^{ac}_{\mu}, \overline{\omega}^{ac}_{\mu}, \omega^{ac}_{\mu})$. Setting

$$(\overline{\varphi}^{ac}_{\mu}, \varphi^{ac}_{\mu}, \overline{\omega}^{ac}_{\mu}, \omega^{ac}_{\mu}) = (\overline{\varphi}^{a}_{i}, \varphi^{a}_{i}, \overline{\omega}^{a}_{i}, \omega^{a}_{i}), \tag{2.4}$$

we get

$$\begin{aligned} S_0 = S_{\rm YM} + \int d^4x (b^a \partial_\mu A^a_\mu + \overline{c}^a \partial_\mu (D_\mu c)^a) \\ + \int d^4x (\overline{\varphi}^a_i \partial_\nu (D_\nu \varphi_i)^a - \overline{\omega}^a_i \partial_\nu (D_\nu \omega_i)^a \\ - g(\partial_\nu \overline{\omega}^a_i) f^{abm} (D_\nu c)^b \varphi^m_i)). \end{aligned} \tag{2.5}$$

For the $U(f)$ invariance we have

$$U_{ij} S_0 = 0,$$

$$U_{ij} = \int d^4x \left(\varphi^a_i \frac{\delta}{\delta \varphi^a_j} - \overline{\varphi}^a_j \frac{\delta}{\delta \overline{\varphi}^a_i} + \omega^a_i \frac{\delta}{\delta \omega^a_j} - \overline{\omega}^a_j \frac{\delta}{\delta \overline{\omega}^a_i} \right). \tag{2.6}$$

The presence of the global $U(f)$ invariance means that one can make use of the composite index $i = (\mu, c)$. By means of the diagonal operator $Q_f = U_{ii}$, the i-valued fields turn out to possess an additional quantum number. As shown in [51–53], the action S_0 is left invariant by the following nilpotent BRST transformations,

$$\begin{aligned} sA^a_\mu = -(D_\mu c)^a, \qquad sc^a = \frac{1}{2} g f^{abc} c^b c^c, \qquad s\overline{c}^a = b^a, \\ sb^a = 0, \qquad s\varphi^a_i = \omega^a_i, \qquad s\omega^a_i = 0, \\ s\overline{\omega}^a_i = \overline{\varphi}^a_i, \qquad s\overline{\varphi}^a_i = 0, \end{aligned} \tag{2.7}$$

with

$$sS_0 = 0. \tag{2.8}$$

For further use, the quantum numbers of all fields entering the action S_0 are displayed in Table I. It is worth noticing that, when $f^{abc} A^a_\mu \varphi^{bc}_\mu$ and $f^{abc} A^a_\mu \overline{\varphi}^{bc}_\mu$ are treated as composite operators, they are introduced in the starting action S_0 coupled to local external sources M^{ai}_μ, V^{ai}_μ, namely

$$-\int d^4x (M^{ai}_\mu (D_\mu \varphi_i)^a + V^{ai}_\mu (D_\mu \overline{\varphi}_i)^a). \tag{2.9}$$

The horizon condition (1.5) is thus obtained from the quantum action by requiring that, at the end of the computation, the sources M^{ai}_μ, V^{ai}_μ attain the physical values,

TABLE I. Quantum numbers of the fields.

	A^a_μ	c^a	$\overline{c}^a$	b^a	φ^a_i	$\overline{\varphi}^a_i$	ω^a_i	$\overline{\omega}^a_i$
dimension	1	0	2	2	1	1	1	1
ghostnumber	0	1	−1	0	0	0	1	−1
Q_f − charge	0	0	0	0	1	−1	1	1

TABLE II. Quantum numbers of the sources.

	U^{ai}_μ	M^{ai}_μ	N^{ai}_μ	V^{ai}_μ	η	τ	K^a_μ	L^a
dimension	2	2	2	2	2	2	3	4
ghostnumber	−1	0	1	0	−1	0	−1	−2
Q_f − charge	−1	−1	1	1	0	0	0	0

obtained by setting

$$M^{ab}_{\mu\nu} = V^{ab}_{\mu\nu} = \gamma^2 \delta^{ab} \delta_{\mu\nu}. \tag{2.10}$$

Indeed, expression (2.9) reduces precisely to that of Eq. (2.2) when the sources M^{ai}_μ, V^{ai}_μ attain their physical value.

III. RENORMALIZATION OF THE ZWANZIGER ACTION IN THE PRESENCE OF THE COMPOSITE OPERATOR $A^a_\mu A^a_\mu$

The purpose of this section is to show that the renormalizability of the local action $S_0$is preserved when, besides the operators $f^{abc} A^a_\mu \varphi^{bc}_\mu$ and $f^{abc} A^a_\mu \overline{\varphi}^{bc}_\mu$, also the local composite operator $A^a_\mu A^a_\mu$ is introduced. This is a remarkable feature of the Zwanziger action, allowing us to discuss the condensation of the operator $A^a_\mu A^a_\mu$ when the restriction to the Gribov region Ω is implemented. To discuss the renormalizability of the model in the presence of A^2_μ, we start from the following complete action

$$\Sigma = S_0 + S_{\rm s} + S_{\rm ext}, \tag{3.1}$$

where $S_{\rm s}$ is the term containing all needed local composite operators with their respective local sources, and is given by

$$\begin{aligned} S_{\rm s} = s \int d^4x \Big(-U^{ai}_\mu (D_\mu \varphi_i)^a - V^{ai}_\mu (D_\mu \overline{\omega}_i)^a - U^{ai}_\mu V^{ai}_\mu \\ + \frac{1}{2} \eta A^a_\mu A^a_\mu - \frac{1}{2} \zeta \tau \eta \Big), \end{aligned} \tag{3.2}$$

where the BRST operator acts as

$$\begin{aligned} sU^{ai}_\mu = M^{ai}_\mu, \qquad sM^{ai}_\mu = 0, \\ sV^{ai}_\mu = N^{ai}_\mu, \qquad sN^{ai}_\mu = 0, \end{aligned} \tag{3.3}$$

and

$$s\eta = \tau, \qquad s\tau = 0. \tag{3.4}$$

Therefore, for $S_{\rm s}$ one gets

$$S_{\rm s} = \int d^4x \Big(-M^{ai}_\mu (D_\mu \varphi_i)^a - g U^{ai}_\mu f^{abc} (D_\mu c)^b \varphi^c_i + U^{ai}_\mu (D_\mu \omega_i)^b - N^{ai}_\mu (D_\mu \overline{\omega}_i)^a - V^{ai}_\mu (D_\mu \overline{\varphi}_i)^a + g V^{ai}_\mu f^{abc} (D_\mu c)^b \overline{\omega}^c_i - M^{ai}_\mu V^{ai}_\mu + U^{ai}_\mu N^{ai}_\mu + \frac{1}{2}\tau A^a_\mu A^a_\mu + \eta A^a_\mu \partial_\mu c^a - \frac{1}{2}\zeta \tau^2 \Big). \tag{3.5}$$

As already noticed, the sources M^{ai}_μ, V^{ai}_μ are needed to introduce the composite operators $(D_\mu \varphi_i)^a$ and $(D_\mu \overline{\varphi}_i)^a$. The sources U^{ai}_μ, N^{ai}_μ define the BRST variations of these operators, given by $(D_\mu \omega_i)^b$ and $(D_\mu \overline{\omega}_i)^a$. The physical value of these sources is given by

$$M^{ab}_{\mu\nu} = V^{ab}_{\mu\nu} = \gamma^2 \delta^{ab} \delta_{\mu\nu}, \qquad U^{ab}_{\mu\nu} = N^{ab}_{\mu\nu} = 0. \tag{3.6}$$

The local composite operator $A^a_\mu A^a_\mu$ and its BRST variation, $A^a_\mu \partial_\mu c^a$, are then introduced by means of the local sources τ, η. We also notice that the complete action Σ contains terms quadratic in the external sources, namely $(M^{ai}_\mu V^{ai}_\mu - U^{ai}_\mu N^{ai}_\mu)$ and $\zeta\tau^2$. These terms, allowed by power counting, are in fact needed for the multiplicative renormalizability of the model. As shown in [3], the dimensionless LCO parameter ζ of the quadratic term in the source τ is needed to account for the divergences present in the correlation function $\langle A^2_\mu(x) A^2_\nu(y)\rangle$ for $x \to y$. It should be remarked that, unlike for the term quadratic in the external source τ, we have not introduced a new free parameter for the quadratic term $(M^{ai}_\mu V^{ai}_\mu - U^{ai}_\mu N^{ai}_\mu)$ in expression (3.5). As we shall see, this term goes through the renormalization without the need of introducing a new parameter for its renormalizability. This is a remarkable feature of the Zwanziger action which plays an important role when the ghost propagator in the presence of the Gribov horizon will be discussed, see Sec. VI.

Finally, the term $S_{\rm ext}$ is the source term needed to define the nonlinear BRST transformations of the gauge and ghost fields, i.e.

$$S_{\rm ext} = \int d^4x \Big(-K^a_\mu (D_\mu c)^a + \frac{1}{2} g L^a f^{abc} c^b c^c \Big). \tag{3.7}$$

The technical details concerning the algebraic renormalization procedure have been worked out in Appendix A. The quantum numbers of the sources are enlisted in Table II. In summary, the Zwanziger action in the presence of the local operator $A^a_\mu A^a_\mu$ is multiplicative renormalizable. In turn, this ensures that the quantum effective action obeys the homogeneous renormalization group equations (RGE). This is an important feature of the model, which will be useful when we shall try to obtain estimates for both the Gribov and mass parameter.

The effective action is defined upon setting the sources $U^{ab}_{\mu\nu}$, $N^{ab}_{\mu\nu}$, K^a_μ, L^a and η equal to zero and implementing the condition (2.10). Doing so, we get

$$S = S_0 + S_\gamma + \int d^4x \Big(\frac{\tau}{2} A^a_\mu A^a_\mu - \frac{\zeta}{2}\tau^2 \Big),$$
$$S_\gamma = \int d^4x [-\gamma^2 g f^{abc} A^a_\mu \varphi^{bc}_\mu - \gamma^2 g f^{abc} A^a_\mu \overline{\varphi}^{bc}_\mu - 4(N^2-1)\gamma^4]. \tag{3.8}$$

The term $-4(N^2-1)\gamma^4$ originates from the quadratic term in the external sources, namely $(-M^{ai}_\mu V^{ai}_\mu + U^{ai}_\mu N^{ai}_\mu)$, in expression (3.5), evaluated at the physical values given by Eq. (2.10). Following [3,12,20,25], we introduce a Hubbard-Stratonovich field σ by means of the following unity

$$1 = \int [d\sigma] e^{-(1/2\zeta)\int d^4x [(\sigma/g) + 1/2 A^a_\mu A^a_\mu - \zeta\tau]^2}, \tag{3.9}$$

to remove the term proportional to τ^2. The source τ is henceforth linearly coupled to the field σ, as can be directly seen from the action, which now reads

$$S = S_0 + S_\gamma + S_\sigma + \int d^4x \Big(-\tau \frac{\sigma}{g} \Big),$$
$$S_\sigma = \frac{\sigma^2}{2g^2\zeta} + \frac{1}{2}\frac{g\sigma}{g^2\zeta} A^a_\mu A^a_\mu + \frac{1}{8\zeta}(A^a_\mu A^a_\mu)^2. \tag{3.10}$$

The following identification is easily derived [3,12,20,25]

$$\langle A^a_\mu A^a_\mu \rangle = -\frac{1}{g}\langle \sigma \rangle, \tag{3.11}$$

from which it follows that a nonvanishing vacuum expectation value of the field σ will result in a nonvanishing condensate $\langle A^a_\mu A^a_\mu \rangle$. The quantum action Γ is obtained through the definition

$$e^{-\Gamma} = \int [d\Phi] e^{-S_0 - S_\gamma - S_\sigma}, \tag{3.12}$$

where Φ is a shorthanded notation for all the relevant fields.

The value for $\langle \sigma \rangle$ is found through the minimization condition

$$\frac{\partial \Gamma}{\partial \sigma} = 0. \tag{3.13}$$

The horizon is implemented by the condition [51,52].

$$\frac{\partial \Gamma}{\partial \gamma^2} = 0. \tag{3.14}$$

Let us show this here. The following equivalence is readily found

$$\frac{\partial \Gamma}{\partial \gamma^2} = 0 \Leftrightarrow \langle g f^{abc} A^a_\mu \varphi^{bc}_\mu \rangle + \langle g f^{abc} A^a_\mu \overline{\varphi}^{bc}_\mu \rangle = -8(N^2-1)\gamma^2, \tag{3.15}$$

From expressions (1.1) and (2.2), it follows that

$$-2\gamma^2\langle h\rangle = \langle g f^{abc} A^a_\mu \varphi^{bc}_\mu\rangle + \langle g f^{abc} A^a_\mu \overline{\varphi}^{bc}_\mu\rangle. \quad (3.16)$$

The combination of Eq. (3.15) with Eq. (3.16) gives rise to the horizon condition equation (1.5). In order to conclude this, it is tacitly assumed that $\gamma \neq 0$. We notice that the condition (3.14) does possess the solution $\gamma = 0$. This is an artefact of the reformulation of the horizon condition in terms of the Eq. (3.14), and must be excluded as it does not lead to the horizon condition (1.5). We shall, however, continue to keep this solution of the gap equation (3.14), as $\gamma \equiv 0$ corresponds to the case where the restriction to the Gribov region Ω would not be implemented. In this case, we must only solve the gap equation stemming from Eq. (3.13) with $\gamma \equiv 0$.

The original Gribov-Zwanziger model, i.e. without the inclusion of the operator A^2_μ, is obtained by only retaining the condition (3.14) with $\sigma \equiv 0$.

Up to now, the LCO parameter ζ is still a free parameter of the theory. We do not intend here to give a complete overview of the LCO formalism, we suffice by saying that ζ is fixed by the demand that the action Γ should obey the homogeneous renormalization group equation

$$\left(\overline{\mu}\frac{\partial}{\partial\overline{\mu}} + \beta(g^2)\frac{\partial}{\partial g^2} + \gamma_{\gamma^2}(g^2)\gamma^2\frac{\partial}{\partial\gamma^2} + \gamma_\sigma(g^2)\sigma\frac{\partial}{\partial\sigma}\right)\Gamma = 0, \quad (3.17)$$

with

$$\overline{\mu}\frac{\partial g^2}{\partial\overline{\mu}} = \beta(g^2), \qquad \overline{\mu}\frac{\partial\gamma^2}{\partial\overline{\mu}} = \gamma_{\gamma^2}(g^2)\gamma^2, \qquad \overline{\mu}\frac{\partial\sigma}{\partial\overline{\mu}} = \gamma_\sigma(g^2)\sigma. \quad (3.18)$$

This can be accommodated for by making ζ a function of the running coupling constant g^2, in which case it is found that

$$\zeta(g^2) = \frac{\zeta_0}{g^2} + \zeta_1 + \zeta_2 g^2 + \cdots. \quad (3.19)$$

We refer to the available literature [3,12,20,22,25,26] for a detailed account of the LCO formalism.

A. Renormalization group invariance of the one-loop effective action in the $\overline{\text{MS}}$ scheme without the inclusion of A^2_μ

Before proceeding with the detailed analysis of the horizon condition in the presence of the local operator $A^a_\mu A^a_\mu$, let us first derive the horizon condition and check the explicit renormalization group invariance of the quantum action Γ by switching off the source τ coupled to the operator $A^a_\mu A^a_\mu$. This amounts to consider the original Gribov-Zwanziger model. We consider thus the action

$$S = S_0 + S_\gamma. \quad (3.20)$$

The one-loop effective action $\Gamma^{(1)}$ is easily obtained from the quadratic part of Eq. (3.20)

$$e^{-\Gamma^{(1)}_\gamma} = \int [D\Phi] e^{-S_{\text{quad}}}, \quad (3.21)$$

with S_{quad} given by

$$S_{\text{quad}} = \int d^4x \Big[\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \frac{1}{2\alpha}(\partial_\mu A^a_\mu)^2 + \overline{\varphi}^{ab}_\mu \partial^2 \varphi^{ab}_\mu - \gamma^2 g(f^{abc} A^a_\mu \varphi^{bc}_\mu + f^{abc} A^a_\mu \overline{\varphi}^{bc}_\mu) - 4(N^2-1)\gamma^4\Big], \quad (3.22)$$

where the limit $\alpha \to 0$ is understood in order to recover the Landau gauge. After a straightforward computation, one gets

$$\Gamma^{(1)} = -4(N^2-1)\gamma^4 + \frac{(N^2-1)}{2}(d-1) \times \int \frac{d^dp}{(2\pi)^d} \ln(p^4 + 2Ng^2\gamma^4). \quad (3.23)$$

Dimensional regularization, with $d = 4 - \varepsilon$, will be employed throughout this work. Taking the derivative of $\Gamma^{(1)}$, one reobtains the original gap equation for the Gribov parameter γ, namely

$$\frac{\partial\Gamma^{(1)}}{\partial\gamma} = 0 \Rightarrow 1 = \frac{N(d-1)}{4} g^2 \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^4 + 2Ng^2\gamma^4)}. \quad (3.24)$$

More precisely, recalling that

$$\int \frac{d^dp}{(2\pi)^d} \ln(p^4+\rho^2) = -\frac{\rho^2}{32\pi^2}\left(\ln\frac{\rho^2}{\overline{\mu}^4} - 3\right) + \frac{1}{\varepsilon}\frac{4\rho^2}{32\pi^2}, \quad (3.25)$$

the one-loop effective action $\Gamma^{(1)}$ reads

$$\Gamma^{(1)} = -4(N^2-1)\gamma^4 - \frac{3(N^2-1)}{64\pi^2}(2Ng^2\gamma^4) \times \left(\ln\frac{2Ng^2\gamma^4}{\overline{\mu}^4} - \frac{5}{3}\right), \quad (3.26)$$

where the $\overline{\text{MS}}$ renormalization scheme has been used. In order to check the renormalization group invariance of $\Gamma^{(1)}$, we need to know the anomalous dimension of the Gribov parameter γ. This is easily obtained from Eq. (A36), yielding

$$\gamma_{\gamma^2}(g^2) = -\frac{1}{2}\left(\frac{\beta(g^2)}{2g^2} - \gamma_A(g^2)\right), \quad (3.27)$$

where $\gamma_A(g^2)$ stands for the anomalous dimension of the gauge field A^a_μ. Thus, at one-loop order,

$$\overline{\mu}\frac{d\Gamma^{(1)}}{d\overline{\mu}} = (4(N^2-1)(\frac{\beta^{(1)}(g^2)}{2g^2} - \gamma_A^{(1)}(g^2)) + \frac{3(N^2-1)}{16\pi^2}2Ng^2)\gamma^4. \tag{3.28}$$

Furthermore, from (see e.g. [65])

$$\beta^{(1)}(g^2) = -\frac{22}{3}\frac{g^4N}{16\pi^2}, \qquad \gamma_A^{(1)}(g^2) = -\frac{13}{6}\frac{g^2N}{16\pi^2}, \tag{3.29}$$

it follows

$$\overline{\mu}\frac{d\Gamma^{(1)}}{d\overline{\mu}} = 0, \tag{3.30}$$

which establishes the RGE invariance of the effective action at the order considered. We are now ready to face the more complex case in which the local composite operator $A^a_\mu A^a_\mu$ is present. This will be the topic of the next section.

IV. ONE-LOOP EFFECTIVE ACTION IN THE $\overline{\text{MS}}$ SCHEME WITH THE INCLUSION OF A^2_μ

A. Calculation of the one-loop effective potential

Let us turn to the explicit one-loop evaluation of the effective action Γ in the presence of A^2_μ. At one-loop, it turns out that[2]

$$\Gamma = -4(N^2-1)\gamma^4 + \frac{\sigma^2}{2g^2\zeta} + \frac{N^2-1}{2}\,\text{lndet}\Big[\, p^2\delta_{\mu\nu} + \frac{2Ng^2\gamma^4}{p^2}\delta_{\mu\nu} - p_\mu p_\nu\Big(1-\frac{1}{\alpha}\Big) + \frac{g\sigma}{g^2\zeta}\delta_{\mu\nu}\Big], \tag{4.1}$$

or

$$\Gamma = -4(N^2-1)\gamma^4 + \frac{\sigma^2}{2g^2\zeta} + \frac{N^2-1}{2}(d-1)\int\frac{d^dp}{(2\pi)^d} \times \ln\Big[\, p^4 + 2Ng^2\gamma^4 + \frac{g\sigma}{g^2\zeta}p^2\,\Big]. \tag{4.2}$$

Before calculating the integral, we quote the two gap equations

$$\begin{aligned}\frac{\partial\Gamma}{\partial\sigma} = 0 &\Leftrightarrow \frac{\sigma}{\zeta_0}\Big(1-\frac{\zeta_1}{\zeta_0}g^2\Big) + \frac{(N^2-1)}{2}\frac{g(d-1)}{\zeta_0} \times\int\frac{d^dp}{(2\pi)^d}\frac{p^2}{p^4+\frac{g\sigma}{\zeta_0}p^2+2Ng^2\gamma^4} = 0,\\ \frac{\partial\Gamma}{\partial\gamma} = 0 &\Leftrightarrow \gamma^3 = \gamma^3\frac{d-1}{4}g^2N \times\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^4+\frac{g\sigma}{\zeta_0}p^2+2Ng^2\gamma^4}.\end{aligned} \tag{4.3}$$

The second gap equation of (4.3), being the horizon condition, gives rise to the one obtained in the previous paper [49], while the first one describes the condensation of A^2_μ when the restriction to the Gribov region Ω is implemented. We notice that the explicit value of the Gribov parameter γ is influenced by the presence of $\langle A^2_\mu\rangle$. It remains to calculate

$$\mathcal{I} = \int\frac{d^dp}{(2\pi)^d}\ln[p^4+bp^2+c], \tag{4.4}$$

with

$$b = \frac{g\sigma}{\zeta_0}, \qquad c = 2Ng^2\gamma^4, \tag{4.5}$$

Since

$$p^4+bp^2+c = (p^2+\omega_1)(p^2+\omega_2), \tag{4.6}$$

with

$$\omega_1 = \frac{b+\sqrt{b^2-4c}}{2}, \qquad \omega_2 = \frac{b-\sqrt{b^2-4c}}{2}, \tag{4.7}$$

one has

$$\mathcal{I} = \int\frac{d^dp}{(2\pi)^d}\ln(p^2+\omega_1) + \int\frac{d^dp}{(2\pi)^d}\ln(p^2+\omega_2). \tag{4.8}$$

To make sense, the expression (4.4) should be real to ensure that the one-loop effective action is real-valued. Therefore, we must demand that $c \geq 0$. If $b \geq 0$, $\mathcal{I}$ is certainly real. However, when $b^2-4c \leq 0$, then also $b < 0$ is allowed. We should thus have a positive Gribov parameter γ^4, while the condensate $\langle A^2_\mu\rangle$ can be negative or positive, depending on the case.

Using

$$\int\frac{d^dp}{(2\pi)^d}\ln(p^2+m^2) = \frac{-m^4}{32\pi^2}\Big(\frac{2}{\varepsilon} - \ln\frac{m^2}{\overline{\mu}^2} + \frac{3}{2}\Big), \tag{4.9}$$

it holds

$$\mathcal{I} = -\frac{\omega_1^2}{32\pi^2}(\frac{2}{\varepsilon} - \ln\frac{\omega_1}{\overline{\mu}^2} + \frac{3}{2}) - \frac{\omega_2^2}{32\pi^2}(\frac{2}{\varepsilon} - \ln\frac{\omega_2}{\overline{\mu}^2} + \frac{3}{2}). \tag{4.10}$$

Finally, in the $\overline{\text{MS}}$ scheme, we obtain

[2]We shall drop from now on the superscript (1) indicating that we are working at one-loop order.

$$\Gamma = -4(N^2-1)\gamma^4 + \frac{\sigma^2}{2\zeta_0}\left(1-\frac{\zeta_1}{\zeta_0}g^2\right) + \frac{3(N^2-1)}{2}\left[\frac{\left(\frac{g\sigma}{\zeta_0}+\sqrt{\frac{g^2\sigma^2}{\zeta_0^2}-8g^2N\gamma^4}\right)^2}{128\pi^2}\left(\ln\frac{\frac{g\sigma}{\zeta_0}+\sqrt{\frac{g^2\sigma^2}{\zeta_0^2}-8g^2N\gamma^4}}{2\overline{\mu}^2}-\frac{5}{6}\right)\right.$$
$$\left.+\frac{\left(\frac{g\sigma}{\zeta_0}-\sqrt{\frac{g^2\sigma^2}{\zeta_0^2}-8g^2N\gamma^4}\right)^2}{128\pi^2}\left(\ln\frac{\frac{g\sigma}{\zeta_0}-\sqrt{\frac{g^2\sigma^2}{\zeta_0^2}-8g^2N\gamma^4}}{2\overline{\mu}^2}-\frac{5}{6}\right)\right]. \tag{4.11}$$

To lighten the notation a bit, let us introduce the new variables[3]

$$\lambda^4 = 8g^2N\gamma^4, \tag{4.12}$$

$$m^2 = \frac{g\sigma}{\zeta_0}. \tag{4.13}$$

in which case the action (4.11) can be rewritten as

$$\Gamma = -\frac{(N^2-1)\lambda^4}{2g^2N} + \frac{\zeta_0 m^4}{2g^2}\left(1-\frac{\zeta_1}{\zeta_0}g^2\right) + \frac{3(N^2-1)}{256\pi^2}$$
$$\times\left[(m^2+\sqrt{m^4-\lambda^4})^2\left(\ln\frac{m^2+\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2}-\frac{5}{6}\right)\right.$$
$$\left.+(m^2-\sqrt{m^4-\lambda^4})^2\left(\ln\frac{m^2-\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2}-\frac{5}{6}\right)\right]. \tag{4.14}$$

We notice that the foregoing expression is also valid, i.e. real-valued, in the case in which $m^4 < \lambda^4$, as $\ell_+(m,\lambda)$ and $\ell_-(m,\lambda)$, defined by,

$$\ell_+(m,\lambda) = (m^2+\sqrt{m^4-\lambda^4})^2\left(\ln\frac{m^2+\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2}-\frac{5}{6}\right),$$
$$\ell_-(m,\lambda) = (m^2-\sqrt{m^4-\lambda^4})^2\left(\ln\frac{m^2-\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2}-\frac{5}{6}\right) \tag{4.15}$$

are complex conjugate[4].

The horizon condition, Eq. (3.14), can be translated to

$$\frac{\partial\Gamma}{\partial\lambda} = 0, \tag{4.16}$$

and the gap equation (3.13) to

$$\frac{\partial\Gamma}{\partial m^2} = 0. \tag{4.17}$$

As a check of this one-loop calculation, the expression (4.14) with $m^2 \equiv 0$ reduces to the result obtained earlier in Eq. (3.26), i.e. the original Gribov-Zwanziger model without the inclusion of A_μ^2. If $\lambda \equiv 0$, i.e. the case where the condensation of A_μ^2 is investigated without implementing the restriction to the Gribov region Ω, Eq. (4.14) coincides with the result of [3,12,25]. From [11], one knows that

$$\overline{\mu}\frac{\partial\langle A_\mu^2\rangle}{\partial\overline{\mu}} = \gamma_{A_\mu^2}(g^2)\langle A_\mu^2\rangle = -\left(\frac{\beta(g^2)}{2g^2}+\gamma_A(g^2)\right)\langle A_\mu^2\rangle, \tag{4.18}$$

or, using the relation (3.11) and the definition (4.13),

$$\overline{\mu}\frac{\partial m^2}{\partial\overline{\mu}} = \gamma_{m^2}(g^2)m^2 = \left(\frac{\beta(g^2)}{2g^2}-\gamma_A(g^2)\right)m^2, \tag{4.19}$$

while from Eq. (3.27), it can be inferred that

$$\overline{\mu}\frac{\partial\lambda}{\partial\overline{\mu}} = \gamma_\lambda(g^2)\lambda = \frac{1}{4}\left(\frac{\beta(g^2)}{2g^2}+\gamma_A(g^2)\right)\lambda. \tag{4.20}$$

We notice the remarkable fact that the anomalous dimensions of the Gribov parameter and of the operator A_μ^2 are proportional to each other, to all orders of perturbation theory.

It can now be checked that Γ is renormalization group invariant, namely

$$\overline{\mu}\frac{d}{d\overline{\mu}}\Gamma = 0. \tag{4.21}$$

Finally, taking the derivatives of the action given in Eq. (4.14) gives rise to

$$\frac{1}{\lambda^3}\frac{\partial\Gamma}{\partial\lambda} = -\frac{2(N^2-1)}{g^2N} + \frac{3(N^2-1)}{256\pi^2}$$
$$\times\left[-4\frac{(m^2+\sqrt{m^4-\lambda^4})}{\sqrt{m^4-\lambda^4}}\ln\frac{m^2+\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2}\right.$$
$$\left.+4\frac{(m^2-\sqrt{m^4-\lambda^4})}{\sqrt{m^4-\lambda^4}}\ln\frac{m^2-\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2}+\frac{8}{3}\right], \tag{4.22}$$

[3]In comparison with the previous article [49], we have the correspondence $\lambda^4 = 4\gamma^4$ with the Gribov parameter γ^4 as defined there. It is actually this γ^4 which will enter the modified propagators, see [49] and further in this paper.

[4]Using $\ln(z) = \ln|z| + i\arg(z)$ with $-\pi < \arg(z) \leq \pi$.

and

$$\frac{\partial\Gamma}{\partial m^2} = \frac{\zeta_0 m^2}{g^2}\left(1-\frac{\zeta_1}{\zeta_0}g^2\right) + \frac{3(N^2-1)}{256\pi^2}\Big[2(m^2+\sqrt{m^4-\lambda^4}) \times\left(1+\frac{m^2}{\sqrt{m^4-\lambda^4}}\right)\ln\frac{m^2+\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2} + 2(m^2-\sqrt{m^4-\lambda^4})\left(1-\frac{m^2}{\sqrt{m^4-\lambda^4}}\right) \times\ln\frac{m^2-\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2} - \frac{8}{3}m^2\Big]. \quad (4.23)$$

B. Solving the gap equations

We have now all the ingredients at hand to search for estimates of the mass parameter m^2 and Gribov parameter λ as solutions of the gap equations (4.22) and (4.23). To avoid misinterpretations due to the suggestive use of the notation m^2, we remark that, due to the presence of λ, the mass parameter does not even appear as a pole in the tree level gluon propagator, see Eq. (6.2). Let us first consider the pure Gribov-Zwanziger case, i.e. we put $m^2 \equiv 0$ in the expression (4.14). The relevant gap equation (horizon condition) reads

$$\frac{\partial\Gamma}{\partial\lambda} = \lambda^3\left(-\frac{2(N^2-1)}{g^2N} - \frac{3(N^2-1)}{64\pi^2}\left(\ln\frac{\lambda^4}{4\overline{\mu}^4} - \frac{5}{3}\right) - \frac{3(N^2-1)}{64\pi^2}\right) = 0. \quad (4.24)$$

We remind here that the solution $\lambda = 0$ must be rejected. The natural choice for the renormalization scale is to set $\overline{\mu}^2 = \frac{\lambda^4}{4}$ to kill the logarithms, and we find

$$\left.\frac{g^2N}{16\pi^2}\right|_{\overline{\mu}^2=\lambda^2/2} = 4. \quad (4.25)$$

In principle, from

$$g^2(\overline{\mu}^2) = \frac{1}{\beta_0\ln\frac{\overline{\mu}^2}{\Lambda^2_{\overline{\rm MS}}}}, \qquad \text{with} \quad \beta_0 = \frac{11}{3}\frac{N}{16\pi^2}, \quad (4.26)$$

Equation (4.25) could be used to determine an estimate for the Gribov parameter, however it might be clear that this is meaningless since the corresponding expansion parameter (4.25) is far too big.

It is interesting to notice that, in a general massless renormalization scheme, the one-loop action with $m^2 \equiv 0$ would read

$$\Gamma = -\frac{(N^2-1)}{2g^2N}\lambda^4 - \frac{3\lambda^4(N^2-1)}{264\pi^2}\left(\ln\frac{\lambda^4}{4\overline{\mu}^4} + a\right), \quad (4.27)$$

with a an arbitrary constant. The corresponding gap equation equals

$$\frac{\partial\Gamma}{\partial\lambda} = \lambda^3\left(-\frac{2(N^2-1)}{g^2N} - \frac{3(N^2-1)}{64\pi^2}\left(\ln\frac{\lambda^4}{4\overline{\mu}^4} + a\right) - \frac{3(N^2-1)}{64\pi^2}\right) = 0. \quad (4.28)$$

Denoting by λ_* a solution of Eq. (4.28), for the vacuum energy corresponding to (4.27) one finds

$$E_{\rm vac} = \Gamma(\lambda_*) = \frac{3(N^2-1)}{64\pi^2}\frac{\lambda_*^4}{4}. \quad (4.29)$$

This expression is valid for all $\overline{\mu}$ and for all a. The vacuum energy is thus always nonnegative at one-loop order in the original Gribov-Zwanziger model.

The gap equation (4.23) with $\lambda \equiv 0$ obviously has the solution already obtained in [3,12,25] where the restriction to the Gribov region Ω was not taken into account. We recall the values

$$\frac{g^2N}{16\pi^2} = \frac{36}{187} \approx 0.193, \quad (4.30)$$

$$m^2 = e^{17/12}\Lambda^2_{\overline{\rm MS}} \approx (2.031\Lambda_{\overline{\rm MS}})^2, \quad (4.31)$$

$$E_{\rm vac} = -\frac{3}{16\pi^2}e^{17/6}\Lambda^4_{\overline{\rm MS}} \approx -0.323\Lambda^4_{\overline{\rm MS}}, \quad (4.32)$$

which were obtained upon setting $\overline{\mu}^2 = m^2$ to kill the logarithms. We shall now show that, in the $\overline{\rm MS}$ scheme, the gap equations (4.22) and (4.23) have no solution with $m^2 > 0$ when the restriction to the horizon is implemented (i.e. when $\lambda \neq 0$). To this purpose, we introduce for $m^2 > 0$ the variable

$$t = \frac{\lambda^4}{m^4}. \quad (4.33)$$

Evidently, we should only consider $t > 0$.

Dividing the gap equations (4.22) and (4.23) by m^2, they can be rewritten as[5]

$$\frac{16\pi^2}{g^2N} = \frac{3}{8}\left(-2\ln\frac{m^2}{2\overline{\mu}^2} + \frac{2}{3} + \frac{1}{\sqrt{1-t}}\ln\frac{t}{(1+\sqrt{1-t})^2} - \ln t\right), \quad (4.34)$$

and

$$-\frac{24}{13}\left(\frac{16\pi^2}{g^2N}\right) + \frac{322}{39} = 4\ln\frac{m^2}{2\overline{\mu}^2} - \frac{4}{3} - \frac{2-t}{\sqrt{1-t}} \times\ln\frac{t}{(1+\sqrt{1-t})^2} + 2\ln t, \quad (4.35)$$

where use has been made of the explicit values of ζ_0 and ζ_1, which can be found in [3,12,25]

[5]We have already factored out m^2 or λ^3 since these are nonzero in the present case.

$$\zeta_0 = \frac{9}{13}\frac{N^2-1}{N}, \qquad \zeta_1 = \frac{161}{52}\frac{N^2-1}{16\pi^2}, \tag{4.36}$$

The Eqs. (4.35) and (4.36) can be combined to eliminate $\ln m^2/2\overline{\mu}^2$, yielding the following condition

$$\frac{68}{39}\left(\frac{16\pi^2}{g^2N}\right) + \frac{161}{39} = \frac{t}{\sqrt{1-t}}\ln\frac{t}{(1+\sqrt{1-t})^2} \equiv F(t). \tag{4.37}$$

It can be checked that $F(t)$ is real-valued and negative for $t > 0$, thus the right-hand side of Eq. (4.37) is always negative. Since the left-hand side of Eq. (4.37) is necessarily positive for a meaningful result (i.e. $g^2 \geq 0$), there is no solution with $m^2 > 0$. As already mentioned, there are *a priori* also possible solutions with $m^2 < 0$.

To investigate the existence of a solution with $m^2 < 0$, it might be instructive to look again at the gap equations (4.22) and (4.23) from another perspective. We recall that, if the horizon is not implemented, i.e. $\lambda \equiv 0$, the gap equation (4.23) has two solutions, a perturbative one corresponding to $m^2 = 0$ (no condensation) and a nonperturbative one corresponding to the m^2 given in Eq. (4.31).

If we momentarily consider λ as a free, adjustable parameter of the theory, Eq. (4.23) dictates how m^2 becomes a function of the parameter λ. From the result at $\lambda = 0$, we could expect that two branches of solutions would evolve, one starting from the perturbative and one from the nonperturbative value of m^2 at $\lambda = 0$. When $\lambda \equiv 0$, the choice for the scale $\overline{\mu}$ is quite obvious from the requirement that all the logarithms $\ln(m^2/\overline{\mu}^2)$ are vanishing. However, when $\lambda \neq 0$, we notice that there are two kinds of logarithms present, being $\ln[(m^2 + \sqrt{m^4 - \lambda^4})/(2\overline{\mu}^2)]$ and $\ln[(m^2 - \sqrt{m^4 - \lambda^4})/(2\overline{\mu}^2)]$. We opt to set

$$\overline{\mu}^2 = \frac{|m^2 + \sqrt{m^4 - \lambda^4}|}{2}. \tag{4.38}$$

This reduces to $\overline{\mu}^2 = m^2$ if $\lambda = 0$, while it allows for[6] $m^2 < 0$. This is possible if $m^4 \leq \lambda^4$, as it was mentioned below Eq. (4.8). In this case, the size of both logarithms, $\ln[(m^2 + \sqrt{m^4 - \lambda^4})/(2\overline{\mu}^2)]$ and $\ln[(m^2 - \sqrt{m^4 - \lambda^4})/(2\overline{\mu}^2)]$, is determined by their arguments, which are complex conjugate.

Let us specify to the case $N = 3$. In Fig. 1, we have plotted the behavior of $m^2(\lambda^4)$. We see that next to the ''nonperturbative'' branch of solutions, starting from $m^2 \neq 0$, also a ''perturbative'' branch of solutions with $m^2 < 0$ is emerging from $m^2 = 0$, in correspondence with our expectation. However, λ^4 is not a free parameter of the theory. We should require that λ^4 is such that the doublet $(\lambda^4, m^2(\lambda^4))$ is a solution of the gap equation (4.22), i.e.

[6]Evidently, $\overline{\mu}^2$ should be real and positive, hence the modulus in Eq. (4.38).

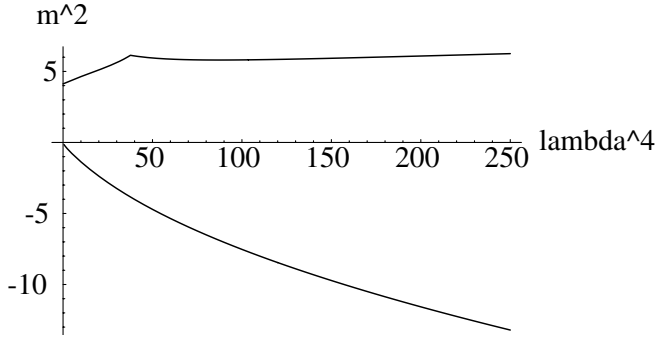

FIG. 1. m^2 as a function of λ^4, in units $\Lambda_{\overline{\mathrm{MS}}} = 1$.

the horizon condition. In Fig. 2, we have plotted the value of the horizon condition equation, as a function of λ^4. It is clear that no solution with $m^2 > 0$ exists as the horizon condition never becomes zero. Of course, this is in correspondence with the foregoing general proof that there is never such a solution, independently of the choice of $\overline{\mu}$. However, we see that there is a single solution with $m^2 < 0$.

The corresponding values for the expansion parameter, for the Gribov and mass parameter, as well as for the vacuum energy are found to be

$$\frac{g^2N}{16\pi^2} \approx 1.18, \tag{4.39}$$

$$\lambda^4 \approx 6.351\Lambda^4_{\overline{\mathrm{MS}}}, \tag{4.40}$$

$$m^2 \approx -0.950\Lambda^2_{\overline{\mathrm{MS}}}, \tag{4.41}$$

$$E_{\mathrm{vac}} \approx 0.043\Lambda^4_{\overline{\mathrm{MS}}}, \tag{4.42}$$

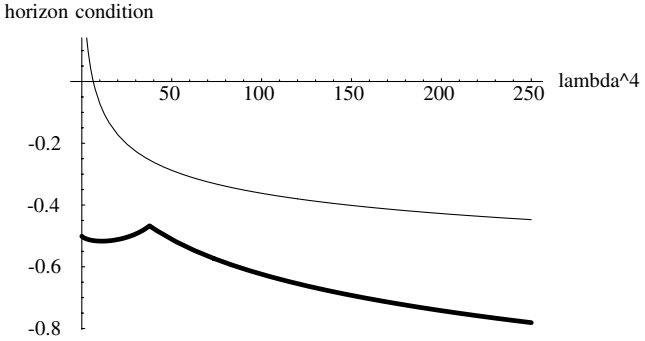

FIG. 2. The horizon condition (4.22) as a function of λ^4, in units $\Lambda_{\overline{\mathrm{MS}}} = 1$. The top curve corresponds to the solutions of (4.23) with $m^2 < 0$ and the lower curve to the solutions with $m^2 > 0$.

C. Intermediate comments

Although the $\overline{\text{MS}}$ expansion parameter (4.39) is too large to speak about reliable results, we nevertheless would like to raise some questions. Apparently, the solution of the coupled gap equations is laying on the perturbative branch, being the one with $m^2 \leq 0$. This gives rise to a positive value for the mass dimension two gluon condensate $\langle A_\mu^2 \rangle$. When the restriction on the domain of integration in the path integral is not implemented, as in the previous papers [3,12,25], $\langle A_\mu^2 \rangle$ was necessarily negative, the reason being that the action should be real-valued, as it was explained below Eq. (4.8). As already explained in the Introduction, a finding a bit unfortunate is that the vacuum energy is positive, Eq. (4.42), which leads to a negative estimate for the gluon condensate $\langle g^2/(4\pi^2)F_{\mu\nu}^2 \rangle$ via the trace anomaly, Eq. (1.8). Essentially, we are thus left with the following questions:

(i) What is the sign and value of m^2 and thus of $\langle A_\mu^2 \rangle$?
(ii) What is the sign and value of E_{vac} and the corresponding value for $\langle g^2/(4\pi^2)F_{\mu\nu}^2 \rangle$?
(iii) Are these values better or not when the operator A_μ^2 is added to the original Gribov-Zwanziger model?

V. CHANGING AND REDUCING THE DEPENDENCE ON THE RENORMALIZATION SCHEME

We have already shown that the vacuum energy obtained in a one-loop approximation is always positive when the condensation of the operator A_μ^2 is left out of the discussion, using whatever renormalization scheme.

To answer the foregoing questions (i.)–(iii.), one could investigate what happens at two-loop order. However, due to the already quite complicated structure of the one-loop effective action and to the fact that the calculations at higher loop order will not get any easier, this task is beyond the scope of the present article. Here, we shall mainly focus on the effects of a change of the renormalization scheme at the one-loop order. It could happen that, in a scheme different from the $\overline{\text{MS}}$ one, the vacuum energy is negative and/or that the coupling constant is small enough to speak about trustworthy results, at least qualitatively.

Since to obtain an optimization of the renormalization scheme and scale dependence is a rather lengthy task, we shall not dwell upon technicalities in this section. The interested reader can find all details in Appendix B. We shall thus focus on the main results obtained after the optimization.

Essentially, what we have done is replace in the effective action (4.14) the quantities m^2 and λ^4 by their order by order renormalization scale and scheme invariant counterparts $\hat{m}^2$ and $\hat{\lambda}^4$. The residual freedom in the choice of renormalization scheme can then be reduced to a single parameter b_0, related to coupling constant renormalization. As the vacuum energy is a physical quantity, it should in principle not depend on b_0. At the same time, the quantities $\hat{m}^2$ and $\hat{\lambda}^4$ should be b_0 independent by construction. This provides one with the interesting opportunity to fix the redundant parameter b_0 by demanding a minimal dependence on it.

The final one-loop action turns out to be given by

$$\Gamma = -\frac{(N^2-1)}{2N} x^{-2b} \hat{\lambda}^4 \left(x + B + (1-2b)\left(\frac{\beta_1}{\beta_0} \ln \frac{x}{\beta_0} - b_0 \right) \right) + \frac{\zeta_0}{2} \hat{m}^4 x^{-2a} \left(x + A - \frac{\zeta_1}{\zeta_0} + (1-2a)\left(\frac{\beta_1}{\beta_0} \ln \frac{x}{\beta_0} - b_0 \right) \right)$$
$$+ \frac{3(N^2-1)}{256\pi^2} \left[(\hat{m}^2 x^{-a} + \sqrt{\hat{m}^4 x^{-2a} - \hat{\lambda}^4 x^{-2b}})^2 \left(\ln \frac{\hat{m}^2 x^{-a} + \sqrt{\hat{m}^4 x^{-2a} - \hat{\lambda}^4 x^{-2b}}}{2\overline{\mu}^2} - \frac{5}{6} \right) \right.$$
$$\left. + (\hat{m}^2 x^{-a} - \sqrt{\hat{m}^4 x^{-2a} - \hat{\lambda}^4 x^{-2b}})^2 \left(\ln \frac{\hat{m}^2 x^{-a} - \sqrt{\hat{m}^4 x^{-2a} - \hat{\lambda}^4 x^{-2b}}}{2\overline{\mu}^2} - \frac{5}{6} \right) \right], \tag{5.1}$$

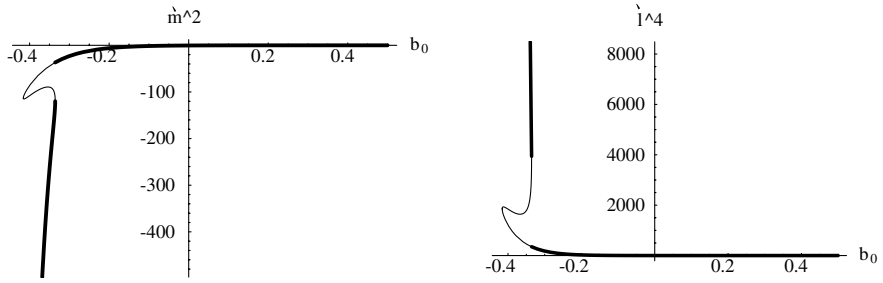

FIG. 3. The quantities $\hat{m}^2$ and $\hat{\lambda}^4$ as a function of b_0, in units $\Lambda_{\overline{\text{MS}}} = 1$.

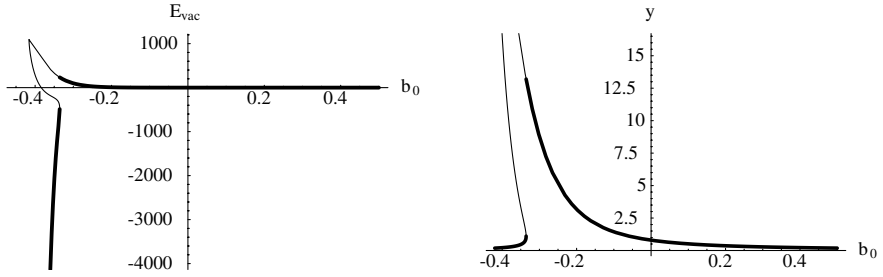

FIG. 4. The vacuum energy $E_{\rm vac}$ and the expansion parameter y as a function of b_0, in units $\Lambda_{\overline{\rm MS}} = 1$.

while the corresponding gap equations read

$$\frac{1}{\hat{\lambda}^3}\frac{\partial\Gamma}{\partial\hat{\lambda}} = 0, \tag{5.2}$$

$$\frac{1}{\hat{m}^2}\frac{\partial\Gamma}{\partial\hat{m}^2} = 0. \tag{5.3}$$

The definitions of all quantities appearing in the above expressions can be found in the Appendix B.

In Fig. 3, we collected the solutions of the scale invariant quantities $\hat{m}^2$ and $\hat{\lambda}^4$ as a function of b_0, while Fig. 4 displays the vacuum energy $E_{\rm vac}$ and the relevant expansion parameter, given by $y \equiv N/(16\pi^2 x)$. For completeness, we have also shown the solutions which correspond to higher values of Γ and are as such describing unstable solutions. These are indicated with the thinner lines.

A. Interpretation of the results

Let us first have a look at the plot of vacuum energy, on the left-hand side of Fig. 4. We notice that for $b_0 < -0.33564\ldots$, the vacuum energy becomes negative. However, we cannot attach any definitive meaning to this result. In fact, as it can be seen from the Figs. 3 and 4, the values of the vacuum energy and the supposedly minimally b_0-dependent quantities $\hat{m}^2$ and $\hat{\lambda}^4$ are extremely b_0-dependent. Very small variations in b_0 induce large fluctuations on e.g. the energy. This is indicative of the fact that the equations we have solved are not yet stable against b_0 variations in the range of the values obtained for b_0. The behavior is better for, let us say $b_0 > -0.2$. However, in this case, we find again that the vacuum energy is positive. The vacuum energy $E_{\rm vac}$, as well as $\hat{m}^2$ and $\hat{\lambda}^4$ fall of to zero for growing b_0.

As an example, we set $b_0 = 0$, which corresponds to use the $\overline{\rm MS}$ coupling constant. Then we find, with the optimized expansion,

$$y \equiv \frac{N}{16\pi^2 x} \approx 0.796, \tag{5.4}$$

$$\hat{\lambda}^4 x^{-2b} \approx 7.939\Lambda^4_{\overline{\rm MS}}, \tag{5.5}$$

$$\hat{m}^2 x^{-a} \approx -0.814\Lambda^2_{\overline{\rm MS}}, \tag{5.6}$$

$$E_{\rm vac} \approx 0.063\Lambda^4_{\overline{\rm MS}}, \tag{5.7}$$

results which are in fair agreement with the naive $\overline{\rm MS}$ results (4.39), (4.40), (4.41), and (4.42). We notice that the expansion parameter y is already smaller than 1, but still relatively large, while the vacuum energy is indeed positive.

The conclusion than can be drawn from this section is that we cannot find a reliable result with negative vacuum energy and hence positive gluon condensate $\langle F^2_{\mu\nu}\rangle$ using a one-loop approximation. We see therefore that, in order to be able to give a reasonable answer to the questions concerning the sign of m^2 and $E_{\rm vac}$ and to get more trustworthy numerical values, the two-loop evaluation of the effective action Γ, at least in the $\overline{\rm MS}$ scheme, would be very useful.

VI. CONSEQUENCES OF A NON-VANISHING GRIBOV PARAMETER

Before turning to the final conclusions, we shall give in this section a brief account of some well-known consequences stemming from the presence of the Gribov parameter, to emphasize the important role of this parameter.

A. The gluon propagator

If there is no generation of a mass parameter due to $\langle A^2_\mu\rangle$, we can consider just the action (2.2). Then the tree level gluon propagator turns out to be

$$\langle A^a_\mu A^b_\nu\rangle_p = \delta^{ab}\frac{p^2}{p^4 + \frac{\lambda^4}{4}}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right). \tag{6.1}$$

This result, first pointed out in [48], was obtained by retaining only the first term of the nonlocal horizon function (1.4), corresponding to the approximation $-\partial D \approx -\partial^2$. The gluon propagator, Eq. (6.1), is suppressed in the infrared region due to the presence of the Gribov parameter λ. In particular, the presence of this parameters implies that $\langle A^a_\mu A^b_\nu\rangle_p$ vanishes at zero momentum, $p = 0$. When the possibility of the existence of a dynamical mass parameter in the gluon propagator is included, by investigating the condensation of A^2_μ, the tree level gluon propa-

gator reads

$$\langle A^a_\mu A^b_\nu \rangle_p \equiv \delta^{ab} \frac{\mathcal{D}(p^2)}{p^2}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) = \delta^{ab} \frac{p^2}{p^4 + m^2 p^2 + \frac{\lambda^4}{4}}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right). \quad (6.2)$$

This type of propagator is sometimes called the Stingl propagator, from the author who used it as an anzatz for solving the Schwinger-Dyson equations, see [66] for more details.

However, it should be realized that Eq. (6.2) describes only the tree level gluon propagator. In particular, to produce a plot of the form factor $\mathcal{D}(p^2)$ as a function of the momentum p, which would allow to make a comparison with the results obtained in lattice simulations, see e.g. [31] for $N = 3$ and [32,34] for $N = 2$, one should go beyond the zeroth order approximation, for example, by including higher order polarization effects and/or trying to perform a renormalization group improvement. In general, these corrections will also be dependent on the external momentum p.

B. The ghost propagator

Even more prominent is the influence of the Gribov parameter on the infrared behavior of the ghost propagator, which can be calculated at one-loop order using the modified gluon propagator (6.1) or (6.2) with their respective gap equations (1.6) and (4.3). In both cases, the infrared behavior of the ghost propagator [48–52] is shown to be

$$\frac{\delta^{ab}}{N^2-1}\langle c^a \bar{c}^b \rangle_{p\approx 0} \equiv \frac{1}{p^2}\mathcal{G}(p^2)\Big|_{p\approx 0} \approx \frac{4}{3Ng^2 J p^4}, \quad (6.3)$$

where J stands for the real, finite integral given by

$$J = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k^4 + m^2k^2 + \frac{\lambda^4}{4})}. \quad (6.4)$$

The original Gribov-Zwanziger model corresponds to $m^2 \equiv 0$. Thus, the ghost propagator is strongly enhanced in the infrared region compared to the perturbative behavior, if the restriction to the first Gribov region is taken into account. It is important to notice that this behavior of the ghost propagator is preserved in the present treatment, due to the peculiar form of the gap equation (4.3) implementing the horizon condition. In particular, from the expression for the effective action in Eq. (4.2), one sees that, while the term quadratic in the field σ, i.e. $\sigma^2/2g^2\zeta$, contains the LCO parameter ζ, the first term which depends on the Gribov parameter, i.e. $-4(N^2-1)\gamma^4$, does not contain any such new LCO parameter. This important feature follows from the fact that no new parameter has to be introduced in order to renormalize the term $(M^{ai}_\mu V^{ai}_\mu - U^{ai}_\mu N^{ai}_\mu)$, as remarked in Eq. (A37). While the parameter ζ is required to take into account the ultraviolet divergences of the vacuum correlator $\langle A^2_\mu(x) A^2_\nu(y)\rangle$, which are proportional to τ^2, no such a parameter is needed for $(M^{ai}_\mu V^{ai}_\mu - U^{ai}_\mu N^{ai}_\mu)$ which, upon setting the external sources to their physical values, gives rise to term $-4(N^2-1)\gamma^4$ in the expression (4.2). Said otherwise, this term is not affected by the presence of a new parameter which would be required if Eq. (A37) would not hold. As a consequence, the factor "1" appearing in the left-hand side of the gap equation (4.3) is, so to speak, left unchanged by the quantum corrections. It is precisely that property which ensures, through a delicate cancellation mechanism, see [48,49,51,52], the infrared enhancement of the ghost propagator.

Analogously to the case of the gluon propagator, a more detailed study of higher order corrections would be needed in order to obtain a plot of the ghost form factor $\mathcal{G}(p^2)$.

C. The strong coupling constant

Usually, a nonperturbative definition of the renormalized strong coupling constant α_R can be written down from the knowledge of the gluon and ghost propagators as, see e.g. [34,39]

$$\alpha_R(p^2) = \alpha_R(\mu)\mathcal{D}(p^2, \mu)\mathcal{G}^2(p^2, \mu), \quad (6.5)$$

where $\mathcal{D}$ and $\mathcal{G}$ stand for the gluon and ghost form factors as defined before. This definition represents a kind of nonperturbative extension of the perturbative results (A35). According to Schwinger-Dyson studies [41–46], those form factors satisfy a power law behavior in the infrared

$$\lim_{p\to 0}\mathcal{D}(p^2) \propto (p^2)^\theta, \qquad \lim_{p\to 0}\mathcal{G}(p^2) \propto (p^2)^\omega, \quad (6.6)$$

where the infrared exponents θ and ω obey the sum rule

$$\theta + 2\omega = 0. \quad (6.7)$$

Such a sum rule suggests the development of an infrared fixed point for the renormalized coupling constant, (6.5), as also pointed out by lattice simulations for the $SU(2)$ as well as for the $SU(3)$ case [34–36],

$$\lim_{p\to 0}\alpha(p^2) = \alpha_c. \quad (6.8)$$

The existence of a fixed point in this reasoning is dependent on the sum rule rather than on the precise value of the exponents. We refer to the already quoted literature for more details on the value of these exponents. We end by noticing that the form factors of the gluon and ghost propagator in our zeroth order approximation give rise to the sum rule (6.7), since we have $\theta = 2$ and $\omega = -1$. Moreover, without Gribov parameter, the sum rule (6.7) is lost, and thus there is no indication for an infrared fixed point.

D. Positivity violation

The behavior of the gluon propagator is sometimes used as an *indication* of confinement of gluons by means of the so-called positivity violation, see e.g. [67,68] and references therein.

Briefly, when the Euclidean gluon propagator $D(p) \equiv \mathcal{D}(p^2)/p^2$ is written through a spectral representation as

$$D(p) = \int_0^{+\infty} dM^2 \frac{\rho(M^2)}{p^2 + M^2}, \tag{6.9}$$

the spectral density $\rho(M^2)$ should be positive in order to have a Källen-Lehmann representation, making possible the interpretation of the fields in term of stable particles. We refer to [67,68] for more details. One can define the temporal correlator [68]

$$C(t) = \int_0^{+\infty} dM \rho(M^2) e^{-Mt}, \tag{6.10}$$

which is certainly positive for positive $\rho(M^2)$. The inverse is not necessarily true. $C(t)$ can be also positive for a $\rho(M^2)$ attaining negative values. However, if $C(t)$ becomes negative for certain t, then *a fortiori* $\rho(M^2)$ cannot be always positive. Using a contour integration argument, it is not difficult to show that $C(t)$ can be rewritten as

$$C(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ipt} D(p) dp. \tag{6.11}$$

Let us consider the function $C(t)$ using the tree level propagator (6.2), thus using

$$D(p) = \frac{p^2}{p^4 + p^2 m^2 + \frac{\lambda^4}{4}}. \tag{6.12}$$

We can consider several cases[7]:

(i) if $\lambda = 0$ (thus $m^2 > 0$), one shall find that

$$C(t) = \frac{e^{-mt}}{2m}. \tag{6.13}$$

This function is always positive.

(ii) if $m^2 = 0$,

$$C(t) = \frac{e^{-Lt/2}}{2L}\left(\cos\frac{Lt}{2} - \sin\frac{Lt}{2}\right), \tag{6.14}$$

and clearly, this function will attain negative values for certain t.

(iii) in any other case, the correlator $C(t)$ is found to be

$$C(t) = \frac{1}{2}\left[\frac{\sqrt{\omega_1}}{\omega_1 - \omega_2} e^{-\sqrt{\omega_1}t} + \frac{\sqrt{\omega_2}}{\omega_2 - \omega_1} e^{-\sqrt{\omega_2}t}\right] \tag{6.15}$$

where the decomposition

$$\frac{p^2}{p^4 + p^2 m^2 + \frac{\lambda^4}{4}} = \frac{\omega_1}{\omega_1 - \omega_2}\frac{1}{p^2 + \omega_1} - \frac{\omega_2}{\omega_1 - \omega_2}\frac{1}{p^2 + \omega_2}, \tag{6.16}$$

has been employed. It is understood that $\sqrt{\omega_1}$ ($\sqrt{\omega_2}$) is the root having a positive real part.

If we assume that $\hat{m}^4 > \lambda^4$, then $\omega_1 > \omega_2$ and $C(t)$ becomes negative for $t > \frac{1}{2}(\ln\frac{\omega_1}{\omega_2})/(\omega_1 - \omega_2)$. In the case that $\hat{m}^4 = \lambda^4$, or $\omega_1 = \omega_2$, one finds that $C(t) = [e^{-\sqrt{\omega_1}t}/(4\sqrt{\omega_1})](1 - \sqrt{\omega_1}t)$, which can also become negative. If $\hat{m}^4 < \lambda^4$, we can reintroduce the complex polar coordinates R and ϕ for the complex conjugate quantities ω_1 and ω_2. If $\cos(\phi/2) \geq 0$, Eq. (6.15) can be rewritten as

$$C(t) = \frac{1}{2\sqrt{R}\sin\phi} e^{-\sqrt{R}\cos(\phi/2)t} \times \sin\left(\frac{\phi}{2} - \sqrt{R}\sin\left(\frac{\phi}{2}\right)t\right) \tag{6.17}$$

By choosing an appropriate value of $t > 0$, this expression can also be made negative. An analogous expression and conclusion can be derived in case that $\cos(\phi/2) < 0$.

We conclude that, when the restriction to the Gribov region Ω is implemented, the function $C(t)$ exhibits a violation of positivity when the tree level propagator is used, with our without the inclusion of $\langle A_\mu^2 \rangle$.

The goal of this section was merely to provide some interesting consequences when the restriction to the first Gribov region Ω is implemented. Higher loop effects, which shall be momentum dependent, would also influence the behavior of the gluon and ghost propagator. Hence, to give a sensible interpretation of the behavior of the form factors and of the strong coupling constant α_R, a more detailed analysis than a tree level one is necessary. This is however far beyond the aim of this work.

VII. CONCLUSION

In this work we have considered $SU(N)$ Euclidean Yang-Mills theories in the Landau gauge, $\partial_\mu A_\mu = 0$. We have studied the condensation of the dimension two composite operator A_μ^2 when the restriction to the Gribov region Ω is taken into account. Such a restriction is needed due to the presence of the Gribov copies [48], which are known to affect the Landau gauge. In a previous work [49], the consequences of the restriction to the region Ω in the presence of a dynamical mass parameter due to the gluon condensate $\langle A_\mu^a A_\mu^a \rangle$ were studied by following Gribov's seminal work [48]. Here, we have relied on Zwanziger's action [51,52], which allows to implement the restriction to the Gribov region Ω within a local and renormalizable framework. We have been able to show that Zwanziger's

[7]Each of the following expressions for $C(t)$ is obtainable via contour integration.

action remains renormalizable to all orders of perturbation theory in the presence of the operator A_μ^2, introduced through the local composite operator technique [3,12,20,25]. The effective action, constructed via the local composite operator formalism [3] obeys a homogeneous renormalization group. The explicit form of the one-loop effective action has been worked out. We have seen that, considering the original Gribov-Zwanziger model, i.e. without including the operator A_μ^2, the vacuum energy is always positive at one-loop order, independently from the choice of the renormalization scheme. A positive vacuum energy would give rise to a negative value for the gauge invariant gluon condensate $\langle F_{\mu\nu}^2 \rangle$, through the trace anomaly. Furthermore, by adding the operator A_μ^2, we have proven that there is no solution of the two coupled gap equations at the one-loop order in the $\overline{\text{MS}}$ scheme with $\langle A_\mu^2 \rangle < 0$. Nevertheless, when $\langle A_\mu^2 \rangle > 0$, a solution of the gap equations was found, although the corresponding expansion parameter was too large and the vacuum energy still positive. In order to find out what happens in other schemes, we performed a detailed study, at lowest order, of the influence of the renormalization scheme. We have been able to reduce the freedom of the choice of the renormalization scheme to two parameters, namely, the renormalization scale $\overline{\mu}$ and a parameter b_0, associated to the coupling constant renormalization. We re-expressed the effective action in terms of the mass parameter $\hat{m}$ and Gribov parameter $\hat{\lambda}$, which are renormalization scheme and scale independent order by order. The resulting gap equations for these parameters have been solved numerically. Although a solution with negative vacuum energy was found, we have been unable to attach any definitive meaning to it. This is due to the fact that the results obtained turn out to be strongly dependent from the parameter b_0. This brought us to the conclusion that we should extend our calculations to a higher order to obtain more sensible numerical estimates. The mass parameters $\hat{m}$ and $\hat{\lambda}$ are of a nonperturbative nature and appear in the gluon and ghost propagator. Even if we lack reliable estimates for these parameters, some already known interesting features can be recovered. For a nonzero mass and Gribov parameter, there is a *qualitative* agreement with the behavior found in lattice simulations and Schwinger-Dyson studies: a suppressed gluon and enhanced ghost propagator in the infrared, while further consequences of the Gribov parameter are the possible existence of an infrared fixed point for the strong coupling constant and the violation of positivity related to the gluon propagator.

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APPENDIX A

In this Appendix, we have collected all details of the multiplicative renormalization of the Zwanziger action in the presence of the operator A_μ^2.

1. Ward identities

In order to begin with the algebraic characterization of the most general counterterm needed for the renormalizability of the complete action Σ of Eq. (3.1), let us first give the set of Ward identities which are fulfilled by Σ. These are

(i) the Slavnov-Taylor identity

$$S(\Sigma) = 0, \tag{A1}$$

with

$$S(\Sigma) = \int d^4x \Big(\frac{\delta \Sigma}{\delta K_\mu^a} \frac{\delta \Sigma}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta c^a} + b^a \frac{\delta \Sigma}{\delta \overline{c}^a} + \overline{\varphi}_i^a \frac{\delta \Sigma}{\delta \overline{\omega}_i^a} + \omega_i^a \frac{\delta \Sigma}{\delta \varphi_i^a} + M_\mu^{ai} \frac{\delta \Sigma}{\delta U_\mu^{ai}} + N_\mu^{ai} \frac{\delta \Sigma}{\delta V_\mu^{ai}} + \tau \frac{\delta \Sigma}{\delta \eta} \Big), \tag{A2}$$

(ii) the Landau gauge condition and the antighost equation

$$\frac{\delta \Sigma}{\delta b^a} = \partial_\mu A_\mu^a, \tag{A3}$$

$$\frac{\delta \Sigma}{\delta \overline{c}^a} + \partial_\mu \frac{\delta \Sigma}{\delta K_\mu^a} = 0, \tag{A4}$$

(iii) the ghost Ward identity

$$\mathcal{G}^a \Sigma = \Delta_{\text{cl}}^a, \tag{A5}$$

with

$$\mathcal{G}^a = \int d^4x \Big(\frac{\delta}{\delta c^a} + g f^{abc} \Big(\overline{c}^b \frac{\delta}{\delta b^c} + \varphi_i^b \frac{\delta}{\delta \omega_i^c} + \overline{\omega}_i^b \frac{\delta}{\delta \overline{\varphi}_i^c} + V_\mu^{bi} \frac{\delta}{\delta N_\mu^{ci}} + U_\mu^{bi} \frac{\delta}{\delta M_\mu^{ci}} \Big) \Big), \tag{A6}$$

and

$$\Delta^a_{\rm cl} = g\int d^4x f^{abc}(K^b_\mu A^c_\mu - L^b c^c). \tag{A7}$$

Notice that the term $\Delta^a_{\rm cl}$, being linear in the quantum fields A^a_μ, c^a, is a classical breaking.

(iv) the linearly broken local constraints

$$\frac{\delta\Sigma}{\delta\overline{\varphi}^{ai}} + \partial_\mu\frac{\delta\Sigma}{\delta M^{ai}_\mu} = gf^{abc}A^b_\mu V^{ci}_\mu, \tag{A8}$$

$$\frac{\delta\Sigma}{\delta\omega^{ai}} + \partial_\mu\frac{\delta\Sigma}{\delta N^{ai}_\mu} - gf^{abc}\overline{\omega}^{bi}\frac{\delta\Sigma}{\delta b^c} = gf^{abc}A^b_\mu U^{ci}_\mu, \tag{A9}$$

$$\frac{\delta\Sigma}{\delta\overline{\omega}^{ai}} + \partial_\mu\frac{\delta\Sigma}{\delta U^{ai}_\mu} - gf^{abc}V^{bi}_\mu\frac{\delta\Sigma}{\delta K^c_\mu} = -gf^{abc}A^b_\mu N^{ci}_\mu, \tag{A10}$$

$$\frac{\delta\Sigma}{\delta\varphi^{ai}} + \partial_\mu\frac{\delta\Sigma}{\delta V^{ai}_\mu} - gf^{abc}\overline{\varphi}^{bi}\frac{\delta\Sigma}{\delta b^c} - gf^{abc}\overline{\omega}^{bi}\frac{\delta\Sigma}{\delta\overline{c}^c} - gf^{abc}U^{bi}_\mu\frac{\delta\Sigma}{\delta K^c_\mu} = gf^{abc}A^b_\mu M^{ci}_\mu, \tag{A11}$$

(v) the integrated Ward identity

$$\int d^4x\left(c^a\frac{\delta\Sigma}{\delta\omega^{ai}} + \overline{\omega}^{ai}\frac{\delta\Sigma}{\delta\overline{c}^a} + U^{ai}_\mu\frac{\delta\Sigma}{\delta K^a_\mu}\right) = 0, \tag{A12}$$

(vi) the exact $\mathcal{R}_{ij}$ symmetry

$$\mathcal{R}_{ij}\Sigma = 0, \tag{A13}$$

with

$$\mathcal{R}_{ij} = \int d^4x\left(\varphi^a_i\frac{\delta}{\delta\omega^a_j} - \overline{\omega}^a_j\frac{\delta}{\delta\overline{\varphi}^a_i} + V^{ai}_\mu\frac{\delta}{\delta N^{ai}_\mu} - U^{ai}_\mu\frac{\delta}{\delta M^{ai}_\mu}\right). \tag{A14}$$

2. Algebraic characterization of the counterterm

Having established all the Ward identities fulfilled by the complete action Σ, we can now turn to the characterization of the most general allowed counterterm Σ^c. Following the algebraic renormalization procedure [21], Σ^c is an integrated local polynomial in the fields and sources with dimension bounded by four, with vanishing ghost number and Q_f charge, obeying the following constraints

$$\frac{\delta\Sigma^c}{\delta\varphi^{ai}} + \partial_\mu\frac{\delta\Sigma^c}{\delta V^{ai}_\mu} - gf^{abc}\overline{\omega}^{bi}\frac{\delta\Sigma^c}{\delta\overline{c}^c} - gf^{abc}U^{bi}_\mu\frac{\delta\Sigma^c}{\delta K^c_\mu} = 0,$$

$$\frac{\delta\Sigma^c}{\delta\overline{\omega}^{ai}} + \partial_\mu\frac{\delta\Sigma^c}{\delta U^{ai}_\mu} - gf^{abc}V^{bi}_\mu\frac{\delta\Sigma^c}{\delta K^c_\mu} = 0,$$

$$\frac{\delta\Sigma^c}{\delta\omega^{ai}} + \partial_\mu\frac{\delta\Sigma^c}{\delta N^{ai}_\mu} = 0, \qquad \frac{\delta\Sigma}{\delta\overline{\varphi}^{ai}} + \partial_\mu\frac{\delta\Sigma}{\delta M^{ai}_\mu} = 0,$$

$$\frac{\delta\Sigma}{\delta\overline{c}^a} + \partial_\mu\frac{\delta\Sigma}{\delta K^a_\mu} = 0, \qquad \frac{\delta\Sigma^c}{\delta b^a} = 0, \tag{A15}$$

$$\mathcal{G}^a\Sigma^c = 0, \tag{A16}$$

$$\int d^4x\left(c^a\frac{\delta\Sigma^c}{\delta\omega^{ai}} + \overline{\omega}^{ai}\frac{\delta\Sigma^c}{\delta\overline{c}^a} + U^{ai}_\mu\frac{\delta\Sigma^c}{\delta K^a_\mu}\right) = 0, \tag{A17}$$

$$\mathcal{R}_{ij}\Sigma^c = 0, \tag{A18}$$

and

$$\mathcal{B}_\Sigma\Sigma^c = 0, \tag{A19}$$

where $\mathcal{B}_\Sigma$ is the nilpotent linearized Slavnov-Taylor operator

$$\mathcal{B}_\Sigma = \int d^4x\left(\frac{\delta\Sigma}{\delta K^a_\mu}\frac{\delta}{\delta A^a_\mu} + \frac{\delta\Sigma}{\delta A^a_\mu}\frac{\delta}{\delta K^a_\mu} + \frac{\delta\Sigma}{\delta L^a}\frac{\delta}{\delta c^a} + \frac{\delta\Sigma}{\delta c^a}\frac{\delta}{\delta L^a} + b^a\frac{\delta}{\delta\overline{c}^a} + \overline{\varphi}^a_i\frac{\delta}{\delta\overline{\omega}^a_i} + \omega^a_i\frac{\delta}{\delta\varphi^a_i} + M^{ai}_\mu\frac{\delta}{\delta U^{ai}_\mu} + N^{ai}_\mu\frac{\delta}{\delta V^{ai}_\mu} + \tau\frac{\delta}{\delta\eta}\right), \tag{A20}$$

$$\mathcal{B}_\Sigma\mathcal{B}_\Sigma = 0. \tag{A21}$$

As it was shown in [51–53], the constraints (A15) imply that Σ^c does not depend on the Lagrange multiplier b^a, and that the antighost $\overline{c}^a$ and the i-valued fields φ^a_i, ω^a_i, $\overline{\varphi}^a_i$, $\overline{\omega}^a_i$ can enter only through the combinations

$$\begin{aligned}
&\tilde{K}^a_\mu = K^a_\mu + \partial_\mu\overline{c}^a - gf^{abc}\tilde{U}^{bi}_\mu\varphi^{ci} - gf^{abc}V^{bi}_\mu\overline{\omega}^{ci},\\
&\tilde{U}^{ai}_\mu = U^{ai}_\mu + \partial_\mu\overline{\omega}^{ai}, \qquad \tilde{V}^{ai}_\mu = V^{ai}_\mu + \partial_\mu\varphi^{ai},\\
&\tilde{N}^{ai}_\mu = N^{ai}_\mu + \partial_\mu\omega^{ai}, \qquad \tilde{M}^{ai}_\mu = V^{ai}_\mu + \partial_\mu\overline{\varphi}^{ai}.
\end{aligned} \tag{A22}$$

Therefore, Σ^c can be parametrized as follows

$$\Sigma^c = S^c(A) + \int d^4x\left(a_1 gf^{abc}L^a c^b c^c + a_2\tilde{K}^a_\mu\partial_\mu c^a + a_3 gf^{abc}\tilde{K}^a_\mu A^b_\mu c^c + a_4 f^{abc}\tilde{V}^{ai}_\mu\tilde{U}^{bi}_\mu c^c + a_5\tilde{V}^{ai}_\mu\tilde{M}^{ai}_\mu + a_6\tilde{U}^{ai}_\mu\tilde{N}^{ai}_\mu + \frac{a_7}{2}\tau A^a_\mu A^a_\mu + \frac{a_8}{2}\zeta\tau^2 + a_9\eta A^a_\mu\partial_\mu c^a + a_{10}\eta c^a\partial A^a\right), \tag{A23}$$

where $S^c(A)$ depends only on the gauge field A^a_μ, and with

$a_1, \ldots, a_{10}$ arbitrary parameters. Notice, however, that there is no mixing in expression (A23) between $\tilde{M}^{ai}_\mu$, $\tilde{N}^{ai}_\mu$, $\tilde{V}^{ai}_\mu$, $\tilde{U}^{ai}_\mu$ and the sources τ, η. This is due to the dimensionality and to the Q_f charge. It is precisely the absence of this mixing that will ensure the renormalizability of the Zwanziger action in the presence of the composite operator $A^a_\mu A^a_\mu$. From the ghost equation (A16) it follows

$$a_1 = a_3 = a_{10} = 0, \qquad a_4 = -g(a_6 + a_5). \tag{A24}$$

From the Eqs. (A17) and (A18) we obtain

$$a_6 = -a_2. \tag{A25}$$

Finally, from Eq. (A19) it turns out that

$$a_5 = a_2, \qquad a_9 = a_7 - a_2, \tag{A26}$$

and

$$S^c(A) = a_0 S_{\rm YM} + a_2 \int d^4x A^a_\mu \frac{\delta S_{\rm YM}}{\delta A^a_\mu}. \tag{A27}$$

In summary, the most general local invariant counterterm compatible with all Ward identities contains four arbitrary parameters, a_0, a_2, a_7, a_8, and reads

$$\begin{aligned}\Sigma^c = a_0 S_{\rm YM} + a_2 \int d^4x \Big(& A^a_\mu \frac{\delta S_{\rm YM}}{\delta A^a_\mu} + \tilde{K}^a_\mu \partial_\mu c^a + \tilde{V}^{ai}_\mu \tilde{M}^{ai}_\mu \\ & - \tilde{U}^{ai}_\mu \tilde{N}^{ai}_\mu \Big) + \int d^4x \Big(\frac{a_7}{2} \tau A^a_\mu A^a_\mu + \frac{a_8}{2} \zeta \tau^2 \\ & + (a_7 - a_2) \eta A^a_\mu \partial_\mu c^a \Big).\end{aligned} \tag{A28}$$

3. Stability and renormalization constants

Having determined the most general local invariant counterterm Σ^c compatible with all Ward identities, it remains to check that the starting action Σ is stable, i.e. that Σ^c can be reabsorbed through the renormalization of the parameters, fields and sources of Σ. According to expression (A28), Σ^c contains four arbitrary parameters a_0, a_2, a_7, a_8, which correspond in fact to a multiplicative renormalization of the gauge coupling constant g, the parameters ζ, and of the fields $\phi = (A^a_\mu, c^a, \overline{c}^a, b^a, \varphi^a_i, \omega^a_i, \overline{\varphi}^a_i, \overline{\omega}^a_i)$ and sources $\Phi = (K^{a\mu}, L^a, M^{ai}_\mu, N^{ai}_\mu, V^{ai}_\mu, U^{ai}_\mu, \tau, \eta)$, according to

$$\Sigma(g, \zeta, \phi, \Phi) + \eta \Sigma^c = \Sigma(g_o, \zeta_o, \phi_o, \Phi_o) + O(\eta^2), \tag{A29}$$

with

$$g_o = Z_g g, \qquad \zeta_o = Z_\zeta \zeta, \tag{A30}$$

and

$$\phi_o = Z^{1/2}_\phi \phi, \qquad \Phi_o = Z_\Phi \Phi. \tag{A31}$$

The coefficients a_0, a_2 are easily seen to be related to the renormalization of the gauge coupling constant g and of the gauge field A^a_μ,

$$Z_g = \Big(1 + \eta \frac{a_0}{2}\Big), \qquad Z^{1/2}_A = \Big(1 + \eta\Big(a_2 - \frac{a_0}{2}\Big)\Big). \tag{A32}$$

From expression (A28) it follows that the Faddeev-Popov ghosts $(c^a, \overline{c}^a)$ and the i-valued fields $(\varphi^a_i, \omega^a_i, \overline{\varphi}^a_i, \overline{\omega}^a_i)$ have a common renormalization constant, given by

$$\begin{aligned}Z_c = Z_{\overline{c}} = Z_\varphi = Z_{\overline{\varphi}} = Z_\omega = Z_{\overline{\omega}} = (1 - \eta a_2) \\ = Z^{-1}_g Z^{-1/2}_A.\end{aligned} \tag{A33}$$

Equation (A33) expresses a well-known renormalization property of the Faddeev-Popov ghosts $(c^a, \overline{c}^a)$ in the Landau gauge, stemming from the transversality of the gauge propagator and from the factorization of the ghost momentum in the ghost-antighost-gluon vertex. We see therefore that, in the present case, this property holds for the i-valued fields $(\varphi^a_i, \omega^a_i, \overline{\varphi}^a_i, \overline{\omega}^a_i)$ as well. Similarly to the ghost and the i-valued fields, the renormalization of the sources $(M^{ai}_\mu, N^{ai}_\mu, V^{ai}_\mu, U^{ai}_\mu)$ is also determined by the renormalization constants Z_g and $Z^{1/2}_A$, being given by

$$Z_M = Z_N = Z_V = Z_U = Z^{-1/2}_g Z^{-1/4}_A. \tag{A34}$$

It is worth noticing here that Eq. (A34) ensures that the counterterm $a_2(V^{ai}_\mu M^{ai}_\mu - U^{ai}_\mu N^{ai}_\mu)$ can be automatically reabsorbed by the term $(-M^{ai}_\mu V^{ai}_\mu + U^{ai}_\mu N^{ai}_\mu)$ in the expression (3.5) without the need of introducing new free parameters. Indeed,

$$\begin{aligned}-M_o V_o &= -MVZ^2_M = -MVZ^{-1}_g Z^{-1/2}_A \\ &= -MV + \varepsilon a_2 MV.\end{aligned} \tag{A35}$$

Concerning now the parameters a_7, a_8, they are easily seen to correspond to a multiplicative renormalization of the local source τ and of the parameter ζ, according to

$$\begin{aligned}\tau_o &= Z_\tau \tau, \qquad Z_\tau = 1 + \eta(a_7 - 2a_2 + a_0), \\ \zeta_o &= Z_\zeta \zeta, \qquad Z_\zeta = 1 + \eta(-a_8 - 2a_7 + 4a_2 - 2a_0).\end{aligned} \tag{A36}$$

Moreover, we would like to underline that there exists even an extra relation, namely

$$Z_\tau = Z_g Z^{-1/2}_A. \tag{A37}$$

It can be proven by introducing the operator A^2_μ through a more sophisticated set of local sources, like it was done in [11]. We will not repeat that analysis here, we only mention that a key ingredient in the proof of relation ((A37)) was the presence of the ghost Ward identity, and since the Zwanziger action possesses that identity, Eq. (A5), one can proceed along the lines of [11]. Thus, there are in

fact only three independent renormalization factors present.

APPENDIX B

In this Appendix, we give the detailed analysis of the procedure used to optimize the renormalization scheme and scale dependence, which was summarized in Sec. V.

1. Preliminaries

Before coming to the actual computations, let us first discuss some results which will turn out to be useful.Consider again the action S of Eq. (3.8). Because of the rich symmetry structure of the model, encoded in the Ward identities (A1)–(A14), and due to the extra relation (A37), only three renormalization factors remain to be fixed, namely Z_g, Z_A and Z_ζ. Apparently, this means that we would need three renormalization conditions in order to fix a particular renormalization scheme. However, taking a look at the bare action associated with expression Eq. (3.8), we would find the following relations

$$\zeta_o = Z_\zeta \zeta, \qquad \zeta_o \tau_o^2 = \overline{\mu}^{-\varepsilon} Z_\zeta \zeta \tau^2, \qquad \tau_o = Z_\tau \tau, \quad \text{(B1)}$$

from which it follows that

$$Z_\zeta \zeta = \overline{\mu}^{\varepsilon} \zeta_o Z_\tau^2. \quad \text{(B2)}$$

Since the bare quantity ζ_o is renormalization scheme and scale independent and since ζ always appears in the combination $Z_\zeta \zeta$ in the action, it follows that only Z_g and Z_A are relevant for the effective action, because Z_τ can be expressed in terms of these two factors. Consequently, we would only need two renormalization conditions to fix the scheme. Obviously, we can equally well choose to make use of, for example, Z_g and Z_τ as the two independent renormalization factors, corresponding to coupling constant and mass renormalization. We will change from the $\overline{\text{MS}}$ to another massless renormalization scheme by means of the following transformations[8]

$$\begin{aligned} \overline{g}^2 &= g^2(1 + b_0 g^2 + b_1 g^4 + \cdots), \\ \overline{\lambda} &= \lambda(1 + c_0 g^2 + c_1 g^4 + \cdots), \\ \overline{m}^2 &= m^2(1 + d_0 g^2 + d_1 g^4 + \cdots), \end{aligned} \quad \text{(B3)}$$

where the parameters b_i, c_i and d_i label the new scheme. However, we should keep in mind that the renormalization of the Gribov parameter λ is not independent of that of g^2 and m^2. Eliminating $\gamma_A(g^2)$ between Eqs. (4.19) and (4.20), yields

$$\gamma_\lambda(g^2) = \frac{1}{4}\left(\frac{\beta(g^2)}{g^2} - \gamma_{m^2}(g^2)\right). \quad \text{(B4)}$$

This relation, valid to all orders of perturbation theory, implies the existence of relationships between the coefficients b_i, c_i and d_i. For further use, we shall explicitly construct the relation between b_0, c_0 and d_0. Let us adopt as parametrization of $\beta(g^2)$, $\gamma_{m^2}(g^2)$ and $\gamma_\lambda(g^2)$

$$\begin{aligned} \beta(g^2) &= -2(\beta_0 g^4 + \beta_1 g^6 + \cdots), \\ \gamma_{m^2}(g^2) &= \gamma_0 g^2 + \gamma_1 g^4 + \cdots, \\ \gamma_\lambda(g^2) &= \lambda_0 g^2 + \lambda_1 g^4 + \cdots, \end{aligned} \quad \text{(B5)}$$

and an analogous one in the case of the $\overline{\text{MS}}$ scheme. Then, one computes

$$\begin{aligned} \overline{\mu}\frac{\partial \overline{\lambda}}{\partial \overline{\mu}} &= \overline{\mu}\frac{\partial}{\partial \overline{\mu}}[\lambda(1 + c_0 g^2 + \cdots)] = \cdots \\ &= \lambda(\lambda_0 g^2 + (\lambda_1 + c_0\lambda_0 - 2\beta_0 c_0) g^4 + \cdots), \end{aligned} \quad \text{(B6)}$$

which can be expressed in terms of γ_i and β_i by exploiting the relation (B4). We find

$$\overline{\mu}\frac{\partial \overline{\lambda}}{\partial \overline{\mu}} = \lambda\left[\frac{-2\beta_0 - \gamma_0}{4} g^2 + \left(\frac{-2\beta_1 - \gamma_1}{4} + c_0\frac{-2\beta_0 - \gamma_0}{4} - 2\beta_0 c_0\right) g^4 + \cdots\right]. \quad \text{(B7)}$$

We can also calculate $\overline{\mu}(d\overline{\lambda}/d\overline{\mu})$ by first exploiting the relation (B4), obtaining

$$\begin{aligned} \overline{\mu}\frac{\partial \overline{\lambda}}{\partial \overline{\mu}} &= \frac{1}{4}[(-2\beta_0 - \gamma_0)\overline{g}^2 + (-2\beta_1 - \overline{\gamma}_1)\overline{g}^4 + \cdots] \\ &\quad \times [\lambda(1 + c_0 g^2 + \cdots)] \\ &= \cdots \\ &= \frac{1}{4}[(-2\beta_0 - \gamma_0) g^2 + (c_0(-2\beta_0 - \gamma_0) \\ &\quad - 2\beta_1 - \gamma_1 - 2\beta_0(-d_0 + b_0)) g^4 + \cdots]. \end{aligned} \quad \text{(B8)}$$

In the previous expression, we had to express $\overline{\gamma}_1$ in terms of γ_1; a task accomplished by using the relation

$$\overline{\gamma}_1 = \gamma_1 - 2\beta_0 d_0 - \gamma_0 b_0, \quad \text{(B9)}$$

which can be obtained along the same lines of the previous calculations. It should also be noted that γ_0, β_0 and β_1 are renormalization scheme independent quantities. Thus, the identification of Eqs. (B7) and (B8) gives the desired relation, given by

$$c_0 = \frac{1}{4}(b_0 - d_0). \quad \text{(B10)}$$

We now perform the transformations (B3) on the action (4.14), which was calculated in the $\overline{\text{MS}}$ scheme, to obtain it in a general scheme.

[8]Barred quantities refer to the $\overline{\text{MS}}$ scheme.

$$\Gamma = -\frac{(N^2-1)\lambda^4}{2g^2N}(1+4c_0g^2-b_0g^2)+\frac{\zeta_0 m^4}{2g^2}\left(1-\frac{\zeta_1}{\zeta_0}g^2+2d_0g^2-b_0g^2\right)+\frac{3(N^2-1)}{256\pi^2}\Bigg[(m^2+\sqrt{m^4-\lambda^4})^2 \times\left(\ln\frac{m^2+\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2}-\frac{5}{6}\right)+(m^2-\sqrt{m^4-\lambda^4})^2\left(\ln\frac{m^2-\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2}-\frac{5}{6}\right)\Bigg], \tag{B11}$$

while the gap equations now read

$$\frac{\partial\Gamma}{\partial\lambda} = -\frac{2(N^2-1)}{g^2N}\lambda^3(1+4c_0g^2-b_0g^2)+\frac{3(N^2-1)\lambda^3}{256\pi^2}\Bigg[\frac{8}{3}-4\frac{(m^2+\sqrt{m^4-\lambda^4})}{\sqrt{m^4-\lambda^4}}\ln\frac{m^2+\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2} + 4\frac{(m^2-\sqrt{m^4-\lambda^4})}{\sqrt{m^4-\lambda^4}}\ln\frac{m^2-\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2}\Bigg], \tag{B12}$$

$$\frac{\partial\Gamma}{\partial m^2} = \frac{\zeta_0 m^2}{g^2}\left(1-\frac{\zeta_1}{\zeta_0}g^2+2d_0g^2-b_0g^2\right)+\frac{3(N^2-1)}{256\pi^2}\Bigg[2(m^2+\sqrt{m^4-\lambda^4})\left(1+\frac{m^2}{\sqrt{m^4-\lambda^4}}\right)\ln\frac{m^2+\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2} + 2(m^2-\sqrt{m^4-\lambda^4})\left(1-\frac{m^2}{\sqrt{m^4-\lambda^4}}\right)\ln\frac{m^2-\sqrt{m^4-\lambda^4}}{2\overline{\mu}^2}-\frac{8}{3}m^2\Bigg].$$

We mention that, in the case in which $m^2 \geq 0$, similar algebraic manipulations as those leading to the condition (4.37), give a more general equation

$$\frac{68}{39}\left(\frac{16\pi^2}{g^2N}\right)+\frac{161}{39}+\frac{16\pi^2}{N}\left(\frac{32}{3}c_0-\frac{68}{39}b_0-\frac{24}{13}d_0\right) = \frac{t}{\sqrt{1-t}}\ln\frac{t}{(1+\sqrt{1-t})^2}, \tag{B13}$$

or, using the relation (B10),

$$\frac{68}{39}\left(\frac{16\pi^2}{g^2N}\right)+\frac{161}{39}+\frac{16\pi^2}{N}\left(\frac{12}{13}b_0-\frac{176}{39}d_0\right) = \frac{t}{\sqrt{1-t}}\ln\frac{t}{(1+\sqrt{1-t})^2}. \tag{B14}$$

From this expression, it is apparent that a sensible solution with $m^2 > 0$ might exist, depending on the values of the renormalization parameters d_0 ($\sim$ mass renormalization) and b_0 ($\sim$ coupling constant renormalization).

Frequently used are the so-called physical renormalization schemes whereby, loosely speaking, one demands that the quantum corrected quantities reduce to the tree level values at a certain scale $\overline{\mu}$. However, it turns out that such an approach is not particularly useful to implement in the current case due to the presence of the several scales. Therefore, the question arises how one can make a somewhat motivated choice for the arbitrary parameters, labeling a certain renormalization scheme. In the next subsection we shall discuss a way to reduce the freedom in the choice of the renormalization parameters. The method relies on the possibility of performing an optimization of the renormalization scheme dependence, as illustrated in [69,70].

2. Optimization of the renormalization scheme

Consider a quantity ϱ that runs according to

$$\overline{\mu}\frac{d\varrho}{d\overline{\mu}} = \gamma_\varrho(g^2)\varrho, \tag{B15}$$

where

$$\gamma_\varrho(g^2) = \gamma_{\varrho,0}g^2+\gamma_{\varrho,1}g^4+\cdots. \tag{B16}$$

To ϱ, we can associate a quantity $\hat{\varrho}$ that does not depend on the choice of the renormalization scheme and which is scale independent. It is defined as

$$\hat{\varrho} = \mathcal{F}_\varrho(g^2)\varrho, \tag{B17}$$

whereby

$$\overline{\mu}\frac{d}{d\overline{\mu}}\mathcal{F}_\varrho(g^2) = -\gamma_\varrho(g^2)\mathcal{F}_\varrho(g^2). \tag{B18}$$

It is apparent that $\hat{\varrho}$ will not depend on the scale $\overline{\mu}$. It can also be checked [69,70] that $\hat{\varrho}$ is left unmodified by a change of the renormalization scheme, implemented through transformations analogous to those of Eqs. (B3). The Eq. (B18) can be solved in a series expansion in g^2 by noticing that

$$\overline{\mu}\frac{d}{d\overline{\mu}}\mathcal{F}_\varrho(g^2) \equiv \beta(g^2)\frac{d}{dg^2}\mathcal{F}_\varrho(g^2). \tag{B19}$$

Then, the above differential equation can be solved in a series expansion in g^2, more precisely by

$$\mathcal{F}_{\varrho}(g^2) = (g^2)^{\gamma_{\varrho,0}/2\beta_0}\left(1 + \frac{1}{2}\left(\frac{\gamma_{\varrho,1}}{\beta_0} - \frac{\beta_1\gamma_{\varrho,0}}{\beta_0^2}\right)g^2 + \cdots\right). \tag{B20}$$

Consider once more the $\overline{\text{MS}}$ action Γ given in Eq. (4.14). We shall now replace the $\overline{\text{MS}}$ variables $\overline{m}^2$ and $\overline{\lambda}$ by their renormalization scheme and scale independent counterparts $\hat{m}^2$ and $\hat{\lambda}$, which are obtained as before. By inverting Eq. (B20), one has

$$\overline{m}^2 = (\overline{g}^2)^{-\gamma_0/2\beta_0}\left(1 - \frac{1}{2}\left(\frac{\overline{\gamma}_1}{\beta_0} - \frac{\beta_1\gamma_0}{\beta_0^2}\right)\overline{g}^2 + \cdots\right)\hat{m}^2, \tag{B21}$$

$$\overline{\lambda} = (\overline{g}^2)^{-\lambda_0/2\beta_0}\left(1 - \frac{1}{2}\left(\frac{\overline{\lambda}_1}{\beta_0} - \frac{\beta_1\lambda_0}{\beta_0^2}\right)\overline{g}^2 + \cdots\right)\hat{\lambda}. \tag{B22}$$

Moreover, introducing the notations

$$a = -\frac{\gamma_0}{2\beta_0}, \qquad b = -\frac{\lambda_0}{\beta_0}, \tag{B23}$$

$$A = -\left(\frac{\overline{\gamma}_1}{\beta_0} - \frac{\beta_1\gamma_0}{\beta_0^2}\right), \qquad B = -2\left(\frac{\overline{\lambda}_1}{\beta_0} - \frac{\beta_1\lambda_0}{\beta_0^2}\right), \tag{B24}$$

the one-loop action is rewritten as

$$\begin{aligned}\Gamma = &-\frac{(N^2-1)}{2N}(\overline{g}^2)^{2b}\hat{\lambda}^4\left(\frac{1}{\overline{g}^2} + B\right) + \frac{\zeta_0}{2}\hat{m}^4(\overline{g})^{2a}\left(\frac{1}{\overline{g}^2} + A - \frac{\zeta_1}{\zeta_0}\right) + \frac{3(N^2-1)}{256\pi^2}\Bigg[(\hat{m}^2(\overline{g}^2)^a + \sqrt{\hat{m}^4(\overline{g}^2)^{2a} - \hat{\lambda}^4(\overline{g}^2)^{2b}})^2 \\ &\times\left(\ln\frac{\hat{m}^2(\overline{g}^2)^a + \sqrt{\hat{m}^4(\overline{g}^2)^{2a} - \hat{\lambda}^4(\overline{g}^2)^{2b}}}{2\overline{\mu}^2} - \frac{5}{6}\right) + (\hat{m}^2(\overline{g}^2)^a - \sqrt{\hat{m}^4(\overline{g}^2)^{2a} - \hat{\lambda}^4(\overline{g}^2)^{2b}})^2 \\ &\times\left(\ln\frac{\hat{m}^2(\overline{g}^2)^a - \sqrt{\hat{m}^4(\overline{g}^2)^{2a} - \hat{\lambda}^4(\overline{g}^2)^{2b}}}{2\overline{\mu}^2} - \frac{5}{6}\right)\Bigg].\end{aligned} \tag{B25}$$

The action (B25) is still written in terms of the $\overline{\text{MS}}$ coupling $\overline{g}^2$. Performing the first transformation of (B3), Γ can be re-expressed as

$$\begin{aligned}\Gamma = &-\frac{(N^2-1)}{2N}(g^2)^{2b}\hat{\lambda}^4\left(\frac{1}{g^2} + B - b_0 + 2bb_0\right) + \frac{\zeta_0}{2}\hat{m}^4(g^2)^{2a}\left(\frac{1}{g^2} + A - b_0 + 2ab_0 - \frac{\zeta_1}{\zeta_0}\right) \\ &+ \frac{3(N^2-1)}{256\pi^2}\Bigg[(\hat{m}^2(g^2)^a + \sqrt{\hat{m}^4(g^2)^{2a} - \hat{\lambda}^4(g^2)^{2b}})^2\left(\ln\frac{\hat{m}^2(g^2)^a + \sqrt{\hat{m}^4(g^2)^{2a} - \hat{\lambda}^4(g^2)^{2b}}}{2\overline{\mu}^2} - \frac{5}{6}\right) \\ &+ (\hat{m}^2(g^2)^a - \sqrt{\hat{m}^4(g^2)^{2a} - \hat{\lambda}^4(g^2)^{2b}})^2\left(\ln\frac{\hat{m}^2(g^2)^a - \sqrt{\hat{m}^4(g^2)^{2a} - \hat{\lambda}^4(g^2)^{2b}}}{2\overline{\mu}^2} - \frac{5}{6}\right)\Bigg].\end{aligned} \tag{B26}$$

So far, we have constructed an action which is written in terms of renormalization scale and scheme independent variables $\hat{\lambda}$ and $\hat{m}^2$ and the coupling constant $g^2(\overline{\mu})$. This is a certain improvement, since we are not faced anymore with a choice of the parameters d_i, related to the renormalization of the Gribov and mass parameter. The remaining freedom in the choice of the renormalization scheme resides in the coupling constant, labeled by the parameters $b_0, b_1, \ldots$, and in the scale $\overline{\mu}$. Of course, the higher order coefficients $b_i, i = 1, \ldots$ do not show up here, since we have restricted ourselves to the one-loop level. Nevertheless, we will perform one more step, since the dependence on the coupling constant renormalization can be reduced to solely b_0, by expanding the perturbative series in inverse powers of

$$x \equiv \beta_0 \ln\frac{\overline{\mu}^2}{\Lambda^2}, \tag{B27}$$

rather than in terms of g^2. For another illustration of this, see e.g. [69,70]. The coupling constant g^2 can be replaced by x since g^2 is explicitly determined by

$$g^2 = \frac{1}{x}\left(1 - \frac{\beta_1}{\beta_0}\frac{\ln\frac{x}{\beta_0}}{x} + \cdots\right). \tag{B28}$$

In [71], the relation between the scale parameter Λ, corresponding to a certain coupling constant renormalization, and that of the $\overline{\text{MS}}$ scheme, $\Lambda_{\overline{\text{MS}}}$, was found to be

$$\Lambda = e^{-b_0/2\beta_0}\Lambda_{\overline{\text{MS}}}. \tag{B29}$$

One finally gets the expression (5.1). We notice that this alternative expansion is correct up to order $(\frac{1}{x})^0$.

In principle, we can solve the two Eqs. (5.2) and (5.3) for the two quantities $\hat{m}_*$ and $\hat{\lambda}_*$, which will be functions of the two remaining parameters $\overline{\mu}$ and b_0. However, by construction, we know that $\hat{m}$ as well as $\hat{\lambda}$ should be indepen-

dent of the renormalization scale and scheme order by order. This gives us an interesting way to fix these parameters by demanding that the solutions $\hat{m}_*(\overline{\mu}, b_0)$ and $\hat{\lambda}_*(\overline{\mu}, b_0)$ depend minimally on b_0 and $\overline{\mu}$. Since this would give a quite complicated set of equations to solve, we can make life somewhat easier by reasonably choosing the scale[9] $\overline{\mu}$ in the gap equations (5.2) and (5.3). In analogy to the choice for $\overline{\mu}^2$ done in the previous Eq. (4.38), we shall now set

$$\overline{\mu}^2 = \left| \frac{\hat{m}^2 x^{-a} + \sqrt{\hat{m}^4 x^{-2a} - \hat{\lambda}^4 x^{-2b}}}{2} \right|, \tag{B30}$$

In order to proceed, we still have two quantities at our disposal to fix the remaining parameter b_0. In fact, we can also take the vacuum energy E_{vac} in consideration since, being a physical quantity, it should depend minimally on the renormalization scheme and scale. Therefore, we could determine the value for b_0 by demanding that

$$\Upsilon(b_0) \equiv \left| \frac{\partial \hat{\lambda}_*^4}{\partial b_0} \right| + \left| \frac{\partial \hat{m}_*^4}{\partial b_0} \right| + \left| \frac{\partial E_{\text{vac}}}{\partial b_0} \right|, \tag{B31}$$

is minimal with respect to the parameter b_0. This seems to be a reasonable candidate. When its dependence on b_0 is small, then the dependence of $\hat{m}$, $\hat{\lambda}$ and E_{vac} on b_0 is necessarily small too. The ideal situation would be that Υ is zero for a certain b_0. If no such an ideal b_0 would exist, we weaken the condition by requiring that Υ is as small as possible. The condition (B31) to fix b_0 can be considered as some kind of *principle of minimal sensitivity* à la Stevenson [72]. An alternative that is sometimes used is a *fastest apparent convergence criterion*, where it is demanded that the quantum corrections are as small as possible compared to the tree level value. For example, if we denote by $\Gamma^{[0]}$ the action to order $(\frac{1}{x})^{-1}$ and by $\Gamma^{[1]}$ to order $(\frac{1}{x})^0$, we could demand that

$$\left| \frac{\Gamma^{[1]} - \Gamma^{[0]}}{\Gamma^{[0]}} \right| \tag{B32}$$

is as small as possible when the parameters fulfill the gap equation describing the vacuum of the theory.

Before continuing with explicit calculations, let us just remark here that the other logarithm, namely $\ln[(\hat{m}^2 x^{-a} - \sqrt{\hat{m}^4 x^{-2a} - \hat{\lambda}^4 x^{-2b}})/(2\overline{\mu}^2)]$, could become large for a small argument, thus when $\hat{\lambda}^4 x^{-2b}$ would be small compared to $\hat{m}^4 x^{-2a}$. However, it is harmless since it appears in the form of $u \ln u$, while we know that $u \ln u|_{u \approx 0} \approx 0$.

[9] This can be motivated thanks to the scale independence of the $\hat{}$ quantities.

3. Numerical results

Let us first give some numerical factors we need. From e.g. [65], we infer that

$$\beta_1 = \frac{34}{3}\left(\frac{N}{16\pi^2}\right)^2, \qquad \gamma_0 = -\frac{3}{2}\frac{N}{16\pi^2}, \qquad \gamma_1 = -\frac{95}{24}\left(\frac{N}{16\pi^2}\right)^2, \tag{B33}$$

and hence, from the relation (B4),

$$\lambda_0 = -\frac{35}{24}\frac{N}{16\pi^2}, \qquad \lambda_1 = -\frac{449}{96}\left(\frac{N}{16\pi^2}\right)^2. \tag{B34}$$

This means that, for any N, the quantities a and b in Eq. (B23) are found to be

$$a = \frac{9}{44}, \qquad b = \frac{35}{88}. \tag{B35}$$

It is instructive to consider once more the original Gribov-Zwanziger model by setting $\hat{m} \equiv 0$ and by solving the gap equation (5.2). If $\hat{\lambda}_*$ is a solution of this equation, then it is not difficult to show that the corresponding vacuum energy is given by

$$E_{\text{vac}} = \frac{3(N^2-1)}{64\pi^2}\frac{\hat{\lambda}_*}{4}, \tag{B36}$$

for any choice of $\overline{\mu}^2$. Thus, also with the improved perturbative expansion, the vacuum energy of the original Gribov-Zwanziger is always nonnegative at the lowest order.

Let us return to the model we were investigating. We solved the gap equations stemming from (5.2) and (5.3) numerically.

Let us first search for a possible solution of the gap equation in the region of space determined by $\hat{m}^4 x^{-2a} \geq \hat{\lambda}^4 x^{-2b}$. Taking a look at the action (5.1), it might be clear that the gap equations derived from it will be coupled and hence quite complicated to solve numerically. From the calculational point of view, it is useful to introduce new variables, defined by

$$\omega_1 = \frac{\hat{m}^2 x^{-a} + \sqrt{\hat{m}^4 x^{-2a} - \hat{\lambda}^4 x^{-2b}}}{2}, \tag{B37}$$

$$\omega_2 = \frac{\hat{m}^2 x^{-a} - \sqrt{\hat{m}^4 x^{-2a} - \hat{\lambda}^4 x^{-2b}}}{2}, \tag{B38}$$

with the inverse transformation

$$\hat{m}^2 x^{-a} = \omega_1 + \omega_2, \qquad \hat{\lambda}^4 x^{-2b} = 4\omega_1\omega_2. \tag{B39}$$

This defines a mapping from the space $\hat{m}^4 x^{-2a} \geq \hat{\lambda}^4 x^{-2b} > 0$ to $\omega_1 \geq \omega_2 > 0$. One checks that the gap equations (5.2) and (5.3) are equivalent to

$$\left(\frac{\omega_1}{\omega_1-\omega_2}\frac{\partial}{\partial\omega_1}-\frac{\omega_2}{\omega_1-\omega_2}\frac{\partial}{\partial\omega_2}\right)\Gamma(\omega_1,\omega_2)=0, \quad \text{(B40)}$$

$$\left(\frac{1}{\omega_1-\omega_2}\frac{\partial}{\partial\omega_1}-\frac{1}{\omega_1-\omega_2}\frac{\partial}{\partial\omega_2}\right)\Gamma(\omega_1,\omega_2)=0. \quad \text{(B41)}$$

We notice that the case in which ω_1 and ω_2 would become equal, i.e. $\hat{m}^4x^{-2a}=\hat{\lambda}^4x^{-2b}$, should be treated with some extra care. Let us therefore first assume that $\omega_1>\omega_2$. Then the two Eqs. (B40) and (B41) can be recombined to

$$\frac{\partial}{\partial\omega_1}\Gamma=0, \quad \text{(B42)}$$

$$\frac{\partial}{\partial\omega_2}\Gamma=0. \quad \text{(B43)}$$

The action $\Gamma(\omega_1,\omega_2)$ is explicitly given by

$$\Gamma=-2\frac{(N^2-1)}{N}\mho_1\omega_1\omega_2+\frac{\zeta_0}{2}\mho_2(\omega_1+\omega_2)^2+\frac{3(N^2-1)}{64\pi^2}\left[\omega_1^2\left(\ln\frac{\omega_1}{\overline{\mu}^2}-\frac{5}{6}\right)+\omega_2^2\left(\ln\frac{\omega_2}{\overline{\mu}^2}-\frac{5}{6}\right)\right]. \quad \text{(B44)}$$

where

$$\mho_1=x+B+(1-2b)\left(\frac{\beta_1}{\beta_0}\ln\frac{x}{\beta_0}-b_0\right), \quad \text{(B45)}$$

$$\mho_2=x+A-\frac{\zeta_1}{\zeta_0}+(1-2a)\left(\frac{\beta_1}{\beta_0}\ln\frac{x}{\beta_0}-b_0\right). \quad \text{(B46)}$$

It is not difficult to work out the gap equations (B42) and (B43), being given by

$$-2\frac{N^2-1}{N}\mho_1\omega_2+\zeta_0\mho_2(\omega_1+\omega_2)+\frac{3(N^2-1)\omega_1}{32\pi^2}\times\left(-\frac{1}{3}+\ln\frac{\omega_1}{\overline{\mu}^2}\right)=0, \quad \text{(B47)}$$

$$-2\frac{N^2-1}{N}\mho_1\omega_1+\zeta_0\mho_2(\omega_1+\omega_2)+\frac{3(N^2-1)\omega_2}{32\pi^2}\times\left(-\frac{1}{3}+\ln\frac{\omega_2}{\overline{\mu}^2}\right)=0. \quad \text{(B48)}$$

From the explicit expression of the gap equations and of the action itself in terms of ω_1 and ω_2, the advantages of using these variables should be obvious, since we can decouple the two gap equations. Explicitly, since $\overline{\mu}^2=\omega_1$, one finds from Eq. (B47),

$$\omega_2=\frac{\frac{N^2-1}{32\pi^2}-\zeta_0\mho_2}{-2\frac{N^2-1}{N}\mho_1+\zeta_0\mho_2}\omega_1, \quad \text{(B49)}$$

which can be substituted in the second gap equation (B48), yielding an equation for ω_1 which does not contain ω_2 anymore. The nominator of Eq. (B49) is different from zero, since filling in the numbers gives

$$-2\frac{N^2-1}{N}\mho_1+\zeta_0\mho_2=\frac{N^2-1}{4576}\left(-\frac{975}{\pi^2}-\frac{5984}{N}x\right)\neq 0. \quad \text{(B50)}$$

where we kept in mind that for a meaningful result, $x\sim\frac{1}{g^2}$, should be positive.

A numerical investigation of the gap equation (B48) using Eq. (B49) revealed that there are no zeros. We conclude that there are no solutions with $\hat{m}^4x^{-2a}>\hat{\lambda}^4x^{-2b}$.

Next, let us find out if a possible solution with $\hat{m}^4x^{-2a}=\hat{\lambda}^4x^{-2b}$ or $\omega_1=\omega_2$ might exist. We explicitly evaluate the gap equations (B40) and (B41), where now $\overline{\mu}^2=\omega_1$,

$$\zeta_0\mho_2(\omega_1+\omega_2)-\frac{N^2-1}{32\pi^2}(\omega_1+\omega_2)-\frac{3(N^2-1)}{32\pi^2}\frac{\omega_2^2}{\omega_1-\omega_2}\ln\frac{\omega_2}{\omega_1}=0, \quad \text{(B51)}$$

$$2\frac{N^2-1}{N}\mho_1-\frac{N^2-1}{32\pi^2}-\frac{3(N^2-1)}{32\pi^2}\frac{\omega_2}{\omega_1-\omega_2}\ln\frac{\omega_2}{\omega_1}=0. \quad \text{(B52)}$$

From the foregoing expressions, we infer that the limit $\omega_1\to\omega_2$ exists, giving rise to

$$2\zeta_0\mho_2-\frac{2(N^2-1)}{32\pi^2}+\frac{3(N^2-1)}{32\pi^2}=0, \quad \text{(B53)}$$

$$2\frac{N^2-1}{N}\mho_1-\frac{N^2-1}{32\pi^2}+\frac{3(N^2-1)}{32\pi^2}=0. \quad \text{(B54)}$$

This means that we have two equations to solve for the single quantity ω_1, which is present in $\mho_1$ and $\mho_2$ through the quantity x. It would be an extreme coincidence if these two different equations, which can be rewritten as

$$\frac{18}{13}\mho_2=-\frac{N}{32\pi^2}, \quad \text{(B55)}$$

$$\mho_1=-\frac{N}{32\pi^2}. \quad \text{(B56)}$$

possess a common solution. That this is not the case can be inferred from the numerical solutions of both Eqs. (B55) and (B56), shown in Fig. 5. As a final step, we should investigate if there is a solution in the region $\hat{m}^4x^{-2a}<\hat{\lambda}^4x^{-2b}$. We can still define the coordinates ω_1 and ω_2 by

$$\omega_1=\frac{\hat{m}^2x^{-a}+i\sqrt{-\hat{m}^4x^{-2a}+\hat{\lambda}^4x^{-2b}}}{2}, \quad \text{(B57)}$$

$$\omega_2=\frac{\hat{m}^2x^{-a}-i\sqrt{-\hat{m}^4x^{-2a}+\hat{\lambda}^4x^{-2b}}}{2}. \quad \text{(B58)}$$

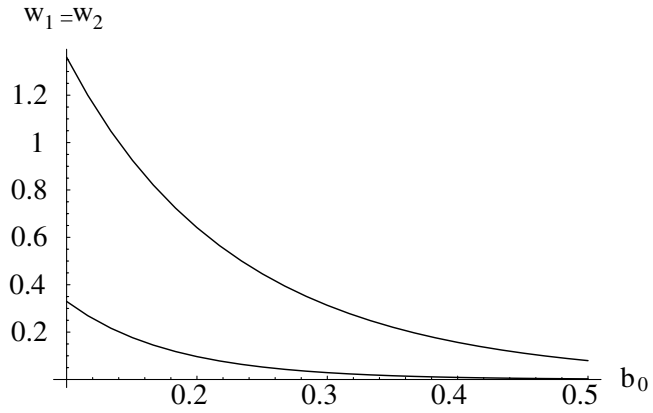

FIG. 5. The solution $\omega_1 = \omega_2$ as a function of b_0 of Eq. (B55), top curve, and Eq. (B56), bottom curve, in units $\Lambda_{\overline{\rm MS}} = 1$. Clearly, these two curves do no coincide.

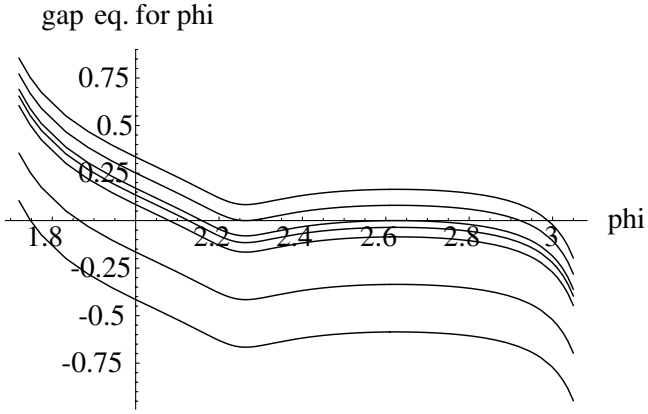

FIG. 6. The gap equation (B67) with $N = 3$ plotted in function of ϕ for the values $b_0 = 0.25, 0, -0.25, -0.3, -0.335\,64\ldots, -0.415\,94\ldots, -0.5$ (from bottom to top).

In this case, ω_1 and ω_2 are complex conjugate. Henceforth, it would be more appropriate to use the modulus R and the argument ϕ, $\phi \in]-\pi, \pi]$, defined by

$$\mathrm{Re}\, e^{i\phi} = \omega_1, \tag{B59}$$

$$\mathrm{Re}\, e^{-i\phi} = \omega_2, \tag{B60}$$

If the argument ϕ is so that $|\phi| > \frac{\pi}{2}$, then $\hat{m}^2 x^{-a} < 0$. As a consequence, the estimate for $\langle A_\mu^2 \rangle$ will be positive. Most of the foregoing analysis can be repeated. The action (B44) is rewritten in terms of R and ϕ by

$$\Gamma = -2\frac{(N^2-1)}{N}\mho_1 R^2 + 2\zeta_0 \mho_2 R^2 \cos^2\phi + \frac{3R^2(N^2-1)}{32\pi^2}\left[\cos(2\phi)\left(\ln\frac{R}{\overline{\mu}^2} - \frac{5}{6}\right) - \phi\sin(2\phi)\right]. \tag{B61}$$

The gap equations (B47) and (B48) reduce to

$$-2\frac{N^2-1}{N}\mho_1 \mathrm{Re}^{-i\phi} + \zeta_0 \mho_2 R(e^{i\phi} + e^{-i\phi}) + \frac{3(N^2-1)\mathrm{Re}^{i\phi}}{32\pi^2}\left(-\frac{1}{3} + i\phi\right) = 0, \tag{B62}$$

and its complex conjugate. With the parametrization (B59), we have $\overline{\mu}^2 = R$. We must solve the following two real equations[10] for ϕ and R.

$$-2\frac{N^2-1}{N}\mho_1 \cos\phi + 2\zeta_0 \mho_2 \cos\phi + \frac{3(N^2-1)}{32\pi^2}\times\left(-\frac{\cos\phi}{3} - \phi\sin\phi\right) = 0, \tag{B63}$$

$$2\frac{N^2-1}{N}\mho_1 \sin\phi + \frac{3(N^2-1)}{32\pi^2}\left(-\frac{\sin\phi}{3} + \phi\cos\phi\right) = 0. \tag{B64}$$

We can divide these equations[11] by $\cos\phi$ to obtain

$$-2\frac{N^2-1}{N}\mho_1 + 2\zeta_0 \mho_2 + \frac{3(N^2-1)}{32\pi^2}\times\left(-\frac{1}{3} - \phi\tan\phi\right) = 0, \tag{B65}$$

$$2\frac{N^2-1}{N}\mho_1 \tan\phi + \frac{3(N^2-1)}{32\pi^2}\left(-\frac{\tan\phi}{3} + \phi\right) = 0. \tag{B66}$$

These equations can also be decoupled. The most efficient way to proceed is to eliminate R between these two equations to obtain an equation for ϕ, as the range in we must search for a solution is limited for this angle. The equation for ϕ finally becomes

$$\frac{-90985N - 107712\pi^2 b_0 + 12N(484\phi\cot\phi + 1734\ln(\frac{-117(50+11\phi\csc\phi\sec\phi)}{8228}) - 1573\phi\tan\phi)}{107712\pi^2} = 0 \tag{B67}$$

[10]The R dependence is hidden in $\mho_1$ and $\mho_2$

[11]We may assume $\cos\phi \neq 0$, otherwise Eqs. (B63) and (B64) would give $\phi = 0$, which is inconsistent with $\cos\phi = 0$.

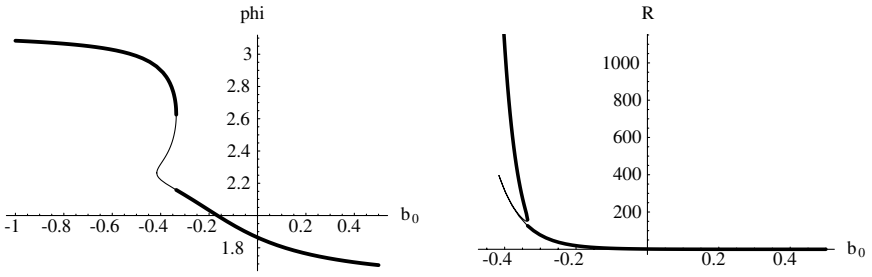

FIG. 7. The angle ϕ and scale R as a function of b_0, in units $\Lambda_{\overline{\rm MS}} = 1$.

while the value of R is obtained from

$$x \equiv \beta_0 \ln\frac{R}{\Lambda^2_{\overline{\rm MS}}} + b_0 = -\frac{1950 + 429\phi\csc\phi\sec\phi}{11968\pi^2}N. \tag{B68}$$

We shall concentrate on the case $N = 3$. Depending on the value of the parameter b_0, there is more than one solution possible. In Fig. 6, we have plotted the expression (B67) for several values of the parameter b_0, namely $b_0 = 0.25, 0, -0.25, -0.3, -0.335\,64\ldots, -0.415\,94\ldots, -0.5$. It is possible to obtain those values of b_0 where the number of solutions change. If we consider the plots of Fig. 6, it is apparent that for each b_0, the corresponding curve possesses two extremal values. The number of solution exactly changes at those values of b_0 where the curve becomes tangent to the ϕ axis. An explicit evaluation learns that his occurs at $b_0 = -0.415\,95\ldots$, where $\phi = 2.264\,07\ldots$ and at $b_0 = -0.335\,64\ldots$ where $\phi = 2.625\,45$. It is important to know these numbers to a high enough accuracy, to instruct the computer in which ϕ interval it can search for a solution. If the initial values are not chosen in an appropriate way, the iterations will jump between the different branches of solutions and there will be no convergence to any of them. There is a single solution ϕ if $b_0 > -0.33564...$ or $b_0 < -0.41595\ldots$. If $-0.41595\ldots < b_0 < -0.335\,64\ldots$, there are three solutions, while for $b_0 = -0.415\,95\ldots$ and $b_0 = -0.335\,64\ldots$ there are two solutions. In Fig. 7, we have displayed the solution for ϕ and R. To determine the solution ϕ which characterizes the vacuum, we should take that one which gives us the absolute minimum of the energy functional Γ, which was shown in Fig. 3. As a final remark, we would like to notice that the same decomposition as in Eq. (4.7) could also be useful for higher loop computations. The effective action Γ will remain symmetric under the exchange of ω_1 and ω_2 and equations like (B40)–(B43) shall remain valid. This should facilitate at least a bit the two-loop evaluation of the effective action and gap equations. Also, one does not need to evaluate any new anomalous dimension, since these are already known, either from previous calculations [3,12,25], or from exploiting relations like Eq. (B4).

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